Strong Artin

A visual atlas of attachment and lifting

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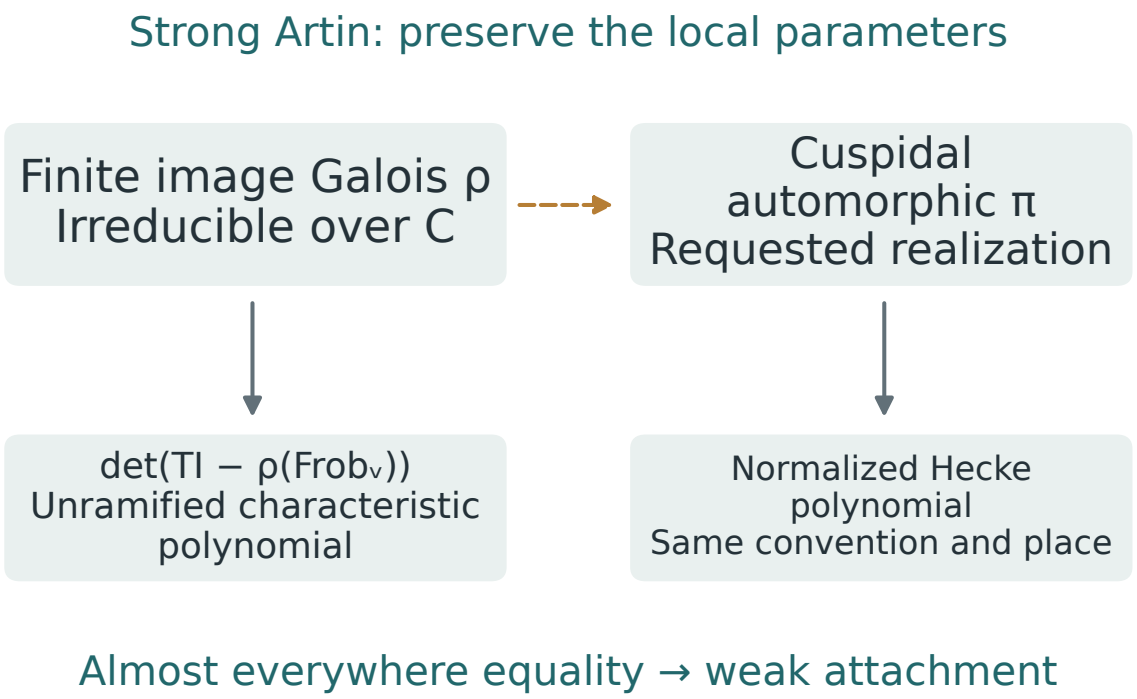
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See how cyclic labels, finite polynomial alphabets, cohomological defects, and exact local lifting fit together. Each example identifies its domain. The global attachment theorem remains an open gate.

WORKED EXAMPLES / EXACT COORDINATES / OPEN PROBLEMS

What Strong Artin asks us to construct

A global correspondence, with local parameters retained



READ THE FIGURE

Start with an irreducible finite image complex representation $\rho : G_F \to \mathrm{GL}_n(\mathbb{C})$. Strong Artin asks for a cuspidal automorphic representation with the corresponding local parameters. Degree one is supplied by class field theory. The trivial character's L function has its familiar pole.

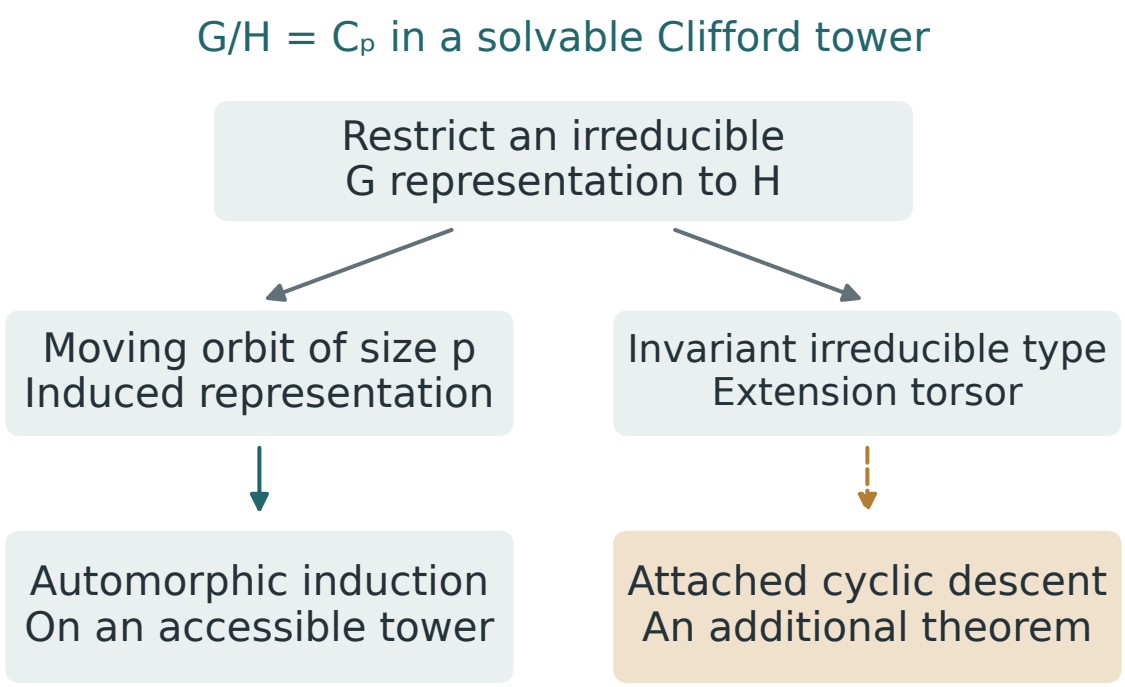
The unramified polynomial is one visible interface. Matching it almost everywhere is weak attachment. All remaining excluded finite places and the archimedean places still require their own comparisons for full attachment.

The dashed arrow is the requested construction. This figure does not supply it. The rest of the guide follows exact reductions and local algebra around that arrow.

Carry forward. Preserve the global object and the normalization of every local test.

A solvable tower has two different steps

Clifford theory separates induction from invariant extension



READ THE FIGURE

At a prime cyclic quotient $G/H \cong C_p$, an irreducible constituent of the restriction either moves in an orbit of size p or is invariant.

The moving case is an induced representation. On an accessible prime cyclic tower, the established automorphic induction machinery mirrors that step under its hypotheses.

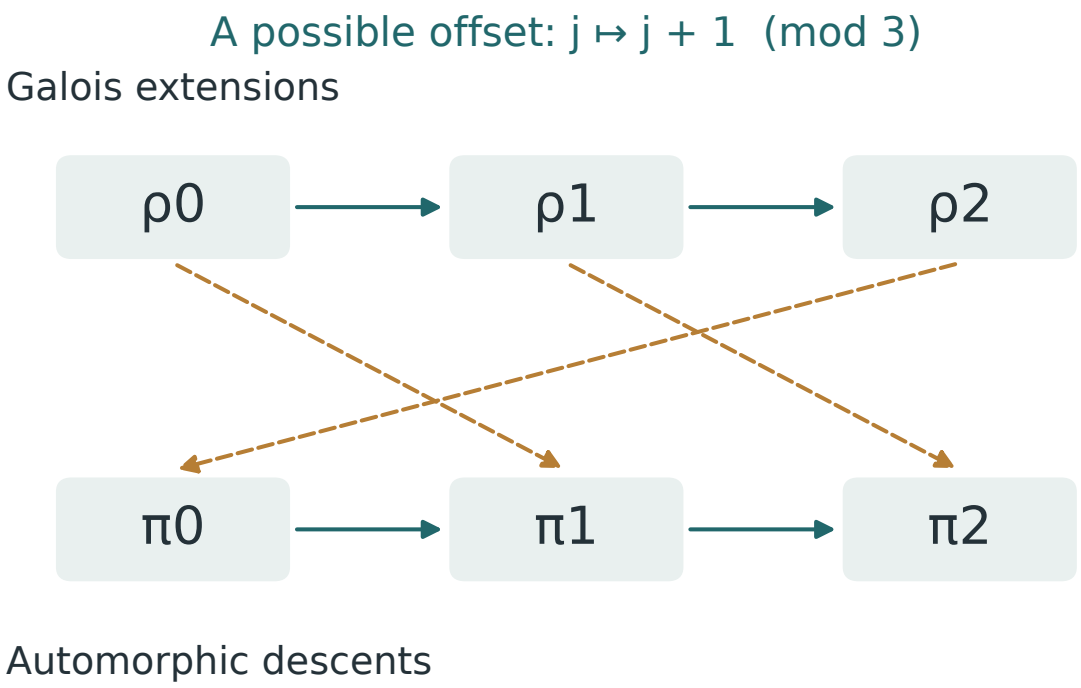
The invariant case produces an extension problem. Cyclic base change may supply descents of the automorphic restriction, but one must still attach the selected descent to the selected Galois extension.

The diagram concerns this solvable recursion. It is not a reduction of every finite group to cyclic quotients.

Carry forward. Induction and attached descent carry different mathematical burdens.

Two torsors share a rotation law

Labels are relative to a chosen origin



READ THE FIGURE

For an invariant irreducible restriction, the p complex extensions are $\rho_0 \otimes \xi^j$. Under cyclic descent hypotheses, the descents are $\pi_0 \otimes \epsilon^j$.

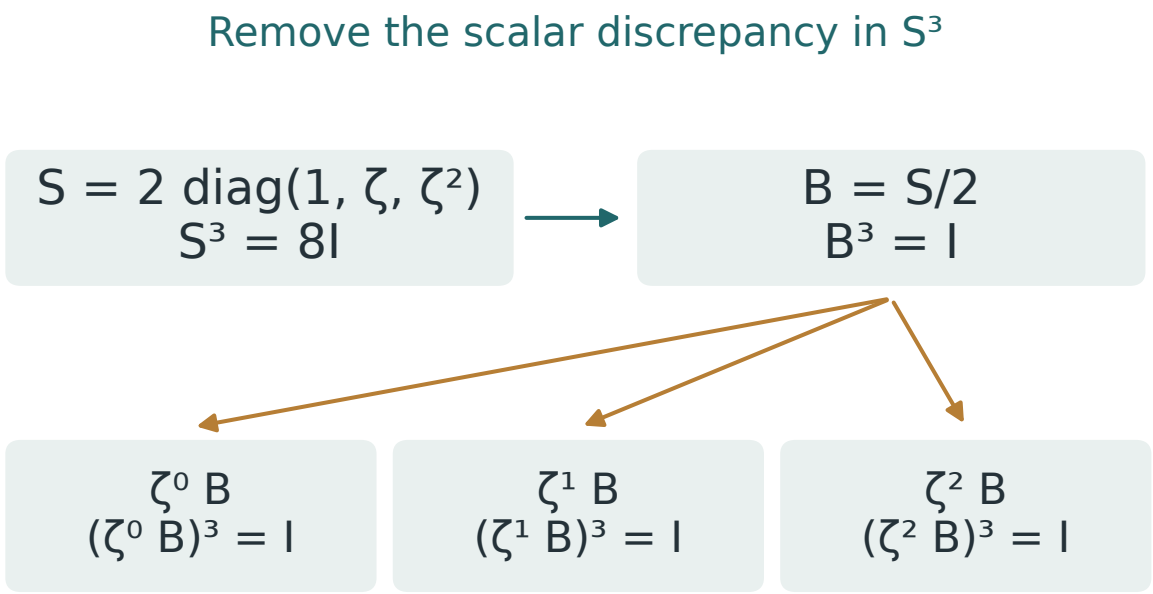
Both rows are acted on freely by $\mathbb{Z}/p\mathbb{Z}$. Choosing an origin labels each row. It does not prove that the two origins are attached.

The figure uses $p = 3$ and a hypothetical offset 1. The gold correspondence is a possible alignment, not an automorphic construction. A simultaneous rotation changes the labels and preserves the same relative offset.

Carry forward. A torsor has a rotation action before it has a preferred origin.

An intertwiner must close the cyclic relation

Normalize its scalar pth power



READ THE FIGURE

An invariant irreducible type gives an intertwiner S. After accounting for the chosen lift of the quotient generator, Schur's lemma leaves a scalar discrepancy in the cyclic relation.

Over $\mathbb{C}$, that nonzero scalar has a pth root. Rescaling S removes the discrepancy. The remaining p choices are the quotient character twists.

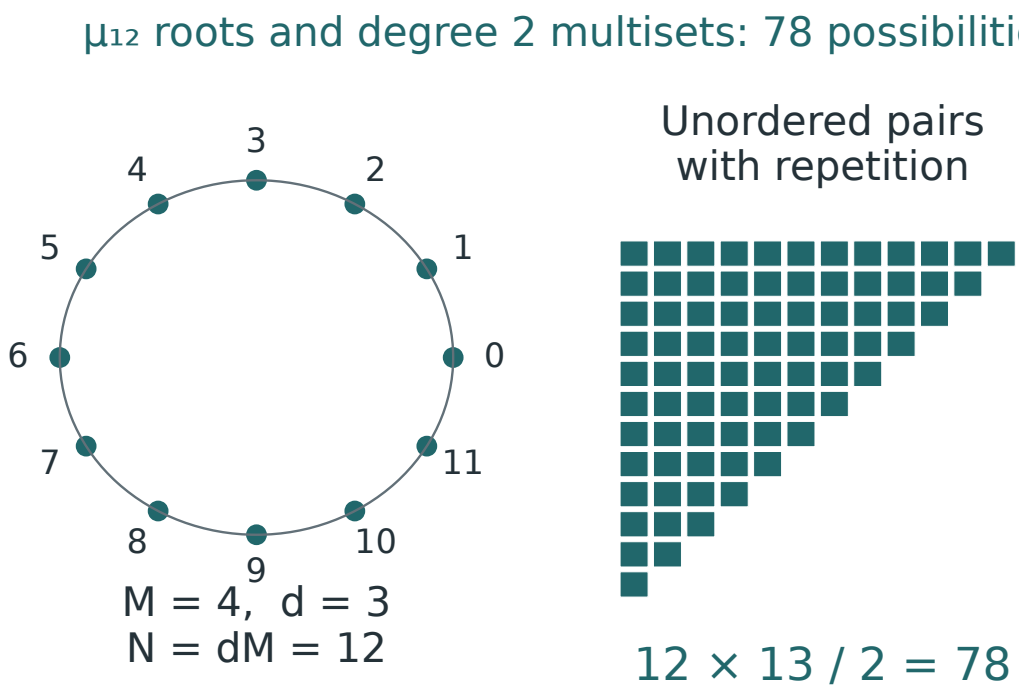
The exact matrix model has $p = 3$, $S = 2\,\mathrm{diag}(1, \zeta, \zeta^2)$, and $S^3 = 8I$. Dividing by 2 gives $B^3 = I$. Multiplying B by ζ^j gives three normalized choices.

This linear algebra illustrates extension normalization. Arithmetic attachment is a separate step.

Carry forward. Normalization supplies extensions and preserves their finite ambiguity.

Finite Satake data form a finite alphabet

A root bound produces a polynomial bound



READ THE FIGURE

At the unramified comparison places outside one fixed exceptional set, if the lower Artin roots lie in μ_M and cyclic base change has degree d, the local root argument puts normalized descent roots in μ_{dM}.

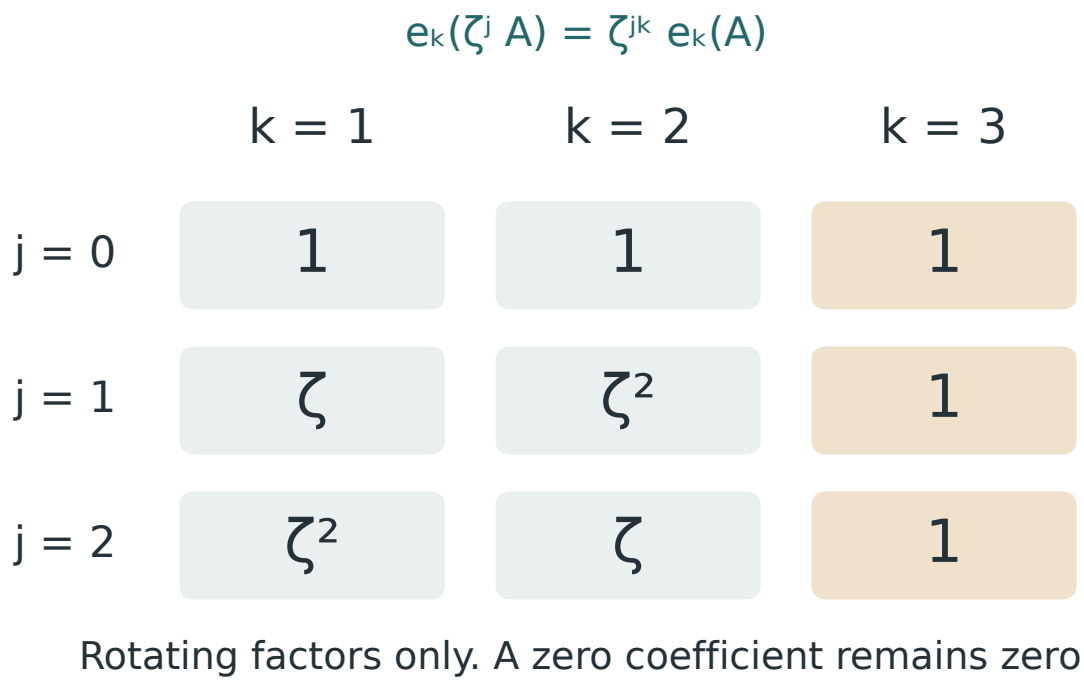
A degree n polynomial is determined by a multiset of n roots. Thus at most $\binom{dM+n-1}{n}$ polynomials can occur. This is a bound on possible local data, not a count of automorphic representations.

The worked alphabet takes $M = 4$, $d = 3$, and $n = 2$. There are 12 root choices and 78 unordered pairs with repetition. The circle shows all 12 roots and the triangular array counts all 78 multisets.

Carry forward. Finitely many local polynomials do not by themselves construct a global representation.

Every coefficient has its own rotation degree

Exterior powers expose the scalar action



Rotating factors only. A zero coefficient remains zero.

READ THE FIGURE

Write $\det(TI - A) = \sum_{k=0}^{n} (-1)^k e_k(A) T^{n-k}$. Scalar multiplication gives $e_k(\zeta^j A) = \zeta^{jk} e_k(A)$.

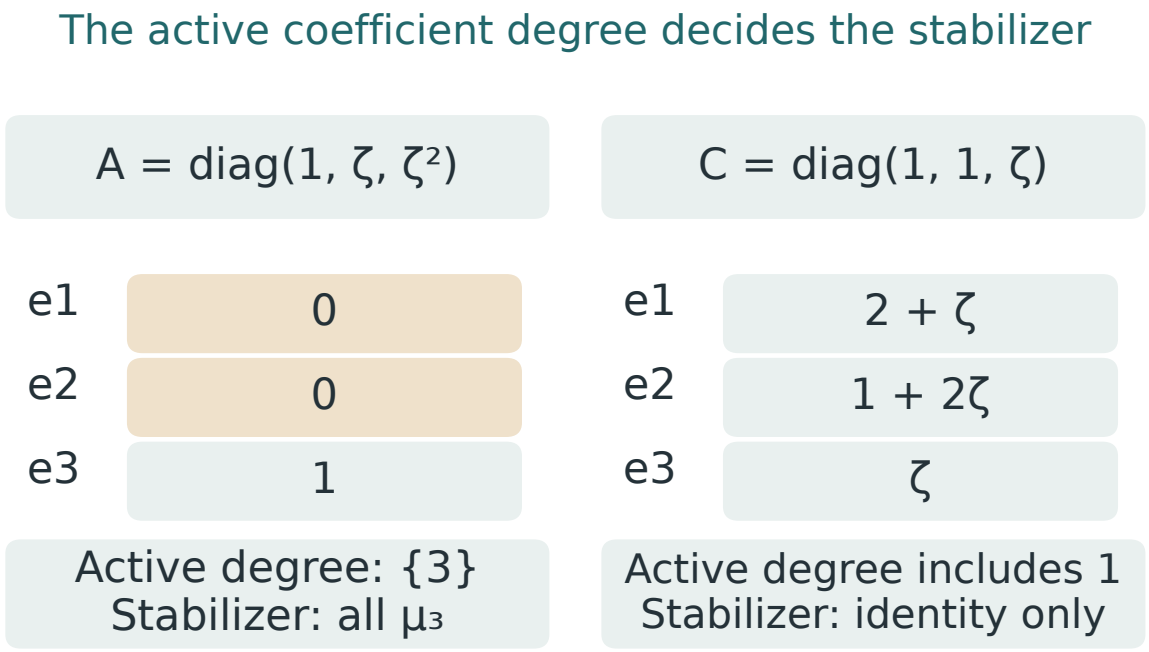
The figure displays the exponents modulo 3 for degrees 1, 2, and 3. The trace rotates once. The second coefficient rotates twice. The determinant does not rotate at rank 3.

A coefficient can detect a label only when it is nonzero and its degree is not divisible by p. The sign convention in the polynomial matters when exact coefficients are compared.

Carry forward. An active coefficient carries both its value and its degree.

Polynomial stabilizers come from active degrees

The example $T^3 - 1$ is completely rotation blind



READ THE FIGURE

For prime p, the stabilizer of the characteristic polynomial is the set of $\zeta \in \mu_p$ with $\zeta^k = 1$ at every active degree k.

For $A = \mathrm{diag}(1, \zeta, \zeta^2)$, the polynomial is $T^3 - 1$. Only degree 3 is active. Multiplying the roots by ζ merely permutes the root multiset.

Contrast $\mathrm{diag}(1, 1, \zeta)$. Its trace is $2 + \zeta \neq 0$, so its three scalar twists have different polynomials.

This coefficient stabilizer concerns a matrix polynomial. Extension spectrum stabilizers concern multiplicity labels and use a different object.

Carry forward. A vanished coefficient supplies no separator, even at a favorable degree.

A determinant can miss a visible label

An exact rank 3 example at the quotient prime 3

$A = \mathrm{diag}(1, 1, \zeta), \quad \zeta^3 = 1, \quad \zeta \neq 1$

label j	trace e_1	coefficient e_2	determinant e_3
0	$2 + \zeta$	$1 + 2\zeta$	ζ
1	$-1 + \zeta$	$1 - \zeta$	ζ
2	$-1 - 2\zeta$	$-2 - \zeta$	ζ

First two columns separate. Last column stays fixed.

READ THE FIGURE

Take $A = \mathrm{diag}(1, 1, \zeta)$ with $\zeta^3 = 1$ and $\zeta \neq 1$. Its elementary coefficients are $2 + \zeta$, $1 + 2\zeta$, and ζ.

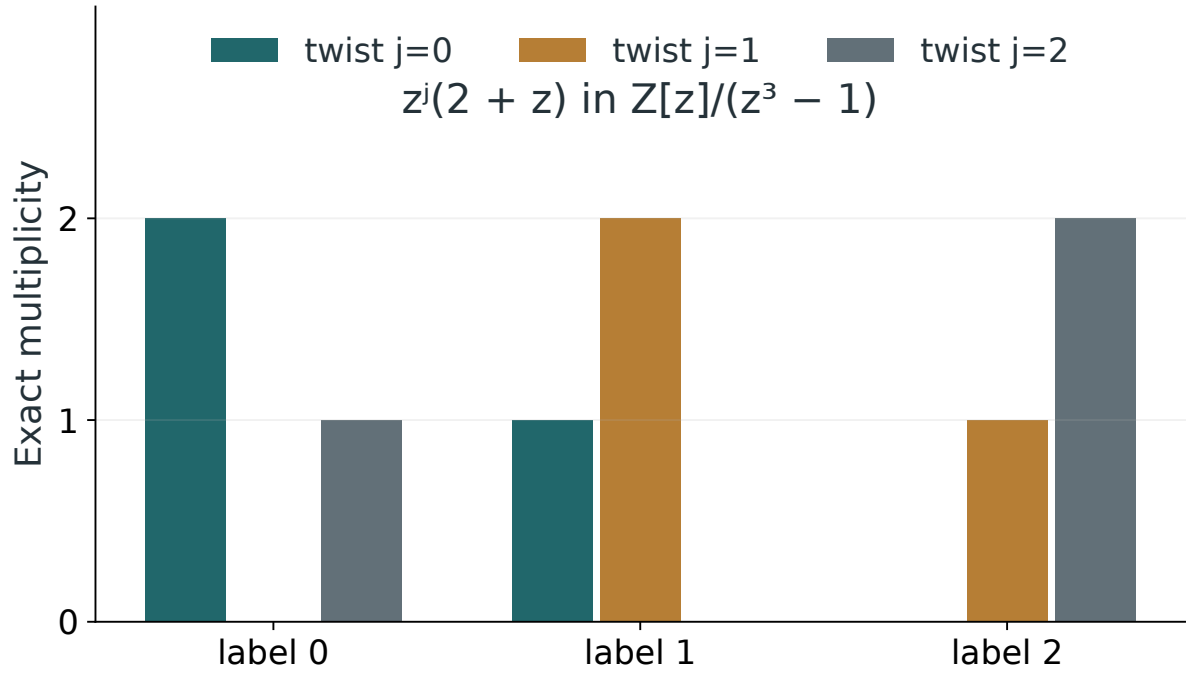
The trace and second coefficient rotate with the quotient label. The determinant stays ζ because multiplying a rank 3 matrix by ζ multiplies its determinant by $\zeta^3 = 1$.

This is the difficult rank divisibility pattern. When $p \nmid n$, determinant data selects a candidate in an already known twist torsor. Candidate selection still requires attachment evidence for the other local parameters.

Carry forward. Determinant blindness is precise and does not imply blindness of all coefficients.

Extension spectra retain multiplicity labels

A stable type records more than its total multiplicity



READ THE FIGURE

For a stable irreducible H type τ, choose one extension. Its multiplicity space decomposes into quotient characters. Record it as $P_{\tau,V}(z) = \sum_a m_{\tau,a} z^a$ in $\mathbb{Z}[z]/(z^p - 1)$.

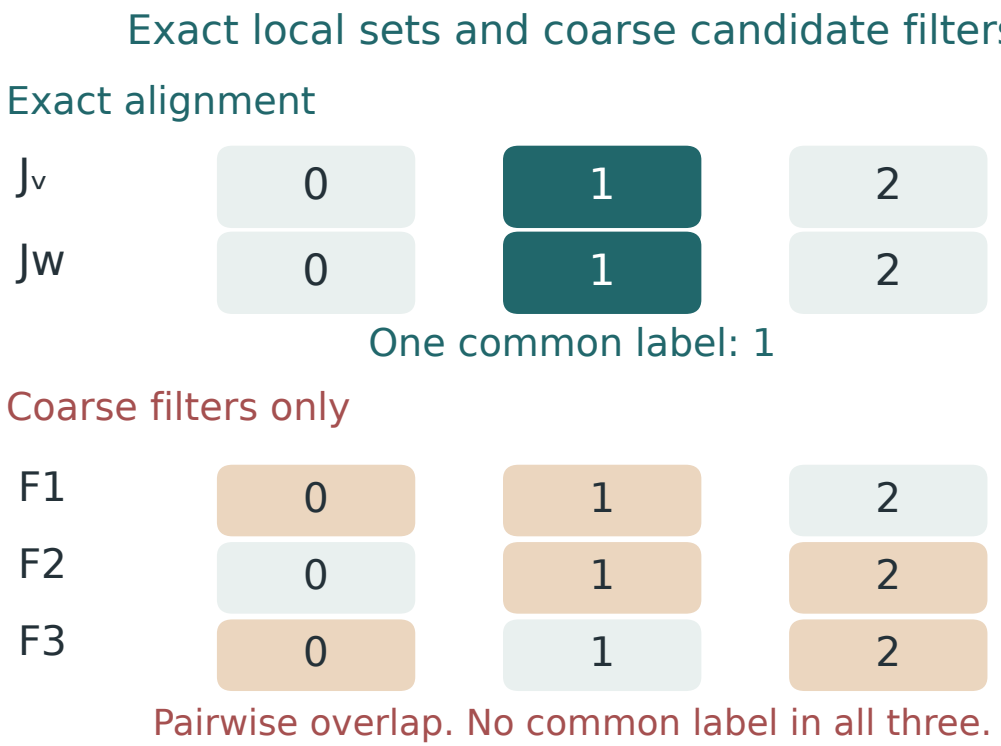
Twisting by ζ^j multiplies this polynomial by z^j. The displayed multiplicities $(2, 1, 0)$ rotate cyclically. Their sum stays 3, while the label distribution changes.

A moving orbit contributes an induced object and is unchanged by quotient twisting. The flat row $(1, 1, 1)$ illustrates rotational invariance of a regular quotient character packet, not an identification of stable and moving types.

Carry forward. Stable extension spectra preserve the labels that total multiplicity forgets.

A global label must pass every local test

Exact alignment and coarse filters have different geometry



READ THE FIGURE

The global criterion is $\bigcap_v J_v \neq \varnothing$, where J_v is the set of quotient shifts giving exact local agreement. In a prime cyclic quotient, an exact J_v is empty, a singleton, or the whole quotient.

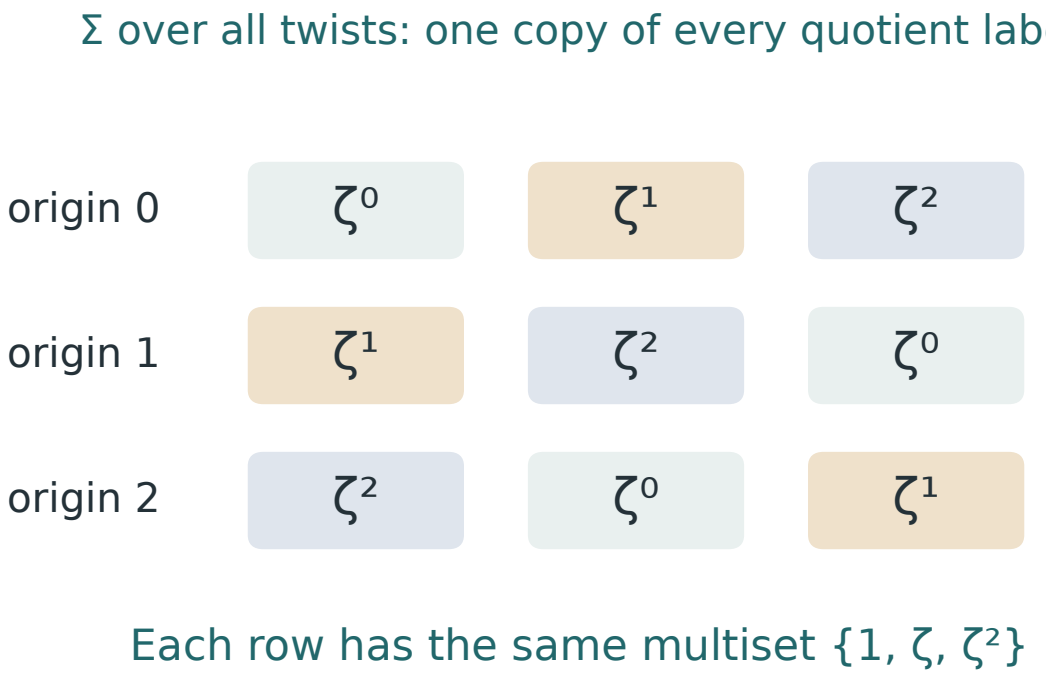
The upper example has $J_v = J_w = \{1\}$ and selects shift 1. Two exact singleton tests $\{1\}$ and $\{2\}$ would instead be incompatible.

The lower example uses coarse filters $\{0,1\}$, $\{1,2\}$, and $\{0,2\}$. Every pair intersects, but their total intersection is empty. These are arbitrary candidate filters, not exact prime cyclic alignment sets.

Carry forward. Check one common offset. Do not replace exact local sets by looser candidate filters.

The total induced object forgets the origin

Rotating a full packet only permutes its constituents



READ THE FIGURE

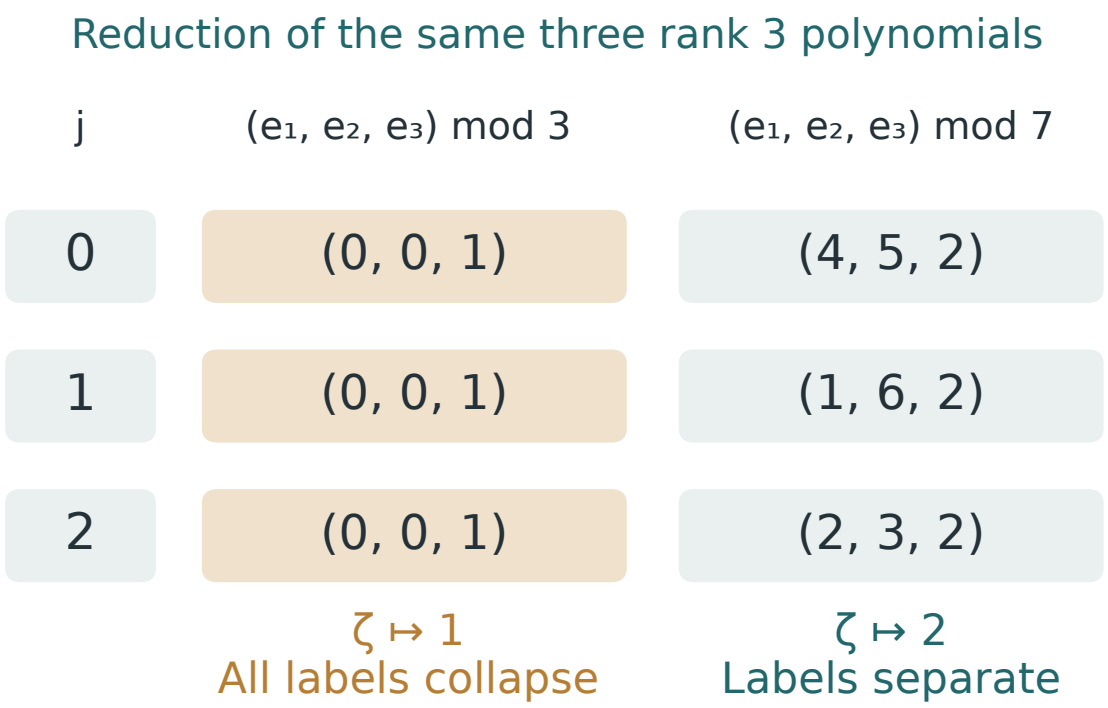
For a quotient character group X, $\bigoplus_{\chi \in X} V \otimes \chi \cong \mathrm{Ind}_H^G \mathrm{Res}_H V$. The automorphic induction identity has the same permutation symmetry in its established domain.

The finite model takes a one dimensional extension with label j. Summing all three twists gives one copy of each cube root for every j. The rows have different origins and identical total multiplicities. Agreement of the total packet therefore cannot tell us which individual constituent matches the target. A product of local factors has the same loss of origin.

Carry forward. A norm identity does not select one factor of the norm.

The quotient prime collapses all labels

Congruence at 3 and separation at 7



READ THE FIGURE

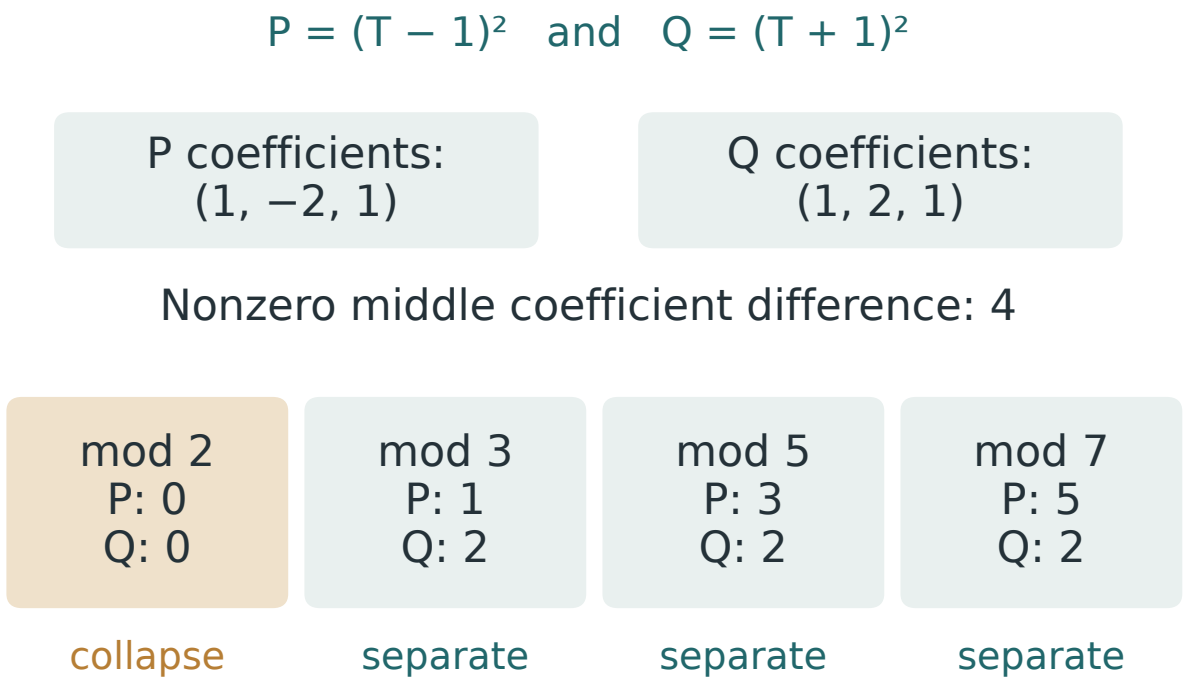
At a coefficient place above p, every pth root of unity reduces to 1. The quotient twists consequently have the same semisimple residual data. For $p = 3$, the three roots collapse in residue characteristic 3. The polynomial $(T-1)^3$ becomes $T^3 - 1$. In $\mathbb{F}_7$, the cube roots $1, 2, 4$ remain distinct. The table reduces the exact three rank 3 polynomials from the previous example. Their coefficients all agree at 3 and separate at 7.

Residual agreement does not prove ordinary equality. It also does not construct a residual Galois carrier from a Hecke eigensystem.

Carry forward. A collapsing prime gives congruence, while a separating prime retains the label.

A finite alphabet has finitely many bad primes

Distinct coefficient vectors survive almost every reduction



READ THE FIGURE

For each pair of distinct integral polynomials, choose a nonzero coefficient difference. Only finitely many coefficient places divide one of those differences. All other places separate the alphabet. The integer example uses $(T-1)^2$ and $(T+1)^2$. Their middle coefficients differ by 4. They coincide modulo 2 and differ modulo 3, 5, and 7.

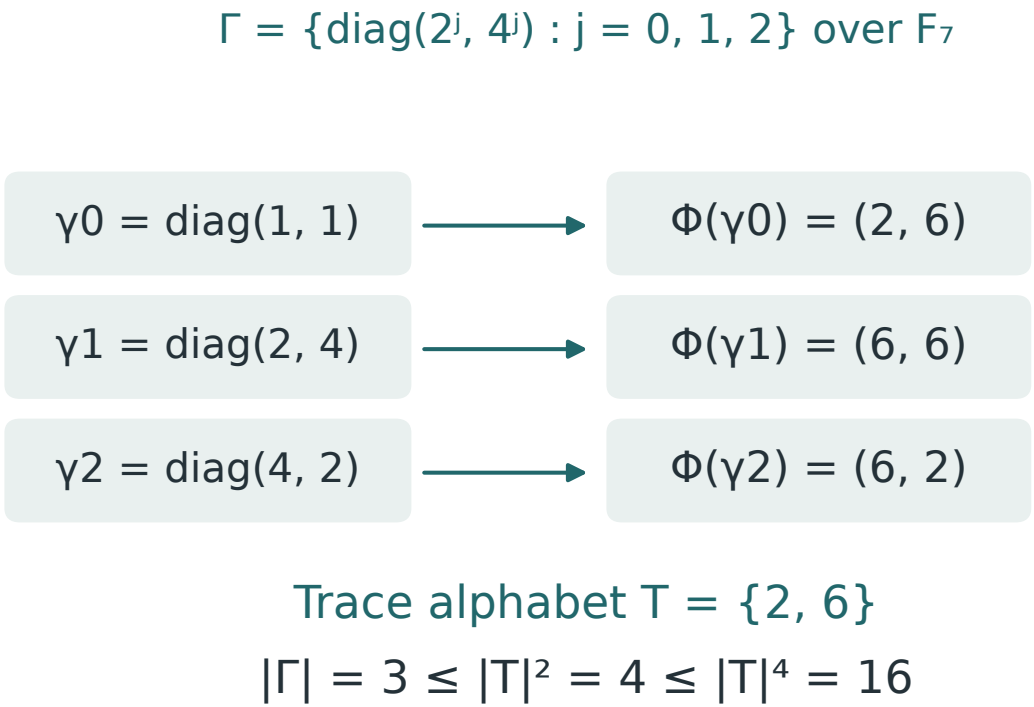
A good prime for the cyclic attachment problem must also avoid the quotient prime and preserve the absolute irreducibility of the restricted Galois representation.

The existence of good primes is established. A Galois realization at a good prime remains an additional input.

Carry forward. Reduction can recover equality only after injectivity on the entire relevant alphabet is proved.

Trace words can encode a finite matrix group

The bound needs semisimplicity and characteristic above the rank



READ THE FIGURE

Let $\Gamma \subset \mathrm{GL}_n(k)$ be finite, with k algebraically closed, characteristic $\ell > n$, and semisimple defining module. Its image algebra has a nondegenerate trace pairing.

Choose $d \le n^2$ group matrices spanning that algebra. The word of traces $\big(\mathrm{Tr}(\gamma\gamma_i)\big)_i$ determines γ.

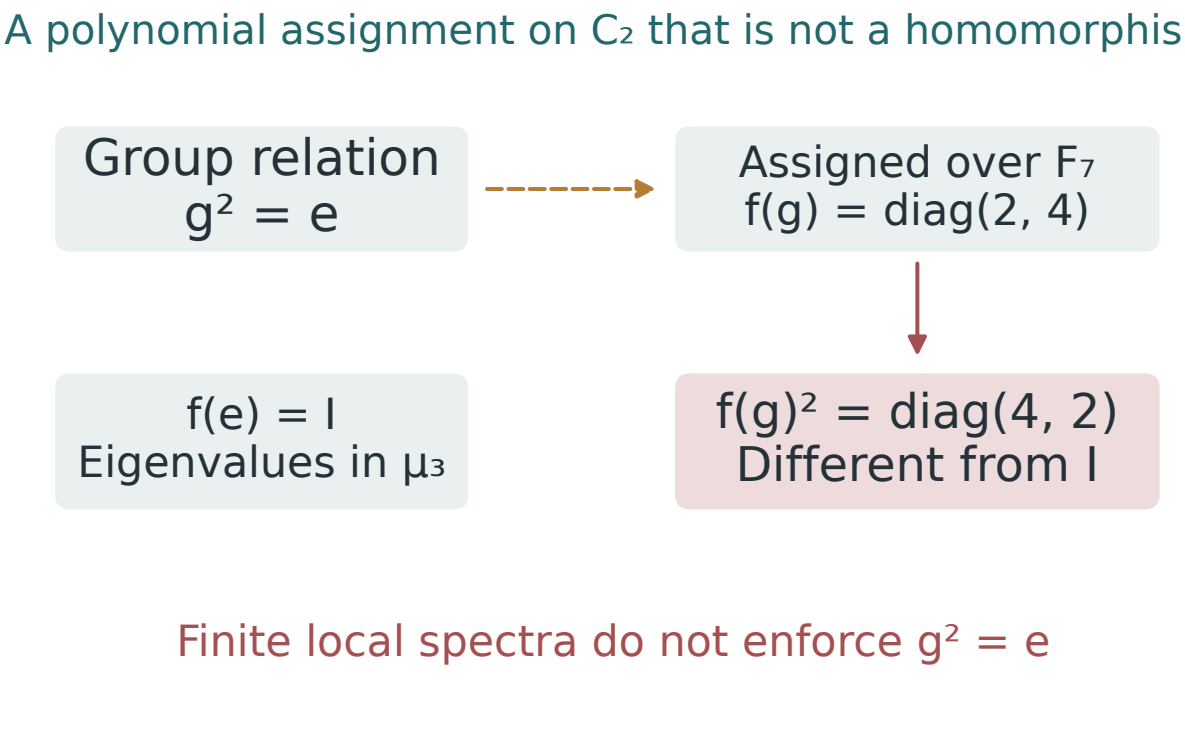
If there are t trace values, $|\Gamma| \le t^d \le t^{n^2}$.

The exact model is the order 3 diagonal group over $\mathbb{F}_7$, viewed over its algebraic closure. It has two trace values and $d = 2$. Three distinct trace words fit inside four available words. The universal rank bound gives 16.

Carry forward. The trace pairing turns a finite alphabet into an injective encoding under explicit hypotheses.

Pointwise spectra do not enforce multiplication

A finite polynomial assignment that is not a representation



READ THE FIGURE

This is a finite algebra model, not an automorphic example. Assign I to the identity of C_2 and $A = \mathrm{diag}(\zeta, \zeta^{-1})$ to its generator.

Both matrices have finite order, and their eigenvalues lie in μ_3. Thus the assigned local polynomial alphabet is finite.

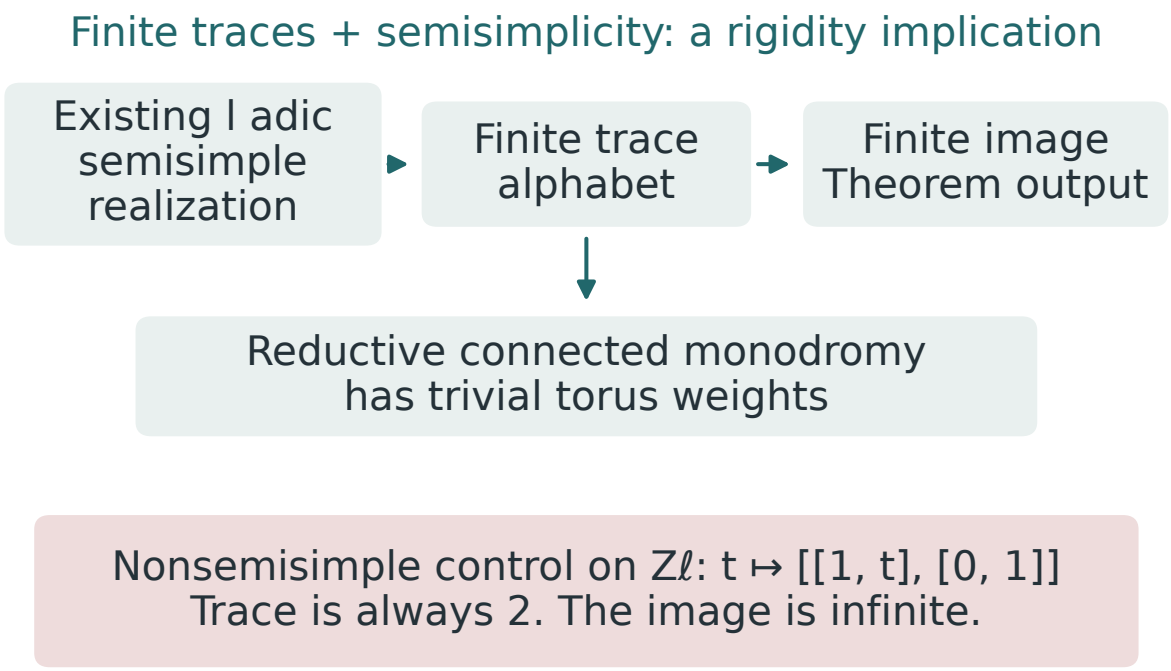
But the generator of C_2 must square to the identity, whereas $A^2 \neq I$. No group representation has this assignment. The bottom row displays the exact failure over $\mathbb{F}_7$ with $\zeta = 2$.

This control isolates why local values need a globally compatible representation or pseudocharacter.

Carry forward. An alphabet constrains values. A representation also satisfies every multiplication relation.

Finite traces force finite semisimple monodromy

One l adic realization is an input, not an output of the alphabet



READ THE FIGURE

For a continuous semisimple Galois representation over a finite extension of $\mathbb{Q}_\ell$, unramified outside a finite set, finitely many Frobenius traces on a density one set force finitely many image traces. In the reductive Zariski closure, the connected component then has constant trace.

Its torus weight characters cannot cancel a nonconstant positive multiplicity character sum in characteristic zero. Every weight is trivial, so the faithful connected monodromy is trivial. The image is finite.

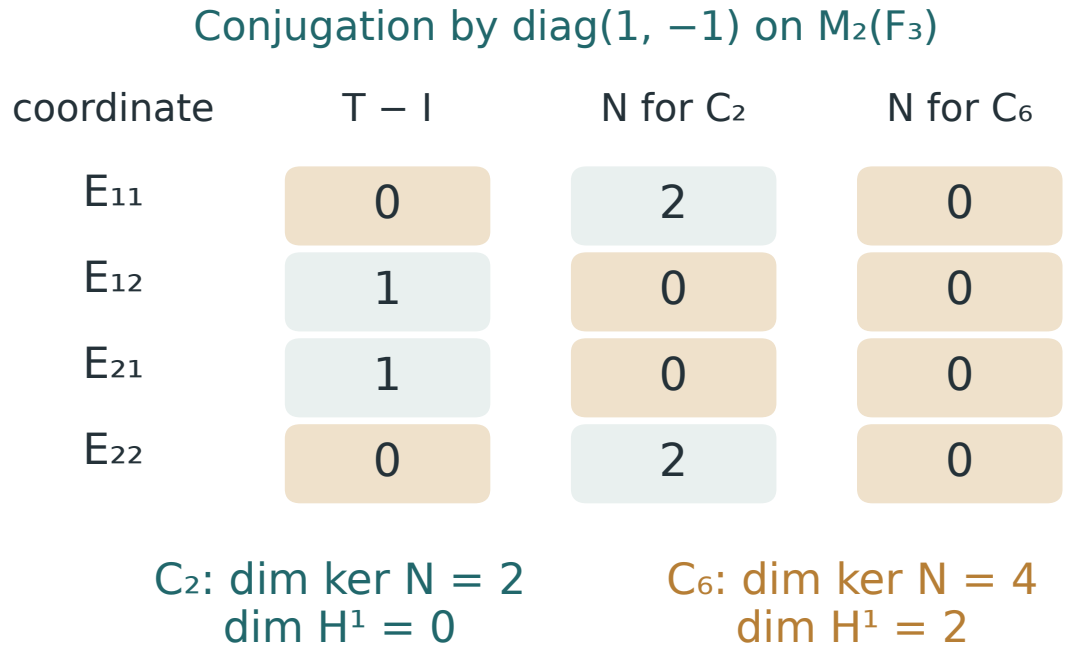
Semisimplicity matters. The lower model is the continuous unipotent representation of $\mathbb{Z}_\ell$ given by $t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Its trace is always 2 and its image is infinite.

The theorem begins with a realization already constructed.

Carry forward. A rigidity theorem upgrades an existing semisimple realization.

Cyclic H^1 keeps the norm condition

A full matrix cohomology group need not be a line



READ THE FIGURE

For a cyclic group of order q acting by T on M, the standard formula is $H^1 = \ker N / \operatorname{im}(T - I)$, with $N = I + T + \cdots + T^{q-1}$.

Over $\mathbb{F}_3$, let T be conjugation by $\operatorname{diag}(1, -1)$ on 2×2 matrices. Diagonal coordinates are fixed. Off diagonal coordinates change sign.

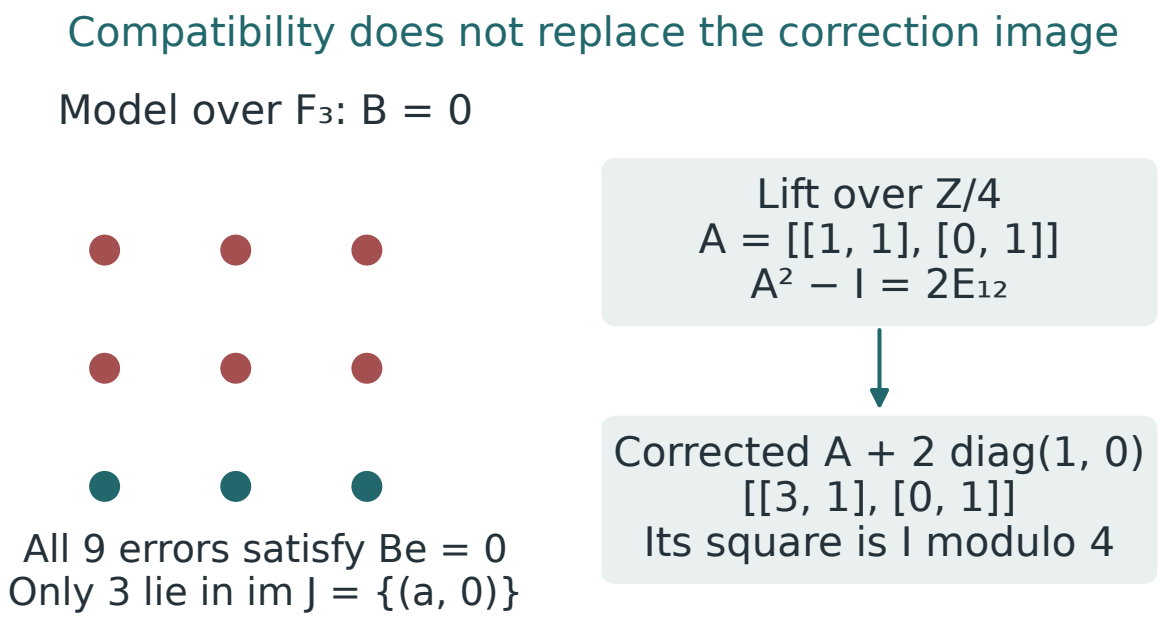
For C_6, the norm is zero. The quotient retains both diagonal coordinates, giving dimension 2. For C_2, the norm kills exactly the off diagonal coordinates, which are also the image of $T - I$, giving dimension 0.

Neither calculation identifies a selected continuous local Tate line.

Carry forward. Name the acting group, coefficient module, norm, and chosen local subquotient.

Square zero defects need an image test

Compatibility of an error is weaker than correctability



READ THE FIGURE

Across a square zero kernel, the multiplicativity error of a chosen representation lift satisfies a cocycle compatibility relation. Changing the lift changes the error by a correction map.

A correction exists exactly when the error class vanishes in the appropriate obstruction quotient. The left finite model has $J(a) = (a, 0)$ and $B = 0$ over $\mathbb{F}_3$. Every error is compatible, but a vertical error is not in im J.

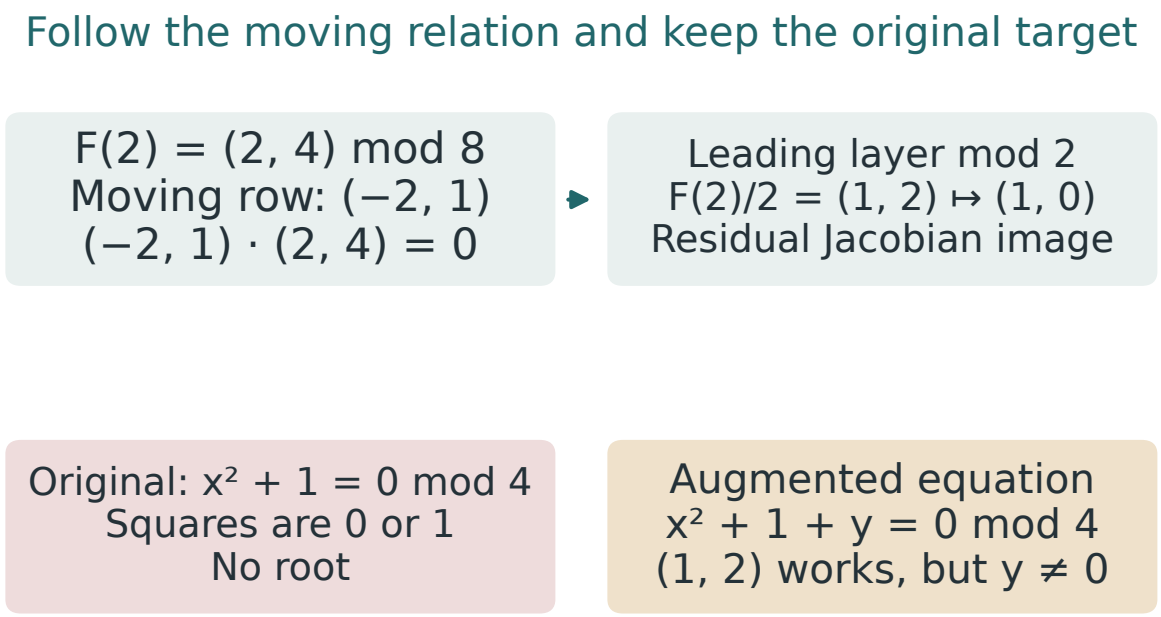
The right example lifts $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ from $\mathbb{F}_2$ to $\mathbb{Z}/4$. Its square has an off diagonal error 2. Adding $2\,\mathrm{diag}(1, 0)$ corrects it.

These models do not construct the required arithmetic H^2 comparison.

Carry forward. A cocycle relation locates an obstruction. Vanishing of its class supplies correction.

A moving syzygy follows the changing point

Correct the original system rather than an enlarged one



READ THE FIGURE

For $F(x) = (x, x^2)$, the exact relation is $(-x, 1)F(x) = 0$. At $x = 2$ modulo 8, the error is $(2, 4)$. The frozen residual row $(0, 1)$ does not annihilate that whole error.

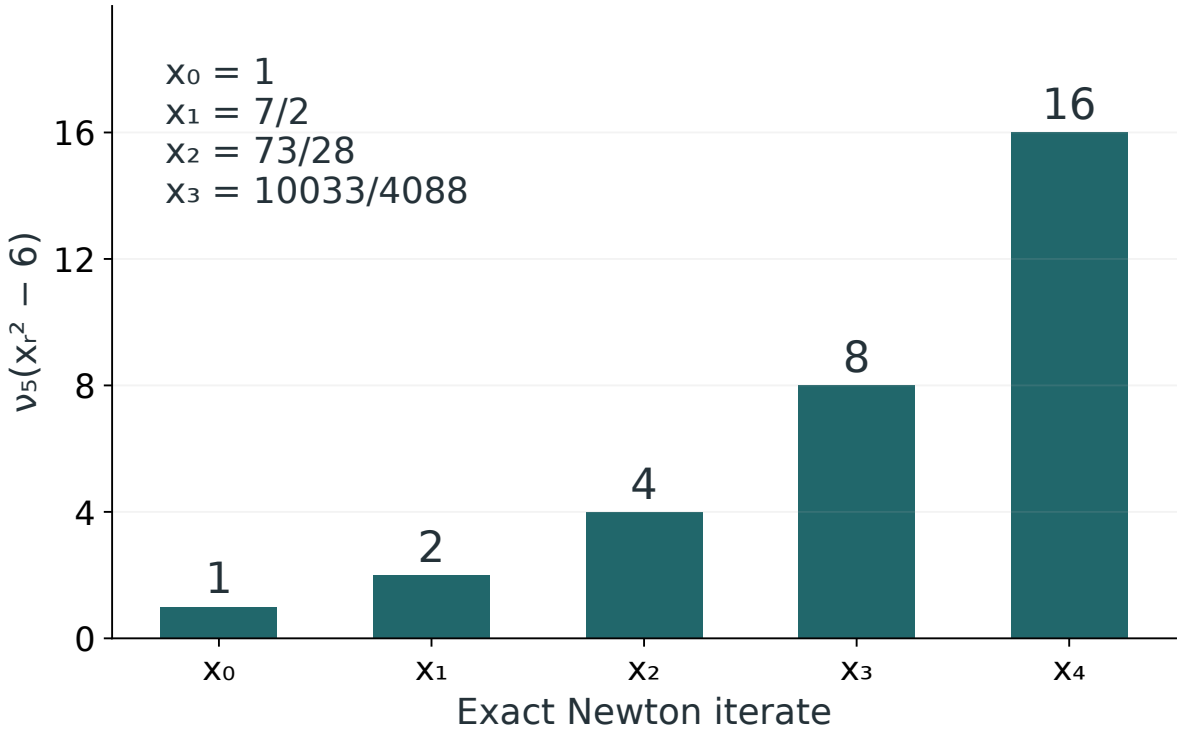
At the leading graded layer, the error divided by 2 reduces to $(1, 0)$ modulo 2 and is in the Jacobian image. This is the layer on which the moving syzygy argument works.

The lower control takes $x^2 + 1$ modulo 4. It has no root. Adding an auxiliary coordinate y gives a root $(1, 2)$ of $x^2 + 1 + y = 0$, but $y \neq 0$. The enlarged point does not solve the original equation.

Carry forward. Prove moving identities and residual exactness for the actual target equations.

A scalar Newton correction gains valuation

Exact rational iterates for $x^2 = 6$ at 5



READ THE FIGURE

Begin at $x_0 = 1$ and use $x_{r+1} = x_r - (x_r^2 - 6)/(2x_r)$. The displayed values are exact rational numbers with 5 adic unit denominators.

The residual valuations are 1, 2, 4, 8, and 16. The identity $x_{r+1}^2 - 6 = (x_r^2 - 6)^2/(4x_r^2)$ explains the doubling, since every x_r remains a 5 adic unit.

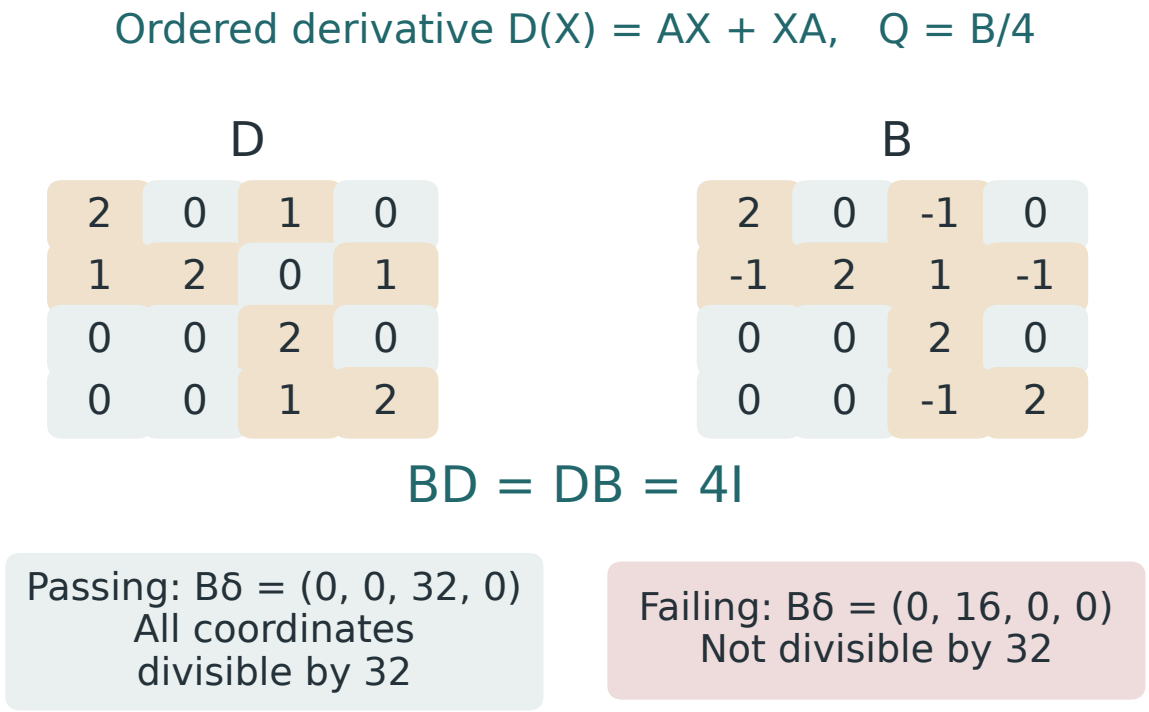
The plotted five iterates are bounded computations. The identity and completeness give the usual local convergence under these hypotheses.

This scalar example illustrates valuation gain. A matrix power requires its ordered derivative, not a scalar substitute.

Carry forward. The error improves because the exact derivative and remainder satisfy the required estimates.

A finite inverse certificate decides a matrix ball

The square map at a unipotent matrix over $\mathbb{Q}_2$



READ THE FIGURE

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The square derivative is $D_A(X) = AX + XA$. With row major coordinates, the displayed integral matrix B satisfies $BD = DB = 4I$. Thus $Q = B/4$ is the exact inverse.

Here $s = 2$ and choose $m = 3 > s$. The local criterion is that $B\,\mathrm{vec}(U - A^2)$ vanish modulo $2^{m+s} = 32$.

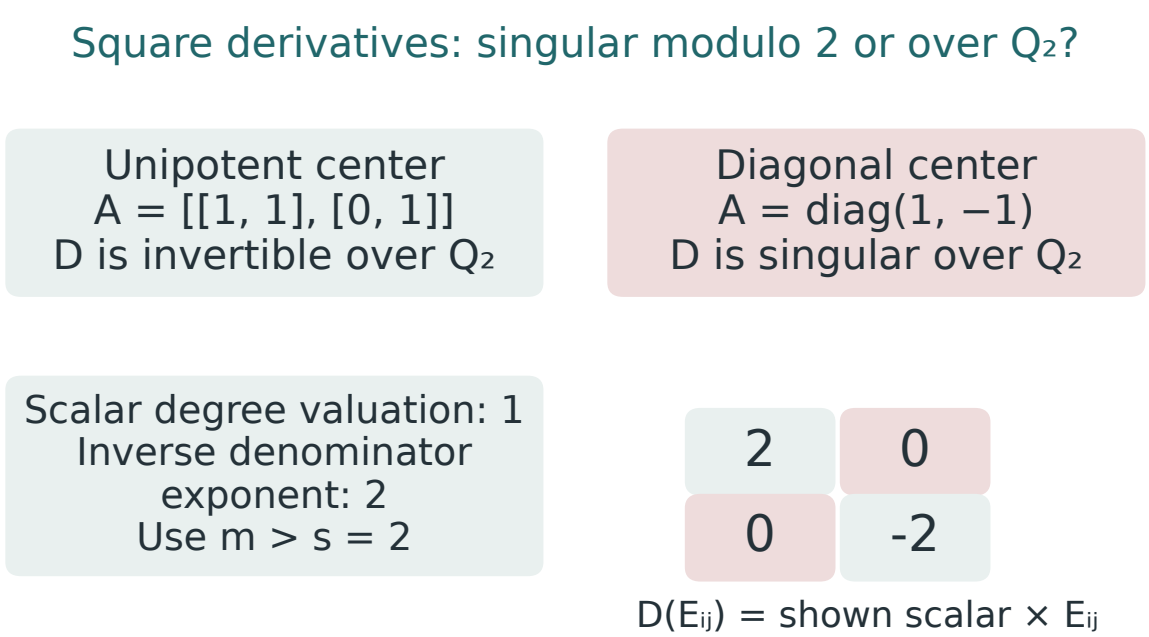
The target $U = \begin{pmatrix} 9 & 2 \\ 16 & 9 \end{pmatrix}$ passes and is the square of $F = \begin{pmatrix} 1 & 1 \\ 8 & 1 \end{pmatrix}$, with $F \equiv A \pmod 8$. The comparison target fails the same finite test.

This is local matrix algebra, not global automorphy.

Carry forward. A supplied two sided inverse certificate makes one finite congruence decisive in the stated ball.

Modular singularity and genuine singularity differ

The inverse denominator can exceed the scalar valuation



READ THE FIGURE

For the unipotent square center, the derivative is invertible over $\mathbb{Q}_2$, even though its reduction modulo 2 is singular. The inverse needs denominator 2^2, so $s = 2 > \nu_2(2) = 1$.

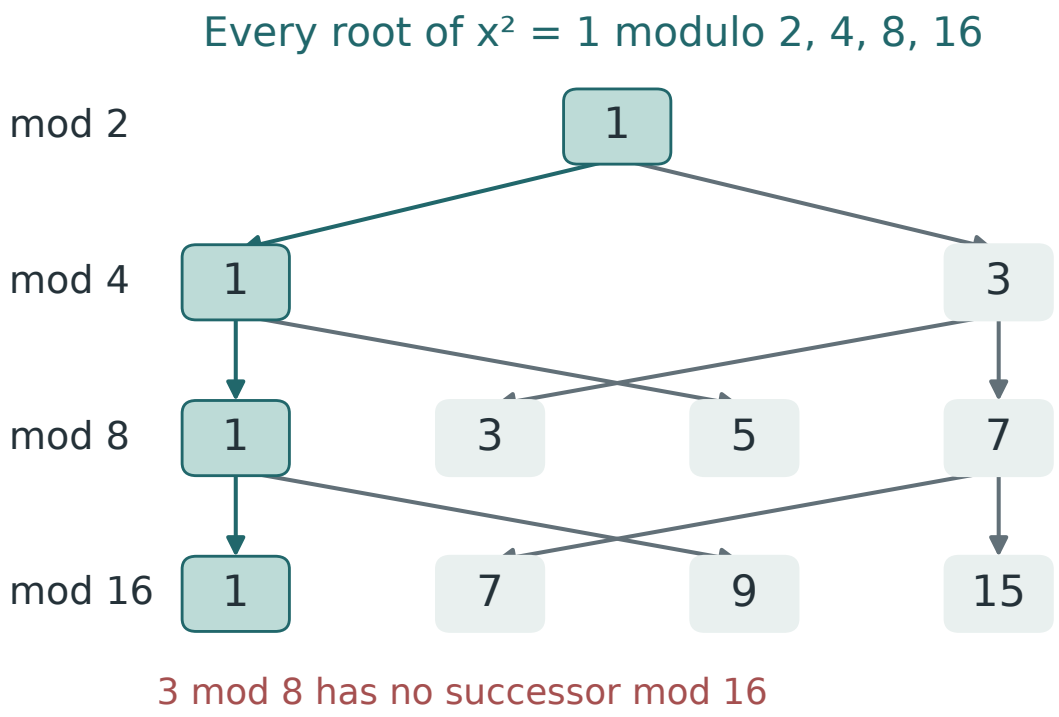
The strict condition for the previous contraction theorem is $m > s$. Replacing s by the scalar degree valuation would enlarge the claimed ball without justification.

At $A = \mathrm{diag}(1, -1)$, the square derivative multiplies the diagonal directions by 2 and -2, and kills both off diagonal directions. It is singular over $\mathbb{Q}_2$. That center cannot satisfy the same two sided inverse certificate.

Carry forward. Audit the full matrix derivative before applying an inverse based lifting theorem.

A tower is a chain of compatible choices

Finite roots illustrate why arbitrary selections can fail



READ THE FIGURE

The tree contains every odd root of $x^2 = 1$ modulo 2, 4, 8, and 16. Each edge is reduction to the preceding level. All node values and edges are exact.

The highlighted branch chooses 1 at every level. The root 3 modulo 8 has no successor modulo 16, although both levels have solutions. Thus a root chosen at one level need not extend.

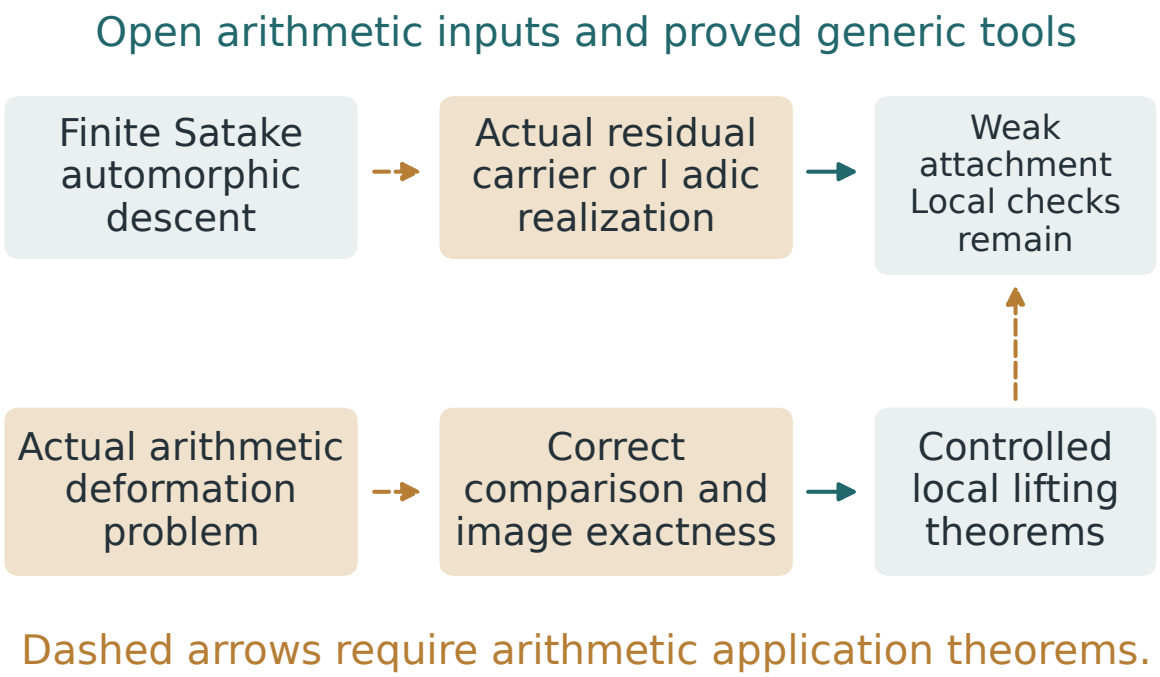
For compact finite solution sets with compatible reduction maps, nonempty levels give an infinite compatible branch by the usual finite branching argument. That conclusion requires the maps and the solution conditions to agree across levels.

A local matrix root already constructed in $\mathbb{Z}_p$ automatically supplies its compatible reductions.

Carry forward. Record transition maps and compatible choices, not just separate existence flags.

The remaining attachment map is concrete

Generic algebraic tools and arithmetic existence have different jobs



READ THE FIGURE

The upper route begins with an actual finite Satake automorphic descent. A residual carrier at a good separating prime, one semisimple l adic realization, or a suitable quotient prime lifting theorem can close the cyclic attachment step under its hypotheses.

The lower route records the generic local toolkit: an actual deformation problem, the correct cochain comparison, compatible defects, correction exactness, and controlled lifting. The square zero and Hensel results apply once those inputs are proved.

The dashed connectors are the arithmetic application theorems still needed. A formal datum equating two spaces does not construct that comparison. An auxiliary root does not construct an automorphic carrier.

Neither route is depicted as completed Strong Artin.

Carry forward. The open theorem must construct the arithmetic input and verify the attachment it claims.