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Signal Reconstruction

Polyphase sampling, unknown edges, and self-calibration

Traditional paper

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Reading scope

Reading set. Traditional paper, visual guide, Lean guide, and complete source bundle. The bundle includes the `.md` files, editable manuscript sources, Python implementations, exact figure generators, related Lean sources, and verification records.

Scope. Exact finite-rate edge recovery and native synthesis; conditional noisy and smooth-tail recovery; finite geometric dictionaries; and explicitly separated adaptive image approximation.

Abstract

Finite signal reconstruction depends on a declared model and an injective observation. This paper develops a packet-based reconstruction chain from complete projector coordinates and transfer kernels to exact off-grid jump recovery, polyphase alias calibration, and native singularity synthesis. Consecutive differentiated Fourier moments determine an unknown finite jump count through Hankel rank, followed by an annihilating polynomial and Vandermonde recovery. Matched adjacent offsets recover the physical phase lost by decimation. Noisy order, root, and branch conclusions retain explicit conditioning and margin gates, with conditional asymptotic rates. A corrected periodic mean-zero jump estimate controls edge synthesis in L^1. Classical Gibbs ringing is derived and localized to the globally truncated edge block. Finite dictionaries and adaptive image experiments are kept distinct from universal continuous-edge recovery. The current numerical replay includes successful controlled checks and reproduced identification failures. Related Lean infrastructure is mapped without claiming a formal certificate for the new recovery chain.

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1 The reconstruction problem

A measured signal is an object seen through a declared observation map. A reconstruction theorem therefore has two obligations: identify the object within a stated class and synthesize it in a representation suited to that class. These obligations differ. A Fourier series can determine a jump signal uniquely while its finite partial sums still ring near the jump. Conversely, a native edge representation can reproduce a known discontinuity exactly while supplying no method to discover an unknown edge.

We treat both obligations. First we organize signals and operators by complete packet systems. We then derive known-edge cancellation, recovery in finite geometric dictionaries, exact off-grid Prony recovery, and self-calibration of polyphase aliases from adjacent offsets. Quantitative conclusions retain their rank, conditioning, separation, and branch hypotheses. The final synthesis error is split into the smooth approximation error and the recovered edge error.

The default periodic domain is $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, with

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx.$$

Signal identities inferred from Fourier coefficients are identities of L^1 classes. For a piecewise-constant edge we select the right-continuous representative. This convention fixes the values at jumps, which Fourier integrals alone cannot determine. When an L^1 norm occurs it is the unnormalized integral over a period. On the unit circle we use chord distance $|z-w|$ and circular angular distance $d_{\mathbb{T}}(\tau,\sigma)$.

1.1 What belongs to this edition

The inherited unknown-edge and self-calibration manuscripts are consolidated here with full proofs and corrected contracts. The Gibbs case studies explain the operator choice. The complex signal/image experiments probe model mismatch. The seed calculus supplies the finite packet language. The infinite packet program supplies a separate adaptive approximation experiment. The raw sources remain unchanged in the bundle. The claim concordance records which source each published result develops and where a printed contract is narrower or corrected.

Prony systems, finite-rate sampling, polyphase decomposition, Fourier Gibbs asymptotics, sparse secant injectivity, and shifted exponential coefficients have established histories. In particular the shifted-coefficient phase relation used below occurs in VEXPA [2]. Our presentation makes those ingredients explicit within one packet analysis, recovery, validation, and synthesis chain. It does not assert that these ingredients were first discovered here.

2 Packets, transfer blocks, and polyphase coordinates

A packet is a component specified by a projector and its image. Complete packets keep every distinction in the declared object space. A transfer block records how an operation sends one packet to another. These definitions are structural: they do not themselves promise that measurements recover the packets. Recovery is established by the later injectivity and conditioning contracts.

Domain and hypotheses. Vector spaces over a field. Let $P_1,\dots,P_m \in \mathrm{End}(X)$ satisfy $P_aP_b = \delta_{ab}P_a$ and $\sum_a P_a = I_X$. Set $X_a = P_aX$.

Theorem 2.1 (SG-01 Complete packet analysis and synthesis)**.** *The maps*

$$\mathcal{A}x = (P_ax)_a, \qquad \mathcal{S}(x_a)_a = \sum_a x_a$$

are inverse isomorphisms between X and $\bigoplus_a X_a$. In particular, $x = \sum_a P_a x$ uniquely.

Proof. Completeness gives $\mathcal{S}\mathcal{A}x = x$. If $x_a \in X_a$, then $P_a x_a = x_a$ and $P_b x_a = 0$ for $b \neq a$. Thus the bth component of $\mathcal{A}\mathcal{S}(x_a)_a$ is $P_b \sum_a x_a = x_b$. Both inverse identities follow. For uniqueness, apply P_b to any proposed sum of components. This proof uses pairwise annihilation, not orthogonality; the packet projections can be oblique.

□

Source lineage: seed, thm:object-reconstruction.

Domain and hypotheses. The packet space of SG-01. Let $S, T \in \mathrm{End}(X)$. Set $T_{ba} = P_b T P_a$, viewed as a map $X_a \to X_b$.

Theorem 2.2 (SG-02 Transfer algebra and packet paths)**.**

$$T = \sum_{a,b} P_b T P_a, \qquad (ST)_{ca} = \sum_b S_{cb} T_{ba}.$$

The block representation is an algebra isomorphism to $\mathrm{End}(\bigoplus_a X_a)$*, with multiplication by contraction of intermediate packets. For every integer $k \geq 1$, iterating the product gives*

$$P_{a_k} T^k P_{a_0} = \sum_{a_1,\ldots,a_{k-1}} (P_{a_k} T P_{a_{k-1}}) \cdots (P_{a_1} T P_{a_0}).$$

Proof. Insert $I = \sum_a P_a$ on both sides of T to obtain reconstruction. Insert that identity between S and T to obtain the product formula. Equivalently, the block map is $T \mapsto \mathcal{A}T\mathcal{S}$; SG-01 shows that its inverse is $B \mapsto \mathcal{S}B\mathcal{A}$ and that it preserves multiplication. The path formula follows by inserting the same identity between every successive pair of operators. These are finite sums, so no convergence hypothesis is hidden in the insertion.

□

Source lineage: seed, thm:operator-reconstruction; thm:block-product; thm:path-power.

Domain and hypotheses. Finite complete packet systems. Let (Q_ρ) refine (P_a), with a partition of fine labels into F_a and $P_a = \sum_{\rho \in F_a} Q_\rho$.

Theorem 2.3 (SG-03 Refinement preserves represented operators)**.**

$$P_b T P_a = \sum_{\sigma \in F_b} \sum_{\rho \in F_a} Q_\sigma T Q_\rho.$$

The coordinate changes $J = \mathcal{A}_Q \mathcal{S}_P$ and $C = \mathcal{A}_P \mathcal{S}_Q$ satisfy $CJ = I$ and $JC = I$ on their packet coordinate spaces. Refinement is exact before any child packet is discarded.

Proof. Expand the two finite projector sums to obtain the block formula. Apply SG-01 twice to compute $CJ = \mathcal{A}_P \mathcal{S}_Q \mathcal{A}_Q \mathcal{S}_P = \mathcal{A}_P \mathcal{S}_P = I$, and similarly $JC = I$. The identities concern the direct sums of packet images, not an unrestricted product with duplicated arbitrary ambient components.

□

Source lineage: seed, prop:block-refinement; thm:coordinate-refinement.

Domain and hypotheses. Vector spaces with an invertible transform. Let $V : X \to Y$ be an isomorphism and (E_a) a finite complete packet family on Y.

Theorem 2.4 (SG-04 Pulled-back transform packets)**.** *The operators $P_a = V^{-1}E_aV$ are a complete packet family on X. For a unitary V and orthogonal coordinate selectors E_a in a Hilbert space,*

$$\|x\|^2 = \sum_a \|P_a x\|^2.$$

Proof. Compute $P_aP_b = V^{-1}E_aE_bV = \delta_{ab}P_a$ and $\sum_a P_a = V^{-1}IV = I$. In the unitary case the pulled-back projections are orthogonal: $P_a^* = P_a$. Their images are mutually orthogonal, and SG-01 writes x as their finite orthogonal sum. The Pythagorean identity gives the energy formula. Without unitarity and orthogonality the algebraic identities remain valid while that norm identity need not hold.

□

Source lineage: seed, thm:transform-atlas; Area II Fourier analysis.

Domain and hypotheses. Absolutely summable sequences on $\mathbb{Z}$. Fix an integer $N \geq 1$. Let $h, x \in \ell^1(\mathbb{Z})$ and $y = h * x$. Define $x_a[q] = x[Nq + a]$ and $y_b[q] = y[Nq + b]$, for $0 \leq a, b < N$.

Theorem 2.5 (SG-05 Polyphase analysis and convolution kernels)**.** *Interleaving the N channels recovers x exactly. With*

$$h_{ba}[k] = h[Nk + b - a],$$

the output channels satisfy

$$y_b[q] = \sum_{a=0}^{N-1} \sum_{t\in\mathbb{Z}} h_{ba}[q - t]x_a[t].$$

Thus convolution becomes an $N \times N$ transfer-kernel system.

Proof. Every integer has a unique Euclidean decomposition $Nt + a$, so the analysis and interleaving maps are inverse. In the convolution sum for $y[Nq + b]$, split the input index m as $Nt + a$. Then the kernel index becomes $Nq + b - m = N(q - t) + b - a$, which is precisely $h_{ba}[q - t]$. Absolute summability permits the regrouping: the total absolute double sum is bounded by the corresponding convolution of $|h|$ and $|x|$. The formula also holds for finitely supported sequences directly. It is the sequence version of SG-02, with the residue projectors selecting addresses.

□

Source lineage: seed, Area II exact polyphase analysis and synthesis.

Domain and hypotheses. Unit-modulus exponential nodes. Let $z \in \mathbb{C}$, $|z| = 1$, and suppose $u = z^N$ and $w_j = z^{r_j}$ are known, with $N \geq 1$ and integer offsets r_j. Let $d = \gcd(N, r_1, \ldots, r_k)$.

Theorem 2.6 (SG-06 The arithmetic of residual alias ambiguity)**.** *Exactly d unit-modulus nodes satisfy the same data. Hence the physical node is unique precisely when $d = 1$. If integers $c, b_1, \ldots, b_k$ obey $Nc + \sum_j r_jb_j = d$, then*

$$u^c \prod_j w_j^{b_j} = z^d.$$

For $d = 1$ this is an explicit exact de-aliasing formula.

Proof. If z' gives the same data, the quotient $v = z'/z$ satisfies $v^N = v^{r_1} = \cdots = v^{r_k} = 1$. Bezout's identity gives $v^d = 1$. Conversely d divides every supplied exponent, so every dth root of unity has all those powers equal to one. Therefore the fiber consists exactly of z multiplied by the d distinct dth roots

of unity. Applying the Bezout identity directly to the measured powers gives the displayed formula. Negative integer powers are legitimate because the data are nonzero.

□

Source lineage: self, Why decimation helps and why it aliases; Offset coefficients and internal phase recovery.

The offset data in SG-06 must be actual powers of the same node. In the multi-edge problem they are obtained only after matching coefficient vectors to common aliased nodes. A gcd computation does not bypass that matching or undo a collision of two edges in a single decimated channel. Adjacent offsets supply the exponent 1, so they remove the scalar branch ambiguity once the individual coefficient progressions have been resolved.

3 Identifiability and its unavoidable boundaries

Domain and hypotheses. An arbitrary measurement model and linear packet classes. Let $M : X \to Y$ be any map and let an edge assignment $J : X \to \mathcal{J}$ be specified. Separately, let $\mathcal{S}, \mathcal{J}$ be linear subspaces of a common vector space.

Theorem 3.1 (SG-07 Measurement and decomposition obstructions)**.** *If $Mf_0 = Mf_1$ but $J(f_0) \neq J(f_1)$, no map $R : Y \to \mathcal{J}$ recovers the correct edge for both. If $\mathcal{S} \cap \mathcal{J} \neq \{0\}$, the sum decomposition $f = s + j$ is not unique. In particular, finitely many point samples on a periodic interval cannot recover every arbitrary interval packet.*

Proof. A recovery map has one value at the common datum, so it cannot equal the two different required edges. For the decomposition obstruction take nonzero $h \in \mathcal{S} \cap \mathcal{J}$: the pairs (s, j) and $(s + h, j - h)$ have the same sum. Finally, the complement of a finite sample set contains a nonempty open interval. A nonzero indicator supported inside that interval vanishes at every sample point. Compare it with the zero signal to obtain the first obstruction concretely.

□

Source lineage: unknown, The measurement obstruction; The decomposition obstruction.

Domain and hypotheses. Two-jump periodic packets of fixed known amplitudes. Fix a finite set $F \subset \mathbb{Z}$ of derivative-moment indices. For $0 < \delta < \pi/4$, compare locations $(0, \delta)$ and $(\pi, \pi + \delta)$, both with amplitudes $(1, -1)$.

Theorem 3.2 (SG-08 No uniform location stability without separation)**.** *Their unordered location sets have permutation-matched bottleneck distance at least $\pi - \delta$, while their measured derivative moments satisfy*

$$|m_n - \widetilde{m}_n| = |(1 - e^{-in\pi})(1 - e^{-in\delta})| \leq 2|n|\delta \qquad (n \in F).$$

There is no Lipschitz inverse for locations with one constant on all distinct two-jump configurations, even at this fixed order and with known amplitudes.

Proof. Every circular distance from a point of $\{0, \delta\}$ to a point of $\{\pi, \pi + \delta\}$ is at least $\pi - \delta$. This bounds every matching. The moments of the first packet are $1 - e^{-in\delta}$; translating by π multiplies them by $e^{-in\pi}$. Subtraction gives the formula. The elementary inequality $|1 - e^{-it}| \leq |t|$ proves the bound. Every fixed finite norm of the measurement difference tends to zero as $\delta \to 0$, whereas the location distance stays bounded away from zero. A common finite Lipschitz constant is therefore impossible. This argument

compares two admissible fixed-order packets; it does not require assigning location parameters to the zero-edge signal.

□

Source lineage: unknown, A collapsing pair of jumps; Collision is the unavoidable hard boundary.

Publication correction. The original collision argument compared an edge parameter vector with the zero packet. The published proof compares two fixed-order configurations and specifies a permutation-invariant location metric.

4 Known edges and the source of Gibbs ringing

The simplest reconstruction problem already separates discovery from synthesis. Suppose a singular packet is known, but its amplitude and the smooth background are not. A high-frequency functional can identify the amplitude exactly on a bandlimited background. The recovered singular packet then has its own synthesis route.

Domain and hypotheses. Periodic integrable signals with one known edge shape. Let $\mathcal{S}_K$ be the trigonometric polynomials of degree at most K. Let $m \in L^1(\mathbb{T})$ and a finite nonempty set $F \subset \{n : |n| > K\}$ satisfy $D_F = \sum_{n\in F} |\widehat{m}(n)|^2 > 0$.

Theorem 4.1 (SG-09 Out-of-band matched identification)**.** *On $X = \mathcal{S}_K + \operatorname{span}\{m\}$, the functional*

$$\ell_F(g) = D_F^{-1} \sum_{n\in F} \overline{\widehat{m}(n)}\widehat{g}(n)$$

satisfies $\ell_F(s + am) = a$. The decomposition is direct. If the measured coefficients have errors η_n, then

$$|\widetilde{a} - a| \le D_F^{-1/2}\Big(\sum_{n\in F} |\eta_n|^2\Big)^{1/2}.$$

Proof. The coefficients of $s \in \mathcal{S}_K$ vanish on F. Substituting $s + am$ into the functional leaves $aD_F/D_F = a$. If m belonged to $\mathcal{S}_K$, all its coefficients on F would vanish, contradicting $D_F > 0$; hence the sum is direct. The amplitude error is the inner product of the coefficient error with the known coefficient vector divided by its squared norm. Cauchy–Schwarz gives the displayed bound. The frequencies must actually see the edge: an all-zero identifying set makes the formula undefined.

□

Source lineage: singularity, An edge functional from out-of-band modes.

Domain and hypotheses. The model space of SG-09. Let $P_J g = \ell_F(g)m$, $P_S = I - P_J$, and let S_M be Fourier truncation of degree $M \ge K$.

Theorem 4.2 (SG-10 The singularity-exact block operator)**.**

$$P_J^2 = P_J, \quad P_S^2 = P_S, \quad P_S P_J = P_J P_S = 0, \quad P_S + P_J = I_X,$$

and the resolved reconstruction obeys

$$\mathcal{R}_M = S_M P_S + P_J = I_X.$$

Proof. For $g = s + am$, SG-09 gives $P_J g = am$ and $P_S g = s$. These coordinate formulas immediately imply idempotence and pairwise annihilation. Since s is bandlimited and $M \geq K$, $S_M s = s$. Thus $\mathcal{R}_M g = s + am = g$. The equality is in the declared signal class; for a selected right-continuous packet it also reproduces that representative. This is a known-packet theorem. An unknown interval still needs the discovery step proved later.

□

Source lineage: singularity, The singularity projector; The singularity-exact reconstruction theorem; *gibbs*, The first exact packet decomposition.

Domain and hypotheses. Periodic signals with a supplied exact edge packet. Let $f = s + j$ with $|\widehat{s}(n)| \leq Ce^{-\rho|n|}$, where $C \geq 0$ and $\rho > 0$. Use the canonical uniformly convergent Fourier-series representative of s. Set $\mathcal{R}_M f = S_M s + j$.

Theorem 4.3 (SG-11 Resolved error for an analytic background)**.** *The reconstruction error is exactly* $f - \mathcal{R}_M f = s - S_M s$, *and*

$$\|f - \mathcal{R}_M f\|_\infty \leq \frac{2Ce^{-\rho(M+1)}}{1 - e^{-\rho}}.$$

The bound is uniform across the jumps of j.

Proof. Cancel j from the defining identity. Absolute summability of the Fourier coefficients gives an everywhere uniformly convergent series for s. The omitted positive and negative frequencies are bounded by $2C\sum_{n>M} e^{-\rho n}$, which is the displayed geometric series. No limiting trigonometric approximation of j occurs in this estimate.

□

Source lineage: gibbs, Analytic smooth background.

Domain and hypotheses. The interval packet $j = \mathbf{1}_{[-a,a)}$ on $\mathbb{T}$. Assume $0 < a < \pi$, let $\lambda_M = M + 1/2$, and use the right edge coordinate $x_M(z) = a - z/\lambda_M$.

Theorem 4.4 (SG-12 The scaled interval-edge Gibbs profile)**.** *Uniformly for* z *in compact real intervals,*

$$S_M j(x_M(z)) \longrightarrow \frac{1}{2} + \frac{1}{\pi}\operatorname{Si}(z), \qquad \operatorname{Si}(z) = \int_0^z \frac{\sin t}{t}\, dt.$$

The first positive maximum of the limiting profile is $1 + G$, *where*

$$G = \frac{\operatorname{Si}(\pi)}{\pi} - \frac{1}{2} = 0.089489872236\ldots.$$

Proof. Direct integration gives $\widehat{j}(0) = a/\pi$ and $\widehat{j}(n) = \sin(na)/(\pi n)$ for $n \neq 0$. The Dirichlet kernel $D_M(t) = \sin(\lambda_M t)/\sin(t/2)$ yields

$$S_M j(x) = \frac{1}{2\pi}\int_{-a}^{a} D_M(x - t)\, dt, \qquad (S_M j)'(x) = \frac{D_M(x + a) - D_M(x - a)}{2\pi}.$$

At $x = a$, the coefficient formula gives

$$S_M j(a) = \frac{a}{\pi} + \frac{1}{\pi}\sum_{n=1}^{M} \frac{\sin(2na)}{n} \longrightarrow \frac{1}{2}.$$

For completeness, the sine-series limit follows by summation by parts from bounded partial sums of e^{2ina} and by taking radial limits in $\sum_{n\geq 1} r^n e^{2ina}/n = -\log(1-re^{2ia})$; its imaginary part tends to $(\pi - 2a)/2$ for $0 < 2a < 2\pi$. Now differentiate the scaled function:

$$\frac{d}{dz} S_M j(x_M(z)) = \frac{D_M(-z/\lambda_M) - D_M(2a - z/\lambda_M)}{2\pi\lambda_M}.$$

The first term converges uniformly on compact z sets to $\sin z/(\pi z)$, with its continuous value at zero. The second tends uniformly to zero because $\sin(a - z/(2\lambda_M))$ stays bounded away from zero there. Integrate from 0 to z and use the base-point limit. The derivative of the limiting profile is $\sin z/(\pi z)$, positive on $(0,\pi)$ and negative on $(\pi, 2\pi)$, so its first positive maximum is at $z = \pi$. Substitution gives G. The constant is relative to a unit jump; scaling a packet scales its overshoot by the jump amplitude.

□

Source lineage: gibbs, The scaled Gibbs limit; *singularity,* Detailed proof of the local scaling limit.

Domain and hypotheses. Periodic bounded real signals. Let $\sigma_M = (M+1)^{-1} \sum_{k=0}^{M} S_k$ be the Fejer mean. Suppose $0 \leq j \leq 1$ almost everywhere.

Theorem 4.5 (SG-13 Positive Fourier averaging preserves range)**.** *Then $0 \leq \sigma_M j \leq 1$ everywhere. For an interval packet, this suppresses overshoot while retaining a continuous transition rather than an exact jump.*

Proof. The averaging kernel is

$$F_M(t) = \frac{1}{M+1} \left| \sum_{k=0}^{M} e^{ikt} \right|^2 \geq 0, \qquad \frac{1}{2\pi} \int_0^{2\pi} F_M(t)\, dt = 1.$$

Its convolution with j is therefore a weighted mean of numbers in $[0,1]$. The resulting finite trigonometric polynomial is continuous. A nonconstant interval indicator is discontinuous, so the mean cannot equal its selected representative at all points or have its exact one-sided jump.

□

Source lineage: gibbs, Fejer averaging.

5 Finite dictionaries and residualized geometric discovery

In a finite dictionary model the unknown is a coefficient vector. The smooth background is eliminated before testing the edge dictionary. This removal is exact only for a declared background subspace. A residualized dictionary can lose distinctions that the original dictionary had, so its kernel is the correct object to inspect.

Domain and hypotheses. Finite real or complex Euclidean spaces. Let $\mathcal{S} \subseteq \mathbb{F}^m$ be a subspace, P its orthogonal projector, $R = I - P$, $D \in \mathbb{F}^{m\times d}$, and $A = RD$. Let $s \geq 0$ be an integer.

Theorem 5.1 (SG-14 Residualized sparse injectivity)**.** *Every vector α with $\|\alpha\|_0 \leq s$ is uniquely determined by $A\alpha$ if and only if*

$$\ker A \cap \{h : \|h\|_0 \leq \min(2s, d)\} = \{0\}.$$

Equivalently every set of at most $\min(2s,d)$ columns is independent. With the convention $\operatorname{spark}(A) = d+1$ when all d columns are independent, the equivalent condition is $\operatorname{spark}(A) > \min(2s,d)$. For $s = 0$ uniqueness is automatic.

Proof. The difference of two s-sparse vectors has support of size at most $2s$ and at most d. Thus the kernel condition implies uniqueness. Conversely, if a nonzero kernel vector h has support size at most $2s$, partition that support into two sets of at most s indices. Let α equal h on the first set and zero elsewhere; let β equal $-h$ on the second set. Then $h = \alpha - \beta$, and $A\alpha = A\beta$ contradicts uniqueness. Column dependence is exactly the existence of a nonzero kernel vector supported on the selected columns. The spark convention then gives the stated capped inequality, including a fully independent matrix when $2s > d$. When $s = 0$, the only admissible vector is zero and the kernel test concerns only zero. If $y = s_0 + D\alpha$ with $s_0 \in \mathcal{S}$, then $Ry = A\alpha$, so the same theorem applies after background elimination.

□

Source lineage: unknown, The residualized secant theorem; *complex*, Two-dimensional finite dictionary model.

Publication correction. The uncapped inequality spark(A)>2s is replaced by the kernel-secant condition, or its capped spark equivalent. This also covers full-column-rank dictionaries with 2s greater than the number of columns.

Domain and hypotheses. The Euclidean model of SG-14. Assume $s \geq 1$, $d \geq 1$, and the injectivity condition of SG-14. Let $y = s_0 + D\alpha_0 + \eta$, $s_0 \in \mathcal{S}$, $\|\alpha_0\|_0 \leq s$, and $\|R\eta\|_2 \leq \epsilon$. Let $\widehat{\alpha}$ be s-sparse with $\|A\widehat{\alpha} - Ry\|_2 \leq \epsilon$.

Theorem 5.2 (SG-15 Sparse noise stability and support thresholding)**.** *Define*

$$\gamma = \min_{\substack{\|h\|_2=1 \\ \|h\|_0 \leq \min(2s,d)}} \|Ah\|_2 > 0.$$

Then

$$\|\widehat{\alpha} - \alpha_0\|_2 \leq 2\epsilon/\gamma, \qquad \|D(\widehat{\alpha} - \alpha_0)\|_2 \leq 2\|D\|_2\epsilon/\gamma.$$

If every nonzero true coefficient has magnitude greater than $4\epsilon/\gamma$, thresholding at $2\epsilon/\gamma$ with a strict greater-than rule recovers the true support.

Proof. The admissible unit vectors form a finite union of compact coordinate spheres. The continuous function $\|Ah\|_2$ attains its minimum. A zero minimum would contradict SG-14, hence $\gamma > 0$. For $h = \widehat{\alpha} - \alpha_0$, subtract the two residuals to obtain

$$\|Ah\|_2 \leq \|A\widehat{\alpha} - Ry\|_2 + \|R\eta\|_2 \leq 2\epsilon.$$

Its support is admissible, so homogeneity gives $\gamma\|h\|_2 \leq \|Ah\|_2$. Apply D for the packet bound. Every coordinate error is at most $2\epsilon/\gamma$. A true zero therefore never passes the strict threshold, while a true coefficient larger than twice that threshold remains strictly above it. For $s = 0$ both vectors vanish and no restricted constant is required.

□

Source lineage: unknown, A deterministic stability theorem.

Domain and hypotheses. Finite real dictionaries with positive amplitudes. Let B have full column rank, P be the orthogonal projector onto its range, $R = I - P$, and let $\mathcal{Q}$ be a finite mask library with $Rq \neq 0$ for every q. Suppose $y = B\theta_* + a_*q_*$, with $a_* > 0$ and $q_* \in \mathcal{Q}$.

Theorem 5.3 (SG-16 Profiled one-packet discovery and its finite gap)**.** *For an unrestricted scalar amplitude the least-squares profile is*

$$\widehat{a}(q) = \frac{\langle Rq, Ry\rangle}{\|Rq\|_2^2}, \qquad J(q) = \|Ry - \widehat{a}(q)Rq\|_2^2.$$

The true mask has $\widehat{a}(q_*) = a_*$ and $J(q_*) = 0$. It is the unique zero-residual positive-amplitude mask exactly when no other residualized mask lies on its positive ray. Define the closed rays $C_q = \{aRq : a \geq 0\}$. If

$$\Gamma = \inf_{q \neq q_*} \operatorname{dist}(Ry, C_q) > 0,$$

with the convention $\inf \varnothing = +\infty$, then every perturbed residualized datum $Ry + e$ with $\|e\|_2 < \Gamma/2$ selects q_* uniquely by nearest-ray distance.

Proof. Differentiate the quadratic function $\|Ry - aRq\|_2^2$ with respect to a. Its unique stationary point is the profile formula and is the global minimum. At the true mask $Ry = a_* Rq_*$, giving the stated values. A wrong mask gives an admissible zero residual precisely when $Ry = aRq$ for some $a > 0$. This is the positive-ray condition. For the noise statement, the distance to any nonempty closed set is 1-Lipschitz by the triangle inequality. The true ray has perturbed distance at most $\|e\|_2$, and every wrong ray has distance at least $\Gamma - \|e\|_2 > \|e\|_2$. Thus the true ray wins strictly. A squared objective gap must be square-rooted before it is used in this norm-distance condition. For positive constraints the optimum on a closed ray uses $\max(0, \widehat{a}(q))$, rather than an unconditionally signed amplitude.

□

Source lineage: gibbs, Profile least squares over interval packets; The finite identifiability gap; *singularity*, Noise stability of finite packet discovery.

Domain and hypotheses. Finite binary masks. Let $q, \widetilde{q} \in \{0,1\}^m$ and $a, \widetilde{a} \in \mathbb{R}$.

Theorem 5.4 (SG-17 Mask error is a support geometry calculation)**.**

$$\begin{aligned}\|aq - \widetilde{a}\widetilde{q}\|_2^2 = {} & |a - \widetilde{a}|^2 |\operatorname{supp} q \cap \operatorname{supp} \widetilde{q}| \\ & + |a|^2 |\operatorname{supp} q \setminus \operatorname{supp} \widetilde{q}| + |\widetilde{a}|^2 |\operatorname{supp} \widetilde{q} \setminus \operatorname{supp} q|.\end{aligned}$$

For $a = \widetilde{a} = 1$ the squared norm is the cardinality of the symmetric support difference.

Proof. Partition the coordinates into common support, first-only support, second-only support, and neither. The difference equals respectively $a - \widetilde{a}$, a, $-\widetilde{a}$, and zero. Sum the squared values. The same pointwise partition gives a continuous-measure formula when support cardinalities are replaced by support measures.

□

Source lineage: gibbs, Packet perturbation identity.

6 Exact off-grid recovery from finitely many Fourier coefficients

Let $\mathcal{U}_{K,L}$ consist of real periodic signals $f = s + j$, where $s \in \mathcal{S}_K$ and j is right-continuous, piecewise constant, mean zero, and has at most L distinct active jumps. An active jump has nonzero amplitude. For an actual jump count r, write

$$Dj = \sum_{\ell=1}^{r} a_\ell \delta_{\tau_\ell}, \quad z_\ell = e^{-i\tau_\ell}, \quad \sum_{\ell} a_\ell = 0.$$

The sum-zero condition is required by periodicity. A single nonzero active jump cannot occur, so the nonconstant real class starts at $r = 2$. The case $r = 0$ is allowed and will be handled explicitly. The

smooth-edge representation is unique with this normalization: a piecewise-constant trigonometric polynomial is constant, and mean zero fixes that constant.

Domain and hypotheses. The class $\mathcal{U}_{K,L}$. Use the Fourier convention in Section 1. Let $n > K$ and $z_\ell = e^{-i\tau_\ell}$.

Theorem 6.1 (SG-18 Differentiation isolates an exponential moment packet)**.**

$$Dj = \sum_\ell a_\ell \delta_{\tau_\ell}, \qquad m_n := 2\pi i n \widehat{f}(n) = \sum_{\ell=1}^{r} a_\ell z_\ell^n.$$

For a non-bandlimited integrable background the same formula holds with an additive defect $e_n = 2\pi i n \widehat{s}(n)$.

Proof. For a smooth periodic test function φ, integrate by parts over each constant segment of j. Boundary terms telescope at each jump to $a_\ell \varphi(\tau_\ell)$, which is exactly the displayed distribution. Evaluate the distribution on e^{-inx}:

$$\langle Dj, e^{-inx} \rangle = -\int j(x)(-in)e^{-inx}\,dx = 2\pi i n \widehat{j}(n).$$

The Dirac sum gives $\sum_\ell a_\ell e^{-in\tau_\ell}$. For $n > K$, the smooth polynomial contributes zero. Without bandlimiting it contributes the stated defect. At $n = 0$ the derivative moment is zero, in agreement with $\sum a_\ell = 0$; it cannot determine the signal mean.

□

Source lineage: unknown, The derivative packet; Removing the unknown smooth background; *self*, Signal class and Fourier convention.

Domain and hypotheses. Finite exponential moments over $\mathbb{C}$. Let $m_{n_0+k} = \sum_{\ell=1}^{r} c_\ell z_\ell^k$, with distinct nonzero nodes and nonzero coefficients. Let $H_{h,w} = (m_{n_0+i+j})_{0\le i<h,\ 0\le j<w}$, with $h, w \ge r$.

Theorem 6.2 (SG-19 Hankel rank detects the actual order)**.** *Writing $V_h = (z_\ell^i)_{i,\ell}$ and $V_w = (z_\ell^j)_{j,\ell}$,*

$$H_{h,w} = V_h \operatorname{diag}(c_\ell) V_w^{\mathsf{T}}, \qquad \operatorname{rank} H_{h,w} = r.$$

For $r = 0$ the Hankel matrix is zero. The transpose is algebraic and is not a conjugate transpose.

Proof. Expand a matrix entry to obtain the factorization. The leading $r \times r$ Vandermonde submatrix has determinant $\prod_{\ell<k}(z_k - z_\ell)$, which is nonzero. Thus both rectangular Vandermonde factors have full column rank, and their transposes have full row rank. The diagonal factor is invertible. Its product with V_w^T is surjective onto $\mathbb{C}^r$, and subsequent multiplication by the injective V_h has image dimension r. For an empty exponential sum every entry is zero and the rank is zero.

□

Source lineage: unknown, The Hankel packet and its rank; *self*, Unknown jump count from a Hankel gap.

Domain and hypotheses. The nonzero-order moment model of SG-19. Assume $r \ge 1$ and $2r$ consecutive moments $m_{n_0}, \dots, m_{n_0+2r-1}$ are known.

Theorem 6.3 (SG-20 Exact Prony annihilation and location recovery)**.** *The unique monic annihilator*

$$Q(z) = \prod_{\ell=1}^{r}(z - z_\ell) = \sum_{k=0}^{r-1} q_k z^k + z^r$$

is recovered from

$$H_{r,r}q = -v, \qquad v = (m_{n_0+r+i})_{i=0}^{r-1}.$$

Its roots are exactly the nodes. Once the nodes are known, r consecutive moments determine the coefficients uniquely by a Vandermonde solve. For unit-modulus nodes the corresponding locations are $\tau_\ell = -\arg z_\ell \pmod{2\pi}$.

Proof. For every available recurrence index,

$$\sum_{k=0}^{r} q_k m_{n+k} = \sum_{\ell} c_\ell z_\ell^{n-n_0} Q(z_\ell) = 0, \qquad q_r = 1.$$

SG-19 makes $H_{r,r}$ invertible. Thus the system has one solution and the true polynomial supplies it. Factoring the resulting polynomial recovers the distinct nodes up to a permutation. The amplitude matrix for consecutive moments is a square Vandermonde multiplied by an invertible diagonal node power matrix, so it too is invertible. The argument convention identifies locations modulo a period, as required. Polynomial root extraction is a finite algebraic step; the theorem does not claim that arbitrary real locations have rational closed forms.

□

Source lineage: unknown, The annihilating polynomial; Recovering jump amplitudes and the edge packet.

Domain and hypotheses. A period of length $T > 0$. Let distinct locations $\tau_\ell \in \mathbb{R}/(T\mathbb{Z})$ and real amplitudes a_ℓ satisfy $\sum_\ell a_\ell = 0$. Define the mean-zero periodic sawtooth $G_T(x) = 1/2 - x/T$ for $0 < x < T$, with the right-continuous endpoint value.

Theorem 6.4 (SG-21 Canonical periodic integration of jump data)**.**

$$j(x) = \sum_{\ell} a_\ell G_T(x - \tau_\ell)$$

is the unique mean-zero right-continuous piecewise-constant packet with those jumps. Its distributional derivative is $\sum_\ell a_\ell \delta_{\tau_\ell}$.

Proof. The sawtooth derivative is $DG_T = \delta_0 - T^{-1}dx$: its slope is $-1/T$ away from the endpoint and its endpoint jump is one. Summing translated copies gives the prescribed Dirac masses and a constant-density term $-T^{-1}\sum a_\ell$, which vanishes. Consequently j is constant between the jump points. Each translated sawtooth has mean zero, so j does too. Any other packet with the same jumps has distributional derivative difference zero. Since both are piecewise constant, their difference is the same constant on every segment and across every jump. The mean-zero condition sets it to zero almost everywhere, and right continuity fixes the endpoint values. This explicit formula also supplies the perturbation estimate later.

□

Source lineage: unknown, Periodic integration.

Domain and hypotheses. The real class $\mathcal{U}_{K,L}$. Let $K \geq 0$ and $L \geq 1$ be known integers. Assume exact Fourier coefficients for $-K \leq n \leq K$ and $K+1 \leq n \leq K+2L$.

Theorem 6.5 (SG-22 Finite exact unknown-order reconstruction)**.** *Those data uniquely determine the actual active order r, every jump location and amplitude up to permutation, the canonical edge packet j, the smooth*

coefficients of s, and the complete signal f as an L^1 class and as its selected right-continuous representative. The high-frequency budget is $2L$ complex coefficients. The low budget is $2K+1$ coefficients, with conjugate symmetry redundancy for a real signal.

Proof. Use SG-18 to form $2L$ exact high derivative moments. With $n_0 = K+1$, form $H_{L,L}$ using the first $2L-1$ of them. SG-19 says its rank is the actual r. If $r = 0$, $j = 0$ and the low coefficients directly give s. Otherwise the first $2r$ moments are available. SG-20 recovers the nodes and amplitudes. SG-21 constructs the unique normalized edge. For each $|n| \le K$, subtract its coefficient from $\widehat{f}(n)$ to obtain $\widehat{s}(n)$; all other smooth coefficients are zero by the model. Their finite trigonometric sum gives s. Every step is uniquely determined, so another signal in the class with those data must have the same edge and background. Finally add $s + j$. Values at jump points are selected by the stated convention, rather than inferred from measure-zero Fourier data.

□

Source lineage: unknown, The universal off-grid recovery theorem.

Publication correction. Exactness from Fourier data is stated for the L1 class and its declared right-continuous representative. Arbitrarily assigned point values at discontinuities are not recoverable from Fourier integrals.

The measurement bound is a sufficient budget for this construction, not a minimal sample theorem. Exact order is an algebraic rank. A floating-point rank threshold requires a quantitative gap, addressed below. Distinct nodes need not have a prescribed minimum separation for exact uniqueness, but stability constants deteriorate as nodes approach one another.

7 Perturbation contracts for the finite recovery steps

Domain and hypotheses. A rectangular Hankel matrix of derivative moments. Let measured Fourier coefficients have errors η_n and moment errors $\epsilon_n = 2\pi i n \eta_n$. Let $E = (\epsilon_{n_0+i+j})$ have $h \ge 1$ rows and $w \ge 1$ columns.

Theorem 7.1 (SG-23 Hankel perturbation from Fourier noise)**.**

$$\|E\|_2 \le \|E\|_F \le \sqrt{hw} \max_{0 \le i < h,\ 0 \le j < w} |\epsilon_{n_0+i+j}|.$$

In particular, an error bound $|\eta_n| \le \eta$ implies

$$\|E\|_2 \le 2\pi\sqrt{hw}\,\eta \max_{i,j} |n_0 + i + j|.$$

Proof. Differentiate the noisy Fourier coefficients to obtain the exact moment error identity. The spectral norm is at most the Frobenius norm. Its square is the sum of hw squared entry magnitudes, each bounded by the squared maximum. Substitute $|\epsilon_n| = 2\pi|n||\eta_n|$. Repeated moments occur at multiple matrix entries; the Frobenius bound correctly includes those repetitions. Smooth background tails can be included in ϵ_n by the same calculation.

□

Source lineage: unknown, Fourier noise becomes derivative-moment noise; Hankel perturbation.

Domain and hypotheses. The invertible square Prony system of SG-20. Write $Hq = -v$, and suppose $\widetilde{H} = H + E$, $\widetilde{v} = v + e$. Assume $\|H^{-1}\|_2 \|E\|_2 < 1$.

Theorem 7.2 (SG-24 Annihilator coefficient perturbation)**.** *The perturbed system is invertible, and its solution satisfies*

$$\|\widetilde{q} - q\|_2 \le \frac{\|H^{-1}\|_2}{1 - \|H^{-1}\|_2\|E\|_2}(\|e\|_2 + \|E\|_2\|q\|_2).$$

Proof. Factor $\widetilde{H} = H(I + H^{-1}E)$. The Neumann series converges in operator norm under the supplied inequality, so $\|\widetilde{H}^{-1}\|_2 \le \|H^{-1}\|_2/(1 - \|H^{-1}\|_2\|E\|_2)$. Subtract the equations to obtain $\widetilde{H}(\widetilde{q} - q) = -e - Eq$. Apply the inverse bound and the triangle inequality. The estimate assumes that the same true order is used; it does not itself choose the order from noisy data.

□

Source lineage: unknown, Hankel perturbation; Root localization by Rouche theorem.

Domain and hypotheses. Monic complex polynomials of the same degree. Let $r \ge 1$ and $Q(z) = \prod_{\ell=1}^{r}(z - z_\ell)$ have distinct roots and $\widetilde{Q} = Q + E_Q$. Choose $0 < \rho < \frac{1}{2}\min_{\ell \ne k}|z_\ell - z_k|$ when $r \ge 2$, or any $\rho > 0$ when $r = 1$.

Theorem 7.3 (SG-25 Conditional simple-root disk localization)**.** *If for every ℓ*

$$\sup_{|z - z_\ell| = \rho} |E_Q(z)| < \min_{|z - z_\ell| = \rho} |Q(z)|,$$

then $\widetilde{Q}$ has exactly one root, counted with multiplicity, in each disk $|z - z_\ell| < \rho$. All its roots are thereby matched with error less than ρ. A sufficient explicit bound on each circle is

$$\sum_{k=0}^{r-1} |\widetilde{q}_k - q_k|(|z_\ell| + \rho)^k < \rho \prod_{j \ne \ell}(|z_\ell - z_j| - \rho).$$

Proof. Each disk contains exactly one root of Q. The strict boundary inequality is the hypothesis of Rouche's theorem, which preserves the number of zeros inside. There are r disjoint disks, each containing one zero of the degree r perturbed polynomial, so they account for every zero. For the explicit version, apply the triangle inequality to the coefficient error polynomial and use $|z| \le |z_\ell| + \rho$. On the circle, $|z - z_\ell| = \rho$ and $|z - z_j| \ge |z_\ell - z_j| - \rho$, giving the lower product bound. For unit true nodes and a recovered root inside a disk with $\rho < 1$, radial normalization changes its chord error to at most 2ρ by the triangle inequality. Its angular location error is then at most $2\arcsin\rho$. The disk radius is a complex-root error, not directly an angular error.

□

Source lineage: unknown, Root localization by Rouche theorem.

Publication correction. This is a conditional stability theorem involving true root centers and true coefficient errors. It is not a certificate computable from noisy data alone, and a complex disk radius is not asserted to equal the angular location error.

Domain and hypotheses. A square or overdetermined Vandermonde amplitude fit. Let V have full column rank, $m = Va$, $\widetilde{V} = V + \Delta V$, and $\widetilde{m} = m + e$. Assume $\sigma_{\min}(V) > \|\Delta V\|_2$. Let $\widehat{a}$ minimize $\|\widetilde{V}b - \widetilde{m}\|_2$.

Theorem 7.4 (SG-26 Amplitude stability with perturbed nodes)**.**

$$\|\widehat{a} - a\|_2 \le \frac{\|e\|_2 + \|\Delta V\|_2\|a\|_2}{\sigma_{\min}(V) - \|\Delta V\|_2}.$$

For unit nodes in rows with nonnegative powers n, matched chord error $|\widehat{z}_\ell - z_\ell| \le \delta$ gives the entry bound $|\widehat{z}_\ell^{\,n} - z_\ell^{\,n}| \le n\delta$.

Proof. The minimum singular value of $\widetilde{V}$ is at least $\sigma_{\min}(V) - \|\Delta V\|_2 > 0$. Thus its least-squares inverse exists. Since $\widetilde{V}a = m + \Delta V a$,

$$\widehat{a} - a = \widetilde{V}^{\dagger}(e - \Delta V a).$$

Apply $\|\widetilde{V}^{\dagger}\|_2 = 1/\sigma_{\min}(\widetilde{V})$. The power bound follows by factoring $\widehat{z}^n - z^n = (\widehat{z} - z)\sum_{k=0}^{n-1} \widehat{z}^{n-1-k} z^k$; each summand has modulus one. The statement assumes aligned nodes and unit normalization. Arbitrary unnormalized roots require extra modulus factors.

□

Source lineage: unknown, Amplitude stability.

8 Self-calibration of polyphase aliases

Decimation reorganizes the differentiated moments, and it also maps each physical node z to z^N. This can improve a selected geometry or create a collision. The physical phase is retained in the offset coefficients. The central recovery step uses those coefficients rather than choosing a principal root by convention. The shifted-exponential ingredient is classical [2]; the contracts below specify what is sufficient for its use in the complete unknown-order pipeline.

Domain and hypotheses. Derivative moments with an additive defect. Fix integers $N \geq 2$, $q_0 \geq 1$, and an offset $r \geq 0$. Suppose $m_n = \sum_\ell a_\ell z_\ell^n + e_n$, with $|z_\ell| = 1$.

Theorem 8.1 (SG-27 Shared-node offset channels)**.**

$$m_{(q_0+q)N+r} = \sum_\ell b_{\ell,r} u_\ell^q + e_{(q_0+q)N+r}, \quad u_\ell = z_\ell^N, \quad b_{\ell,r} = a_\ell u_\ell^{q_0} z_\ell^r.$$

For each matched edge $b_{\ell,r+1} = z_\ell b_{\ell,r}$. If $|\widehat{s}(n)| \leq C_s(1+|n|)^{-p-2}$ with $p \geq 0$, then for nonnegative q, r

$$|e_{(q_0+q)N+r}| \leq 2\pi C_s (q_0 N)^{-p-1}.$$

Proof. Split the exponent as $(q_0+q)N + r = q_0 N + r + qN$ to obtain the channel identity. Incrementing the offset multiplies only z_ℓ^r by z_ℓ. For the tail bound, use $|e_n| = 2\pi|n||\widehat{s}(n)| \leq 2\pi C_s(1+|n|)^{-p-1}$ and $n \geq q_0 N > 0$. The channel identity is valid even when aliased nodes collide, but a Prony solve on that channel then observes combined coefficients and cannot separate those nodes using the simple distinct-node theorem.

□

Source lineage: self, Why decimation helps and why it aliases; *complex*, One-dimensional finite-rate packet model.

Domain and hypotheses. Finite rectangular complex matrices. Let $H = H_0 + E$, with H_0 of rank $r \geq 1$, $k = \min(h, w) \geq r$, and $\epsilon = \|E\|_2 < \sigma_r(H_0)/2$. Supply a threshold t with $\epsilon < t < \sigma_r(H_0) - \epsilon$.

Theorem 8.2 (SG-28 Order recovery with a supplied singular threshold)**.** *Exactly r singular values of H exceed t. If $r = 0$, a supplied threshold $t > \|E\|_2$ selects zero. If $r = k$, all k singular values exceed the threshold and no nonexistent σ_{k+1} is used.*

Proof. The singular-value perturbation inequality [1] gives $|\sigma_j(H) - \sigma_j(H_0)| \leq \|E\|_2$ for $1 \leq j \leq k$. It holds equally over the complex field: in the min–max characterization

$$\sigma_j(A) = \min_{\dim V = w-j+1} \max_{x \in V,\, \|x\|=1} \|Ax\|.$$

the triangle inequality bounds each restricted maximum for $A + E$ by that for A plus $\|E\|_2$. Minimize and then interchange the two matrices. For $j \le r$ this yields $\sigma_j(H) \ge \sigma_r(H_0) - \epsilon > t$. For $j > r$, when such indices exist, it gives $\sigma_j(H) \le \epsilon < t$. For rank zero the same upper bound applies to every singular value. For full rectangular rank the lower bound already accounts for every singular value. The theorem assumes a threshold satisfying the inequalities; it does not prove that every heuristic gap selector finds it.

□

Source lineage: self, Unknown jump count from a Hankel gap.

8.1 Why an empirical rank rule is a separate algorithm

The native implementation estimates order from singular-value ratios and validation heuristics. A largest adjacent ratio is not implied to select the true order by SG-28. An exact witness makes the distinction concrete. Take four-by-four Hankel matrices of moments beginning at $n_0 = 4$ from the nodes $(1, i, -1)$ and real amplitudes $(100, -99, -1)$. The four-point Fourier columns are orthogonal, so the true singular values are $(400, 396, 4, 0)$. Add the moment perturbation $0.1(-i)^n$. It contributes a fourth orthogonal column and produces singular values $(400, 396, 4, 0.4)$ with perturbation norm 0.4. The hypothesis $0.4 < 4/2$ holds. A threshold in $(0.4, 3.6)$ selects the true order three, while the largest adjacent ratio $396/4 = 99$ selects order two. Thus a valid quantitative threshold theorem and a successful practical heuristic remain different claims.

Domain and hypotheses. One matched unit-circle coefficient progression. Let $R \ge 1$ be an integer counting adjacent products, and let $c_r = bz^r$ for $0 \le r \le R$, with $b \ne 0$ and $|z| = 1$. Define $S(c) = \sum_{r=0}^{R-1} \overline{c_r} c_{r+1}$.

Theorem 8.3 (SG-29 Exact phase from adjacent offset products)**.**

$$S(c) = R|b|^2 z \ne 0, \qquad \Phi(c) := S(c)/|S(c)| = z.$$

Two channels already give $z = c_1/c_0$ exactly. More channels provide repeated adjacent relations, without an unconditional deterministic averaging gain.

Proof. Each summand is $\overline{bz^r}\, bz^{r+1} = |b|^2 z$ because $|z| = 1$. There are R summands and $R \ge 1$, so normalization returns z. For two channels direct division cancels b. No statistical independence is assumed, so one cannot improve a worst-case noise order merely by increasing R; the next bound shows the cancellation of the number of products.

□

Source lineage: self, Offset coefficients and internal phase recovery.

Publication correction. R denotes the number of adjacent products, so R+1 offset channels are used and R is at least one. An empty sum has no phase.

Domain and hypotheses. A noisy progression as in SG-29. Let $c_r = bz^r + e_r$ for $0 \le r \le R$, with $R \ge 1$, $b \ne 0$, $|z| = 1$, and $|e_r| \le \epsilon$. Put $A = R|b|^2$ and $\Delta = R(2|b|\epsilon + \epsilon^2)$. Assume $\Delta < A$.

Theorem 8.4 (SG-30 Deterministic offset-phase perturbation)**.** *The phase sum is nonzero and*

$$|\Phi(c) - z| \le \frac{2\Delta}{A} = \frac{4\epsilon}{|b|} + 2\left(\frac{\epsilon}{|b|}\right)^2.$$

Proof. Expand each product:

$$\overline{c_r}c_{r+1} = |b|^2 z + \overline{bz^r}e_{r+1} + \overline{e_r}bz^{r+1} + \overline{e_r}e_{r+1}.$$

The three error terms have total magnitude at most $2|b|\epsilon + \epsilon^2$. Summing gives $S(c) = Az + E$ with $|E| \le \Delta$. Therefore $|S(c)| \ge A - \Delta > 0$. For any nonzero v and unit z,

$$\left|\frac{v}{|v|} - z\right| \le \left|\frac{v}{|v|} - \frac{v}{A}\right| + \left|\frac{v}{A} - z\right| = \frac{\big||A| - |v|\big|}{A} + \frac{|v - Az|}{A} \le \frac{2|v - Az|}{A}.$$

Apply this inequality to $v = S(c)$ and substitute the definitions. The number of products cancels, as appropriate for coherent adversarial errors.

□

Source lineage: self, Deterministic phase perturbation; Detailed algebra for the multi-offset estimator.

Domain and hypotheses. Finite complex offset data. Let $R \ge 1$, $c_0 \ne 0$, and $|\zeta| = 1$. Define $\nu(c, \zeta) = \left(\sum_{r=0}^{R-1} |c_{r+1} - \zeta c_r|^2\right)^{1/2} / \left(\sum_{r=0}^{R-1} |c_r|^2\right)^{1/2}$.

Theorem 8.5 (SG-31 A zero offset residual tests a progression)**.** *Then $\nu(c, \zeta) = 0$ if and only if $c_r = c_0\zeta^r$ for $0 \le r \le R$.*

Proof. The denominator is positive because it includes $|c_0|^2$. A sum of nonnegative squared magnitudes is zero precisely when each recurrence $c_{r+1} = \zeta c_r$ holds. Induction gives the progression. The reverse implication follows by substitution. This is a consistency check for the fitted offsets. Another physical model or structured background could produce the same progression; zero residual alone does not certify the true edge model or a true-location error bound. Unused offset channels provide a stronger practical check because they are not part of the fit, but no universal probability guarantee is asserted here.

□

Source lineage: self, A posteriori phase validation.

Domain and hypotheses. Nth roots of a unit-modulus estimate. Let $N \ge 2$, $|z| = 1$, $u = z^N$, and $\overline{u} = \widehat{u}/|\widehat{u}|$ with $\widehat{u} \ne 0$. Let ρ be a root of $\overline{u}$ continuously associated with z, and $\widetilde{z}$ a coarse phase estimate.

Theorem 8.6 (SG-32 Unique branch selection under a strict margin)**.** *If*

$$|\widetilde{z} - z| + |\rho - z| < \sin(\pi/N),$$

then ρ is the unique root of $\overline{u}$ nearest to $\widetilde{z}$. It equals the branch an oracle chooses by the true-node neighborhood.

Proof. The N candidates lie on the unit circle at angles spaced by $2\pi/N$. The smallest distance between distinct candidates is $2\sin(\pi/N)$. The triangle inequality gives $|\widetilde{z} - \rho| \le |\widetilde{z} - z| + |z - \rho| < \sin(\pi/N)$. For any other candidate ρ', $|\widetilde{z} - \rho'| \ge |\rho' - \rho| - |\rho - \widetilde{z}| > \sin(\pi/N)$. Thus the nearest candidate is unique. The same spacing argument places every wrong root farther from the true node than the associated root under the given margin. The oracle and data-based choices therefore coincide. This is conditional on the margin; the true-node terms are not available directly to the algorithm.

□

Source lineage: self, Branch lifting from aliased nodes.

Publication correction. The candidate-spacing and branch-margin statement explicitly assumes N at least two.

Domain and hypotheses. Unit-modulus nodes and their associated root branches. Let $N \geq 1$, $z^N = u$, $|z| = |u| = |\overline{u}| = 1$, and $|\overline{u} - u| \leq \eta < 2$. Let ρ be the associated root along the shortest angular perturbation.

Theorem 8.7 (SG-33 Associated roots divide the angular error)**.**

$$|\rho - z| \leq 2 \sin\left(\frac{1}{N} \arcsin\frac{\eta}{2}\right) \leq \frac{2}{N} \arcsin\frac{\eta}{2}.$$

For a nonzero raw estimate, $|\widehat{u} - u| \leq \epsilon$ implies $|\widehat{u}/|\widehat{u}| - u| \leq 2\epsilon$. Thus when the raw error tends to zero, correct lifting gains a factor $1/N$ in chord and angular accuracy.

Proof. If $\theta \in (-\pi, \pi)$ is the shortest angular difference, the unit-circle chord identity gives $|\overline{u} - u| = 2\sin(|\theta|/2)$. Hence $|\theta| \leq 2\arcsin(\eta/2)$. On the associated branch the root angle changes by θ/N. Convert to chord distance and use $\sin t \leq t$. For normalization, add and subtract $\widehat{u}$:

$$|\widehat{u}/|\widehat{u}| - u| \leq |1 - |\widehat{u}|| + |\widehat{u} - u| \leq 2|\widehat{u} - u|.$$

The shortest-angle ambiguity at an antipodal estimate is excluded by $\eta < 2$. The gain applies to an associated branch, so a branch-selection argument such as SG-32 is needed to transfer it to an estimator.

□

Source lineage: self, One-order root gain; Detailed algebra for the branch lift.

Domain and hypotheses. Bandlimited background with a bounded number of jumps. Let $f \in \mathcal{U}_{K,L_{\max}}$, with $L_{\max} \geq 1$. Supply $N \geq 2$, $q_0 \geq 1$ with $q_0 N > K$, and assume the active aliased nodes $u_\ell = z_\ell^N$ are distinct.

Theorem 8.8 (SG-34 An explicit exact self-calibration budget)**.** *The Fourier coefficients at the high indices*

$$\mathcal{I}_0 = \{(q_0 + q)N : 0 \leq q \leq 2L_{\max}\}, \qquad \mathcal{I}_1 = \{(q_0 + q)N + 1 : 0 \leq q < L_{\max}\}$$

determine the unknown edge order, every physical node, and every jump amplitude exactly. The high-frequency budget is $3L_{\max} + 1$. Low modes $-K \leq n \leq K$ then give the complete signal. No supplied jump neighborhood is required.

Proof. All high indices exceed K, so the derivative moments have no smooth defect. The base channel is an exponential sum in the distinct u_ℓ with coefficients $b_{\ell,0} = a_\ell u_\ell^{q_0} \neq 0$. Its $2L_{\max} + 1$ moments form an $(L_{\max} + 1)$ square Hankel matrix; SG-19 gives its rank r. If $r = 0$, the edge is zero and the low modes finish the recovery. Otherwise SG-20 recovers the u_ℓ and then the base coefficients. The adjacent channel supplies at least r moments of $\sum_\ell b_{\ell,0} z_\ell u_\ell^q$. Using the same recovered nodes, its Vandermonde system recovers $b_{\ell,1} = b_{\ell,0} z_\ell$. Division gives $z_\ell = b_{\ell,1}/b_{\ell,0}$, and $a_\ell = b_{\ell,0} u_\ell^{-q_0}$. SG-21 constructs j, and subtraction from the low modes constructs s. The two index sets are disjoint because $N \geq 2$ and their residues are zero and one. Their sizes add to $3L_{\max} + 1$. This is an upper bound for the construction and excludes the low reconstruction coefficients.

□

Source lineage: self, Self-calibrating reconstruction theorem; Explicit finite innovation budget.

Publication correction. The budget is conditional on one supplied noncolliding decimation. A finite search over candidate decimations is not guaranteed to contain such a decimation without an additional existence hypothesis.

Domain and hypotheses. A selected sequence of decimation scales. Let $p \geq 0$, fix q_0 and the finite number of channel samples and offsets, and take a sequence $N \to \infty$. Assume distinct aliased nodes, uniform rank-threshold gaps/local inverse bounds for the aligned Prony and coefficient fits, $|b_{\ell,0}| \geq b_{\min} > 0$, uniformly bounded amplitudes, and the strict branch margin of SG-32. Suppose the moment defect is $O(N^{-p-1})$.

Theorem 8.9 (SG-35 Conditional asymptotic rates after calibrated lifting)**.** *The threshold-selected order is eventually correct. The aligned lifted nodes and locations satisfy*

$$|\widehat{z}_\ell - z_\ell| = O(N^{-p-2}), \qquad d_{\mathbb{T}}(\widehat{\tau}_\ell, \tau_\ell) = O(N^{-p-2}).$$

The recovered amplitudes satisfy $|\widehat{a}_\ell - a_\ell| = O(N^{-p-1})$.

Proof. Uniform local inverse bounds transfer the moment error to aliased node and matched coefficient errors of order N^{-p-1}. The supplied rank gap and threshold select the correct order by SG-28. Nonvanishing base coefficients allow SG-30 to give the same order for the coarse phase error. The strict branch margin selects the associated root by SG-32. Normalize the aliased estimate and use SG-33: its root error is N^{-1} times an error of order N^{-p-1}, giving N^{-p-2}. On the unit circle, $d_{\mathbb{T}} = 2\arcsin(|\widehat{z} - z|/2)$, which has the same order for small errors. For the amplitude use $\widehat{a} = \widehat{b}_0\, \overline{u}^{-q_0}$ and $a = b_0 u^{-q_0}$. Since q_0 is fixed and both powers have unit modulus, SG-26's telescoping power estimate bounds the power error by $q_0|\overline{u} - u|$. Together with the coefficient error this gives the stated amplitude order. The smooth-tail assumption in SG-27 supplies the moment defect. The conclusion is uniform only on the selected scales and parameter families for which the inverse, rank, and branch bounds are uniform.

□

Source lineage: self, Rate-preserving polyphase lifting; Perturbed self-calibration for a non-bandlimited background.

Publication correction. For p=0 the coarse phase error and root spacing are both of order 1/N, so the branch margin is retained as an explicit hypothesis. Physical separation alone does not imply uniform aliased separation for every N; rates are asserted only along admissible selected scales.

The native pipeline does not receive the true order or jump neighborhoods. It tests several decimations, fits common nodes, uses offsets for branch selection, and performs validation and optional nonlinear refinement. Those algorithmic choices are supported by the displayed conditional contracts and finite experiments. The rate theorem does not assert that a largest-gap heuristic, every decimation, or every non-bandlimited signal satisfies its hypotheses.

9 Native edge synthesis and a corrected perturbation estimate

Domain and hypotheses. A normed signal space. Let $f = s + j$, let $\widehat{j}$ belong to the same space, and let A_M be a bounded linear operator with norm at most C_M. Define $\widehat{f}_M = A_M(f - \widehat{j}) + \widehat{j}$.

Theorem 9.1 (SG-36 Resolved synthesis splits the two error sources)**.**

$$f - \widehat{f}_M = (s - A_M s) + (I - A_M)(j - \widehat{j}),$$

and therefore

$$\|f - \widehat{f}_M\| \leq \|s - A_M s\| + (1 + C_M)\|j - \widehat{j}\|.$$

If $\widehat{j} = j$*, the complete error equals the smooth approximation error.*

Proof. Substitute $f = s + j$ into the definition and collect the two terms. Linearity of A_M is used in that collection. The triangle inequality and $\|I - A_M\| \le 1 + \|A_M\|$ give the estimate. With the exact edge, its error term vanishes as an object identity. The theorem holds for the declared norm; a misplaced discontinuity need not be small in supremum norm even when its location error is small. In L^1, the next theorem quantifies that error. A nonlinear clipping operator is not an instance of this linear identity.

□

Source lineage: self, A stable synthesis theorem; *complex*, Resolved synthesis.

Domain and hypotheses. Mean-zero periodic packets of period $T > 0$. Let $j, \widehat{j}$ have matched real jump data (a_ℓ, τ_ℓ) and $(\widehat{a}_\ell, \widehat{\tau}_\ell)$, with both amplitude sums zero. Use their canonical representatives and let $d_\ell = d_T(\tau_\ell, \widehat{\tau}_\ell) \le T/2$.

Theorem 9.2 (SG-37 Sharp additive L1 control for normalized jump packets)**.**

$$\|j - \widehat{j}\|_1 \le \frac{T}{4} \sum_\ell |a_\ell - \widehat{a}_\ell| + 2 \sum_\ell |\widehat{a}_\ell| d_\ell (1 - d_\ell / T).$$

In particular the simpler location term $2\sum_\ell |\widehat{a}_\ell| d_\ell$ is valid. The printed location coefficient one in the inherited self-calibration manuscript is not valid for mean-zero normalization.

Proof. SG-21 writes both packets as translated sawtooth sums. Subtract in the form

$$j - \widehat{j} = \sum_\ell (a_\ell - \widehat{a}_\ell) G_T(\cdot - \tau_\ell) + \sum_\ell \widehat{a}_\ell [G_T(\cdot - \tau_\ell) - G_T(\cdot - \widehat{\tau}_\ell)].$$

Direct integration gives $\|G_T\|_1 = T/4$. For a shift by $d \in [0, T/2]$, on an interval of length $T - d$ the difference has magnitude d/T, and on an interval of length d it has magnitude $1 - d/T$. Hence its L^1 norm is $(T - d)d/T + d(1 - d/T) = 2d(1 - d/T)$. Apply the triangle inequality term by term. Periodic translation invariance makes the calculation depend only on circular shift distance. To refute coefficient one, take

$$j = \mathbf{1}_{[0,T/2)} - \tfrac{1}{2}, \qquad \widehat{j} = \mathbf{1}_{[T/4,T/2)} - \tfrac{1}{4}.$$

Both have jumps $(1, -1)$, with one jump shifted by $T/4$. Their norm difference is $3T/8$, while the inherited bound gives $T/4$. The sharper location term is attained in this example. The amplitude coefficient $T/4$ is also attained by changing opposite antipodal jump amplitudes: the difference packet has constant magnitude half the amplitude change on the two half-periods.

□

Source lineage: self, A stable synthesis theorem.

Publication correction. The old L1 estimate is refuted by two mean-zero interval packets. The published bound includes the normalization-induced constant shift and uses the exact translated-sawtooth norm.

Domain and hypotheses. The rate family of SG-35 with $T = 2\pi$. Assume a fixed finite jump count, bounded jump amplitudes, the hypotheses of SG-35, and a bounded linear smooth reconstruction A_M acting on $L^1(\mathbb{T})$. Require $\widehat{j}$ to be the canonical packet of real matched estimated amplitudes whose sum is zero. Let $b_M = \|s - A_M s\|_1$ and $C_M = \|A_M\|_{L^1 \to L^1}$.

Theorem 9.3 (SG-38 A conditional global L1 reconstruction rate)**.**

$$\|j - \widehat{j}\|_1 = O(N^{-p-1}), \qquad \|f - \widehat{f}_M\|_1 \le b_M + O((1 + C_M) N^{-p-1}).$$

For Fourier truncation, $C_M = O(1 + \log(M + 1))$. If also $|\widehat{s}(n)| \le C_s (1 + |n|)^{-p-2}$, its smooth remainder satisfies $b_M = O(M^{-p-1})$. Taking M comparable to N yields $O((1 + \log N) N^{-p-1})$.

Proof. SG-35 gives an amplitude error of order N^{-p-1} and a location error of order N^{-p-2}. If the fitted amplitudes are complex or do not sum to zero, take their real parts and subtract their common arithmetic mean. Since the true real amplitudes sum to zero, this changes each amplitude error by at most another maximum amplitude error, preserving its order. Apply SG-21 to the resulting real zero-sum data to define $\widehat{j}$. Insert these errors into SG-37, with finitely many bounded amplitudes, to obtain the edge norm bound. SG-36 then gives the general synthesis estimate. For Fourier truncation, its convolution norm is bounded by the normalized L^1 norm of D_M. On $[-\pi,\pi]$ use $|D_M(t)| \le \min(2M+1, \pi/|t|)$, with the removable value at zero. Split the integral at $1/(M+1)$ to obtain the logarithmic bound. The smooth Fourier tail has supremum bound $2C_s\sum_{n>M}(1+n)^{-p-2} = O(M^{-p-1})$ by an integral comparison; multiplication by 2π gives the L^1 estimate. This conclusion accounts for the norm growth of partial sums instead of assuming a uniformly bounded Fourier truncation operator on L^1.

□

Source lineage: self, A stable synthesis theorem; Limitations and next theorem.

10 Shared edges, jump jets, and geometric image models

Domain and hypotheses. A finite collection of exponential moment channels. Let $m_n^{(c)} = \sum_{\ell=1}^{r} a_{c\ell} z_\ell^n$, with distinct common unit-modulus nodes. At each node at least one channel amplitude is nonzero. Form the stacked linear recurrence system for a monic polynomial of degree r and assume its coefficient matrix has column rank r.

Theorem 10.1 (SG-39 Shared-location multichannel recovery)**.** *The stacked system uniquely recovers the common annihilator $Q(z) = \prod_\ell (z - z_\ell)$. Its roots recover the common locations. With r consecutive moments per channel, a channelwise Vandermonde solve recovers every amplitude, including amplitudes that vanish in that channel.*

Proof. The polynomial Q annihilates every exponential term and therefore every channel, as in SG-20. Full column rank of the stacked recurrence matrix means that a consistent coefficient vector is unique; the true Q provides one. Its distinct roots give the nodes. Once known, the same nonsingular Vandermonde system solves each channel independently. No individual channel has to see all nodes, but the stated stacked rank is essential. Merely collecting channels does not prove that rank condition automatically.

□

Source lineage: unknown, Several channels with shared edge locations.

Domain and hypotheses. Polynomial-exponential sequences. Let $m_n = \sum_{\ell=1}^{r} z_\ell^n P_\ell(n)$ with distinct nonzero nodes and $\deg P_\ell \le d$. Let E be the forward shift. For an offset ratio assume $N \ge 1$ and the leading coefficient of P_ℓ at degree d is nonzero.

Theorem 10.2 (SG-40 Confluent annihilation and the leading offset phase)**.**

$$\prod_{\ell=1}^{r} (E - z_\ell)^{d+1} m = 0.$$

For offset $v \ge 0$, writing a decimated channel as

$$m_{(q_0+q)N+v} = \sum_\ell u_\ell^q \sum_{h=0}^{d} \gamma_{\ell,v,h} q^h, \qquad u_\ell = z_\ell^N,$$

the highest-degree coefficients satisfy $\gamma_{\ell,1,d}/\gamma_{\ell,0,d} = z_\ell$. This ratio applies to separated, matched confluent terms with nonzero leading coefficient. The displayed annihilator is not asserted to have minimal order when leading coefficients vanish.

Proof. For one term, $(E - z)(z^n P(n)) = z^{n+1}(P(n+1) - P(n))$. A finite difference lowers a nonconstant polynomial's degree by one, and annihilates a constant. After $d + 1$ differences the term vanishes. The shift factors commute, so the product annihilates every term in the sum. If $P_\ell(n)$ has leading coefficient $c_\ell \neq 0$ at degree d, substitution of $(q_0 + q)N + v$ has leading q coefficient $c_\ell N^d$. The remaining exponential prefactor is $z_\ell^{q_0 N+v}$; hence $\gamma_{\ell,v,d} = c_\ell N^d z_\ell^{q_0 N+v}$ and the ratio is z_ℓ. Repeated roots require confluent recovery and stability arguments; the simple-root disk theorem SG-25 cannot be used to claim one perturbed root per physical node or a Lipschitz individual-root bound at a multiple root.

□

Source lineage: unknown, Piecewise-polynomial packets; *self*, Higher jump jets and confluent Prony systems.

Publication correction. Annihilation is proved without a minimal-order assertion. Phase division requires a nonzero highest-degree coefficient, and simple-root perturbation claims are not transferred to repeated roots.

Domain and hypotheses. A finite-dimensional background-edge model. Use $A = RD$ from SG-14 and assume $s \geq 1$, $d \geq 1$, and its restricted constant $\gamma > 0$. Suppose $y = s_0 + D\alpha_0 + e + \eta$, with $s_0 \in \mathcal{S}$, $\|\alpha_0\|_0 \leq s$, and $\|R(e + \eta)\|_2 \leq \epsilon$. Let $\widehat{\alpha}$ be s-sparse with $\|A\widehat{\alpha} - Ry\|_2 \leq \epsilon$.

Theorem 10.3 (SG-41 Model mismatch enters the same sparse stability bound)**.**

$$\|\widehat{\alpha} - \alpha_0\|_2 \leq 2\epsilon/\gamma.$$

If an actual edge j obeys $\|j - D\alpha_0\|_2 \leq \delta$, then

$$\|j - D\widehat{\alpha}\|_2 \leq \delta + 2\|D\|_2\epsilon/\gamma.$$

The theorem applies to finite image dictionaries under these printed sparsity, residual, and restricted-injectivity hypotheses.

Proof. The nuisance $e + \eta$ has exactly the role of the residualized noise in SG-15, so its proof gives the coefficient bound. Add and subtract $D\alpha_0$ in the edge difference and apply the operator norm bound to the coefficient error. A best approximation can be substituted when it exists; otherwise any admissible approximate minimizer gives the same statement. The theorem does not show that a greedy finite dictionary selector outputs an s-sparse vector with the specified tolerance. Nor does it establish restricted injectivity for every geometric dictionary; both remain hypotheses to verify for a concrete model.

□

Source lineage: complex, Two-dimensional finite dictionary model; A theorem for model mismatch.

10.1 What the image experiments test

The complex benchmark uses a low-frequency DCT background and 318 geometric atoms: rectangles, disks, ellipses, and horizontal and vertical bars. The named exact-model images and forty archived random dictionary cases select the intended supports under their deterministic settings. Those are finite results. SG-14 and SG-41 explain the contract a universal statement about such a library would require;

the benchmark does not establish all its restricted singular values or make a greedy selector universally exact.

The seven-point star, text over a gradient, and nested rings lie outside the fixed dictionary. Their comparisons are retained as model-mismatch tests. Four natural images are exploratory examples, not instances of a finite-rate exact edge theorem. In the natural-background hybrid the selector returns five atoms for three true overlays. Background structure outside the chosen DCT space is being explained as extra geometry. This exposes a background model failure rather than just an edge-library failure.

11 Countable packet grammars and finite image approximation

A countable grammar can describe an unlimited ideal family while a program uses a finite prefix or a finite store. The infinite packet code in this bundle performs adaptive approximation of an available pixel image. Its constructor receives the full image, and residual seeds are generated from that image minus the current reconstruction. This is a different observation model from the finitely measured Fourier recovery theorem. The exact statements below specify what survives in the ideal mathematical model and what changes under the native ridge and clipping operations.

Domain and hypotheses. A Hilbert space with finite selected packet spans. Let $V_1 \subseteq V_2 \subseteq \cdots$ be finite-dimensional subspaces. Let P_k be the orthogonal projector onto V_k.

Theorem 11.1 (SG-42 Ideal nested-span refitting decreases residual)**.**

$$\|f - P_{k+1}f\| \le \|f - P_k f\|, \qquad f - P_k f \perp V_k.$$

If $\bigcup_k V_k$ is dense, then $P_k f \to f$ in norm.

Proof. Orthogonal projection minimizes distance to its subspace. Since $P_k f$ belongs to the larger subspace V_{k+1}, its residual is a feasible upper bound on the next minimum. Orthogonality follows from the defining projection identity, or by varying the quadratic norm along every vector in V_k. For convergence, fix $\epsilon > 0$ and choose $v \in V_K$ within ϵ of f by density. For all $k \ge K$, minimum distance is at most $\|f - v\| < \epsilon$. Thus the residual converges to zero. This theorem assumes dense actual selected spans. A countably dense candidate pool alone does not show that a particular greedy selection sequence becomes dense.

□

Source lineage: infinite, InfinitePacketReconstructor._refit; InfinitePacketGenerator.

Domain and hypotheses. Finite-dimensional real packet designs. Let $M \in \mathbb{R}^{m\times d}$, $y \in \mathbb{R}^m$, and $\lambda > 0$. Let c minimize $\|y - Mc\|_2^2 + \lambda\|c\|_2^2$ and set $r = y - Mc$.

Theorem 11.2 (SG-43 Ridge refitting has a different stationarity law)**.** *The minimizer is unique and*

$$(M^{\mathsf{T}}M + \lambda I)c = M^{\mathsf{T}}y, \qquad M^{\mathsf{T}}r = \lambda c.$$

Adding candidate columns cannot increase the minimized ridge objective, but exact orthogonality $M^{\mathsf{T}}r = 0$ is not the ridge stationarity law. Even a complete pixel basis need not give zero residual when $\lambda > 0$.

Proof. The Hessian is twice $M^TM + \lambda I$, which is positive definite because $v^T(M^TM + \lambda I)v = \|Mv\|^2 + \lambda\|v\|^2 > 0$ for nonzero v. Setting the gradient to zero gives the unique minimizer and rearrangement

gives $M^T r = \lambda c$. When columns are added, extending the previous coefficient vector by zero is a feasible point with exactly the previous objective, so the new optimum is no larger. This concerns the ridge objective, not an automatic monotonicity theorem for the unpenalized residual alone. For the full pixel basis $M = I$, $c = y/(1+\lambda)$ and $r = \lambda y/(1+\lambda)$, which is nonzero for $y \neq 0$. The native default $\lambda = 10^{-9}$ makes this residual small, not mathematically zero. Numerical tolerances must remain numerical tolerances.

□

Source lineage: infinite, InfinitePacketReconstructor._refit; default ridge=1e-9.

Publication correction. Claims of exact unregularized refitting are separated from the implemented positive-ridge solve. The native normal-equation law includes the ridge term.

Domain and hypotheses. Real pixel arrays with Euclidean norms. Define $C(x)$ coordinatewise by $C(t) = \min(1, \max(0, t))$. Suppose the truth f has every coordinate in $[0, 1]$.

Theorem 11.3 (SG-44 Clipping is nonexpansive toward a bounded truth)**.**

$$\|C(x) - C(y)\|_2 \leq \|x - y\|_2, \qquad \|C(x) - f\|_2 \leq \|x - f\|_2.$$

Clipping is nonlinear; it is not the bounded linear synthesis operator in SG-36. Its truth-error guarantee depends on the stated box constraint.

Proof. For real scalars the clipping function is nondecreasing with slope between zero and one on each of its three affine pieces. A direct case split at zero and one gives $|C(x) - C(y)| \leq |x - y|$ for every pair. Square and sum the scalar inequalities. Since $C(f) = f$, substitute $y = f$ to obtain the second estimate. Nonlinearity is explicit: $C(1+1) = 1$ whereas $C(1) + C(1) = 2$. If the truth lies outside the box, clipping can increase its error; for instance $x = f = 2$ becomes $C(x) = 1$.

□

Source lineage: infinite, bounded_synthesis.

Domain and hypotheses. Any normed signal space. Let an approximation q to an available object f be supplied, and retain the complete residual $r = f - q$.

Theorem 11.4 (SG-45 An exact residual archive preserves the object)**.** *The synthesis $q + r = f$ is exact. This identity supplies no sparsity, compression, or recovery-from-fewer-measurements conclusion. In finite pixel space the full pixel basis likewise spans every image in the ideal unregularized model.*

Proof. The first statement is the defining subtraction identity. Retaining r can carry every component omitted from q, so its data cost is not bounded by a small number of packet coefficients. For an image $f \in \mathbb{R}^m$, the standard pixel vectors satisfy $f = \sum_{j=1}^{m} f_j e_j$ and form a complete finite basis. Those coefficients are all the pixels themselves. A program using the truth residual thus certifies faithful storage of its available input, rather than uniqueness under a smaller measurement map.

□

Source lineage: infinite, exact_archive_residual; benchmark_case.

Domain and hypotheses. Normed spaces and bounded linear maps. Let D be a dense linear subspace of X, and let $A, B : X \to Y$ be bounded linear operators agreeing on D.

Theorem 11.5 (SG-46 Bounded identities extend from a dense core)**.** *Then $A = B$ on X. Consequently a finite packet identity established on a dense core extends to a completion when both represented operators have bounded extensions. An infinite projector sum additionally requires a specified convergent synthesis; bounded finite identities alone do not supply that convergence.*

Proof. For $x \in X$ choose $x_n \in D$ with $x_n \to x$. Continuity gives $Ax = \lim_n Ax_n = \lim_n Bx_n = Bx$. This proves uniqueness of bounded extensions. For an infinite packet sum one would need partial sums to converge to the declared operator, in a stated operator or strong topology. Neither density nor the finite identity by itself proves such a limit. In SG-42 density of the actual selected spans provides a specific strong convergence theorem; a capped dictionary store or an arbitrary greedy prefix does not inherit it automatically.

□

Source lineage: seed, thm:dense-extension; cor:packet-completion.

The native infinite-packet generator uses finite stores, including mask caps of 120 or 180 in reported settings. Its discrete masks, quadtree refinements, and residual-based seeds are a finite adaptive realization, not an implemented unbounded continuous atlas. The code's clipping stage should be interpreted through SG-44 and its ridge refit through SG-43. Case-level wall time is meaningful in the existing benchmark record; per-method runtime fields that remain zero because of method-key bookkeeping do not supply valid method timing comparisons.

12 A closed four-jump calculation

Take a bandlimited background

$$s(x) = \frac{1}{4} + \frac{2}{5}\cos(2x) - \frac{1}{3}\sin(3x) + \frac{1}{5}\cos(5x)$$

and a mean-zero edge with locations and jumps

$$(\tau_\ell) = (0, \pi/2, \pi, 3\pi/2), \qquad (a_\ell) = (1, -2, -1, 2).$$

The physical nodes are $(1, -i, -1, i)$. Their derivative moments are

$$m_n = 1 - 2(-i)^n - (-1)^n + 2i^n,$$

so in residues modulo four they are

$$m_{4q} = 0, \quad m_{4q+1} = 2 + 4i, \quad m_{4q+2} = 0, \quad m_{4q+3} = 2 - 4i.$$

Since the smooth degree is five, the first clean consecutive moment is $n_0 = 6$. With an order bound four, coefficients at $n = 6, \dots, 13$ determine the edge. The Hankel factorization has four distinct nodes and four nonzero amplitudes, hence rank four. Its recurrence is $m_{n+4} - m_n = 0$, with monic polynomial $Q(z) = z^4 - 1$. Recovering those roots and solving the Vandermonde system returns the displayed jumps. SG-21 then constructs the edge, and coefficients at $-5, \dots, 5$ return the smooth background.

The same object demonstrates internal phase recovery. Choose $N = 3$ and $q_0 = 2$. The nodes are permuted by cubing, so the aliased nodes remain distinct. The sufficient self-calibration sets for $L_{\max} = 4$ are

$$\mathcal{I}_0 = \{6, 9, 12, 15, 18, 21, 24, 27, 30\}, \qquad \mathcal{I}_1 = \{7, 10, 13, 16\}.$$

For each matched term the base coefficient is $b_{\ell,0} = a_\ell z_\ell^6$, and the adjacent coefficient is $b_{\ell,1} = b_{\ell,0} z_\ell$. Their ratio is the original physical node, not the principal cube root. The total high budget is thirteen, agreeing with $3L_{\max} + 1$. The un-decimated construction uses eight high coefficients; the polyphase construction trades additional channel information for its chosen decimation geometry and branch recovery. Neither budget is claimed minimal. Both reconstruct the same canonical object exactly.

13 Evidence, implementations, and unresolved scope

13.1 Current finite checks

The current release replay contains ten bounded suites with 288 recorded checks. After excluding the seven repeated checks in the SPD variant of the same infinite-packet tests, the count is 281 distinct recorded checks under the stated counting convention. A named check can compare a whole vector or family; these numbers are neither theorem counts nor claims of independent proofs. The manifests include source hashes, fixed seeds or deterministic inputs, tolerances, resource limits, and scientific result records.

Suite	Recorded checks	Counted distinct checks
Exact invariants	134	134
Unknown edge	24	24
Self-calibration	31	31
Singularity dictionary	26	26
Gibbs case	8	8
Complex benchmark	23	23
Infinite packet	7	7
Infinite packet SPD	7	0
Rank-aware execution copy	16	16
Rank-ratio boundary	12	12
Total	288	281

The controlled finite-recovery checks pass in their printed ranges and tolerances. The replay also reproduces two recovery failures, and its status explicitly includes them. In the localized-burst signal the true order six is inferred as nine. Location and jump comparison errors are nonfinite because there is no same-cardinality match, while reconstructed waveform RMSE remains finite and improved. Improved waveform error therefore does not establish correct edge identification. The rank-ratio witness in Section 7 returns order two rather than three under the native heuristic, although the supplied-threshold rule returns three.

An initial replay check mistakenly required every complex stress example to have correct order and finite parameter errors. It failed on the burst case. That failed record is retained. The follow-up uses the actual scientific scope: controlled examples pass their recovery criteria and stress failures are reported as failures. The scientific claim was narrowed; the burst outcome was not changed or hidden.

The self-calibration replay includes one hundred deterministic separated random cases with correct order in the tested settings. The archived wider benchmark figures and natural-image comparisons remain historical finite evidence unless their particular experiment is covered by a fresh manifest. A bounded current replay is not a universal probability statement and does not formalize the Python implementation.

13.2 Executable boundaries and a separate repaired copy

The original unknown-edge implementation's generic background projector uses an unpivoted reduced QR construction. Its usual full-column-rank design is appropriate, but dependent background columns need numerical-rank handling. The original restricted-singular-value helper also needs an explicit zero case for supports wider than the number of residual rows. A separate rank-aware execution copy supplies those two local repairs and has sixteen finite regression controls. The immutable source payload is preserved; the repair provenance records the changed copy and its hashes.

The zero-edge signal is a valid trivial branch of SG-22 and SG-34. The nonconstant native edge-model interface does not thereby acquire a zero-edge wrapper. Exact model assertions and accepted input domains of a program are listed separately. Floating-point root, rank, and amplitude tolerances are finite checks, not exact kernel certificates.

13.3 Formal coverage

The included related Paper V checkpoint contains 225 theorem declarations in twenty-three theorem-bearing modules. It supports packet, mask, polyphase, finite transfer, and known-edge reconstruction infrastructure under its own printed types. It is an earlier intact checkpoint, and its inherited logs belong to that checkpoint. It does not certify the July unknown-edge and self-calibration claims of this paper. The Lean guide maps each SG result to related infrastructure or an explicit uncovered formalization obligation. No Signal-specific Lean proof or fresh Lean compilation is claimed in this edition. The manuscript proofs and the finite numerical evidence remain separate from that formal coverage.

13.4 What the paper closes

For a bandlimited periodic background plus finitely many distinct off-grid jumps, finitely many exact Fourier coefficients recover the complete canonical signal. Under a supplied noncolliding decimation, adjacent offsets internally recover physical phase with a stated finite budget. Under uniform local conditioning and a strict branch margin, smooth Fourier tails yield the printed conditional location and amplitude rates. Native edge synthesis removes the edge's global-truncation error, and SG-37 supplies a valid mean-zero periodic L^1 perturbation estimate.

13.5 What remains open here

The paper does not provide model-free unknown-edge recovery, a universal noise bound without conditioning, an automatic proof that an empirical order heuristic meets the rank-threshold gate, or a data-only root enclosure from arbitrary noisy Fourier coefficients. It does not guarantee that every finite decimation search contains a noncolliding candidate. General piecewise-smooth backgrounds can imitate extra edges. Continuous arbitrary two-dimensional boundaries need a geometry-adaptive identifiability and approximation theorem. A sparse convergent adaptive infinite grammar needs a selection theorem, not just a dense candidate description or a truth-residual archive.

These boundaries are mathematical distinctions exposed by the proofs and negative controls. They guide the next work: robust unknown-order selection, validated root enclosures, model-mismatch control, and adaptive geometry with a measured-data observation model.

A Reading the source and claim concordance

Every SG result has a stable identifier, a domain, printed hypotheses, a statement and proof, dependency identifiers, and hash-pinned source lineage in `Claim_Concordance.json`. Source-section labels identify the inherited mathematical material; a lineage anchor is not an assertion that every sentence of the new proof appeared verbatim in that source. The CSV is a compact index of the same results. Reconstructed proofs and publication corrections are marked explicitly. All SG results are manuscript results rather than Signal-specific Lean-certified declarations.

The bundle includes the full `.md` manuscript files, editable LaTeX sources, Python implementations and figure generators, original source archives, related Lean sources, fresh and inherited verification records, and the two companion guides. The raw inherited manuscripts are retained alongside the corrected edition so that changes remain reviewable.

B Principal proof dependencies

SG-01–SG-06 supply exact finite coordinates, transfer multiplication, polyphase kernels, and alias arithmetic. SG-07–SG-08 state the identifiability boundaries. Known-edge reconstruction SG-09–SG-11 precedes the Gibbs profile SG-12 and positive averaging SG-13. Finite dictionary recovery follows the kernel-secant path SG-14 to SG-15; SG-16 gives the one-packet profiled search and its independent finite gap.

The off-grid chain is differentiation SG-18, Hankel rank SG-19, annihilation SG-20, canonical integration SG-21, and complete recovery SG-22. SG-23–SG-26 quantify its perturbed linear and root steps. Self-calibration uses SG-27's offset identity and SG-28's supplied threshold, then SG-29–SG-33 for phase and branch control. SG-34 is the exact finite budget; SG-35 is the conditional asymptotic theorem. SG-36–SG-38 return those parameters to a normed signal with the corrected mean-zero edge bound. SG-39–SG-41 extend only the declared shared-edge, confluent, and dictionary contracts. SG-42–SG-46 keep ideal projection, ridge, clipping, residual storage, and completion as separate operations.

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The finite-rate paper is cited for historical attribution. Exact sampling hypotheses from it are not imported: the consecutive-moment theorem is proved directly in this manuscript. The literature review records the checked versions and the limits of the source search. No global novelty claim follows from that search.