Komorebi Seeded Two-Axis Calculus

Transformation, Resolution, Residual Generation, and Reconstruction

Working Formalization v0.11.0

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Abstract

This manuscript develops a seed-generated calculus of transformations, resolution changes, and residuals of their exchange maps. Its categorical core uses indexed categories, coherent event diagrams, stable cofibers, and constructor-compatible quotients, while retaining history-sensitive filtrations. Arithmetic test beds include nonidentity coefficient towers, completion defects, lattice reconstruction, intrinsic derived exchanges, and a finite lifting horizon for a fixed finite group.

Version 0.11.0 retains the actual profinite source. Continuous relation cohomology characterizes next-level lifts, and source inflation identifies when a finite-quotient obstruction disappears after that source resolution is refined. Compatible towers reconstruct continuously, and free pro-p sources admit arbitrary coefficient lifts of their generator matrices.

For continuous representations of $\mathbb{Z}_p^r$, removing a common scalar precision exposes normalized commutators. When one normalized generator is cyclic modulo p, their vanishing is necessary and sufficient for an integral lift, constructed by finite polynomial certificates. This condition covers every rank-two input. It yields a complete finite-matrix decision and a sharp uniform horizon $2n-1$ for lifting a representation modulo p^n. Explicit positive and negative examples distinguish genuine abelian relations from torsion relations introduced by fixing a finite quotient. A mixed-variance groupoid records coefficient reduction and source inflation without identifying their directions.

Written proofs, self-review, exact regression tests, and a usable rank-two certificate tool accompany the construction. No Lean certification, independent review, historical novelty, universal profinite lifting theorem, prescribed geometric deformation condition, automorphic existence, or Langlands reciprocity theorem is claimed.

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1 Purpose and status

The goal is to formalize the idea that a mathematical theory may contain two distinct ambient directions:

transformation: change of representation	and	resolution: change of distinguishability.

The distinction is operational rather than rhetorical. A Fourier transform, a duality, or a change of coordinates may alter representation while preserving the level of information being resolved. Coarsening a filtration, passing from modulus p^{n+1} to modulus p^n, truncating a spectral expansion, or forgetting fine ramification data changes the level at which distinctions are visible.

The central hypothesis of this note is that a large class of mathematical constructions can be organized by a two-axis calculus in which:

1. objects occur at resolutions;
2. each resolution has its own transformation category;
3. resolution changes induce functors between those categories;
4. mixed transformation-resolution squares can commute exactly, commute up to equivalence, or leave a structured residual;
5. residuals can themselves be promoted to new seeds;
6. compatible local or resolved data can sometimes be reconstructed into global objects.

Status convention. Results tagged "proved here" have written proofs under the stated hypotheses. They are not asserted to be Lean-checked. Results tagged "standard machinery" are direct applications of established category, order, or rewriting theory. Statements tagged "program theorem" are precise targets whose Komorebi-specific hypotheses remain to be supplied in applications. No claim of mathematical novelty is made merely because standard machinery is expressed in Komorebi language.

Categorical regimes and size. The initial indexed construction uses small ordinary categories and a strict contravariant functor. Every section involving functorial cofibers, higher coherence, stable envelopes, or stable Verdier quotients uses stable ∞-categories, or a specified enhancement presenting them. These are small unless a section explicitly uses an unbounded ambient derived category in a fixed universe. All subcategories in closure constructions are full and replete. "Stable" means closed under finite limits and colimits, while "thick" additionally means closed under retracts. A stable seed zoo is not automatically idempotent-complete. Thick results are applied to a separately designated thick zoo or to its idempotent completion. Homotopy categories support the corresponding triangulated statements, not arbitrary canonical choices of cones. A fixed small presentation or larger ambient universe is understood whenever object collections are used as sets.

What the two axes do not assert. An admissible transformation need not be invertible or information-preserving. A coarsening functor need not have an inverse. A factorization in a constructed fibration does not establish a canonical resolution system for every mathematical theory. These are separate semantic requirements, not consequences of the notation.

2 Resolution systems

2.1 The resolution base

Definition 2.1 (Resolution category)**.** A *resolution category* is a small poset $(R, \leq)$ regarded as a thin category $\mathbf{R}$. An arrow $r \to s$ exists exactly when $r \leq s$. We interpret s as at least as fine as r.

The poset assumption is intentionally restrictive for the first formalization. A later version may replace $\mathbf{R}$ by a general category, a site, a filtered category, or a multi-parameter resolution geometry.

Definition 2.2 (Indexed transformation system)**.** An *indexed transformation system* over $\mathbf{R}$ is a strict functor

$$F : \mathbf{R}^{op} \longrightarrow \mathbf{Cat}.$$

For $r \in R$, write

$$\mathbf{T}_r := F(r).$$

For $r \leq s$, write

$$C_{s \to r} := F(r \leq s) : \mathbf{T}_s \longrightarrow \mathbf{T}_r.$$

We call $C_{s \to r}$ the *coarsening functor*.

Strictness gives

$$C_{r \to r} = \mathrm{id}_{\mathbf{T}_r}$$

and, for $r \leq s \leq t$,

$$C_{t \to r} = C_{s \to r} \circ C_{t \to s}.$$

A pseudofunctor version is more natural in some applications. The strict version isolates the main structure without coherence bookkeeping.

Definition 2.3 (Resolved object)**.** A *resolved object* is a pair (r, X) with $X \in \mathrm{Ob}(\mathbf{T}_r)$.

2.2 Resolution equivalence

Let $X, Y \in \mathbf{T}_t$ and let $r \leq t$.

Definition 2.4 (Indistinguishability at resolution r)**.** We write

$$X \sim_r Y$$

when

$$C_{t \to r}(X) \cong C_{t \to r}(Y)$$

in $\mathbf{T}_r$.

For a fixed ambient fine level t, each $\sim_r$ is an equivalence relation on isomorphism classes of objects of $\mathbf{T}_t$.

Theorem 2.5 (Resolution monotonicity)**.** *Let $r \leq s \leq t$ and $X, Y \in \mathbf{T}_t$. If*

$$X \sim_s Y,$$

then

$$X \sim_r Y.$$

Proof. An isomorphism

$$C_{t\to s}(X) \cong C_{t\to s}(Y)$$

is carried by the functor $C_{s\to r}$ to an isomorphism

$$C_{s\to r}C_{t\to s}(X) \cong C_{s\to r}C_{t\to s}(Y).$$

By functorial coherence this is

$$C_{t\to r}(X) \cong C_{t\to r}(Y).$$

Hence $X \sim_r Y$. □

Corollary 2.6 (Nested distinguishability)**.** *For $r \le s \le t$, the equivalence relation $\sim_s$ refines $\sim_r$.*

This is the first exact form of the statement that coarsening may erase distinctions but cannot create distinctions that were absent at finer resolution.

2.3 Refinement as a fiber rather than a function

Definition 2.7 (Lift set)**.** For $r \le s$ and $X \in \mathbf{T}_r$, define the set of isomorphism classes of lifts

$$\mathrm{Lift}_{r\to s}(X) := \{Y \in \mathrm{Ob}(\mathbf{T}_s) : C_{s\to r}(Y) \cong X\} / \cong .$$

Three basic cases occur:

$$|\,\mathrm{Lift}_{r\to s}(X)| = 0, \qquad |\,\mathrm{Lift}_{r\to s}(X)| = 1, \qquad |\,\mathrm{Lift}_{r\to s}(X)| > 1.$$

They correspond respectively to obstruction to refinement, uniqueness of the fine object's isomorphism class, and hidden fine-scale multiplicity. The middle case does not imply a contractible space of refinements. It forgets the identification $C_{s\to r}Y \cong X$ and all automorphisms. When these matter, use the groupoid of pairs (Y, θ) with $\theta : C_{s\to r}Y \xrightarrow{\sim} X$, and isomorphisms commuting with θ. In the enhanced setting this is the homotopy fiber of the functor on cores.

Definition 2.8 (Resolution branching number)**.** When $\mathrm{Lift}_{r\to s}(X)$ is finite, define

$$b_{r\to s}(X) := |\,\mathrm{Lift}_{r\to s}(X)|.$$

When $b_{r\to s}(X) > 0$, its logarithm

$$H_{r\to s}(X) := \log b_{r\to s}(X)$$

may be regarded as a crude *resolution information* measuring how much ambiguity is hidden by coarsening.

The empty lift set is recorded as an obstruction, not as an ordinary finite entropy. This quantity is only a first model. In probabilistic, measurable, or derived settings one would replace cardinality by entropy, measure, dimension, rank, or a homotopical invariant.

3 The total two-axis category

3.1 Grothendieck construction

Construction 3.1 (Total Komorebi category)**.** Given $F : \mathbf{R}^{op} \to \mathbf{Cat}$, define the category

$$\mathbf{K} := \int_{\mathbf{R}} F$$

as follows.

Its objects are pairs (r, X) with $X \in \mathbf{T}_r$.

For $r \leq s$, a morphism

$$(r, X) \longrightarrow (s, Y)$$

is a morphism

$$\alpha : X \longrightarrow C_{s\to r}(Y)$$

in $\mathbf{T}_r$. If $r \not\leq s$, there is no such morphism in this directed version.

Given

$$(r, X) \xrightarrow{\alpha} (s, Y) \quad \text{and} \quad (s, Y) \xrightarrow{\beta} (t, Z),$$

define

$$\beta \circ_{\mathbf{K}} \alpha := C_{s\to r}(\beta) \circ \alpha : X \to C_{t\to r}(Z).$$

Theorem 3.2 (Total category)**.** *The construction above defines a category* $\mathbf{K}$*.*

Proof. Identity at (r, X) is $\mathrm{id}_X : X \to C_{r\to r}(X) = X$. For associativity, let

$$(r, X) \xrightarrow{\alpha} (s, Y) \xrightarrow{\beta} (t, Z) \xrightarrow{\gamma} (u, W).$$

One composite gives

$$C_{t\to r}(\gamma) \circ C_{s\to r}(\beta) \circ \alpha.$$

The other gives the same expression by functoriality and

$$C_{t\to r} = C_{s\to r} C_{t\to s}.$$

Thus composition is associative. □

Define

$$\pi : \mathbf{K} \to \mathbf{R}, \qquad \pi(r, X) = r.$$

The fiber over r is canonically equivalent to $\mathbf{T}_r$.

3.2 Canonical factorization

For $r \leq s$ and $Y \in \mathbf{T}_s$, define the canonical morphism

$$\kappa_{r,s,Y} : (r, C_{s\to r} Y) \longrightarrow (s, Y)$$

represented by the identity map

$$\mathrm{id}_{C_{s\to r} Y}\,.$$

Theorem 3.3 (Two-axis factorization)**.** *Every morphism*

$$f : (r, X) \longrightarrow (s, Y)$$

in $\mathbf{K}$ *factors uniquely as*

$$(r, X) \xrightarrow{f_v} (r, C_{s\to r}Y) \xrightarrow{\kappa_{r,s,Y}} (s, Y),$$

where f_v *lies entirely in the fiber* $\mathbf{T}_r$*.*

Proof. By definition, f is exactly a morphism

$$\alpha : X \to C_{s\to r}Y$$

in $\mathbf{T}_r$. Take $f_v = \alpha$. Composition with $\kappa_{r,s,Y}$ is represented by

$$\mathrm{id}_{C_{s\to r}Y} \circ \alpha = \alpha,$$

so the composite is f. Any factorization through the fixed cartesian target $(r, C_{s\to r}Y)$ must use the same fiber morphism α, hence is unique. □

Remark 3.4. This is the formal version of the two-axis statement

$$\boxed{\text{mixed movement} = \text{within-resolution transformation} + \text{resolution movement.}}$$

Categorically, $\pi : \mathbf{K} \to \mathbf{R}$ is the Grothendieck fibration associated with F. Indeed, a map $(u, Z) \to (s, Y)$ above $u \le r \le s$ is exactly a map $Z \to C_{s\to u}Y = C_{r\to u}C_{s\to r}Y$. This uniquely factors through $\kappa_{r,s,Y}$ as a morphism to $(r, C_{s\to r}Y)$ above $u \le r$, which is the cartesian lifting property. Notice the direction: the lift starts from an already specified fine object Y. It is not a procedure that selects a refinement of an arbitrary coarse object.

4 Resolution geometry

The nested equivalence relations from theorem 2.6 induce a genuine geometry when the resolution set is linearly ordered.

Definition 4.1 (Separation depth)**.** Let S be a set of states, with equivalence relations $\sim_n$ for every $n \in \mathbb{N} = \{0, 1, \ldots\}$. Assume

$$\sim_{n+1} \subseteq \sim_n, \qquad \sim_0 = S \times S.$$

Define

$$\nu(X, Y) := \sup\{n \in \mathbb{N} : X \sim_n Y\} \in \mathbb{N} \cup \{\infty\}.$$

The universal zeroth level makes this set nonempty. Assume separation, meaning that $X \sim_n Y$ for every n implies $X = Y$. The states can, for example, be coherent towers equipped with the chosen levelwise equivalence relations. No finiteness of the individual resolutions is required.

Theorem 4.2 (Resolution ultrametric)**.** *Define*

$$d_R(X, Y) := \begin{cases} 0, & X = Y, \\ 2^{-\nu(X,Y)}, & X \neq Y. \end{cases}$$

Then d_R *is an ultrametric:*

$$d_R(X, Z) \le \max\{d_R(X, Y), d_R(Y, Z)\}.$$

Proof. Let

$$m = \min\{\nu(X,Y), \nu(Y,Z)\}.$$

For every $n \le m$, we have $X \sim_n Y$ and $Y \sim_n Z$. Since $\sim_n$ is an equivalence relation, $X \sim_n Z$. Hence

$$\nu(X,Z) \ge m.$$

Therefore

$$2^{-\nu(X,Z)} \le 2^{-m} = \max\{2^{-\nu(X,Y)}, 2^{-\nu(Y,Z)}\}.$$

The identity and symmetry axioms are immediate, and separation gives positivity for distinct states. Without separation the same formula, with $2^{-\infty} = 0$, is an ultrapseudometric. □

Thus the resolution axis can carry an intrinsic non-Archimedean geometry. Two objects are close when many levels of resolution are required before they can be distinguished.

Definition 4.3 (First distinguishing scale)**.** When finite, define

$$\delta(X,Y) := \nu(X,Y) + 1.$$

This is the first resolution at which X and Y become distinguishable.

This quantity may be useful in p-adic, dyadic, symbolic, filtration, and machine-representation settings.

5 Seeds and generated zoos

5.1 Seed signatures

Let U be a set of isomorphism classes of resolved objects under consideration. Smallness is imposed only to use ordinary powerset fixed-point arguments.

Definition 5.1 (Seed signature)**.** A *seed signature* on U is data

$$\mathfrak{S} = (\Sigma_0, \mathscr{T}, \mathscr{C}, \mathscr{L}, \mathscr{D}),$$

where:

- $\Sigma_0 \subseteq U$ is the initial seed set;
- $\mathscr{T}$ is a collection of partial unary transformation constructors on U;
- $\mathscr{C}$ is the collection of allowed coarsening constructors;
- $\mathscr{L}$ assigns to an object its admissible refinement classes;
- $\mathscr{D}$ is a collection of partial residual constructors.

The use of partial constructors is intentional. An operation need not be defined on every object or every resolution.

Definition 5.2 (One-step zoo operator)**.** For $A \subseteq U$, define

$$\Phi(A) := A \cup \mathscr{T}[A] \cup \mathscr{C}[A] \cup \mathscr{L}[A] \cup \mathscr{D}[A],$$

where each term denotes all defined outputs of the corresponding constructors applied to inputs from A.

Lemma 5.3 (Monotonicity)**.** *If $A \subseteq B$, then*

$$\Phi(A) \subseteq \Phi(B).$$

Proof. Every constructor output available from inputs in A is also available from the same inputs viewed as elements of B. □

Theorem 5.4 (Seed zoo existence)**.** *There exists a least Φ-closed set containing Σ_0. Equivalently, there is a least fixed point of the closure process above Σ_0.*

Proof. The powerset $\mathcal{P}(U)$ is a complete lattice. The map

$$\Psi(A) := \Sigma_0 \cup \Phi(A)$$

is monotone by theorem 5.3. By the Knaster–Tarski fixed-point theorem, Ψ has a least fixed point. This is exactly the least constructor-closed set containing Σ_0. □

Definition 5.5 (Generated zoo)**.** The least fixed point in theorem 5.4 is denoted

$$\mathcal{Z}(\Sigma_0).$$

Proposition 5.6 (Finitary iteration)**.** *Assume each constructor is given by a fixed input/output relation with finitely many inputs, independent of the stage A. Then Φ preserves increasing unions of ω-chains. Then*

$$\mathcal{Z}(\Sigma_0) = \bigcup_{n<\omega} \Psi^n(\varnothing).$$

Proof. For a constructor output from $\bigcup_n A_n$, its finitely many inputs all belong to a common A_N. The output therefore already belongs to $\Phi(A_N)$. This proves preservation of increasing unions. The union on the right is a fixed point because Ψ preserves that union. It contains Σ_0 and is contained in every fixed point by induction on n, hence is least. □

This turns the seed calculus into an explicit generative process:

$$\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \cdots, \qquad \Sigma_{n+1} = \Psi(\Sigma_n).$$

5.2 Zoo growth

Definition 5.7 (Growth function)**.** When each finite stage contains finitely many isomorphism classes, define

$$g_{\Sigma_0}(n) := |\Psi^n(\varnothing)|.$$

The asymptotic class of $g(n)$ may distinguish finite, polynomial, exponential, and faster-growing presentations. It depends on the chosen generator signature and cost convention, not merely on the generated category. This is a proposed invariant of a generated mathematical grammar rather than a theorem of the present note.

6 Transformation-resolution exchange

6.1 Resolution-indexed transformations

Definition 6.1 (Lax resolution-compatible transformation)**.** A *lax resolution-compatible transformation* consists of endofunctors

$$A_r : \mathbf{T}_r \to \mathbf{T}_r$$

for every $r \in R$, together with natural transformations

$$\eta^A_{s,r} : A_r C_{s\to r} \Longrightarrow C_{s\to r} A_s \qquad (r \le s),$$

satisfying identity and composition coherence:

$$\eta^A_{r,r} = \mathrm{id}$$

and, for $r \le s \le t$,

$$\eta^A_{t,r} = (C_{s\to r}\eta^A_{t,s}) \circ (\eta^A_{s,r} C_{t\to s}).$$

If every $\eta^A_{s,r}$ is a natural isomorphism, A is called *invertibly resolution-compatible*. This is distinct from exactness of a functor between stable categories.

The square is

$$A_r C_{s\to r}(X) \xrightarrow{\eta^A_{s,r,X}} C_{s\to r} A_s(X).$$

It compares transforming after coarsening with coarsening after transforming.

Theorem 6.2 (Lift to the total category)**.** *Every lax resolution-compatible transformation A induces an endofunctor*

$$\widehat{A} : \mathbf{K} \to \mathbf{K}$$

defined on objects by

$$\widehat{A}(r, X) = (r, A_r X).$$

Proof. A morphism $(r, X) \to (s, Y)$ in $\mathbf{K}$ is represented by

$$\alpha : X \to C_{s\to r} Y.$$

Define $\widehat{A}(\alpha)$ to be

$$A_r X \xrightarrow{A_r(\alpha)} A_r C_{s\to r} Y \xrightarrow{\eta^A_{s,r,Y}} C_{s\to r} A_s Y.$$

Identity preservation follows from $\eta^A_{r,r} = \mathrm{id}$. Composition preservation is exactly the coherence condition in theorem 6.1, together with naturality of η^A. □

Thus coherent exchange data suffice to assemble the family into a functor of the total system. A converse requires the functor to lie over the resolution base and to have the specified fiberwise restrictions.

6.2 Mixed residuals

Definition 6.3 (Stable mixed residual)**.** Assume each $\mathbf{T}_r$ is a small stable ∞-category and the coarsening and transformation functors are exact, with homotopy-coherent exchange data. For $r \le s$ and $X \in \mathbf{T}_s$, define

$$\Delta^A_{s,r}(X) := \mathrm{Cofib}\big(\eta^A_{s,r,X}\big) \in \mathbf{T}_r.$$

Then there is a cofiber sequence

$$A_r C_{s\to r} X \longrightarrow C_{s\to r} A_s X \longrightarrow \Delta^A_{s,r}(X).$$

Proposition 6.4 (Residual vanishing criterion)**.** *In a stable fiber,*

$$\Delta^A_{s,r}(X) \simeq 0$$

if and only if

$$\eta^A_{s,r,X}$$

is an equivalence.

Proof. In a stable category a morphism is an equivalence exactly when its cofiber is a zero object. □

In an abelian fiber one may instead record

$$\Delta^-_{s,r}(X) = \ker \eta^A_{s,r,X}, \qquad \Delta^+_{s,r}(X) = \mathrm{coker}\, \eta^A_{s,r,X}.$$

These vanish simultaneously exactly when the comparison is an isomorphism.

Definition 6.5 (Residual promotion)**.** If X is a seed-generated object and $\Delta^A_{s,r}(X)$ is nonzero, the operation

$$X \longmapsto \Delta^A_{s,r}(X)$$

is called *residual promotion* when the residual is added to the active seed set.

This creates the recursive pattern

$$\text{seed} \to \text{mixed square} \to \text{residual} \to \text{new seed} \to \text{expanded zoo}.$$

7 Normal forms for mixed words

To make the two-axis calculus computational, mixed transformation-resolution expressions should admit controlled rewriting.

7.1 Typed word system

Consider a typed alphabet containing transformation symbols T and resolution symbols C. A mixed word is a composable expression such as

$$W = T_1 C_1 T_2 T_3 C_2 T_4.$$

Assume every well-typed adjacent transformation/coarsening inversion admits at least one length-preserving exchange rule of the form

$$TC \;\Rightarrow\; C\,T',$$

where T' is the correctly reindexed transformation at the new resolution.

Definition 7.1 (Inversion number)**.** For a word W, let $\iota(W)$ be the number of ordered pairs (T, C) in which a transformation symbol occurs to the left of a resolution symbol when the chosen normal form requires all resolution symbols first.

Lemma 7.2 (Exchange termination)**.** *Every exchange rewrite*

$$TC \Rightarrow CT'$$

reduces ι by one. Therefore every sequence of such rewrites terminates.

Proof. The rewritten adjacent pair contributes one inversion before the rewrite and none afterward. The relative order of every other transformation-resolution pair is unchanged. □

Theorem 7.3 (Exact two-axis normal form)**.** *Assume the exchange rewriting system is locally confluent. Then every well-typed mixed word has a unique normal form up to the equality notion of the rewriting system. The normal form has all resolution operations grouped before all within-resolution transformations.*

Proof. Induct on the inversion number. Every rewrite lowers it, so normal forms are reached. If two paths start with different rewrites, local confluence gives joining paths from their first successors to a common word. Normalize that word. The induction hypothesis at each first successor identifies its normal form with the common descendant's normal form. Thus the two original paths have the same normal form. Since every adjacent inversion has a rule, a normal word has no inversion and therefore has all coarsening symbols to the left of all transformation symbols. This is the finite-height form of Newman's argument. □

Status: proved here from standard rewriting theory under local confluence.

The nontrivial application problem is to prove local confluence for the concrete labelled signature. This theorem concerns word syntax, not equality of the semantic comparison maps along different paths. The pure two-sort grammar is treated more explicitly in theorem 8.6.

7.2 Split residual expansion

Exact exchange is too restrictive for many applications. A first rigorous residual model can be obtained in an additive setting.

Definition 7.4 (Split residual exchange)**.** A chosen exchange morphism $\eta : TC \Rightarrow CT'$ is *split residual* if it is naturally identified with the projection in a decomposition

$$TC(X) \cong CT'(X) \oplus E_{T,C}(X).$$

Here $E_{T,C}$ is the split *source-side remainder*. It is not denoted by Δ, because in the stable convention of this paper

$$\mathrm{Cofib}(\eta_X) \simeq E_{T,C}(X)[1], \qquad E_{T,C}(X) \simeq \Delta_{T,C}(X)[-1].$$

Theorem 7.5 (Finite split residual expansion)**.** *Fix a finite principal normalization path for a well-typed mixed word. Assume every step has a chosen split residual exchange and every surrounding operator preserves finite direct sums. Then*

$$W(s) \cong W_{\mathrm{nf}}(s) \oplus \bigoplus_{i=1}^{m} E_i(s),$$

where $E_i(s)$ is the surrounding context applied to the source-side remainder of step i. In the stable case, $E_i(s) \simeq D_i[-1]$ for the step cofibers D_i defined below.

Proof. At step i, the split exchange and additivity of the surrounding context give $X_{i-1} \cong X_i \oplus E_i(s)$. Substituting these decompositions successively yields $X_0 \cong X_m \oplus E_m(s) \oplus \cdots \oplus E_1(s)$. The direct sum is finite because the chosen principal path is finite. The identification with shifted cofibers follows from the cofiber of a split projection. □

This proof does not recursively normalize the residual branches. Such a recursive residual forest needs its own well-founded measure. Termination of the principal word alone does not prove termination of every newly generated residual word.

Definition 7.6 (Normalization path)**.** Let $\mathcal{S}$ be a stable ∞-category, or a stable enhancement with functorial cofibers. A *normalization path* for a mixed word W evaluated on s is a finite chain

$$X_0 = W(s) \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_m} X_m = W_{\mathrm{nf}}(s),$$

where every f_i is obtained by applying an exact surrounding context to one elementary exchange morphism. Write

$$D_i := \mathrm{Cofib}(f_i), \qquad D_{\mathrm{tot}} := \mathrm{Cofib}(X_0 \to X_m).$$

Theorem 7.7 (Stable residual filtration along a normalization path)**.** *For every finite normalization path in a stable enhancement there is a canonical finite filtration*

$$0 = F_0 \longrightarrow F_1 \longrightarrow \cdots \longrightarrow F_m = D_{\mathrm{tot}}$$

with cofiber sequences

$$F_{i-1} \longrightarrow F_i \longrightarrow D_i.$$

Consequently the associated graded pieces of the total normalization defect are precisely the transported elementary exchange residuals D_i.

Proof. Define

$$F_i := \mathrm{Cofib}(X_0 \longrightarrow X_i).$$

Then $F_0 \simeq 0$ and $F_m = D_{\mathrm{tot}}$. For the composable maps

$$X_0 \longrightarrow X_{i-1} \xrightarrow{f_i} X_i,$$

the octahedral cofiber sequence gives

$$\mathrm{Cofib}(X_0 \to X_{i-1}) \longrightarrow \mathrm{Cofib}(X_0 \to X_i) \longrightarrow \mathrm{Cofib}(f_i).$$

Thus

$$F_{i-1} \longrightarrow F_i \longrightarrow D_i$$

is a cofiber sequence. Since f_i is obtained by placing an elementary exchange morphism inside an exact context, exactness identifies D_i with the same context applied to the elementary exchange cofiber. Iterating gives the stated filtration. □

Status: proved for a chosen normalization path in a stable enhancement.

Corollary 7.8 (Source-to-normal-form residual control)**.** *There is a cofiber sequence*

$$D_{\mathrm{tot}}[-1] \longrightarrow W(s) \longrightarrow W_{\mathrm{nf}}(s),$$

and $D_{\mathrm{tot}}[-1]$ carries the shifted filtration with successive pieces $D_i[-1]$. Hence the failure of the original word to equal its normal form is an iterated extension of shifted elementary exchange residuals.

The theorem is path-relative. If two normalization paths are related by coherent higher exchange cells, their total comparison maps may agree up to homotopy, but path-independence of the induced filtration is a separate problem. In an ordinary triangulated category without an enhancement, the octahedral axiom gives existence of such filtrations but not a canonical functorial choice.

Open Problem 7.9 (Order-independent residual filtration)**.** Identify an intrinsic replacement for the family of path-relative filtrations. Mere higher coherence cannot force their ordered stages to be equivalent: theorem 8.1 gives a counterexample. The residual diagram of theorem 8.9 is a replacement when a coherent event diagram is supplied. Additional hypotheses for equivalence of selected ordered filtrations must constrain the intermediate objects, not only the comparison maps.

8 Normalization histories and intrinsic residual diagrams

There are three different invariance questions. A word may have a unique normal-form *object*, the normalization paths may induce the same *comparison map*, and the cofiber of that map may carry several different *ordered filtrations*. None of these statements should be silently substituted for the next.

8.1 A no-go theorem for ordered filtration independence

Proposition 8.1 (Coherent diamonds do not determine one ordered filtration)**.** *Even a strictly commuting bicartesian square of objects in a stable category can yield two normalization paths with the same total comparison map and inequivalent ordered residual filtrations.*

Proof. Work in the stable category of perfect complexes over a field k. Put $M = k[0]$ and $N = k^2[0]$, and consider the square

$$\begin{array}{ccc} 0 & \longrightarrow & M \\ \downarrow & & \downarrow \\ N & \hookrightarrow & M \oplus N. \end{array}$$

The maps into $M \oplus N$ are the summand inclusions. This square is a pushout, hence also a pullback. Both composites are the unique map $0 \to M \oplus N$.

The residual filtrations of its two boundary paths are

$$0 \longrightarrow M \longrightarrow M \oplus N, \qquad 0 \longrightarrow N \longrightarrow M \oplus N.$$

An equivalence of these objects of $\mathrm{Fun}([2], \mathrm{Perf}(k))$ would induce an equivalence $M \simeq N$ at the middle index. That is impossible because their degree-zero cohomology dimensions are 1 and 2. Nevertheless both total residuals are $M \oplus N$. □

Example 8.2 (The counterexample occurs inside the two-axis grammar)**.** The preceding square is not merely an unrelated diagram. Let the resolution base be $0 < 1 < 2$, put

$$\mathbf{T}_0 = \mathbf{T}_1 = \mathbf{T}_2 = \mathrm{Perf}(k) \times \mathrm{Perf}(k),$$

and take every coarsening functor to be the identity between labelled copies. Define the exact coordinate projections

$$P_1(x, y) = (x, 0), \qquad P_2(x, y) = (0, y).$$

Use transformation families

$$A_0 = P_2, \quad A_1 = A_2 = \mathrm{id}, \qquad B_0 = B_1 = P_1, \quad B_2 = \mathrm{id}\,.$$

For A, the comparison across $1 \to 0$ is the inclusion $P_2 \Rightarrow \mathrm{id}$, and across $2 \to 1$ it is the identity. For B, these choices are respectively the identity and $P_1 \Rightarrow \mathrm{id}$. The long comparisons are their composites. Units and transitivity therefore hold.

On the seed $x = (k[0], k^2[0])$, the word

$$W = A_0 C_{1\to 0} B_1 C_{2\to 1}$$

evaluates to zero. Exchanging its left inversion first yields $P_1 x$. Exchanging its right inversion first yields $P_2 x$. The second exchange in either path yields x. Both paths then perform the same identity exchange to reach the normal word

$$C_{1\to 0} C_{2\to 1} A_2 B_2.$$

Thus the complete paths have residual filtrations

$$0 \to P_1 x \to x \xrightarrow{\mathrm{id}} x, \qquad 0 \to P_2 x \to x \xrightarrow{\mathrm{id}} x.$$

Their first nonzero stages are not equivalent. The transformations, contexts, and coarsenings are exact, and all comparison data are coherent. Consequently extra coherence alone cannot repair the general claim of ordered filtration independence.

Status: general ordered-filtration independence refuted; the path-relative theorem remains valid.

The correction is not to discard the filtration. It is to retain the collection of compatible intermediate stages in a diagram richer than a single linear order.

8.2 What suffices for independence of the total residual

The normal-form endpoint alone does not determine a residual. For example, the maps id_k and $0 : k \to k$ have the same source and target, but cofibers 0 and $k \oplus k[1]$. Comparison maps therefore belong to the data.

Theorem 8.3 (Semantic diamond criterion)**.** *Let a rewrite graph have a height h in $\mathbb{N}$ that decreases on every edge, and suppose every vertex x has a unique reachable normal vertex $n(x)$. Assign an object X_x in a stable ∞-category to each vertex and a morphism to each edge. Assume every elementary peak*

$$x \xrightarrow{e} y, \qquad x \xrightarrow{f} z$$

has joining paths $a : y \rightsquigarrow u$ and $b : z \rightsquigarrow u$ whose semantic composites satisfy

$$[a]\,[e] = [b]\,[f] \quad \textit{in the homotopy category.}$$

Then every normalization path from x to $n(x)$ induces the same homotopy class of comparison map. In particular, the total residuals of any two such paths are equivalent.

Proof. Induct on $h(x)$. A normal vertex has only the empty normalization path. Otherwise take two normalization paths with initial edges e and f. Choose the joining paths in the hypothesis and any normalization path $t : u \rightsquigarrow n(x)$. Such a path exists, and its terminal vertex is $n(x)$ by uniqueness of normal forms.

Since $h(y), h(z) < h(x)$, the induction hypothesis replaces the two original suffixes by ta and tb respectively without changing their homotopy classes. The full comparison classes are therefore

$$[t][a][e] = [t][b][f].$$

This proves equality for all normalization paths from x. Equality in the homotopy category gives a homotopy between the corresponding maps with fixed endpoints. A homotopy identifies the two arrows up to equivalence, and the cofiber functor preserves that equivalence. □

Remark 8.4 (Equivalence is not a chosen canonical identification). The theorem asserts existence of equivalent total cofibers. A specified equivalence requires a specified homotopy between comparison maps. Independence of all such choices requires higher coherence. Neither conclusion identifies the intermediate ordered filtrations in theorem 8.1.

8.3 The event poset of pure exchange

The pure exchange grammar has enough combinatorial structure to index all normalization histories without choosing their order. This subsection concerns only adjacent swaps $TC \Rightarrow CT$ that retain the identity of each symbol. More elaborate rules that duplicate, delete, or replace generators are not included.

Definition 8.5 (Exchange event poset)**.** Fix a finite word W in two types of labelled symbols. Number its transformation symbols $T_1, \ldots, T_a$ and its coarsening symbols $C_1, \ldots, C_b$ in their original left-to-right order. An *exchange event* is a pair

$$(i, j) \in E(W) \quad \Longleftrightarrow \quad T_i \text{ occurs to the left of } C_j \text{ in } W.$$

The event swaps the relative order of this pair exactly once. Define P_W as the partial order generated by

$$\begin{aligned} (i, k) &\prec (i, j) && \text{when } k < j \text{ and both events exist,} \\ (h, j) &\prec (i, j) && \text{when } h > i \text{ and both events exist.} \end{aligned}$$

These relations are acyclic: along a generating relation the transformation index decreases or the coarsening index increases, with at least one strict change. Let $J(P_W)$ be the poset of order ideals, ordered by inclusion.

The first relation says that one transformation crosses the coarsenings from left to right. The second says that one coarsening crosses the transformations from right to left.

Theorem 8.6 (Normalization states are order ideals)**.** *For the pure adjacent-swap grammar, the following correspondences are bijective:*

1. *words reachable from W and ideals $I \in J(P_W)$;*
2. *enabled elementary swaps at a reachable word and minimal elements of $P_W \setminus I$;*
3. *complete normalization histories and linear extensions of P_W.*

The ideal attached to a reachable word is the set of events already performed. The original word corresponds to $\varnothing$, and the normal word to P_W.

Proof. Every swap changes the order of exactly one transformation/coarsening pair, never to be reversed. The relative orders of the transformations among themselves and of the coarsenings among themselves remain fixed.

Before event (i, j) can occur, every initially intervening C_k with $k < j$ must already have crossed T_i, and every initially intervening T_h with $h > i$ must already have crossed C_j. Thus all predecessors of a performed event have already occurred. The performed events form an ideal.

That ideal determines the reachable word uniquely. It specifies whether each originally inverted pair has crossed, while pairs originally of the form C_jT_i never change order. Together with the fixed orders inside each type, this determines every pairwise order of labelled symbols.

Conversely, suppose an ideal I has been performed and (i, j) is a minimal event outside it. In the current word T_i still precedes C_j. If another symbol lies between them, it is either some C_k with $k < j$ or some T_h with $h > i$. In the first case (i, k) is an unperformed predecessor. In the second case (h, j) is an unperformed predecessor. The relevant pair was already inverted in the original word, since swaps never turn an original CT pair into TC. Both cases contradict minimality. Hence T_i and C_j are adjacent and the event is enabled.

Any ideal can now be constructed by adding its events in an order compatible with P_W. This proves surjectivity and the statement about edges. A complete path adds each event exactly once, so it is exactly a linear extension. Conversely each linear extension prescribes enabled swaps and therefore a complete path. □

Lemma 8.7 (All histories are related by independent interchanges)**.** *Any two linear extensions of a finite poset are connected by swaps of adjacent incomparable elements. Consequently any two pure normalization histories are related by commuting independent-event diamonds.*

Proof. Let e be the first element of the second linear extension. It is minimal in the poset. In the first extension, every element appearing before e is incomparable with it: it cannot precede e in the poset by minimality, and it cannot follow e in the poset while occurring before it in a linear extension. Move e left across these incomparable elements. The two extensions now begin with the same element. Delete it and apply induction to the remaining finite poset.

In the exchange poset, incomparable events share neither a transformation symbol nor a coarsening symbol. When both are enabled their adjacent swaps are disjoint. Performing them in either order gives the same word, namely the diamond for adjoining both events to the current ideal. □

It follows that the category of rewrite paths modulo these diamond relations is the thin category $J(P_W)$. The conclusion concerns the pure syntax. In a typed application, labels after crossing must be defined by the specified reindexing rules, and their semantic exchange cells must satisfy the corresponding comparisons.

8.4 A diagram that retains the residual histories

Definition 8.8 (Coherent exchange-event diagram)**.** For a finite event poset P, a coherent exchange-event diagram is a functor

$$X : NJ(P) \longrightarrow \mathcal{S}$$

into a stable ∞-category. Here N denotes the nerve. Its value X_I is the semantic state after the ideal I of events has been completed. Values on cover arrows realize the elementary exchanges, and its higher simplices supply the compatibility data.

Strictly commuting exchange diagrams in a concrete enhanced model, such as complexes, supply examples. Merely checking squares in the homotopy category does not automatically supply all the higher data in this definition.

Theorem 8.9 (Ideal-indexed residual diagram)**.** *A coherent exchange-event diagram determines a functor*

$$\mathcal{F}_X : NJ(P) \longrightarrow \mathcal{S}, \qquad \mathcal{F}_X(I) = \mathrm{Cofib}(X_\varnothing \longrightarrow X_I),$$

with the following properties:

1. $\mathcal{F}_X(\varnothing) \simeq 0$*;*
2. *along every maximal chain of ideals it restricts to the residual filtration of theorem 7.7;*
3. *for every cover $I \subset I \cup \{e\}$ there is a cofiber sequence*
$$\mathcal{F}_X(I) \longrightarrow \mathcal{F}_X(I \cup \{e\}) \longrightarrow \mathrm{Cofib}(X_I \to X_{I\cup\{e\}});$$
4. *its terminal object is the total residual* $\mathrm{Cofib}(X_\varnothing \to X_P)$*, independent of the linear extension used to reach it;*
5. *exact functors preserve the entire diagram, naturally in the specified coherent input.*

Proof. The arrows from the initial ideal $\varnothing$ induce a natural transformation from the constant diagram at $X_\varnothing$ to X. Take its cofiber pointwise in the stable functor category. This constructs $\mathcal{F}_X$ together with its coherence, not just an assignment of objects.

At $\varnothing$ the map is the identity, so its cofiber is zero. Restriction to a maximal chain gives exactly the pointwise formula defining the earlier path-relative filtration. For an inclusion of ideals, apply the cofiber sequence for the composable maps

$$X_\varnothing \longrightarrow X_I \longrightarrow X_{I\cup\{e\}}.$$

This gives the displayed cover cofiber sequence. At the terminal ideal the same formula gives the total residual. Finally an exact functor preserves these finite colimits pointwise and hence preserves the diagram and all its structure maps. □

Status: proved relative to a specified coherent exchange-event diagram.

This is the appropriate replacement for a single path-independent ordered filtration. In theorem 8.1, P has two incomparable events, so $J(P)$ is the four-object diamond. Its two chains remember different intermediate residuals without disagreeing about the total residual. The richer diagram preserves information that the proposed identification of ordered filtrations would have erased.

Corollary 8.10 (History-independent total obstruction)**.** *Let $q : \mathcal{S} \to \mathcal{O}$ be exact. For a fixed coherent exchange-event diagram, the object*

$$q\mathcal{F}_X(P)$$

is independent of the normalization history. Along every maximal chain it has the filtration of theorem 11.11. Therefore its nonvanishing forces a nonzero elementary quotient residual on each *normalization history. The particular event witnessing nonvanishing may depend on the history.*

Proof. The terminal object is fixed by theorem 8.9. Apply theorem 11.11 separately to the finite filtration supplied by each maximal chain. □

The converse is not asserted. Nonzero elementary residuals can participate in extensions whose total residual vanishes. Nor does the corollary exhibit one event that must witness the obstruction in every history.

9 Constructing semantic exchange diagrams

The coherent diagram in theorem 8.9 was an input. We now construct it for a concrete enhanced realization of the typed pure grammar. This does not strictify arbitrary higher-categorical exchange data.

Definition 9.1 (Strict chain-natural realization)**.** Fix a finite resolution chain $r_0 \leq \cdots \leq r_b$. At each level choose a category $\mathcal{E}_j$ of complexes and a functor $L_j : \mathcal{E}_j \to \mathcal{S}_j$ into a stable ∞-category. A model example is bounded complexes of finitely generated projective modules mapping to their derived category. Specify chain-level functors

$$C_j : \mathcal{E}_j \to \mathcal{E}_{j-1}, \qquad T_{i,j} : \mathcal{E}_j \to \mathcal{E}_j,$$

which induce exact functors in the chosen stable realizations, and strict natural transformations

$$\chi_{i,j} : T_{i,j-1}C_j \Longrightarrow C_j T_{i,j} \qquad (1 \leq j \leq b).$$

The components of χ are actual chain maps and their naturality squares commute as chain maps. Functor composition and whiskering are taken in the ordinary strict 2-category of these models.

Coarsening symbols occur in their fixed order $C_1, \ldots, C_b$. A transformation symbol with exactly j coarsening symbols to its left is interpreted as $T_{i,j}$. Hence every reachable word represents a functor from $\mathcal{E}_b$ to $\mathcal{E}_0$. Moving T_i across C_j reindexes its level rather than creating a new untyped symbol. No rule in this subsection contracts coarsenings or introduces a new exchange relation.

Theorem 9.2 (Strict semantic realization of the event diagram)**.** *For a typed pure word W in a strict chain-natural realization and a seed $s \in \mathcal{E}_b$, evaluation gives a functor*

$$X_W : NJ(P_W) \longrightarrow \mathcal{S}_0.$$

Its vertices evaluate the reachable words and its cover arrows evaluate the whiskered exchange maps. Thus the residual diagram in theorem 8.9 is constructed, rather than assumed, for this class of realizations.

Proof. First work in the ordinary category of functors $\mathcal{E}_b \to \mathcal{E}_0$ and natural transformations. Assign to a rewrite edge its exchange transformation, whiskered by the surrounding word. Two simultaneously enabled swaps are disjoint. The two composites around their diamond are equal by the interchange law for natural transformations. This equality uses strict naturality, not invertibility of χ.

For ideals $I \subseteq J$, two paths adding the events in $J \setminus I$ are linear extensions of that interval's induced event order. The argument of theorem 8.7 connects them by adjacent interchanges of incomparable events. Each interchange is one of the commuting diamonds just checked. Hence all such paths induce the same natural transformation. Evaluation therefore factors through the thin category $J(P_W)$.

Evaluate this ordinary functor on s, take its nerve, and compose with L_0. An ordinary functor gives all simplices of its nerve, so the output is a coherent diagram, not merely a diagram in a homotopy category. The prescribed vertices and edges are unchanged by this construction. □

Remark 9.3 (Scope of coherence). Strict models are sufficient, not necessary. This theorem does not infer strict naturality from a collection of commuting squares in a homotopy category. Arbitrary weak exchange cells, additional rewrite relations, and higher reindexing coherence still need their own realization theorem. For the fixed pure grammar, however, no independent braid relation is needed: the only changes of history exchange disjoint events.

10 Interaction residuals and event reconstruction

An exchange square can commute coherently while changing the residual of one event after another event is performed. The appropriate measurement is a second cofiber. It measures failure of excision, not failure of commutativity.

10.1 The interaction of two exchanges

Let a specified coherent square in a stable ∞-category $\mathcal{S}$ be

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v'} \\ C & \xrightarrow[u']{} & D. \end{array}$$

Write $E = \mathrm{Cofib}(u)$ and $E' = \mathrm{Cofib}(u')$. The square induces $E \to E'$.

Definition 10.1 (Interaction residual)**.** The interaction residual of this specified square is

$$\Theta = \mathrm{Cofib}\big(\mathrm{Cofib}(A \to B) \longrightarrow \mathrm{Cofib}(C \to D)\big).$$

The specified square, including its coherent homotopy, is part of the input.

Theorem 10.2 (Interaction is the total cofiber)**.** *There are natural equivalences*

$$\Theta \simeq \mathrm{Cofib}\big(B \amalg_A C \longrightarrow D\big) \simeq \mathrm{Cofib}\big(\mathrm{Cofib}(A \to C) \longrightarrow \mathrm{Cofib}(B \to D)\big).$$

Consequently $\Theta \simeq 0$ if and only if the square is a homotopy pushout, equivalently a homotopy pullback. An exact functor preserves these equivalences.

Proof. Cofibers are pushouts along maps to zero. Interchanging the finite colimits in the two successive pushout constructions identifies the first expression with the cofiber of the map from the punctured-square colimit $B \amalg_A C$ to D. Performing the same construction in the other coordinate gives the second expression. These identifications arise from the universal properties of the same finite diagrams and are natural in the given square.

A map in a stable category has zero cofiber exactly when it is an equivalence. Thus vanishing says precisely that the punctured-square colimit maps equivalently to D. Stability identifies pushout squares with pullback squares. Exact functors preserve zero objects and these finite colimits. □

In a chain model, interchanging coordinates includes the usual graded signs. The theorem is a natural equivalence, not an assertion that differently ordered cone matrices are literally identical. The cone conventions are recorded in [6]; the finite-colimit argument supplies the enhanced statement used here.

10.2 An event's residual before and after unrelated events

Fix a finite poset P and a coherent diagram $X : NJ(P) \to \mathcal{S}$. For $e \in P$, put

$$D_e = \{p \in P : p < e\}, \qquad \downarrow e = D_e \cup \{e\}.$$

The eligible contexts for e form the interval of ideals

$$\mathcal{A}_e = \{I \in J(P) : D_e \subseteq I,\ e \notin I\}.$$

Its initial object is D_e. Define the residual functor on this interval by

$$E_e(I) = \mathrm{Cofib}(X_I \to X_{I\cup\{e\}}).$$

For incomparable events e, f enabled at I, write $\Theta_{e,f}(I)$ for the interaction residual of their elementary square.

Theorem 10.3 (Criterion for context-independent event residuals)**.** *The following conditions are equivalent.*

1. *Every elementary independent-event square of X is a homotopy pushout.*
2. *Every $\Theta_{e,f}(I)$ vanishes.*
3. *For each event e, the natural comparison*

$$\mathrm{const}_{\mathcal{A}_e} E_e(D_e) \longrightarrow E_e$$

is an equivalence of diagrams.

In this regime the successive cofibers of every normalization history are the event-labelled objects

$$d_e := \mathrm{Cofib}(X_{D_e} \to X_{\downarrow e}),$$

listed in the order in which that history performs the events. Their labels and equivalence classes are independent of context, but their ordering and extension data need not disappear.

Proof. The equivalence of (1) and (2) is theorem 10.2. A cover in $\mathcal{A}_e$ adds an event f incomparable with e. Indeed, predecessors of e already lie in I, while a successor of e cannot belong to an ideal omitting e. On this cover the map of event residuals has cofiber $\Theta_{e,f}(I)$. Condition (2) therefore makes every cover map an equivalence. Any inclusion in the finite interval factors into covers. The comparisons from its initial object consequently give the objectwise, hence diagrammatic, equivalence in (3).

Conversely, (3) makes every cover comparison of event residuals an equivalence, so its cofiber vanishes. The final statement follows by restricting the residual diagram to a maximal chain and applying the same comparison to each enabled event. □

10.3 Reconstructing the full diagram from principal ideals

The preceding criterion gives more than a common list of residuals. It makes the diagram recoverable from its values on the individual events together with their predecessors.

Lemma 10.4 (Elementary excision implies ideal excision)**.** *If every elementary independent-event square of $X : NJ(P) \to \mathcal{S}$ is a homotopy pushout, then for all ideals I, J the square*

$$\begin{array}{ccc} X_{I\cap J} & \longrightarrow & X_I \\ \downarrow & & \downarrow \\ X_J & \longrightarrow & X_{I\cup J} \end{array}$$

is a homotopy pushout.

Proof. Elements of $I \setminus J$ and $J \setminus I$ are pairwise incomparable across the two sets. If $p \in I \setminus J$ satisfied $p \leq q \in J \setminus I$, downward closure of J would imply $p \in J$, a contradiction. The reverse comparison is excluded similarly.

Choose linear extensions for both exclusive sets, beginning from $I \cap J$. Adjoining their successive initial segments gives a rectangular grid of ideals. Each cell adjoins one enabled event from each side and is an elementary independent-event square. It is a pushout by hypothesis. Repeated pasting of pushout squares gives the outer square displayed above. Empty exclusive sets give identity squares and cause no exception. □

Theorem 10.5 (Principal-ideal reconstruction)**.** *Let P be finite and $\mathcal{S}$ stable. Write $\mathrm{Exc}_0(J(P), \mathcal{S})$ for the full subcategory of coherent diagrams $F : NJ(P) \to \mathcal{S}$ such that $F(\varnothing) \simeq 0$ and every elementary independent-event square is a homotopy pushout. Restriction along*

$$j : P \longrightarrow J(P), \qquad e \longmapsto {\downarrow} e$$

induces an equivalence

$$\mathrm{Exc}_0(J(P), \mathcal{S}) \simeq \mathrm{Fun}(NP, \mathcal{S}).$$

Its inverse sends $G : NP \to \mathcal{S}$ to

$$(j_! G)(I) = \operatorname*{colim}_{e \in I} G(e), \qquad (j_! G)(\varnothing) = 0.$$

All colimits are finite. In particular, for an excisive event diagram X its reduced residual diagram satisfies

$$\boxed{\mathcal{F}_X(I) \simeq \operatorname*{colim}_{e \in I} \mathcal{F}_X({\downarrow} e).}$$

Proof. The displayed formula is the pointwise left Kan extension along j. It follows directly from the colimit universal property: maps from $j_! G$ to F are the same as compatible maps $G(e) \to F({\downarrow} e)$. For a principal ideal ${\downarrow} e$, its indexing poset has terminal object e, so $(j_! G)({\downarrow} e) \simeq G(e)$.

For arbitrary ideals I, J, every chain in $I \cup J$ lies wholly in I or wholly in J. Its maximal element lies in one ideal and all its predecessors lie there too. Hence the nerve of their union is the union of their nerves along the intersection. Cocones on $G|_{I \cup J}$ are precisely compatible cocones on $G|_I$ and $G|_J$. The resulting colimit is the pushout of the two colimits over the colimit on $I \cap J$. Thus $j_! G$ is reduced and excisive.

Now let F be reduced and excisive. We prove the counit $j_! j^* F \to F$ is an equivalence on an ideal I by induction on $|I|$. The empty ideal is the zero object on both sides. Choose a maximal element $e \in I$. Put $I' = I \setminus \{e\}$. Then

$$I = I' \cup {\downarrow} e, \qquad I' \cap {\downarrow} e = D_e.$$

The ideal-excision lemma expresses $F(I)$ as the pushout of $F(I')$ and $F({\downarrow} e)$ over $F(D_e)$. The same holds for $j_! j^* F$. Induction handles I' and D_e, and the terminal-object observation handles ${\downarrow} e$. Therefore the counit is an equivalence at I. The unit and counit prove the asserted equivalence of functor categories, including natural transformations.

Finally, the total-cofiber construction is exact on square diagrams. Applied to the cofiber sequence of diagrams $\mathrm{const}\, X_\varnothing \to X \to \mathcal{F}_X$, it shows that the elementary total cofibers of X and $\mathcal{F}_X$ agree, since a constant square has zero total cofiber. Thus $\mathcal{F}_X$ is reduced and excisive, and the first part applies. □

This is finite-poset reconstruction by left Kan extension and excision. It is a theorem about the specified event diagram, not the missing arithmetic globalization theorem K15. Its general categorical ingredients are standard, and no historical novelty claim is made.

Corollary 10.6 (Independent events split, dependencies retain extensions)**.** *Under theorem 10.5, if P is an antichain then*

$$\mathcal{F}_X(P) \simeq \bigoplus_{e \in P} d_e.$$

For a general event poset, a direct-sum formula does not follow.

Proof. For an antichain, each principal ideal is a singleton, $D_e = \varnothing$, and a finite colimit over the discrete poset is a finite direct sum. For the negative statement, take P to be a two-element chain and a reduced diagram in $\mathrm{Perf}(\mathbb{Z})$

$$0 \longrightarrow \mathbb{Z}/p \xrightarrow{p} \mathbb{Z}/p^2$$

for a prime p, where the second map sends the class of x to the class of px. Both successive cofibers are $\mathbb{Z}/p$ in degree zero, but the terminal object is not $\mathbb{Z}/p \oplus \mathbb{Z}/p$: the latter is annihilated by p and the former is not. There are no independent-event squares in a chain, so excision holds without imposing a split extension. This diagram is also the residual diagram of $\mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z}$. □

The replacement for ordered-filtration independence is therefore now explicit: the event poset controls which attachments depend on earlier events, excision controls independence from unrelated events, and the principal-ideal diagram retains the extension data needed for reconstruction.

11 Obstruction theory

Residual and obstruction should not be identified. A residual is a structured discrepancy. An obstruction is a discrepancy that cannot be removed by the admissible calculus.

Definition 11.1 (Killable residual)**.** Let $\delta \in \mathcal{Z}(\Sigma_0)$. Call δ *killable* if there exists an admissible finite constructor word W such that

$$W(\delta) \simeq 0.$$

Definition 11.2 (Persistent obstruction)**.** A residual seed δ is a *persistent obstruction* relative to a constructor zoo if it is not killable by any admissible finite word.

This definition is intentionally operational. More refined applications should replace it by an obstruction class in a quotient, derived functor, cohomology group, or localization.

A first taxonomy is:

$$\boxed{\Omega = (\Omega_R, \Omega_T, \Omega_{TR}, \Omega_G)}$$

with:

- Ω_R: no admissible refinement exists;
- Ω_T: a required within-resolution transformation does not exist;
- Ω_{TR}: exchange cannot be made coherent or its residual persists;
- Ω_G: locally compatible data fail to reconstruct globally.

11.1 Universal obstruction quotient in the stable calculus

Let $\mathcal{Z}$ be a small idempotent-complete stable ∞-category representing a thick seed zoo, and let $\mathscr{M}$ be a collection of exact constructor endofunctors of $\mathcal{Z}$ containing the identity and closed under composition.

Definition 11.3 (Operationally killable subcategory)**.** Define

$$\mathcal{K}_{\mathscr{M}} := \{X \in \mathcal{Z} : \exists W \in \mathscr{M},\ W(X) \simeq 0\}.$$

Let

$$\mathcal{N}_{\mathscr{M}} := \mathrm{Thick}(\mathcal{K}_{\mathscr{M}})$$

be the smallest thick stable subcategory containing all operationally killable objects.

The thick closure is necessary in general: two objects may be killed by unrelated constructor words with no single constructor word killing an extension of them.

Definition 11.4 (Stable obstruction category and obstruction class)**.** The *stable obstruction category* is the Verdier quotient

$$\mathcal{O}_{\mathscr{M}} := \mathcal{Z}/\mathcal{N}_{\mathscr{M}},$$

with quotient functor

$$q : \mathcal{Z} \longrightarrow \mathcal{O}_{\mathscr{M}}.$$

For $X \in \mathcal{Z}$, define its *obstruction class*

$$\mathrm{Obs}_{\mathscr{M}}(X) := q(X).$$

Theorem 11.5 (Universal stable obstruction quotient)**.** *The quotient q is exact and satisfies:*

1. *every operationally killable object maps to zero;*
2. *$q(X) \simeq 0$ if and only if $X \in \mathcal{N}_{\mathscr{M}}$;*
3. *for every exact functor $E : \mathcal{Z} \to \mathcal{D}$ into a stable ∞-category such that $E(K) \simeq 0$ for every $K \in \mathcal{K}_{\mathscr{M}}$, there is an exact factorization, unique up to contractible choice together with its comparison to E,*

$$\mathcal{Z} \xrightarrow{q} \mathcal{O}_{\mathscr{M}} \xrightarrow{\bar{E}} \mathcal{D}.$$

Thus $\mathcal{O}_{\mathscr{M}}$ is universal among exact targets in which all operationally killable residuals vanish.

Proof. The Verdier quotient by the thick subcategory $\mathcal{N}_{\mathscr{M}}$ exists and the quotient functor is exact. Its kernel is exactly $\mathcal{N}_{\mathscr{M}}$. This gives (1)–(2). For (3), the kernel of an exact functor is thick. Since E kills $\mathcal{K}_{\mathscr{M}}$, it kills the thick subcategory generated by that class, namely $\mathcal{N}_{\mathscr{M}}$. The universal property of the stable Verdier quotient gives the factorization. More precisely, precomposition with q identifies the category of exact functors out of $\mathcal{O}_{\mathscr{M}}$ with the full subcategory of exact functors out of $\mathcal{Z}$ annihilating $\mathcal{N}_{\mathscr{M}}$. This also specifies the sense of uniqueness. See Drew [4] for the stable localization and [3] for the kernel calculation in the triangulated homotopy category. □

Status: proved in the stable regime using the standard Verdier quotient.

A zero obstruction here means membership in the chosen thick kernel. It need not mean that one constructor word kills the object. Nor is $q(X)$ a chosen irreducible representative back inside $\mathcal{Z}$. A section or reflective localization would be extra data. The quotient also need not inherit the constructors automatically, as theorem 11.12 will show.

Example 11.6 (Why thick closure is genuinely necessary)**.** Let $\mathcal{C}$ be any nonzero stable category and set

$$\mathcal{Z} = \mathcal{C}^3.$$

Define exact idempotent endofunctors

$$U(X_1, X_2, X_3) = (0, 0, X_2 \oplus X_3), \qquad V(X_1, X_2, X_3) = (0, 0, X_1 \oplus X_3).$$

They satisfy

$$U^2 \simeq U, \qquad V^2 \simeq V, \qquad U \circ V \simeq V, \qquad V \circ U \simeq U,$$

so $\mathscr{M} = \{\mathrm{id}, U, V\}$ is closed under composition. Choose $0 \not\simeq A \in \mathcal{C}$ and set

$$X = (A, 0, 0), \qquad Y = (0, A, 0).$$

Then $U(X) \simeq 0$ and $V(Y) \simeq 0$, so both X and Y are operationally killable. However

$$X \oplus Y = (A, A, 0),$$

and both U and V send this object to $(0, 0, A)$, which is fixed by U and V. Hence no constructor word kills $X \oplus Y$. Thus $\mathcal{K}_{\mathscr{M}}$ need not be additive, much less thick.

This example shows that replacing $\mathcal{K}_{\mathscr{M}}$ by its thick closure in theorem 11.4 is forced if the quotient is required to be universal for exact functors.

Definition 11.7 (Directed annihilator property)**.** The constructor system $\mathscr{M}$ has the *directed annihilator property* if for every $U, V \in \mathscr{M}$ there exist $P, Q, W \in \mathscr{M}$ and natural equivalences

$$W \simeq P \circ U \simeq Q \circ V.$$

Thus any two constructor words admit a common postcomposition. With multiplication written as composition, $P \circ U = Q \circ V$ is a common left multiple. The displayed equation, rather than a left/right name, fixes the convention.

Theorem 11.8 (When operational killability is already thick)**.** *If $\mathscr{M}$ consists of exact endofunctors and has the directed annihilator property, then $\mathcal{K}_{\mathscr{M}}$ is a thick stable subcategory. Hence*

$$\mathcal{N}_{\mathscr{M}} = \mathcal{K}_{\mathscr{M}}$$

and

$$\boxed{\mathrm{Obs}_{\mathscr{M}}(X) \simeq 0 \iff X \text{ is operationally killable.}}$$

Proof. The zero object is killed by every exact constructor. If $U(X) \simeq 0$, exactness gives $U(X[n]) \simeq U(X)[n] \simeq 0$ for every integer n, so the class is closed under both suspensions and desuspensions. It is closed under retracts because exact functors preserve retract diagrams.

For cofibers, suppose X is killed by U and Y is killed by V. Choose a common postcomposition

$$W \simeq P \circ U \simeq Q \circ V.$$

Then $W(X) \simeq 0 \simeq W(Y)$. For any morphism $f : X \to Y$, exactness gives

$$W(\mathrm{Cofib}(f)) \simeq \mathrm{Cofib}(Wf) \simeq \mathrm{Cofib}(0 \to 0) \simeq 0.$$

Thus the killable objects form a stable subcategory closed under retracts, hence a thick subcategory. □

Example 11.9 (Idempotent projection model)**.** Let $\mathcal{Z} = \mathcal{A} \times \mathcal{B}$ for stable categories $\mathcal{A}, \mathcal{B}$, and let

$$L(a, b) = (a, 0).$$

With constructor monoid generated by the idempotent exact functor L, an object is operationally killable exactly when it lies in $0 \times \mathcal{B}$. The obstruction quotient is equivalent to $\mathcal{A}$, and

$$\mathrm{Obs}(a, b) \simeq a.$$

Thus the quotient literally retains the component that cannot be annihilated by the admissible constructor.

Theorem 11.10 (Functorial transport of obstruction)**.** *Let $(\mathcal{Z}_A, \mathscr{M}_A)$ and $(\mathcal{Z}_B, \mathscr{M}_B)$ be stable obstruction systems, and let*

$$\Lambda : \mathcal{Z}_A \longrightarrow \mathcal{Z}_B$$

be exact. If

$$\Lambda(\mathcal{K}_{\mathscr{M}_A}) \subseteq \mathcal{N}_{\mathscr{M}_B},$$

then Λ induces an exact functor

$$\bar{\Lambda} : \mathcal{O}_{\mathscr{M}_A} \longrightarrow \mathcal{O}_{\mathscr{M}_B}.$$

If Λ is an equivalence and carries $\mathcal{N}_{\mathscr{M}_A}$ onto $\mathcal{N}_{\mathscr{M}_B}$, then $\bar{\Lambda}$ is an equivalence. In that case obstruction vanishing is preserved and reflected.

Proof. The hypothesis and exactness imply

$$\Lambda(\mathcal{N}_{\mathscr{M}_A}) \subseteq \mathcal{N}_{\mathscr{M}_B},$$

because exact functors preserve the operations used to generate a thick subcategory. Therefore $q_B \Lambda$ kills $\mathcal{N}_{\mathscr{M}_A}$, so the universal property in theorem 11.5 gives $\bar{\Lambda}$. If Λ is an equivalence identifying the two thick kernels, a quasi-inverse descends in the same way and yields a quasi-inverse to $\bar{\Lambda}$. □

This theorem gives a precise version of the intended local-global principle: a correspondence need not merely preserve local invariants; it can transport the quotient in which removable residual structure has been annihilated.

Theorem 11.11 (Obstruction localization along normalization)**.** *Let*

$$X_0 \xrightarrow{f_1} X_1 \to \cdots \xrightarrow{f_m} X_m$$

be a normalization path as in theorem 7.7, with elementary transported residuals $D_i = \mathrm{Cofib}(f_i)$ and total residual

$$D_{\mathrm{tot}} = \mathrm{Cofib}(X_0 \to X_m).$$

Let

$$q : \mathcal{Z} \to \mathcal{O}_{\mathscr{M}}$$

be the stable obstruction quotient. Then:

1. *$q(D_{\mathrm{tot}})$ inherits a finite filtration whose successive cofibers are $q(D_i)$;*
2. *if $q(D_i) \simeq 0$ for every i, then*

$$q(D_{\mathrm{tot}}) \simeq 0;$$

3. equivalently,

$$q(D_{\mathrm{tot}}) \not\simeq 0 \Longrightarrow \exists i,\ q(D_i) \not\simeq 0.$$

If the directed annihilator property of theorem 11.8 holds, the phrase "$q(D) \simeq 0$" can be replaced throughout by "D is operationally killable."

Proof. The residual filtration in theorem 7.7 consists of cofiber sequences

$$F_{i-1} \longrightarrow F_i \longrightarrow D_i, \qquad F_0 \simeq 0, \qquad F_m \simeq D_{\mathrm{tot}}.$$

Because the Verdier quotient functor q is exact, applying q gives cofiber sequences

$$q(F_{i-1}) \longrightarrow q(F_i) \longrightarrow q(D_i).$$

This proves (1). If every $q(D_i)$ vanishes, then $q(F_{i-1}) \to q(F_i)$ is an equivalence for every i. Starting from $q(F_0) \simeq 0$ yields $q(F_m) \simeq 0$, proving (2). Statement (3) is its contrapositive. Under the directed annihilator hypothesis, theorem 11.8 identifies quotient-vanishing with operational killability. □

Status: proved from the stable residual filtration and exactness of the obstruction quotient.

This result gives the calculus a precise diagnostic property: a persistent obstruction in a composite transformation-resolution word cannot appear from nowhere. Relative to a chosen normalization path, it must be visible in at least one elementary exchange residual after passage to the obstruction quotient.

11.2 When the obstruction quotient must retain the constructors

The universal property of $\mathcal{Z}/\mathcal{N}_{\mathscr{M}}$ concerns exact functors that kill a prescribed class. It does not automatically preserve the full transformation grammar. To make the quotient a new environment for that grammar, the kernel must be invariant under the constructors.

Example 11.12 (A constructor can fail to descend, and invariant closure can collapse)**.** Return to theorem 11.6, with $\mathcal{Z} = \mathcal{C}^3$ and $\mathscr{M} = \{\mathrm{id}, U, V\}$. An object is killed by U precisely when its second and third components vanish, and killed by V precisely when its first and third components vanish. Indeed, a summand of a zero direct sum is zero. Consequently

$$\mathcal{N}_{\mathscr{M}} = \mathcal{C} \times \mathcal{C} \times 0, \qquad \mathcal{Z}/\mathcal{N}_{\mathscr{M}} \simeq \mathcal{C}.$$

Nevertheless

$$U(0, A, 0) = (0, 0, A)$$

for every $A \in \mathcal{C}$. The input lies in $\mathcal{N}_{\mathscr{M}}$ and the output generally does not. If U descended to an exact $\bar{U}$ on the quotient, then $qU(0, A, 0) \simeq \bar{U} q(0, A, 0) \simeq 0$, a contradiction for $A \not\simeq 0$.

Moreover, any constructor-invariant thick subcategory containing all killable objects contains both the first and second coordinate subcategories. Applying U to the second coordinate forces it to contain the third. Closure under direct sums then gives all of $\mathcal{Z}$. Thus the constructor-compatible quotient in this example is the zero category.

Definition 11.13 (Constructor-invariant thick saturation)**.** For a small stable constructor system $(\mathcal{Z}, \mathscr{M})$ as above, put

$$\mathcal{J}_{\mathscr{M}} := \mathrm{Thick}\{W(K) : W \in \mathscr{M},\ K \in \mathcal{K}_{\mathscr{M}}\}.$$

Its quotient is denoted

$$q^{\mathrm{inv}} : \mathcal{Z} \longrightarrow \mathcal{O}^{\mathrm{inv}}_{\mathscr{M}} := \mathcal{Z}/\mathcal{J}_{\mathscr{M}}.$$

The identity constructor ensures $\mathcal{N}_{\mathscr{M}} \subseteq \mathcal{J}_{\mathscr{M}}$.

Theorem 11.14 (Universal constructor-compatible obstruction quotient)**.** *The subcategory $\mathcal{J}_{\mathscr{M}}$ is the least thick subcategory that contains $\mathcal{K}_{\mathscr{M}}$ and is preserved by every constructor. Each $T \in \mathscr{M}$ descends to an exact endofunctor of $\mathcal{O}^{\mathrm{inv}}_{\mathscr{M}}$. If the constructor action is supplied with coherent unit and composition data, those data descend as well.*

The quotient is universal among exact functors that annihilate every object $W(K)$ in the displayed generating family. In particular, every exact realization that annihilates killable objects and intertwines the given constructors with exact constructors of its target factors through q^{inv}.

Proof. Fix $T \in \mathscr{M}$. It takes a generator $W(K)$ to $(T \circ W)(K)$, again a generator because $\mathscr{M}$ is closed under composition. Exactness and preservation of retracts imply

$$T(\mathcal{J}_{\mathscr{M}}) \subseteq \mathcal{J}_{\mathscr{M}}.$$

Conversely any invariant thick subcategory containing $\mathcal{K}_{\mathscr{M}}$ contains every $W(K)$ and therefore their thick closure. This proves minimality.

Now $q^{\mathrm{inv}}T$ annihilates $\mathcal{J}_{\mathscr{M}}$, so the quotient universal property gives $\bar{T}$ with

$$\bar{T}q^{\mathrm{inv}} \simeq q^{\mathrm{inv}}T.$$

The full faithfulness of precomposition in the functor-category universal property descends any supplied unit, composition, and higher coherence data. It does not create action data absent from the original system.

The last universal property follows directly from thick closure and Verdier localization. If E is equivariant and kills K, then $E(WK) \simeq W'E(K) \simeq W'(0) \simeq 0$. Thus it kills all the generators and factors through the quotient. □

Status: proved for small stable exact constructor systems.

There is no general nontriviality conclusion. The zero category is a legitimate universal quotient. A useful application must show that the chosen admissible repairs do not annihilate everything after invariant thick closure.

Theorem 11.15 (Directed annihilators also give constructor invariance)**.** *Under the directed annihilator hypothesis of theorem 11.8,*

$$\mathcal{J}_{\mathscr{M}} = \mathcal{N}_{\mathscr{M}} = \mathcal{K}_{\mathscr{M}}.$$

In particular the same quotient both supports the admitted constructors and detects operational killability exactly.

Proof. The earlier theorem shows that $\mathcal{K}_{\mathscr{M}}$ is thick. It remains to prove invariance. Suppose $U(X) \simeq 0$ and fix a constructor T. Directedness supplies constructors P, Q, W with

$$W \simeq P \circ U \simeq Q \circ T.$$

Therefore

$$Q(TX) \simeq P(UX) \simeq P(0) \simeq 0.$$

Thus TX is killable. The killable subcategory is invariant and thick, so minimality of $\mathcal{J}_{\mathscr{M}}$ gives the asserted equalities. □

Proposition 11.16 (A noncollapse certificate)**.** *Suppose $E : \mathcal{Z} \to \mathcal{D}$ is an exact equivariant realization into a stable constructor system, E kills every operationally killable object, and $E(X) \not\simeq 0$ for some X. Then*

$$q^{\mathrm{inv}}(X) \not\simeq 0, \qquad \mathcal{O}^{\mathrm{inv}}_{\mathscr{M}} \not\simeq 0.$$

Proof. By theorem 11.14, write $E \simeq \bar{E} q^{\mathrm{inv}}$. Exactness of $\bar{E}$ would imply $E(X) \simeq 0$ if $q^{\mathrm{inv}}(X) \simeq 0$, a contradiction. □

For the projection example $\mathcal{Z} = \mathcal{A} \times \mathcal{B}$ and $L(a, b) = (a, 0)$, projection to $\mathcal{A}$ is such a detector whenever $\mathcal{A}$ is nonzero. This gives a nontrivial model, in contrast with theorem 11.12. Admissibility matters: allowing the zero functor as a repair makes every object killable and necessarily collapses the quotient.

The refined distinction is therefore

$$\begin{aligned} \mathcal{K}_{\mathscr{M}} &: \text{ objects killed by a single admitted word,} \\ \mathcal{N}_{\mathscr{M}} &: \text{ their thick closure,} \\ \mathcal{J}_{\mathscr{M}} &: \text{ their constructor-invariant thick closure.} \end{aligned}$$

Only under additional hypotheses can these three classes be identified.

Open Problem 11.17 (Nonstable intrinsic obstruction)**.** Construct an analogue of theorem 11.4 before stable/additive completion, or prove that the stable quotient is the correct universal recipient for the applications of interest.

12 Universal excisive localization

The constructor-invariant quotient can now be attached to a precise target property: making the event diagram excisive. This is different from declaring every residual removable.

Let $\mathcal{Z}$ be a small stable category, let $\mathscr{M}$ be a coherently acting monoid of exact endofunctors, and let $\mathscr{X}$ be a small family of finite coherent event diagrams in $\mathcal{Z}$. Denote by $\mathscr{H}$ the family of interaction residuals of all their elementary independent-event squares. Put

$$\mathcal{J}_{\square} = \mathrm{Thick}\{T(H) : T \in \mathscr{M},\ H \in \mathscr{H}\}.$$

This notation does not assert that H was operationally killable before localization.

Theorem 12.1 (Universal constructor-compatible excisive quotient)**.** *The stable Verdier quotient*

$$q_{\square} : \mathcal{Z} \longrightarrow \mathcal{Q}_{\square} := \mathcal{Z}/\mathcal{J}_{\square}$$

has the following properties.

1. *The constructors descend, with their supplied coherent action data.*
2. *Every diagram $q_{\square}X$, for $X \in \mathscr{X}$, has context-independent event residuals and satisfies principal-ideal reconstruction.*
3. *Every exact equivariant functor $E : \mathcal{Z} \to \mathcal{D}$ for which all the diagrams EX are excisive factors through $q_{\square}$, with the enhanced universal uniqueness of Verdier localization.*
4. *The kernel is the least constructor-invariant thick subcategory forcing those squares to be excisive.*

The quotient is not automatically nonzero. If such an E has $E(Y) \not\simeq 0$, then $q_{\square}(Y) \not\simeq 0$.

Proof. The proof of theorem 11.14 applies to any designated family of objects, not only killable objects. A constructor sends a generator $T(H)$ to another generator. Exactness then preserves its thick closure. The Verdier universal property therefore descends constructors and their already specified coherence.

Since $q_\square(H) \simeq 0$ and exact functors preserve interaction residuals, every prescribed elementary square in $q_\square X$ is a pushout. Apply theorems 10.3 and 10.5 to obtain (2).

If EX is excisive, then $E(H) \simeq 0$. Equivariance implies $E(T(H)) \simeq T'(E(H)) \simeq 0$. Exactness annihilates the thick closure, so E factors through the quotient. Conversely, the kernel of any exact equivariant localization making the diagrams excisive must contain $\mathscr{H}$ and its invariant thick closure. This proves the universal and minimality assertions. Finally a factorization $E \simeq \bar{E} q_\square$ would send Y to zero if $q_\square(Y)$ were zero. □

The external input is stable Verdier localization, as in [4, 3]. The new use here is the explicitly chosen family of square interaction residuals. It does not supply an arithmetic interpretation for a quotient unless that interpretation is proved separately.

13 An integral seed model and prime-overlap calculus

We now give a completely specified arithmetic instance. It uses perfect complexes over $\mathbb{Z}$, not Galois representations or automorphic representations. Its purpose is to make the coherence, residual, excision, noncollapse, and finite-generator obligations explicit in one model.

13.1 Tensor cubes and iterated residuals

Theorem 13.1 (Tensor-cube residual formula)**.** *Let $\mathcal{S}$ be a symmetric monoidal stable category whose tensor product is exact separately in each variable. For $m \geq 1$ and arrows $f_i : A_i \to B_i$, form the coherent m-cube*

$$Y(S) = \bigotimes_{i=1}^{m} Z_i(S), \qquad Z_i(S) = \begin{cases} B_i & i \in S, \\ A_i & i \notin S. \end{cases}$$

Its total cofiber, obtained by taking the cofiber in every coordinate, satisfies

$$\operatorname{tcofib}(Y) \simeq \bigotimes_{i=1}^{m} \operatorname{Cofib}(f_i).$$

Equivalently it is the cofiber of the canonical map from the colimit of the punctured cube to its terminal vertex. The equivalences are natural in the arrows, with the coherent symmetries of the monoidal structure, rather than literal equality of chain matrices for different coordinate orders.

Proof. For $m = 1$ this is the definition. Taking cofibers in the last coordinate replaces its factor $A_m \to B_m$ by $\operatorname{Cofib}(f_m)$ because tensoring by the other factors is exact. Repeat for the remaining coordinates. Interchanging finite colimits shows that this iterated construction is the cofiber of the punctured-cube colimit map. The universal properties give naturality and coherent interchange. In a complex model, these interchange maps incorporate the tensor differential's graded signs. □

For a commutative ring R and $a \in R$, set

$$C_R(a) = [R \xrightarrow{a} R],$$

with the source in cohomological degree -1 and the target in degree 0. The tensor product $\bigotimes_i C_R(a_i)$ is the Koszul complex in this cohomological convention. Its tensor, cone, and change-of-basis properties are classical; see [7]. We use them with an explicit sign convention and direct proofs below.

13.2 Scalar exchange diagrams

Fix a finite poset P and assign a positive integer n_e to every event $e \in P$. For $M \in \mathrm{Perf}(\mathbb{Z})$ define

$$X_M(I) = M, \qquad X_M(I \subseteq J) = \left(\prod_{e \in J \setminus I} n_e \right) \mathrm{id}_M .$$

Empty products are 1. For nested ideals these products multiply exactly, so X_M is a strict chain-level diagram followed by the stable realization. When $P = P_W$, take identity transformation and coarsening functors and let the exchange cell for event e be multiplication by n_e. This realizes the diagram inside the typed two-axis grammar of theorem 9.2. The object at every vertex is the same, but its comparison maps retain nontrivial arithmetic.

For an enabled event e,

$$E_e(I) \simeq M \otimes^{\mathbf{L}}_{\mathbb{Z}} C_{\mathbb{Z}}(n_e).$$

The comparison after an unrelated event f is multiplication by n_f on that complex. Consequently

$$\boxed{\Theta_{e,f}(I) \simeq M \otimes^{\mathbf{L}}_{\mathbb{Z}} C_{\mathbb{Z}}(n_e) \otimes^{\mathbf{L}}_{\mathbb{Z}} C_{\mathbb{Z}}(n_f).}$$

The complexes $C_{\mathbb{Z}}(n)$ are bounded and free, so ordinary tensor complexes compute these derived tensors. The total normalization residual is instead

$$\mathcal{F}_{X_M}(P) \simeq M \otimes^{\mathbf{L}}_{\mathbb{Z}} C_{\mathbb{Z}}(N), \qquad N = \prod_{e \in P} n_e.$$

This is not the total cofiber of the entire event cube. The normalization residual compares the initial and final vertices, whereas a cube's total cofiber measures their higher interaction. These are different constructions.

Theorem 13.2 (Integral interaction is supported at the gcd)**.** *For positive integers $a_1, \dots, a_m$, $m \geq 1$, put $g = \gcd(a_1, \dots, a_m)$. Then*

$$K(a_1, \dots, a_m) := \bigotimes_{i=1}^{m} C_{\mathbb{Z}}(a_i) \simeq \bigoplus_{j=0}^{m-1} C_{\mathbb{Z}}(g)[j]^{\oplus \binom{m-1}{j}}.$$

The displayed splitting depends on a unimodular change of basis and is not claimed canonical. Therefore

$$H^{-j} K(a_1, \dots, a_m) \cong (\mathbb{Z}/g)^{\oplus \binom{m-1}{j}} \quad (0 \leq j \leq m-1),$$

and every other cohomology group vanishes. In particular the interaction vanishes exactly when $g = 1$.

Proof. The Euclidean algorithm produces a unimodular matrix $U \in \mathrm{GL}_m(\mathbb{Z})$ with

$$(a_1, \dots, a_m)U = (g, 0, \dots, 0).$$

The Koszul complex is the exterior algebra on a free module in degree -1, with differential the derivation determined by this row vector. Hence the maps $\bigwedge^j U$ give an isomorphism of complexes from $K(g, 0, \dots, 0)$ to $K(a_1, \dots, a_m)$.

Each zero entry contributes $C_{\mathbb{Z}}(0) \cong \mathbb{Z} \oplus \mathbb{Z}[1]$ with zero differential. Distributing these $m-1$ factors gives the stated shifted copies, with the standard shift isomorphisms. Since $g > 0$, the map $g : \mathbb{Z} \to \mathbb{Z}$ is injective. Thus $C_{\mathbb{Z}}(g)$ has only $H^0 = \mathbb{Z}/g$, proving the cohomology formula and the vanishing criterion. □

For two events the entire calculation is visible:

$$K(a,b) = \left[\mathbb{Z} \xrightarrow{\binom{-b}{a}} \mathbb{Z}^2 \xrightarrow{(a\ \ b)} \mathbb{Z}\right], \qquad \text{degrees } -2,-1,0.$$

The product of the two differentials is zero. The kernel of $(a\ b)$ is generated by $(-b/g, a/g)$, and the incoming image is g times that generator. Both H^{-1} and H^0 are $\mathbb{Z}/g$. Equivalently, if $au + bv = g$, then

$$U = \begin{pmatrix} u & -b/g \\ v & a/g \end{pmatrix}, \qquad \det U = 1, \qquad (a\ b)U = (g\ 0).$$

Thus a strictly commuting scalar exchange square is a homotopy pushout on the seed $\mathbb{Z}$ exactly when the two exchange integers are coprime. Its commutativity alone says nothing about this condition.

Example 13.3 (Higher vanishing does not remove lower interactions)**.** For $(6, 10, 15)$, all pair interactions are nonzero: their gcds are $2, 3, 5$. But the threefold interaction is zero because the joint gcd is 1. Hence vanishing of a higher total cofiber does not imply vanishing of the lower square interactions. It cannot replace the square-by-square excision condition in theorem 10.3.

13.3 A nontrivial universal quotient with an exact detector

Let $\mathscr{M}$ consist of tensoring by perfect complexes, closed under composition with its coherent tensor action. This is a constructor system, not the repair monoid defining operational killability in the earlier sections. We now kill only the designated interaction residuals.

For the fixed labelled poset define the finite prime set

$$S_P = \bigcup_{e \| f} \{p \text{ prime} : p \mid \gcd(n_e, n_f)\},$$

where $e \parallel f$ means incomparable. Let $\mathcal{J}_P$ be the invariant thick closure of the square interaction residuals for the unit seed $\mathbb{Z}$.

Theorem 13.4 (Prime-overlap classification of the excisive kernel)**.** *In* $\mathrm{Perf}(\mathbb{Z})$*,*

$$\boxed{\mathcal{J}_P = \mathrm{Thick}\{\mathbb{Z}/p : p \in S_P\}.}$$

Equivalently, $\mathcal{J}_P$ consists exactly of perfect complexes whose cohomology groups are finite abelian groups with prime divisors in S_P. For the stable Verdier quotient q_P, and every perfect complex M,

$$q_P(M) \simeq 0 \quad \Longleftrightarrow \quad M \otimes^{\mathbf{L}}_{\mathbb{Z}} \mathbb{Z}[S_P^{-1}] \simeq 0.$$

The quotient is nonzero. Its constructor action is defined, every $q_P X_M$ is excisive, and its reduced residual diagram is reconstructed from principal ideals.

Proof. First, every pair of incomparable events occurs in an elementary square. Take the ideal $D_e \cup D_f$. It contains neither e nor f, contains their predecessors, and enables both. The square residual at the unit is $K(n_e, n_f)$.

By theorem 13.2, this complex has the same thick closure as $C_{\mathbb{Z}}(g)$ for $g = \gcd(n_e, n_f)$: it is a direct sum of that complex and its shift, and contains it as a retract. The thick closure of any family in $\mathrm{Perf}(\mathbb{Z})$ is already a tensor ideal. Indeed $\mathrm{Perf}(\mathbb{Z}) = \mathrm{Thick}(\mathbb{Z})$, so tensoring a fixed generator by a perfect complex builds its image using finite cones, shifts, and retracts from that same generator.

For $p \mid g$, tensor $C_{\mathbb{Z}}(g)$ by $C_{\mathbb{Z}}(p)$. The gcd theorem identifies the result with $C_{\mathbb{Z}}(p) \oplus C_{\mathbb{Z}}(p)[1]$. Hence $\mathbb{Z}/p \simeq C_{\mathbb{Z}}(p)$ belongs to the thick closure. Conversely, the Chinese remainder decomposition of $\mathbb{Z}/g$ and the short exact sequences building $\mathbb{Z}/p^j$ from copies of $\mathbb{Z}/p$ put $C_{\mathbb{Z}}(g) \simeq \mathbb{Z}/g$ in the thick subcategory generated by its prime divisors. This proves the boxed equality.

Having finite S_P-primary cohomology is preserved by finite triangles and retracts, by the long exact sequence in cohomology. Conversely, each such finite abelian cohomology group is built by finite extensions from the indicated $\mathbb{Z}/p$. A perfect complex is bounded, and its finite truncation triangles then build it from these groups. These truncations are perfect over $\mathbb{Z}$: every finitely generated abelian group has a finite free resolution of length at most one. This proves the cohomological characterization.

Localization of abelian groups is exact. Thus tensoring with $\mathbb{Z}[S_P^{-1}]$ has cohomology obtained by localizing each cohomology group. A finitely generated abelian group localizes to zero exactly when it is finite with all prime divisors in S_P. The localization kernel is therefore $\mathcal{J}_P$, as is the kernel of the Verdier quotient by this thick subcategory. This proves the equivalence of vanishing tests on every source object.

Finally $\mathbb{Z}[S_P^{-1}]$ is a nonzero ring, so localization detects that $q_P(\mathbb{Z}) \not\simeq 0$. It is exact and intertwines tensor constructors with their localized counterparts. The universal excisive-quotient theorem supplies the remaining assertions. For general M, each square residual is its unit-seed counterpart tensored with M, so it lies in the same invariant kernel. □

The positivity hypothesis excludes a real collapse case. If two independent labels are both zero, then $K(0,0) \simeq \mathbb{Z} \oplus \mathbb{Z}[1]^{\oplus 2} \oplus \mathbb{Z}[2]$ contains the unit as a retract. Killing that interaction forces the whole perfect category to vanish. Thus noncollapse is proved for the specified positive-label model, not for arbitrary scalar labels.

No equivalence of the entire quotient category with a category of localized perfect complexes is needed for this theorem. The proved statement is equality of kernels, a nonzero exact detector, and the universal excisive property. This avoids inserting an unproved full-faithfulness or idempotent-completion claim.

Example 13.5 (Remove shared torsion without deleting the primary residuals)**.** Take two independent events labelled 6 and 10. Their interaction is

$$K(6,10) \simeq \mathbb{Z}/2 \oplus (\mathbb{Z}/2)[1].$$

Here $S_P = \{2\}$. Inverting 2 kills this interaction while retaining

$$C_{\mathbb{Z}}(6) \otimes \mathbb{Z}[1/2] \simeq \mathbb{Z}/3, \qquad C_{\mathbb{Z}}(10) \otimes \mathbb{Z}[1/2] \simeq \mathbb{Z}/5.$$

The total normalization residual is $C_{\mathbb{Z}}(60)$. After localization it is $\mathbb{Z}/15$, and principal-ideal reconstruction gives

$$\mathbb{Z}/15 \simeq \mathbb{Z}/3 \oplus \mathbb{Z}/5.$$

This is a classical Chinese-remainder instance of the calculus. Before localization the two paths have first residuals $\mathbb{Z}/6$ and $\mathbb{Z}/10$. After localization those first stages remain different, so the original ordered-filtration independence claim is still not being restored.

13.4 A finite certificate for an infinite scalar grammar

Corollary 13.6 (Finite-signature arithmetic certification)**.** *Fix finitely many positive integer exchange labels $n_1, \ldots, n_t$ and a specified symmetric relation $\mathcal{I}$ of label pairs allowed to occur on*

incomparable events. The relation may contain a diagonal pair. Put

$$S_{\mathcal{I}} = \bigcup_{(i,j)\in\mathcal{I}} \{p : p \mid \gcd(n_i, n_j)\}.$$

Killing the finitely many unit-seed interactions $K(n_i, n_j)$ for $(i, j) \in \mathcal{I}$, with invariant thick closure, makes every scalar event diagram on every finite labelled poset satisfying this independence restriction excisive, for every perfect input M. The quotient is nonzero and its kernel is the finite-prime torsion class supported on $S_{\mathcal{I}}$.

Proof. There are only finitely many prescribed pair types. Every elementary square in every admitted diagram has one of these types, and its residual is the corresponding unit-seed Koszul complex tensored with M. All such residuals lie in the same invariant thick kernel. Apply theorems 12.1 and 13.4. The latter proof depends only on the finite family of gcds, so it applies unchanged to the signature-wide family. □

This is a genuine finite reduction of an unbounded word-length obligation, but only in the scalar integral regime with the stated independence relation. If a label can occur independently of itself, its primes belong to $S_{\mathcal{I}}$ and its primary torsion is erased. Declaring independence therefore changes the quotient's information content. Dependent events can share a prime without forcing its removal, as the p-then-p chain in theorem 10.6 demonstrates.

The construction resolves a concrete semantic and obstruction problem in arithmetic. It does not identify these scalar kernels with automorphic occupancy, a ramification obstruction, or a global Langlands parameter.

13.5 Beyond scalar exchanges: an integral matrix criterion

The noncollapse test extends to tensor cubes of finite integer matrices. This is a different concrete realization from identity functors with scalar exchange cells: its independent directions are tensor factors of actual two-term complexes.

Theorem 13.7 (Finite integral matrix collapse criterion)**.** *Let $f_i : \mathbb{Z}^{a_i} \to \mathbb{Z}^{b_i}$ be finitely many integer matrices. Specify which pairs of directions may occur independently. For each matrix fix a Smith normal form, with rank r_i and positive nonzero diagonal entries $d_{i,1}, \dots, d_{i,r_i}$. Put*

$$\epsilon_i = \begin{cases} 1 & a_i + b_i - 2r_i > 0, \\ 0 & a_i + b_i - 2r_i = 0, \end{cases} \qquad P_i = \{p : p \mid d_{i,k} \text{ for some } k\}.$$

Let $\mathcal{J}$ be the tensor-invariant thick subcategory generated by $\mathrm{Cofib}(f_i) \otimes^{\mathbf{L}} \mathrm{Cofib}(f_j)$ over the allowed independent pairs.

1. *If an allowed pair has $\epsilon_i = \epsilon_j = 1$, then $\mathcal{J} = \mathrm{Perf}(\mathbb{Z})$ and its quotient is zero.*
2. *Otherwise the quotient is nonzero and*

$$\mathcal{J} = \mathrm{Thick}\{\mathbb{Z}/p : p \in S\},$$

where S is the union over allowed pairs of

$$S_{ij} = \begin{cases} P_i \cap P_j & \epsilon_i = \epsilon_j = 0, \\ P_j & \epsilon_i = 1,\ \epsilon_j = 0, \\ P_i & \epsilon_i = 0,\ \epsilon_j = 1. \end{cases}$$

Localization at S is an exact nonzero detector with that kernel. It makes the prescribed tensor-cube squares excisive.

A supplied Smith form is a finite certificate: its row and column changes must be unimodular and their matrix product must give the claimed diagonal form. The existence of Smith forms over $\mathbb{Z}$ follows from the Euclidean algorithm; the argument below needs only such a verified form.

Proof. Changing bases in the source and target yields a chain isomorphism

$$\mathrm{Cofib}(f_i) \simeq \mathbb{Z}^{b_i - r_i} \oplus \mathbb{Z}^{a_i - r_i}[1] \oplus \bigoplus_{k=1}^{r_i} C_{\mathbb{Z}}(d_{i,k}).$$

Unit diagonal entries contribute contractible complexes. Thus $\epsilon_i = 1$ means that a shift of the unit occurs as a direct summand. If two independent cones each have such a summand, their tensor product contains a shift of the unit. A thick subcategory containing it is all of $\mathrm{Perf}(\mathbb{Z})$.

If no allowed pair has two free summands, every generating tensor is torsion. A free summand tensored with a torsion cone preserves that torsion's prime support. Two torsion summands have tensor product $K(d_{i,k}, d_{j,l})$, whose thick support is the prime intersection by theorem 13.2. Taking finite direct sums gives exactly the listed sets S_{ij}. The argument of theorem 13.4 identifies the generated thick subcategory with finite S-primary cohomology and identifies its kernel with localization at S. The localized unit is nonzero. The tensor-cube formula and the excisive-quotient theorem finish the proof. □

This criterion shows why noncollapse cannot be inferred from commutativity. Two commuting tensor directions with free cofiber components can force total collapse when their interaction is declared zero. Pure torsion interactions, in contrast, admit the precise finite-prime localization just proved.

14 Arithmetic resolution towers and nilpotent interactions

The integral model of section 13 has nontrivial exchange maps but identity coarsening functors. We now replace those coarsenings by derived reduction between different coefficient rings. Three questions must be kept separate: whether an exchange is canonically exact, what a specified noninvertible exchange cell leaves behind, and whether forcing that residual to vanish destroys the coefficient category.

14.1 Derived reduction is a genuine resolution operation

Fix a finite nonempty set $\mathcal{P}$ of primes and put

$$D = \prod_{p \in \mathcal{P}} p, \qquad B_n = \mathbb{Z}/D^n\mathbb{Z} \quad (n \geq 1).$$

Write $B_{p,n} = \mathbb{Z}/p^n\mathbb{Z}$. The Chinese remainder isomorphisms

$$B_n \cong \prod_{p \in \mathcal{P}} B_{p,n}$$

are compatible with reduction. Use the enhanced category $\mathcal{S}_n = \mathrm{Perf}(B_n)$ and the coarsening

$$C_{m,n}(M) = M \otimes^{\mathbf{L}}_{B_m} B_n, \qquad m \geq n.$$

Derived tensor associativity supplies coherent equivalences $C_{l,n} \simeq C_{m,n}C_{l,m}$. These are the data of an enhanced indexed system. Literal equalities are not asserted between arbitrarily chosen derived-functor models. On specified bounded projective complexes the operation is termwise reduction. Preservation of perfect complexes under derived base change is standard [8]; the pointwise cone model also shows exactness directly.

Theorem 14.1 (Nonidentity conservative resolution tower)**.** *The functors $C_{m,n}$ are exact and conservative on perfect complexes. They need not be faithful. In particular, for $m > n$, the nonzero endomorphism $D^n : B_m \to B_m$ becomes zero under $C_{m,n}$.*

Proof. Exactness follows by reducing bounded projective cone diagrams. For conservativity it suffices, by the Chinese remainder decomposition and composition with reduction to level one, to prove that a perfect complex over $R = \mathbb{Z}/p^m$ is zero if its derived reduction to $\mathbb{F}_p$ is zero.

A finite projective module over R is free. Indeed, lift a basis modulo (p) to obtain a surjection $R^r \to P$. It is surjective because its cokernel Q satisfies $Q = pQ$, whence $Q = p^m Q = 0$. The surjection splits, and its kernel has zero reduction modulo p, so the same argument kills the kernel. A perfect complex therefore has a bounded finite free representative.

If a differential has a unit entry, elementary changes of bases split off the contractible two-term complex $R \xrightarrow{1} R$. The adjacent differentials have zero components into and out of that summand because their composites are zero. Repeating strictly decreases the total rank and terminates in a complex all of whose differential entries lie in (p). Its reduction has zero differentials. If that reduction is acyclic, every remaining rank is zero. The original complex is contractible. Applied to a cone, this also proves that reduction reflects equivalences.

Finally D^n is not zero modulo D^m, but is zero modulo D^n. Since $\mathrm{Hom}_{D(B_m)}(B_m, B_m) = B_m$, this is genuine loss of morphism information, not just a change of representatives. □

Conservativity detects zero objects and equivalences. It does not recover every morphism, a preferred lift, or the full information in a fine presentation.

Example 14.2 (Why ordinary tensor and cohomology truncation are not substitutes)**.** Over $R = \mathbb{Z}/p^2$, the exact sequence $0 \to (p) \to R \to \mathbb{F}_p \to 0$ ceases to be left exact after ordinary tensor with $\mathbb{F}_p$: the first map becomes zero. The module $\mathbb{F}_p$ has the infinite periodic free resolution with every differential multiplication by p. Tensoring that resolution with $\mathbb{F}_p$ gives zero differentials, hence $\mathrm{Tor}_i^R(\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p$ for every $i \geq 0$.

In contrast, $C_R(p) = [R \xrightarrow{p} R]$ is perfect, but has both $H^{-1} = \mathbb{F}_p$ and $H^0 = \mathbb{F}_p$. It is not the module $\mathbb{F}_p$ placed in degree zero. We retain the actual two-term complex throughout the tower. Derived tensor is computed on these projective representatives, not on a truncated cohomology surrogate [9].

14.2 Coherent cells and their actual source of residuals

For a fixed perfect integral complex L, put

$$A_{L,n}(M) = (L \otimes^{\mathbf{L}}_{\mathbb{Z}} B_n) \otimes^{\mathbf{L}}_{B_n} M.$$

Base change gives a natural equivalence

$$\theta_{L;m,n} : A_{L,n}C_{m,n} \simeq C_{m,n}A_{L,m}.$$

Thus the canonical tensor/base-change exchange already has zero residual. The tower alone does not force a defect. To test a noninvertible exchange, we explicitly add a positive integer label b and use the cell

$$\eta_{L,b;m,n} = b^{m-n}\theta_{L;m,n}.$$

For scalar unit templates take $L = \mathbb{Z}$, so that the transformation functors are identities but the coarsening functors are not.

Theorem 14.3 (Coherent weighted exchange and residual transport)**.** *The cells $\eta_{L,b;m,n}$ are natural and coherent with composition of resolution changes. Their residual at $M \in \mathcal{S}_m$ is*

$$\mathrm{Cofib}(\eta_{L,b;m,n,M}) \simeq A_{L,n}C_{m,n}(M) \otimes^{\mathbf{L}}_{B_n} C_{B_n}(b^{m-n}).$$

For a fixed scalar a, subsequent base change transports its cone exactly:

$$C_{n,k}(N \otimes^{\mathbf{L}}_{B_n} C_{B_n}(a)) \simeq C_{n,k}(N) \otimes^{\mathbf{L}}_{B_k} C_{B_k}(a).$$

At a common target level n, two independent scalar events of positive lengths d, e and labels a, b have interaction

$$N \otimes^{\mathbf{L}}_{B_n} K_{B_n}(a^d, b^e), \qquad K_R(u, v) = C_R(u) \otimes^{\mathbf{L}}_R C_R(v).$$

These formulas also hold with compatible tensor templates absorbed into N.

Proof. Naturality follows from central scalar multiplication. For $l \geq m \geq n$, the two-step cell has scalar $b^{m-n}b^{l-m} = b^{l-n}$. The associativity coherence of derived tensor identifies its underlying θ with $\theta_{L;l,n}$. Identity intervals have scalar one. Tensoring transformation templates gives their usual associative action, with scalar weights multiplied; no additional cells are inferred without these specified data.

After using θ to identify the target, the exchange is multiplication by b^{m-n} on $A_{L,n}C_{m,n}M$. Exact tensor identifies its cone with the first formula. Associativity of derived tensor proves the second. The two-variable interaction formula follows by taking the two cofibers, as in theorem 13.1. For a pure adjacent-swap word, tensor templates keep their relative order and all coarsenings combine into the same final reduction. Identifying those endpoints by the supplied tensor associativities leaves precisely the products of the scalar cells. This constructs the corresponding coherent scalar event diagram at the target level. □

Remark 14.4 (Specified cells, not a claimed arithmetic correspondence)*.* Nonzero residuals in this model come from the chosen noninvertible scalar cells. They are not defects of the canonical base-change equivalence, and they are not identified with a ramification or automorphy obstruction. The construction supplies a reproducible nonidentity resolution test bed with explicit semantics.

14.3 The finite-level Koszul calculation

Theorem 14.5 (Prime-power interaction cohomology)**.** *Let $R = \mathbb{Z}/p^n$, let $a_1, \ldots, a_t$ be positive integers with $t \geq 1$, and put*

$$h = \min\{n, v_p(a_1), \ldots, v_p(a_t)\}, \qquad Q_h = R/(p^h).$$

Here $Q_0 = 0$ and $Q_n = R$. There is a generally noncanonical chain isomorphism

$$K_R(a_1, \ldots, a_t) \cong \bigoplus_{j=0}^{t-1} C_R(p^h)[j]^{\oplus\binom{t-1}{j}}.$$

Consequently

$$\boxed{H^{-j}K_R(a_1,\dots,a_t) \cong Q_h^{\oplus\binom{t}{j}} \quad (0 \le j \le t),}$$

and all other cohomology groups vanish. The complex is zero in $D(R)$ exactly when at least one label is a unit modulo p.

Proof. If $h = n$, all entries are zero in R. Otherwise choose an entry of minimal valuation, scale its unit part away, and subtract suitable multiples of it from the other entries. These operations form an invertible change of the coefficient row and produce $(p^h, 0, \dots, 0)$. The exterior powers of the basis change give a Koszul chain isomorphism, with the convention of theorem 13.2 and [7].

Each zero entry contributes $R \oplus R[1]$. This proves the complex decomposition. Multiplication by p^h on R has cokernel Q_h and kernel $p^{n-h}R$, the latter isomorphic to Q_h by multiplication by p^{n-h}. Hence $C_R(p^h)$ has Q_h in degrees -1 and 0. Counting the shifted copies and using Pascal's identity gives the formula. The unit case is contractible, and otherwise $H^0 = Q_h \neq 0$. □

For two events the cohomology ranks over Q_h are $1, 2, 1$ in degrees $-2, -1, 0$, rather than the $0, 1, 1$ pattern over the integral or complete domain. These extra classes are real finite-level cohomology. They may not be discarded while retaining the same complex. In particular, a cohomology group appearing here is not automatically a perfect R-module.

Theorem 14.6 (Nilpotent interaction forces excisive collapse)**.** *For any commutative ring R and nilpotent $x \in R$, the thick subcategory generated by $C_R(x)$ is all of* $\mathrm{Perf}(R)$*. Therefore, over $R = \mathbb{Z}/p^n$, forcing the interaction of two nonunit scalar labels to vanish in an exact quotient forces that quotient to be zero.*

Proof. Choose N with $x^N = 0$. For composable scalar multiplications, the octahedral cofiber relation builds $C_R(x^k)$ from $C_R(x^{k-1})$ and $C_R(x)$. Induction puts $C_R(x^N)$ in $\mathrm{Thick}(C_R(x))$. But

$$C_R(x^N) = C_R(0) \cong R \oplus R[1].$$

Retract closure therefore includes R, and R thickly generates $\mathrm{Perf}(R)$. This cone-of-a-product mechanism is also part of the standard Koszul calculus [7].

For two nonunit labels, theorem 14.5 makes their interaction a sum of $C_R(p^h)$ and its shift, with $1 \le h \le n$. Thus it contains $C_R(p^h)$ as a retract. This element is nilpotent. Any exact functor annihilating the interaction annihilates the thick subcategory it generates, hence every perfect object. □

This is stronger than the zero-label counterexample over $\mathbb{Z}$. At finite p-power resolution, even nonzero labels such as p and p^2 can force total collapse. Localizing $\mathbb{Z}/p^n$ at p gives the zero ring. Noncollapse over the integral coefficients does not survive this base change automatically.

14.4 Several primes give a nonzero, but selective, quotient

Specify finitely many positive scalar labels and a collection $\mathcal{I}$ of allowed independent pairs. Include their unit-seed interaction diagrams as the designated tests. This hypothesis ensures that the minimal kernel below is not being inferred from templates that accidentally annihilate an interaction already. Put

$$S = \{p \in \mathcal{P} : \text{some } (a,b) \in \mathcal{I} \text{ satisfies } p \mid a \text{ and } p \mid b\}, \qquad D_0 = \prod_{p \in \mathcal{P} \setminus S} p.$$

Let $\mathcal{J}_n$ be the tensor-invariant thick closure of those interactions in $\mathrm{Perf}(B_n)$. Positive lengths in theorem 14.3 do not change S.

Theorem 14.7 (Prime-component quotient and tower compatibility)**.** *Under the Chinese remainder equivalence,*

$$\boxed{\mathcal{J}_n = \prod_{p\in S} \mathrm{Perf}(\mathbb{Z}/p^n) \times \prod_{p\in\mathcal{P}\setminus S} 0.}$$

In particular there is an equivalence of stable categories

$$\mathrm{Perf}(B_n)/\mathcal{J}_n \simeq \prod_{p\in\mathcal{P}\setminus S} \mathrm{Perf}(\mathbb{Z}/p^n) \simeq \mathrm{Perf}(\mathbb{Z}/D_0^n).$$

The equivalence is realized by projection to the retained prime components. Its kernel is constructor-invariant and preserved by every coarsening. If q_n denotes projection, then

$$q_n C_{m,n} \simeq C^0_{m,n} q_m$$

with the supplied base-change coherence. The quotient is nonzero exactly when $S \neq \mathcal{P}$. When $S = \mathcal{P}$, use the zero category and the zero ring $\mathbb{Z}/1$.

Proof. At a prime outside S, each designated interaction has a unit label and is acyclic. At a prime in S, at least one pair has both labels nonunit. Tensor by the corresponding Chinese remainder idempotent to isolate that component. The nilpotent-collapse theorem then puts its unit in $\mathcal{J}_n$. Its whole perfect category follows. These two observations prove equality of the kernel with the displayed product subcategory.

Projection to the complementary factors is exact and has an exact fully faithful section. Every object is a direct sum of its selected and retained components. In the quotient the selected summand is zero, so that object becomes isomorphic to its retained summand. Projection and section therefore induce inverse equivalences. This proves the whole quotient assertion directly, without an unproved identification of an arbitrary Verdier quotient with localized perfect complexes.

Tensor products and reduction respect the Chinese remainder idempotents. Projection consequently commutes with reduction, and the same tensor associativities give coherence for composite reductions. All prescribed square interactions vanish after projection, so the earlier excisive reconstruction results apply at every level. □

This repair does not retain the selected prime's finer torsion while deleting only one common exponent. It removes the entire selected prime component. Thus its minimality is a statement about exact excision, not a guarantee that it preserves all arithmetic information desired by a later application.

15 Completion defects and arithmetic reconstruction

15.1 A controlled reconstruction of a resolution tower

Write

$$\widehat{B} = \varprojlim_n B_n \cong \prod_{p\in\mathcal{P}} \mathbb{Z}_p.$$

The isomorphism follows by taking inverse limits of the compatible Chinese remainder decompositions. Here $\mathbb{Z}_p$ is the ring of compatible residue sequences modulo p^n, and $\mathbb{Q}_p = \mathbb{Z}_p[1/p]$.

Theorem 15.1 (Uniform based-complex tower reconstruction)**.** *Suppose M_n are based finite free cochain complexes over B_n, concentrated in a common finite interval with fixed ranks in each degree. Suppose their differential matrices reduce exactly from level m to level n for $m \geq n$. Then there is a unique based complex $\widehat{M}$ with those ranks over $\widehat{B}$ whose matrices reduce to the supplied ones.*

Compatible chain maps and specified compatible chain homotopies reconstruct coefficientwise. Moreover, as complexes of abelian groups,

$$\widehat{M} \simeq R\varprojlim_n M_n.$$

Finite tensor products, finite direct sums, shifts, and cones of compatible based maps commute with this reconstruction. In particular this holds for the scalar residual and interaction complexes with fixed integral labels.

Proof. Every matrix entry is a compatible residue sequence and therefore determines a unique element of $\widehat{B}$. The equation $d^{i+1}d^i = 0$ holds because it holds modulo every D^n and the inverse-limit ring is separated. The same argument proves the equations for a reconstructed chain map or homotopy.

Degreewise transition maps of the finite free modules are surjective. In the product model of the homotopy inverse limit, the map

$$1 - \text{shift} : \prod_n M_n^i \longrightarrow \prod_n M_n^i$$

is surjective in each degree: choose an initial entry and successively lift the required difference across the surjective transition map. Its kernel is the ordinary inverse limit. Thus the homotopy fiber is represented by that kernel complex. This is the degreewise Mittag–Leffler case of derived inverse limits [10].

All the listed finite constructions have matrices formed by finite sums and products, or by fixed block arrangements of entries. These operations commute with the coefficient inverse limit. This proves the final assertions. □

This theorem does not assert that arbitrary objects chosen independently at every level have a compatible system of bases, bounded ranks, or a coherent lift. It also does not claim an equivalence between all categories of perfect complexes and an inverse limit of categories. Those stronger effectivity statements require additional work. The precise input here is a compatible, uniformly finite, based tower.

Proposition 15.2 (Transient negative cohomology in the resolution direction)**.** *Fix p and an integer $a \geq 1$. For $n \geq a$ put $R_n = \mathbb{Z}/p^n$. Then*

$$H^0 C_{R_n}(p^a) \cong \mathbb{Z}/p^a, \qquad H^{-1} C_{R_n}(p^a) \cong \mathbb{Z}/p^a.$$

Under reduction from m to n with $m \geq n \geq a$, the map on H^0 is the identity in these coordinates. On H^{-1} it is multiplication by p^{m-n}. The negative-cohomology system is therefore pro-zero, although every one of its terms is nonzero. The completed complex is

$$C_{\mathbb{Z}_p}(p^a), \qquad H^{-1} = 0, \quad H^0 = \mathbb{Z}/p^a.$$

Proof. Represent H^{-1} by the annihilator $p^{n-a}R_n$ and identify $\mathbb{Z}/p^a$ with it by $x \mapsto p^{n-a}x$. Reduction of $p^{m-a}x$ gives $p^{n-a}p^{m-n}x$. Thus for each target level n, every source level $m \geq n + a$ maps to zero. This is exactly the pro-zero assertion. The cokernel coordinates reduce identically. The preceding theorem reconstructs the complex coefficientwise. Multiplication by p^a on $\mathbb{Z}_p$ is injective, so its completed kernel vanishes. □

This calculation concerns the actual transition maps on homology, not merely the group appearing at each level. A list of isomorphism types $(\mathbb{Z}/p^a, \mathbb{Z}/p^a, \ldots)$ cannot distinguish the identity system from the pro-zero system. The two-axis data must retain those maps.

15.2 Completion and excisive repair do not generally commute

Let $S \subseteq \mathcal{P}$ be the interaction-prime set in theorem 14.7, and set $T = \prod_{p \in S} p$, with $T = 1$ if S is empty. Localizing the finite rings and then completing gives

$$\varprojlim_n B_n[1/T] \cong \prod_{p \in \mathcal{P} \setminus S} \mathbb{Z}_p.$$

Completing and then localizing instead gives

$$\widehat{B}[1/T] \cong \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \in \mathcal{P} \setminus S} \mathbb{Z}_p.$$

The latter is also an exact realization that annihilates the specified scalar interactions, because each selected prime becomes invertible. There is a canonical comparison map between these rings induced by the reductions. Localization is the usual algebraic operation [11], not an additional choice of quotient semantics.

Theorem 15.3 (Explicit completion–repair defect)**.** *The canonical map*

$$\chi : \widehat{B}[1/T] \longrightarrow \varprojlim_n B_n[1/T]$$

is the projection that forgets the factors $\mathbb{Q}_p$ for $p \in S$. As $\widehat{B}$-modules it fits into the split exact sequence

$$0 \longrightarrow \prod_{p \in S} \mathbb{Q}_p \longrightarrow \widehat{B}[1/T] \xrightarrow{\chi} \prod_{p \in \mathcal{P} \setminus S} \mathbb{Z}_p \longrightarrow 0.$$

The splitting is a module splitting, not a claimed unital ring section.

For every perfect $\widehat{B}$-complex M, the natural comparison

$$\chi_M : M \otimes^{\mathbf{L}}_{\widehat{B}} \widehat{B}[1/T] \longrightarrow R\varprojlim_n (M \otimes^{\mathbf{L}}_{\widehat{B}} B_n[1/T])$$

has fiber

$$\boxed{\operatorname{Fib}(\chi_M) \simeq \bigoplus_{p \in S} M \otimes^{\mathbf{L}}_{\widehat{B}} \mathbb{Q}_p.}$$

Thus χ_M is an equivalence exactly when every displayed generic fiber is zero. Its cofiber is the shift by [1] of that fiber. These statements take place in $D(\widehat{B})$; the field summands need not be perfect over $\widehat{B}$.

Proof. In the p-primary factor of B_n, inverting T gives zero if $p \in S$ and does nothing otherwise. In the p-primary factor of $\widehat{B}$, it gives $\mathbb{Q}_p$ if $p \in S$ and leaves $\mathbb{Z}_p$ otherwise. This proves both ring formulas. The induced map is zero on the first set of factors and the identity on the others, proving the split sequence.

For $M = \widehat{B}$, the derived inverse limit on the right is the ordinary one, since the retained residue-module tower is degreewise surjective. The asserted comparison and fiber are therefore exactly the split sequence. For a general perfect M, all functors in the claimed identity are exact

and preserve finite sums and retracts. Homotopy inverse limits are exact in a stable derived category. The natural comparison identifies its right side with the retained summand of M: this holds on the unit by the just-computed inverse limit and hence on its thick closure, which is $\mathrm{Perf}(\widehat{B})$. Tensoring the split sequence now identifies the fiber with the indicated finite sum. A morphism in a stable category is an equivalence precisely when its fiber is zero. □

Example 15.4 (A nonzero generic fiber is lost at every finite repair)**.** For $\mathcal{P} = \{p\}$ and two independent labels divisible by p, every finite excisive quotient is zero, whereas

$$(\varprojlim_n \mathbb{Z}/p^n)[1/p] = \mathbb{Q}_p \neq 0.$$

For a nonzero finite-level quotient, take $D = 30$ and labels $6, 10$. Only the prime 2 is shared, so the finite repair is

$$B_n[1/2] \cong \mathbb{Z}/15^n.$$

The two completed outputs are $\mathbb{Q}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$ and $\mathbb{Z}_3 \times \mathbb{Z}_5$, with fiber $\mathbb{Q}_2$. This is an explicit failure of an unconditional bi-completion principle, even though the finite quotient commutes with every finite coarsening.

The generic-fiber criterion also gives a positive case. For $M = C_{\widehat{B}}(u)$ with a nonzero integer u, its selected generic fibers vanish, because u becomes invertible in each $\mathbb{Q}_p$. Thus this comparison is an equivalence on that torsion residual even though it is not an equivalence on the unit. Commutation depends on the object, not only on the existence of coherent finite-level squares. The distinction is consistent with the standard derived-completion fact that a complex on which a defining element is invertible has zero completion [12].

15.3 Reconstruction that retains the overlap instead of deleting it

There is a different operation from killing an interaction: retain the localized and completed descriptions together with their common generic information. In the present arithmetic setting this yields an actual pullback theorem.

Put

$$A = \mathbb{Z}[1/D], \qquad B = \prod_{p \in \mathcal{P}} \mathbb{Z}_p, \qquad G = \prod_{p \in \mathcal{P}} \mathbb{Q}_p.$$

Theorem 15.5 (Arithmetic fracture reconstruction)**.** *The diagonal maps and their difference give an exact sequence of abelian groups*

$$0 \longrightarrow \mathbb{Z} \longrightarrow A \oplus B \xrightarrow{(a,b)\mapsto a-b} G \longrightarrow 0.$$

Consequently, for every perfect integral complex M there is a natural homotopy pullback square in $D(\mathbb{Z})$,

$$\begin{array}{ccc} M & \longrightarrow & M \otimes^{\mathbf{L}}_{\mathbb{Z}} A \\ \downarrow & & \downarrow \\ M \otimes^{\mathbf{L}}_{\mathbb{Z}} B & \longrightarrow & M \otimes^{\mathbf{L}}_{\mathbb{Z}} G. \end{array}$$

Its bottom-left term is $R\varprojlim_n (M \otimes^{\mathbf{L}}_{\mathbb{Z}} B_n)$. In particular M is zero if and only if its localized and completed terms are both zero.

Proof. An element of A integral at every $p \in \mathcal{P}$ is an integer: its possible denominator primes all lie in $\mathcal{P}$, and integrality removes each of them. This identifies the kernel with the diagonal copy of $\mathbb{Z}$.

For surjectivity take $(y_p) \in G$. Choose N large enough that $D^N y_p \in \mathbb{Z}_p$ for every p. For each p, choose the residue of $D^N y_p$ modulo p^N. The ordinary Chinese remainder theorem provides $z \in \mathbb{Z}$ with all these residues. Set $a = z/D^N \in A$. Then $a - y_p \in \mathbb{Z}_p$ for every p, so $(a, (a - y_p)_p)$ maps to $(y_p)_p$. This proves exactness.

The exact sequence gives a cofiber sequence in the derived category. Derived tensor with M is exact, so it gives the displayed homotopy pullback. For a bounded finite free representative of M, the tower of reductions has uniform ranks and compatible matrices. The based-tower theorem identifies its derived limit with $M \otimes B$. Finally if the localized and completed terms are zero, so is their generic base change, and the pullback is zero. The converse is immediate. □

This is the classical arithmetic-square mechanism, proved directly here using the Chinese remainder theorem and derived tensor. It is not claimed as a new local–global correspondence. It does show exactly which overlap data must accompany these arithmetic representations to reconstruct a supplied perfect complex.

Corollary 15.6 (Residuals reconstruct through the same arithmetic square)**.** *Let X be a finite coherent event diagram in* $\mathrm{Perf}(\mathbb{Z})$*. Every edge residual, square interaction, and initial-to-terminal residual of X satisfies theorem 15.5. Its terms are the corresponding residuals of the diagrams obtained by base change to A, B, G. Each such integral residual vanishes if and only if both its localized and completed versions vanish.*

Proof. Derived base change is exact and therefore preserves all the finite cofibers defining those residuals. Apply the arithmetic fracture theorem to the resulting perfect integral complex. Naturality gives the maps on the whole residual diagram, not merely equality of its cohomology dimensions. □

15.4 Effective gluing of lattices from independent local input

The previous theorem starts with a global complex. The following finite-rank theorem instead constructs a global object from supplied local and localized data.

Theorem 15.7 (Effective arithmetic lattice gluing)**.** *Let V be a finite free $A = \mathbb{Z}[1/D]$-module of rank r. For each $p \in \mathcal{P}$ let L_p be a finite free $\mathbb{Z}_p$-module, with an isomorphism*

$$\theta_p : V \otimes_A \mathbb{Q}_p \xrightarrow{\sim} L_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Then

$$\Lambda = \{v \in V : \theta_p(v) \in L_p \text{ for every } p \in \mathcal{P}\}$$

is a finite free $\mathbb{Z}$-module of rank r, and the natural maps identify

$$\Lambda[1/D] \cong V, \qquad \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong L_p.$$

It is the unique lattice inside V with these properties and these specified embeddings. Compatible maps of gluing data restrict uniquely to maps of lattices. Hence finite free integral modules are equivalent to this category of localized free modules, local lattices, and specified generic identifications.

Proof. Choose a basis $V = A^r$, and let $E_p = \theta_p^{-1}(L_p)$ be the corresponding lattice in $\mathbb{Q}_p^r$. Bounds on the entries of θ_p and its inverse give a common $N \geq 1$ such that, with $b = D^N$,

$$b\mathbb{Z}_p^r \subseteq E_p \subseteq b^{-1}\mathbb{Z}_p^r \quad \text{for all } p.$$

The intersection defining Λ therefore satisfies

$$b\mathbb{Z}^r \subseteq \Lambda \subseteq b^{-1}\mathbb{Z}^r.$$

The right inclusion uses the same integrality-of-denominators argument as the arithmetic fracture theorem, coordinatewise. Consequently Λ is a finite-rank free abelian lattice of rank r, and localizing the bounds at D gives $\Lambda[1/D] = A^r$.

Here is also an explicit finite construction proving the local identifications. Put $F_p = bE_p$, so

$$b^2\mathbb{Z}_p^r \subseteq F_p \subseteq \mathbb{Z}_p^r.$$

The Chinese remainder isomorphism identifies

$$\mathbb{Z}^r/b^2\mathbb{Z}^r \cong \prod_{p\in\mathcal{P}} \mathbb{Z}_p^r/b^2\mathbb{Z}_p^r.$$

Let $H \subseteq \mathbb{Z}^r$ be the inverse image of the product of the subgroups $F_p/b^2\mathbb{Z}_p^r$. Directly from the definitions, $H = b\Lambda$. The quotient $\mathbb{Z}^r/H$ is the product of the corresponding finite p-primary quotients. Tensoring with $\mathbb{Z}_p$ retains exactly the p component. Thus $H \otimes \mathbb{Z}_p$ is the inverse image of $F_p/b^2\mathbb{Z}_p^r$, namely F_p. Dividing by b gives $\Lambda \otimes \mathbb{Z}_p = E_p$, and θ_p identifies it with L_p.

For completeness, tensoring the displayed subgroup inclusion is exact: $\mathbb{Z}_p$ is torsion-free over $\mathbb{Z}$, so it is flat; alternatively the assertion follows from a Smith diagonal presentation of the finite-index subgroup H. The latter proves the same statement using only integer bases.

Any other lattice with the specified generic and local identifications has analogous bounds by powers of D. After a common scaling, its finite quotient is determined by the same primary subgroups in the Chinese remainder decomposition. It therefore equals Λ. A compatible A-linear map $V \to V'$ sends every θ_p-integral element to a θ'_p-integral element and hence restricts to $\Lambda \to \Lambda'$. Conversely a lattice map induces the compatible generic and local maps. Uniqueness follows from the embedding in V. These constructions are inverse on objects up to the specified identifications and on morphisms, proving the equivalence. The zero-rank case gives the zero module throughout. □

This is an effective local–global existence theorem for lattices, not for Galois representations or automorphic representations. The generic identifications are essential input. The construction does not derive them from matching coarse invariants. Nor does it assert gluing of arbitrary nonlinear structures, group actions, filtrations, or analytic conditions. Those would have to descend through the lattice construction with their own compatibility proofs.

Consequences for the calculus. The new example separates three operations that a universal slogan would conflate. Exact finite-level excision can erase an entire prime component. Completion can retain a generic fiber that all those finite repairs erased. Effective reconstruction instead retains the localized object, the completed objects, and their overlap identifications. The lattice theorem shows that this latter pattern can genuinely construct a global arithmetic object from local input, in a precisely specified class.

Corollary 15.8 (Compatible operators glue with the lattice)**.** *In theorem 15.7, suppose* $f_1, \dots, f_s \in \mathrm{End}_A(V)$ *and* $g_{p,1}, \dots, g_{p,s} \in \mathrm{End}_{\mathbb{Z}_p}(L_p)$ *satisfy the exact intertwining equations*

$$\theta_p(f_i \otimes 1) = (g_{p,i} \otimes 1)\theta_p.$$

Then they induce unique integral endomorphisms of Λ*. Every specified noncommutative polynomial relation with integer coefficients holding on* V *holds on* Λ*. If the operators and their inverses are supplied compatibly, they glue as automorphisms.*

Proof. For $v \in \Lambda$, the displayed identity puts $\theta_p(f_i v) = g_{p,i}\theta_p(v)$ in L_p for every p. Hence $f_i v \in \Lambda$. Restriction is unique because Λ spans V after localization. The polynomial relations hold by restriction from V, and applying the argument to the supplied inverses gives automorphisms. □

This corollary glues an already compatible operator action. It does not construct a generic action, a continuous Galois action, or an automorphic realization from scalar or Frobenius invariants alone.

16 Intrinsic arithmetic exchange and certified lifting

The weighted comparisons of theorem 14.3 leave a concrete question open: can a nonzero defect arise from the arithmetic functors themselves? This section gives two answers and keeps their categorical regimes separate. Ordinary invariants against reduction have a first-cohomology defect. Fully derived invariants against a perfect coefficient change commute. Against a specified nonperfect coefficient change they can fail even as exact functors, with a non-scalar representation and no weight inserted into the comparison.

Throughout the lattice part, fix a prime p, write $O = \mathbb{Z}_p$ and $O_n = O/p^n O$, and take $n \geq 1$. A lattice means a finite free O-module. The exponent in $Q[p^n]$ denotes annihilation by p^n, not a quotient. We distinguish a residual module, a selected element of that module obstructing a particular lift, and a cone in a specified derived category.

16.1 A natural kernel–reduction comparison

For a map $d : V \to W$ of lattices, put

$$K = \ker d, \qquad Q = \operatorname{coker} d, \qquad K_n = \ker(d \otimes_O O_n).$$

The module K is saturated in V: if $p^n v \in K$, then $p^n d(v) = 0$ in the torsion-free module W, so $v \in K$. It is therefore a finite free direct summand as an O-module, although a splitting is not canonical. Reduction supplies a natural injection $\alpha_n : K/p^n K \to K_n$.

Theorem 16.1 (Intrinsic kernel exchange and its boundary)**.** *There is a natural short exact sequence*

$$\boxed{0 \longrightarrow K/p^n K \xrightarrow{\alpha_n} K_n \xrightarrow{\delta_n} Q[p^n] \longrightarrow 0.}$$

For $\bar{v} \in K_n$, choose a lift $v \in V$. Since $d(v) \in p^n W$, set

$$\delta_n(\bar{v}) = \left[\frac{d(v)}{p^n}\right] \in Q.$$

This formula is independent of the lift. For $m \geq n$, ordinary reduction $r_{m,n} : K_m \to K_n$ satisfies

$$\boxed{\delta_n r_{m,n} = p^{m-n}\delta_m.}$$

Consequently the cokernel of the natural comparison α_n is $Q[p^n]$, with these specified transition maps. Its cone, viewing both modules in degree zero in $D(O_n)$, is canonically equivalent to that cokernel in degree zero.

Proof. If v is replaced by $v + p^n u$, the displayed quotient changes by $d(u)$, which is zero in Q. Moreover $p^n[d(v)/p^n] = [d(v)] = 0$. If the class vanishes, write $d(v)/p^n = d(u)$ and replace v by $v - p^n u \in K$. This proves exactness at K_n. Conversely, if $[w] \in Q[p^n]$, there is $v \in V$ with $p^n w = d(v)$, so $\bar{v} \in K_n$ maps to $[w]$. Saturation proves injectivity of α_n.

For the transition formula, use the same lift v to represent an element first at level m and then at level n. The quotients obey $d(v)/p^n = p^{m-n} d(v)/p^m$. Functoriality for a commuting square between two maps d follows from this same formula. Finally the cone of an injection between degree-zero modules has just its cokernel in degree zero. □

This is a direct coefficient-boundary argument, not a new general homological-algebra principle. The modules on the two sides need not be perfect over O_n. The operation of taking a kernel is not an exact functor on arbitrary short exact sequences. Thus this theorem by itself does not instantiate the earlier exact-endofunctor assertions.

16.2 The exact amount of additional resolution needed

Choose Smith data for d over the discrete valuation ring O. After invertible row and column operations its nonzero entries are $p^{a_1}, \dots, p^{a_r}$, with $a_i \geq 0$. Units have been absorbed into the basis changes. Let

$$h = \max\{a_1, \dots, a_r, 0\}.$$

Equivalently, h is the least integer with $p^h Q_{\mathrm{tors}} = 0$. If $Q_{\mathrm{tors}} = 0$, set $h = 0$.

Theorem 16.2 (Residual profile and sharp finite-resolution lifting)**.** *Let $v = \mathrm{rank}_O V$. Then*

$$Q[p^n] \cong \bigoplus_{i=1}^{r} O/p^{\min(a_i,n)}O, \qquad \ell_O(K_n) = n(v-r) + \sum_{i=1}^{r} \min(a_i, n).$$

The splittings require chosen Smith bases and are not canonical. For every $n \geq 1$,

$$\boxed{\mathrm{im}(K_{n+h} \longrightarrow K_n) = \mathrm{im}(K/p^n K \longrightarrow K_n).}$$

Thus a particular solution at precision n is the reduction of an exact p-adic solution if and only if it lifts, for the same fixed map d, to precision $n + h$. No smaller nonnegative delay works uniformly for all n when $h > 0$.

The residual system $(Q[p^n], p^{m-n})$ is pro-zero, and

$$K \xrightarrow{\sim} \varprojlim_n K_n.$$

Here pro-zero means that for each target level some later transition is the zero map. It does not mean that the individual residual modules vanish.

Proof. In Smith coordinates a nonzero diagonal entry imposes $p^{a_i} x_i = 0$ modulo p^n, giving $\min(a_i, n)$ units of O-length. Each of the $v - r$ zero columns contributes a free O_n coordinate. This proves the formulas.

By theorem 16.1, the image of an element of K_{n+h} has boundary $p^h \delta_{n+h} = 0$, so lies in the image of $K/p^n K$. The reverse inclusion follows by reducing any exact lift at level $n + h$.

If $c < h$, choose an element of exact order p^h in Q_{tors} and take $n \geq h$. Surjectivity of δ_{n+c} supplies a preimage. Its reduction to level n has nonzero boundary after multiplication by p^c, so it is not the reduction of an exact solution. This proves sharpness. Multiplication by p^h also makes every transition with gap at least h zero on residuals.

Finally a compatible family of vectors modulo p^n has a unique lift to V, since V is finite free and complete. If every reduction solves the equation, its image under d lies in $\bigcap_n p^n W = 0$. This proves the inverse-limit assertion directly, without interchanging an arbitrary kernel and an unspecified completion. □

Remark 16.3 (What a finite certificate must contain). For integer input matrices, a verified integer Smith decomposition gives the p-adic data by taking the valuations of its nonzero diagonal entries. Verification requires the two change-of-basis matrices to be unimodular and their product with d to equal the displayed diagonal matrix. For arbitrary p-adic entries, a finite truncation alone does not certify a claimed zero diagonal entry or an exact rank. Such a claim requires an exact input presentation or a separate rank certificate. The theorem does not supply a universal numerical decision procedure from unbounded-precision oracles.

16.3 Symmetry provides the comparison without imposed weights

Let G be a finite group acting O-linearly on a lattice M. The natural comparison is

$$\alpha_{G,n} : M^G/p^n M^G \longrightarrow (M/p^n M)^G.$$

The first path takes invariant vectors and then reduces. The second reduces the representation and then takes invariant vectors. Its direction is fixed by reduction of an invariant vector; no arbitrary scalar enters this map.

Theorem 16.4 (Cohomological identification of the symmetry residual)**.** *There is a natural exact sequence*

$$\boxed{0 \longrightarrow M^G/p^n M^G \longrightarrow (M/p^n M)^G \xrightarrow{\partial_n} H^1(G, M)[p^n] \longrightarrow 0.}$$

For a fixed residue class $\bar{m}$ and a lift $m \in M$, the obstruction is represented by the crossed homomorphism

$$c_{\bar{m}}(g) = \frac{gm - m}{p^n}.$$

It vanishes exactly when $\bar{m}$ has a G-invariant lift in M. The group $H^1(G, M)$ is finite and killed by $|G|$. In particular it is killed by p^e, where $e = v_p(|G|)$, and the delay e is a valid uniform lifting bound.

More precisely, for

$$d_G : M \longrightarrow \prod_{g \in G} M, \qquad d_G(m) = (gm - m)_g,$$

the torsion subgroup of $\operatorname{coker} d_G$ *is canonically $H^1(G, M)$. Therefore the sharp bound is computed by theorem 16.2, and can be strictly smaller than e.*

Proof. The formula for c satisfies $c(gh) = gc(h) + c(g)$, because this identity holds before division by p^n in the torsion-free module M. Changing m by $p^n b$ changes c by $gb - b$. Hence it defines a cohomology class. The usual long exact coefficient sequence gives the displayed short sequence, but it can also be proved directly. If $p^n[c] = 0$, there is $m \in M$ with $p^n c(g) = gm - m$. Its reduction is invariant and maps to $[c]$. If $[c] = 0$, write $c(g) = gb - b$ and observe that $m - p^n b$ is the required invariant lift. Saturation gives injectivity on the left.

For completeness, let $c : G \to M$ be a cocycle and $b = \sum_{g \in G} c(g)$. For $u \in G$,

$$ub = b - |G|c(u), \qquad |G|c(u) = u(-b) - (-b).$$

Thus $|G|[c] = 0$. The cocycle and coboundary modules are subquotients of finite free O-modules, so H^1 is finitely generated and hence finite when annihilated by this nonzero integer.

Write $d^1(c)(g,u) = gc(u) - c(gu) + c(g)$. If a class $[c]$ in $\operatorname{coker} d_G$ is torsion, then $p^a c = d_G(m)$ for some a. Applying d^1 gives $p^a d^1(c) = 0$, whence $d^1(c) = 0$ since the target is torsion-free. Thus its class lies in H^1. Conversely the preceding annihilation argument shows that every H^1 class is torsion. This proves the canonical identification and the sharp-bound assertion. □

The long exact sequence, explicit boundary, and cyclic-group cohomology used here are classical. Our conventions agree with [13, Chapter II, 1.9, 1.18–1.21]. The proof above is included to fix naturality, the coefficient regime, and the transition maps. If p does not divide $|G|$, all these residuals vanish. Divisibility of the group order alone does not imply a nonzero residual: the trivial action on a lattice has $H^1(G, M) = 0$.

16.4 Lifting transformations, not just invariant vectors

Theorem 16.5 (Certified lifting of equivariant maps)**.** *Let L and M be G-lattices and put*

$$V = \operatorname{Hom}_O(L, M), \qquad (g \cdot f) = \rho_M(g) f \rho_L(g)^{-1}.$$

Then

$$\begin{aligned} 0 \longrightarrow &\operatorname{Hom}_{O[G]}(L, M)/p^n \longrightarrow \operatorname{Hom}_{O_n[G]}(L/p^n L, M/p^n M) \\ &\longrightarrow H^1(G, V)[p^n] \longrightarrow 0 \end{aligned}$$

is exact and natural. A modular equivariant map lifts to an integral equivariant map if and only if its boundary class vanishes. If p^h annihilates the p-primary cohomology as in theorem 16.2, this is equivalent to lifting the same map equivariantly to precision $n + h$.

If L and M have the same rank and the given modular map is invertible, every integral lift supplied in this way is an isomorphism. Thus existence of a G-equivariant isomorphism at precision $n + h$ implies existence of an integral one whose reduction at precision n is the specified map. The coarser bound $v_p(|G|)$ is always valid.

Proof. Finite freeness identifies $V/p^n V$ with the full module of linear maps between the reduced lattices. Its G-invariants are exactly the equivariant maps. Apply theorems 16.2 and 16.4. Explicitly, for a lift $\widetilde{f}$ the boundary cocycle is

$$g \longmapsto \frac{\rho_M(g)\widetilde{f}\rho_L(g)^{-1} - \widetilde{f}}{p^n}.$$

If this is $g \cdot b - b$, then $\widetilde{f} - p^n b$ is an integral equivariant lift. In the equal-rank case its determinant is a unit because it is a unit modulo p. Its inverse is integral and equivariant. No chosen modular intertwiner is replaced silently by another at the target precision. □

The theorem concerns two already supplied lattices with an actual action of the same finite group. It does not construct a Galois representation, provide a finite quotient for an arbitrary profinite action, or produce an automorphic representation. It addresses a concrete transformation-lifting obligation once those inputs exist.

16.5 A non-scalar family with a sharp obstruction lifespan

Let $m \geq 2$, and put

$$N_m(t) = 1 + t + \cdots + t^{m-1}, \qquad M_m = O[t]/(N_m(t)).$$

This is free of rank $m-1$. Let a generator σ of C_m act by multiplication by t. Since $(t-1)N_m(t) = t^m - 1$, this is a genuine cyclic-group action.

Theorem 16.6 (Norm-quotient lattice and transient invariants)**.** *For this action,*

$$M_m^{C_m} = 0, \qquad H^1(C_m, M_m) \cong O/mO.$$

Consequently, with $a = v_p(m)$,

$$(M_m/p^n M_m)^{C_m} \cong O/p^{\min(a,n)}O.$$

Under the boundary identification with $(O/mO)[p^n]$, reduction from level s to level n is multiplication by p^{s-n}. For $a > 0$ the sharp uniform delay is a. Every nonzero finite invariant fails to lift integrally, although the action itself is defined over O.

Proof. The operator $\sigma - 1$ is multiplication by $t - 1$. Its cokernel is $O/(N_m(1)) = O/mO$. Over $\mathbb{Q}_p$, the element $t - 1$ is invertible modulo $N_m(t)$ because $N_m(1) = m \neq 0$. Hence its kernel over O is zero. The norm operator is multiplication by $N_m(t)$ and therefore zero on M_m.

For a cyclic group, a crossed homomorphism is determined by $c(\sigma) = x$ with norm zero, and a coboundary has $x = (\sigma - 1)y$. This gives $H^1 = \ker N/(\sigma - 1)M_m = O/mO$ directly, agreeing with [13, Chapter II, 1.20]. The remaining assertions follow from the preceding theorems. □

For $p = m = 3$, in the basis $1, t$,

$$A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \qquad A^3 = I, \qquad \det(A - I) = 3.$$

The fixed vectors modulo 3^n are exactly

$$\left\{ b\, 3^{n-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} : b \in \{0, 1, 2\} \right\}.$$

The three vectors at level $n+1$ all reduce to zero at level n. Thus a nonzero residual appears at every level, while its inverse system has zero limit. This is not persistent nonexistence of the lattice; it is the failure of a particular finite-level vector to be the reduction of an invariant integral vector.

17 Fully derived exchange: a positive theorem and an exact counterexample

17.1 The comparison and its direction

For a commutative ring R, a finite group G, and $M \in D(R[G])$, define

$$\Gamma_R(M) = \mathbf{R}\,\mathrm{Hom}_{R[G]}(R, M) \in D(R).$$

This is the derived invariant functor. Derived categories in this section are the enhanced unbounded derived categories. They are stable. Their exact functors preserve finite cofiber sequences but are not asserted to preserve every infinite limit or colimit.

For a ring map $R \to S$ there is a canonical comparison

$$\boxed{\beta_{R,S,M} : \Gamma_R(M) \otimes^{\mathbf{L}}_R S \longrightarrow \Gamma_S(M \otimes^{\mathbf{L}}_R S).}$$

It is adjoint to base change of the evaluation map from the object of homotopy invariants, regarded as carrying the trivial group action, to M. Its construction is natural in M and uses the actual coefficient map. No exchange weight is selected.

This direction is reduction-after-invariants to invariants-after-reduction. It is the reverse of the optional $A_r C \to C A_s$ convention used for some earlier signatures. We use the direction supplied by the arithmetic comparison, not an unproved inverse. No earlier orientation-dependent normalization theorem is applied without changing its signature explicitly.

A two-fiber exact model can be written entirely with endofunctors: set

$$\mathcal{K}_R = D(R[G]) \times D(R), \qquad A_R(M, N) = (0, \Gamma_R(M)),$$

and let $C_{R,S}$ apply derived coefficient change in both components. The second component of $C_{R,S} A_R \to A_S C_{R,S}$ is precisely β. This places the example in a genuine two-axis exact system, rather than identifying differently typed functors by notation. A fixed ambient universe is understood. A small stable subdiagram can instead be generated by the stated inputs and their images, without making a claim that all outputs are perfect.

Theorem 17.1 (Perfect coefficient changes have zero derived exchange defect)**.** *If S is perfect as an R-module, then $\beta_{R,S,M}$ is an equivalence for every $M \in D(R[G])$. In particular, reduction $\mathbb{Z}_p \to \mathbb{Z}/p^n$ commutes with fully derived invariants.*

Proof. For a complex $P \in D(R)$ with trivial G-action, evaluation defines a natural comparison

$$\Gamma_R(M) \otimes^{\mathbf{L}}_R P \longrightarrow \Gamma_R(M \otimes^{\mathbf{L}}_R P).$$

Both sides are exact functors of P. The right side is exact because homotopy invariants preserve finite limits, and finite limits agree with finite colimits in stable categories. The comparison is the identity for $P = R$. The class of P on which it is an equivalence is closed under finite cofibers, shifts, and retracts, hence contains $\mathrm{Perf}(R) = \mathrm{Thick}(R)$. For $P = S$, the right side has its natural S-module structure and is the stated derived invariant object over S.

Finally $\mathbb{Z}/p^n$ has the two-term finite free resolution $[\mathbb{Z}_p \xrightarrow{p^n} \mathbb{Z}_p]$. It is perfect over $\mathbb{Z}_p$. This uses the ordinary derived tensor construction, not termwise tensor of an arbitrary unbounded non-K-flat complex [9]. □

For a lattice the earlier nonzero $H^1[p^n]$ therefore records the effect of extracting degree-zero invariants, not a failure of this fully derived comparison over $\mathbb{Z}_p$. This positive theorem rules out a tempting but incorrect source of a new exact residual. It also gives an information-preserving way to handle that particular truncation defect: retain derived invariant data rather than require degree-zero invariants to commute with reduction.

17.2 A natural non-scalar defect at finite arithmetic resolution

The perfectness hypothesis cannot be removed. Fix an odd prime p, let

$$R = \mathbb{Z}/p^2, \qquad k = \mathbb{F}_p, \qquad G = C_p,$$

and choose a matrix $J \in M_d(\mathbb{Z})$ whose reduction modulo p is invertible. Put $M = R^d$ and let the chosen generator act by

$$A = I + pJ \pmod{p^2}.$$

For $d = 2$, the choice $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ makes this action non-scalar. It mixes the two coordinates. The group action is input, but the exchange map remains the canonical β, not a scalar multiple of another comparison.

Theorem 17.2 (Unbounded intrinsic derived exchange defect)**.** *For the preceding data, $A^p = I$ and the norm is $N = I + A + \cdots + A^{p-1} = pI$. Moreover,*

$$\Gamma_R(M) \simeq pM[0] \cong k^d[0].$$

Let $U = \Gamma_R(M) \otimes^{\mathbf{L}}_R k$ and $V = \Gamma_k(M \otimes^{\mathbf{L}}_R k)$. Their cohomology is

$$H^i(U) \cong \begin{cases} k^d & i \le 0, \\ 0 & i > 0, \end{cases} \qquad H^i(V) \cong \begin{cases} 0 & i < 0, \\ k^d & i \ge 0. \end{cases}$$

The natural comparison $\beta : U \to V$ is zero on H^0. Its functor-intrinsic cofiber satisfies

$$\boxed{H^i \operatorname{Cofib}(\beta) \cong k^d \quad \text{for every } i \in \mathbb{Z}.}$$

Both composites defining β are exact. The nonzero residual is therefore a genuine defect between fully derived exact functors, not solely a degree-zero truncation effect.

Proof. The binomial expansion gives $(I + pJ)^p = I$ modulo p^2. Also $A^i = I + ipJ$ modulo p^2, so

$$N = pI + p\frac{p(p-1)}{2}J = pI \pmod{p^2},$$

using that p is odd. The standard cyclic free resolution of the trivial $R[G]$-module gives the cochain complex

$$M \xrightarrow{pJ} M \xrightarrow{pI} M \xrightarrow{pJ} M \xrightarrow{pI} \cdots$$

in nonnegative degrees. The cyclic resolution is obtained from the integral one, whose exact sequences split as abelian groups, so it remains a free resolution after coefficient change. Its alternating difference and norm maps agree with [13, Chapter II, 1.20 and 3.4].

Since J is invertible over R, each displayed differential has image and kernel pM. All positive cohomology vanishes and $H^0 = pM$. Inclusion of that kernel in degree zero is a quasi-isomorphism $pM[0] \to \Gamma_R(M)$.

The R-module k has an infinite free resolution, in nonpositive cochain degrees, with each differential multiplication by p. It is exact because $\ker(p : R \to R) = pR = \operatorname{im}(p)$. Tensoring this bounded-above free resolution with k gives zero differentials. Since $pM \cong k^d$, this computes $H^i(U)$ as claimed.

On the other side M is free over R, so $M \otimes^{\mathbf{L}}_R k = k^d[0]$. Its action is trivial. In the cyclic cochain resolution over k, both $A - I$ and the norm are zero. This gives the displayed cohomology of V.

The evaluation map on degree-zero invariants is the inclusion $pM \hookrightarrow M$. On tensoring with k its degree-zero map is zero. Evaluation on $H^0(V) = (k^d)^G = k^d$ is an isomorphism. Naturality of evaluation defining β therefore proves $H^0(\beta) = 0$. The long exact cohomology sequence of its cofiber now gives k^d in every degree: the groups in negative degrees come from $H^{i+1}(U)$, those in positive degrees from $H^i(V)$, and the two middle groups use the zero degree-zero comparison.

Exactness of derived coefficient change and derived invariants proves exactness of both composites. Their canonical natural transformation has a functorial cofiber in the stable enhancement. No illicit termwise tensor of the nonnegative invariant complex is used in computing U. □

Example 17.3 (The two-dimensional $\mathbb{Z}/9$ instance)**.** Take

$$R = \mathbb{Z}/9, \qquad A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}, \qquad G = C_3.$$

Then $A^3 = I$ over R, $I + A + A^2 = 3I$, and $M^G = 3M$. The invariant cochain complex is quasi-isomorphic to $(\mathbb{F}_3)^2$ in degree zero. Derived reduction of that invariant object has cohomology in every nonpositive degree, whereas taking derived invariants after reduction has cohomology in every nonnegative degree. The comparison cone has $(\mathbb{F}_3)^2$ in every degree. Ordinary tensor of the original nonnegative complex would incorrectly reproduce only the right-hand answer.

Corollary 17.4 (Erasing this residual loses the coefficient unit)**.** *Let $\Delta = \mathrm{Cofib}(\beta)$ in theorem 17.2. Any exact functor out of $D(k)$ that annihilates Δ annihilates every perfect k-complex. If that functor also preserves all coproducts, it annihilates every object of $D(k)$.*

Proof. Every complex of vector spaces admits a generally noncanonical splitting into its cohomology and contractible pieces, obtained by choosing complements to boundaries inside cycles and to cycles inside each degree. Since $H^0(\Delta) = k^d \neq 0$, the unit $k[0]$ is a retract of Δ in $D(k)$. Exact functors preserve retracts, finite sums, shifts, and cofibers, so annihilating Δ annihilates $\mathrm{Thick}(k) = \mathrm{Perf}(k)$. If coproducts are preserved, every split complex is a coproduct of shifts of vector spaces and every vector space is a coproduct of copies of k. This proves the stronger conclusion. □

Thus exactness alone does not make the comparison invertible, and existence of its cofiber does not make a quotient repair harmless. The arithmetic coefficient module k is not perfect over R. Also $\Gamma_R(M) \simeq k^d$ is not perfect over R, and the resulting residual is unbounded over k. These facts explain exactly why neither the preceding perfect-base-change theorem nor a blanket perfect-fiber model applies.

17.3 Finite-level validity does not provide a finer action

Theorem 17.5 (A one-step representation-lifting obstruction)**.** *In theorem 17.2, the representation of C_p on $(\mathbb{Z}/p^2)^d$ has no lift to a representation on $(\mathbb{Z}/p^3)^d$ with that reduction. More explicitly, every matrix lift $\widetilde{A}$ obeys*

$$\boxed{\widetilde{A}^p - I \equiv p^2 J \not\equiv 0 \pmod{p^3}.}$$

This obstruction is unchanged by the choice of the lift. There is also no finite free $\mathbb{Z}_p$-representation with this reduction, even up to isomorphism of the reduced representation.

Proof. Every lift has the form $I + pJ + p^2B$ modulo p^3. Put $X = pJ + p^2B$ and use the binomial identity for $I + X$. The linear term is $pX = p^2J$ modulo p^3. For $2 \leq j < p$, the coefficient $\binom{p}{j}$ is divisible by p and X^j by p^j, so those terms vanish modulo p^3. The last term X^p also vanishes

because $p \geq 3$. Since J is invertible modulo p, the surviving term is nonzero. Hence the relation $\sigma^p = 1$ cannot hold at the next precision.

A $\mathbb{Z}_p$-lift would reduce to such a level-three lift, which is impossible. Finally an invertible change of basis modulo p^2 lifts to an invertible matrix modulo p^3. Conjugating by that lift reduces an alleged representation lifting only up to isomorphism to the already refuted matrix-lifting problem. □

This is a concrete relation obstruction for a prescribed deformation problem. It is not asserted to classify arbitrary deformation obstructions or to identify the entire unbounded exchange cone with an H^2 obstruction group. In contrast, the norm-quotient actions in theorem 16.6 genuinely exist over $\mathbb{Z}_p$ at all precisions, even though some of their finite-level invariant vectors do not lift. These are different failures.

17.4 Exact preservation on coefficient-lifted inputs

The preceding negative result has a complementary positive theorem. A coefficient quotient may be nonperfect over an intermediate ring, yet the canonical exchange can be exact on inputs that genuinely come from an earlier coefficient level.

Theorem 17.6 (Lifted-input exchange and a necessary obstruction to lifting)**.** *Let $O \to R \to S$ be maps of commutative rings. Assume R and S are perfect as O-modules. For every $N \in D(O[G])$, put $M = N \otimes^{\mathbf{L}}_O R$. Then*

$$\boxed{\beta_{R,S,M} : \Gamma_R(M) \otimes^{\mathbf{L}}_R S \xrightarrow{\sim} \Gamma_S(M \otimes^{\mathbf{L}}_R S).}$$

No perfectness of S over R is required. Consequently, if $\mathrm{Cofib}(\beta_{R,S,M})$ *is nonzero, there is no $N \in D(O[G])$ together with an equivalence $N \otimes^{\mathbf{L}}_O R \simeq M$.*

For $O = \mathbb{Z}_p$, $R = \mathbb{Z}/p^m$, and $S = \mathbb{Z}/p^n$ with $m \geq n \geq 1$, this applies at every pair of levels. It applies in particular to reductions of a supplied integral G-lattice.

Proof. The canonical base-change comparisons obey transitivity. After the tensor associativities, the composite

$$\Gamma_O(N) \otimes^{\mathbf{L}}_O S \xrightarrow{\beta_{O,R,N} \otimes_R S} \Gamma_R(N \otimes^{\mathbf{L}}_O R) \otimes^{\mathbf{L}}_R S \xrightarrow{\beta_{R,S,M}} \Gamma_S(N \otimes^{\mathbf{L}}_O S)$$

is $\beta_{O,S,N}$. To check this equality with its natural coherence, take the adjoints defining the comparisons: both are base change of the same evaluation map from homotopy invariants to N. The coefficient extension adjunctions compose by derived tensor associativity. Their mates therefore agree. No arbitrary homotopy-category cone is chosen in this argument.

By theorem 17.1, $\beta_{O,R,N}$ and $\beta_{O,S,N}$ are equivalences. Tensoring preserves an equivalence, so the two-out-of-three property proves that $\beta_{R,S,M}$ is an equivalence. Naturality makes the conclusion invariant under a specified equivalence between M and a lifted object. The obstruction assertion is the contrapositive. For the arithmetic specialization, both quotients have two-term finite free resolutions over $\mathbb{Z}_p$. □

The cofiber of $\beta_{R,S,-}$ is an exact functor. Its vanishing class is therefore thick and contains every coefficient-lifted input, as well as the thick closure of those inputs. The example in theorem 17.2 lies outside that class. This is a functor-intrinsic necessary lifting obstruction, not merely a failed scalar equality.

The converse is false in general: theorem 18.3 supplies a zero comparison cone for a nonliftable finite-group representation. The positive theorem preserves information on the class with supplied lifts. A sufficient criterion for a fixed finite source instead comes from the actual relation obstruction and the finite lifting horizon in theorems 18.1 and 18.6.

17.5 The resulting boundary of the arithmetic calculus

There are now four precisely separated regimes.

1. For a supplied integral lattice, ordinary invariants and reduction can differ. The residual is $H^1[p^n]$, and a finite amount of additional precision detects and repairs liftability of a specified vector or intertwiner without taking a destructive quotient.
2. For a perfect coefficient change, fully derived invariants and reduction agree. Passing to derived data resolves the degree-zero truncation problem in this regime.
3. A supplied coefficient lift over a common ring makes the intermediate comparison exact when the two coefficient quotients are perfect over that common ring. Nonvanishing of the comparison cone is therefore a necessary obstruction to such a lift.
4. For the exhibited nonperfect change $\mathbb{Z}/p^2 \to \mathbb{F}_p$, the fully derived canonical exchange itself has a nonzero, unbounded cofiber. Killing that cofiber destroys the target's perfect coefficient theory. Its input representation also fails to extend one coefficient level further.

These statements distinguish intrinsic exchange from actual existence. The following section refutes sufficiency of the single comparison-cone test and proves a finite necessary-and-sufficient lifting criterion for a fixed finite group. That correction preserves the requested finite-level representation without killing the coefficient unit. It does not close unrestricted profinite-tower existence or the Langlands-facing globalization condition.

18 Sufficient lifting for a fixed finite group

This section closes a specified version of the sufficient-lifting question left by K61. The source group G is fixed and finite, the rank d is fixed, the coefficient ring is $\mathbb{Z}_p$, and every lift must satisfy the same group multiplication table. We first exhibit a zero-exchange-defect representation that nevertheless does not lift. We then give a complete one-step obstruction and an explicit finite horizon after which convergent correction constructs an integral action. None of these assertions concerns arbitrary continuous representations of a changing system of finite Galois quotients.

Attribution. The one-step obstruction is the classical group-extension class in H^2, with the standard inhomogeneous conventions of [13, Chapter II, 1.17–1.19]. Finite-group ultrametric stability and the quantitative bound used below are known: [14, Lemma 7.3, Proposition 7.4, Example 7.5] supply the same valuation threshold, in a broader setting. We provide a direct matrix-averaging proof, translate it into an exact finite lifting criterion, and connect it to the present residual calculus. Historical novelty is not claimed for that bound or the deformation-theoretic mechanism.

18.1 The relation residual and its repair class

Put $R_n = \mathbb{Z}/p^n$, $k = \mathbb{F}_p$, and fix

$$\rho_n : G \longrightarrow \mathrm{GL}_d(R_n), \qquad n \geq 1.$$

Let $V = \mathrm{End}_k(k^d)$ with action

$$g \cdot X = \bar{\rho}(g) X \bar{\rho}(g)^{-1}, \qquad \bar{\rho} = \rho_n \bmod p.$$

Choose matrices $A_g \in \mathrm{GL}_d(R_{n+1})$ lifting $\rho_n(g)$, with $A_1 = I$. They need not constitute a representation. Define the normalized relation residual

$$c_A(g,h) = \frac{A_g A_h A_{gh}^{-1} - I}{p^n} \pmod p \quad \in V.$$

Division here identifies the square-zero ideal $p^n R_{n+1}$ with k. It is not inversion of p in R_{n+1}. The square-zero assertion uses $2n \geq n+1$.

For a normalized 1-cochain $b : G \to V$, set

$$(\delta b)(g,h) = b(g) + g \cdot b(h) - b(gh).$$

For a normalized 2-cochain set

$$(\delta c)(g,h,t) = g \cdot c(h,t) - c(gh,t) + c(g,ht) - c(g,h).$$

A cochain is normalized when it vanishes whenever an argument is the identity.

Theorem 18.1 (Complete one-step relation obstruction)**.** *The residual c_A is a 2-cocycle. Its cohomology class*

$$o_n(\rho_n) = [c_A] \in H^2(G, V)$$

is independent of the chosen matrix lifts. Moreover,

$$\boxed{o_n(\rho_n) = 0 \quad \Longleftrightarrow \quad \rho_n \text{ lifts to } \mathrm{GL}_d(R_{n+1}).}$$

More precisely, all lifts with the fixed reduction have the form

$$A_g^b = (I + p^n b(g)) A_g,$$

where representatives of $b(g)$ may be chosen arbitrarily, and the necessary and sufficient equation is

$$\boxed{\delta b = -c_A.}$$

Proof. Associativity of the threefold product gives the exact identity

$$(I + p^n c_A(g,h))(I + p^n c_A(gh,t)) = (I + p^n g \cdot c_A(h,t))(I + p^n c_A(g,ht))$$

in R_{n+1}, where the conjugation can be reduced modulo p. All products of two p^n terms vanish. Dividing the resulting linear identity by p^n gives $\delta c_A = 0$.

Any alternative matrix lift is uniquely of the form $(I + p^n b(g))A_g$, for a normalized cochain b. Expanding its multiplicativity residual and discarding square-zero terms gives

$$c_{A^b} = c_A + \delta b.$$

Thus the class is unchanged and the new matrices satisfy every group relation exactly if and only if $\delta b = -c_A$. Such a cochain exists precisely when $[c_A] = 0$. Invertibility is automatic because $I + p^n b(g)$ has inverse $I - p^n b(g)$ at this level. □

Proposition 18.2 (Repair ambiguity, conjugacy, and finite certificates)**.** *If the one-step lift set is nonempty, its framed lifts form a torsor for $Z^1(G,V)$. Conjugation by matrices $I+p^nX$, $X\in V$, quotients this torsor by coboundaries. Hence the strict-conjugacy classes form a torsor for $H^1(G,V)$, and the stabilizer of a lift under this conjugation is $H^0(G,V)$ as an additive group.*

For a fixed multiplication table and residual action, deciding existence is a finite linear problem over k. A solution b is a positive certificate. A linear functional λ on normalized 2-cochains with

$$\lambda\circ\delta=0,\qquad \lambda(c_A)\neq 0$$

is a negative certificate.

Proof. If b_0 is one solution, every other solution is b_0+z with $\delta z=0$. Conjugating the matrices by $I+p^nX$ replaces $b(g)$ by $b(g)+X-g\cdot X$. These are exactly the negatives of the usual 0-coboundaries and yield the same quotient subgroup. Conjugation fixes the action precisely when X is invariant. The cochain spaces have dimensions $(|G|-1)d^2$ and $(|G|-1)^2d^2$. Gaussian elimination decides whether $-c_A$ is in the image of δ. If it is not, elementary linear algebra supplies a functional annihilating that image but not c_A. □

This is an actual obstruction to realizing the finer group action, not merely failure of a selected invariant to commute with coefficient change. The repair changes matrices within their prescribed residue classes and keeps the coefficient unit. It does not take a quotient of the coefficient theory.

18.2 A vanishing exchange cone can miss nonliftability

Theorem 18.3 (A central sign hides the obstruction from derived invariants)**.** *Let p be odd, $R=\mathbb{Z}/p^2$, and let $J\in M_d(\mathbb{Z})$ be invertible modulo p. On $M=R^d$ let*

$$G=C_p\times C_2=\langle\sigma,\tau\rangle,\qquad \sigma\mapsto I+pJ,\quad \tau\mapsto -I.$$

For the canonical comparison of theorem 17.6,

$$\Gamma_R(M)\simeq 0,\qquad \Gamma_k(M\otimes^{\mathbf{L}}_R k)\simeq 0,\qquad \mathrm{Cofib}(\beta_{R,k,M})\simeq 0.$$

Nevertheless this representation does not lift to $\mathbb{Z}/p^3$, and hence does not lift as a finite free $\mathbb{Z}_p$-representation of this fixed group.

Proof. The two displayed matrices commute and satisfy the defining relations. Since 2 is invertible in R, the central orthogonal idempotents

$$e_+=(1+\tau)/2,\qquad e_-=(1-\tau)/2$$

split $R[G]$ into a product of rings. The trivial module R is in the plus factor and M in the minus factor. Every derived Hom between the two factors is zero, as is seen by resolving within either direct factor. The same decomposition persists after reduction to k. Since M is free over R, its derived reduction is its ordinary reduction. Both endpoints of β therefore vanish.

Any lift of this G-action would restrict to a lift of its C_p-action. By theorem 17.5, each lift of $I+pJ$ has pth power $I+p^2J\neq I$ modulo p^3. This is impossible. □

The example refutes sufficiency of the single Γ-comparison test in the finite-group class. It does not merely leave the converse unproved. The loss is a loss of detection: a functor can see none of an input while the input still has inconsistent lifting relations. In contrast, conjugation by $-I$ is trivial on $\mathrm{End}_k(k^d)$, so the adjoint relation obstruction is not hidden by this sign. Twisting by the integral sign character also restores the earlier nonzero exchange example. Neither observation asserts that a finite family of such probe comparisons classifies all lifting obstructions.

18.3 Quadratic correction of an approximate action

Let $N = |G|$ and write $e = v_p(N)$. For a matrix X over $\mathbb{Q}_p$, the assertion $v_p(X) \geq a$ means every entry lies in $p^a\mathbb{Z}_p$. Invertible integral matrices and their inverses preserve this divisibility under multiplication. Let $A : G \to \mathrm{GL}_d(\mathbb{Z}_p)$ be a normalized map, not necessarily a homomorphism. Define

$$E(g,h) = A_g A_h A_{gh}^{-1} - I, \qquad \alpha_g(X) = A_g X A_g^{-1}.$$

Suppose $E(g,h) \in p^m M_d(\mathbb{Z}_p)$ for every pair. The operators α_g are not assumed to give a group action. This distinction is essential before the repair has succeeded.

Lemma 18.4 (Averaged correction with controlled precision loss)**.** *If $m > e$, define in $M_d(\mathbb{Q}_p)$*

$$b(g) = \frac{1}{N}\sum_{t\in G} E(g,t), \qquad A'_g = (I - b(g))A_g.$$

Then $b(g) \in p^{m-e}M_d(\mathbb{Z}_p)$, $b(1) = 0$, and A' is a normalized map into $\mathrm{GL}_d(\mathbb{Z}_p)$. Its defect satisfies

$$\boxed{A'_g A'_h (A'_{gh})^{-1} - I \in p^{2m-2e}M_d(\mathbb{Z}_p).}$$

Also $A'_g - A_g \in p^{m-e}M_d(\mathbb{Z}_p)$.

Proof. Division by $N = p^e u$, $u \in \mathbb{Z}_p^\times$, loses at most e digits, proving the bound on b. Normalization gives $b(1) = 0$. Because $m - e \geq 1$, each $I - b(g)$ is invertible.

Associativity gives an identity without assuming an exact adjoint action:

$$(I + E(g,h))(I + E(gh,t)) = (I + \alpha_g E(h,t))(I + E(g,ht)).$$

Consequently

$$E(g,h) + E(gh,t) - \alpha_g E(h,t) - E(g,ht) \in p^{2m}M_d(\mathbb{Z}_p).$$

Sum over t and divide by N. The permutation $t \mapsto ht$ identifies the last average with $b(g)$, giving

$$E(g,h) - b(g) - \alpha_g b(h) + b(gh) \in p^{2m-e}M_d(\mathbb{Z}_p).$$

Put $a = m - e > 0$. The repaired multiplicativity product is exactly

$$(I - b(g))(I - \alpha_g b(h))(I + E(g,h))(I - b(gh))^{-1}.$$

Modulo p^{2a} its difference from I is the linear expression just displayed: products of two corrections vanish at that precision, and $(I - b)^{-1} \equiv I + b \pmod{p^{2a}}$. That expression is divisible by p^{2m-e}, which is contained in p^{2a}. This proves the new defect bound. The change in A_g is $-b(g)A_g$ and has the asserted precision. □

This proof exhibits the residual as input to a repair operator. The correction is admissible because its denominators are absorbed by the defect's valuation. It requires no prior integral action and does not delete any module or coefficient information.

Theorem 18.5 (Convergent finite-group action repair)**.** *If the initial defect depth satisfies $m > 2e$, the preceding averaging operation iterates to a representation*

$$\rho : G \longrightarrow \mathrm{GL}_d(\mathbb{Z}_p) \quad \textit{with} \quad \rho(g) \equiv A_g \pmod{p^{m-e}}.$$

After j iterations the certified defect depth is

$$m_j = 2e + 2^j(m - 2e),$$

and the next change has depth at least $m_j - e$. In particular, the sequence converges uniformly on G and all prescribed multiplication relations hold exactly in the limit.

Proof. Set $m_0 = m$ and $m_{j+1} = 2m_j - 2e$. Induction gives the displayed formula, so $m_j > 2e$ at every stage and $m_j \to \infty$. The correction lemma applies repeatedly. Differences of consecutive matrices have entry valuations tending to infinity, hence their telescoping sums converge in the complete ring $M_d(\mathbb{Z}_p)$. Every correction preserves reduction modulo p, so the limits remain invertible. Their reduction modulo p^{m-e} is the original one.

Matrix multiplication and inversion on $\mathrm{GL}_d(\mathbb{Z}_p)$ are continuous. The defects converge to zero because their depths are at least m_j. Passing to the limit gives $\rho(g)\rho(h) = \rho(gh)$ and $\rho(1) = I$. □

The recurrence is a quantitative certificate of convergence, not a claim that arbitrary small residuals can always be removed. At $m = 2e$ this estimate does not improve the defect depth. We do not assert that the threshold is optimal for every group, rank, or representation.

18.4 A finite necessary-and-sufficient lifting horizon

Theorem 18.6 (Finite horizon for fixed finite-group representations)**.** *Let G be a finite group, $d, n \geq 1$, and $e = v_p(|G|)$. Set*

$$\boxed{h_G(n) = \max\{n + e,\, 2e + 1\}.}$$

For every $\rho_n : G \to \mathrm{GL}_d(\mathbb{Z}/p^n)$, the following are equivalent:

1. *There is a representation $\rho : G \to \mathrm{GL}_d(\mathbb{Z}_p)$ with reduction ρ_n.*
2. *There is a representation $\rho_h : G \to \mathrm{GL}_d(\mathbb{Z}/p^{h_G(n)})$ with reduction ρ_n.*

The implication from (2) to (1) is constructive to arbitrary requested precision by theorem 18.5. It preserves the original level n, not necessarily every digit of the chosen level-h witness.

Proof. Reduction proves (1) implies (2). Conversely, lift every matrix of ρ_h arbitrarily to an invertible integral matrix, keeping the identity equal to I. Its multiplicativity defects are divisible by p^h. Since $h > 2e$, convergent repair applies and returns an action agreeing with these matrices modulo p^{h-e}. Because $h - e \geq n$, it returns precisely the specified ρ_n on reduction. □

Thus a supplied high-precision action is a finite certificate for existence of an entire integral action. A level-h witness may itself have an obstruction at its next level: the repair is allowed to alter the top e digits while keeping the requested lower data. There is no claim that a positive certificate yields an exact lift of that witness at every digit.

A terminating decision method follows. Begin with ρ_n and use theorems 18.1 and 18.2 to enumerate all one-step solutions, retaining their actual framed matrices, until level $h_G(n)$. A surviving branch is a positive witness. If all branches die before that finite level, no integral lift exists. Every layer is finite, and Gaussian elimination provides checkable obstruction certificates at dead nodes. Choosing a single convenient branch is not sufficient for a negative verdict. Quotienting branches by conjugacy is optional and requires preserving their lifting data.

For $p \nmid |G|$, one has $e = 0$ and $h_G(n) = n$. Every finite-level representation in this fixed finite-group problem then lifts. When $e > 0$, the test is still finite, but its number of branches may be enormous. No polynomial-time or uniformly practical algorithm is claimed.

Example 18.7 (A positive witness need not lift itself)**.** Take $p = 2$, $G = C_2$, and rank one. The level-two representation with generator $3 \pmod 4$ has the level-three witness $3 \pmod 8$, since $3^2 \equiv 1 \pmod 8$. Here $e = 1$ and $h_G(2) = 3$. Nevertheless neither lift of $3 \pmod 8$ to $\mathbb{Z}/16$ has square one: both 3^2 and 11^2 are $9 \pmod{16}$.

Starting with the integer matrix $a_0 = 3$, the averaged repair becomes

$$a_{j+1} = \frac{3 - a_j^2}{2}\, a_j.$$

The first values are $3, -9, 351$, whose relation defects have valuations $3, 4, 6$. The convergence theorem gives an exact 2-adic involution congruent to 3 (mod 4). In the integral domain $\mathbb{Z}_2$, the equation $a^2 = 1$ has only the solutions $1, -1$. Thus the repaired limit is -1, which is 7 (mod 8), not the selected witness 3 (mod 8). This is exactly the distinction between preserving the requested seed and preserving every exploratory refinement.

18.5 Controlled uniqueness and a finite catalogue

Theorem 18.8 (Close integral actions are conjugate)**.** *Let $\rho, \sigma : G \to \mathrm{GL}_d(\mathbb{Z}_p)$ be representations satisfying*

$$\sigma(g) \equiv \rho(g) \pmod{p^s} \quad (g \in G), \qquad s > e.$$

Then

$$T = \frac{1}{|G|} \sum_{g \in G} \sigma(g)\rho(g)^{-1} \in I + p^{s-e} M_d(\mathbb{Z}_p)$$

is invertible and satisfies

$$\sigma(g)T = T\rho(g), \qquad \sigma(g) = T\rho(g)T^{-1}.$$

Proof. Every summand is congruent to I modulo p^s. Its average is therefore I modulo p^{s-e}, so it is integral and invertible. For $h \in G$,

$$\sigma(h)T\rho(h)^{-1} = \frac{1}{|G|} \sum_{g} \sigma(hg)\rho(hg)^{-1} = T,$$

by reindexing the finite sum. This is the required intertwining identity. □

This is the averaging mechanism underlying finite-precision rigidity of integral finite-group representations, closely related to the classical rigidity of integral finite-group lattices. The proof above is independent of any such external rigidity theorem. The strict inequality matters: for $p = 2$, $G = C_2$, and rank one, the trivial and sign representations agree modulo 2 but are not conjugate. Here $s = e = 1$.

Corollary 18.9 (A finite catalogue of integral action classes)**.** *Let $s = e + 1$ and $h = 2e + 1$. Denote by $\mathscr{L}_s$ the image of*

$$\mathrm{Hom}(G, \mathrm{GL}_d(\mathbb{Z}/p^h)) \longrightarrow \mathrm{Hom}(G, \mathrm{GL}_d(\mathbb{Z}/p^s)).$$

Reduction gives a bijection

$$\boxed{\frac{\mathrm{Hom}(G, \mathrm{GL}_d(\mathbb{Z}_p))}{\mathrm{GL}_d(\mathbb{Z}_p)} \longleftrightarrow \frac{\mathscr{L}_s}{\mathrm{GL}_d(\mathbb{Z}/p^s)}.}$$

Consequently this class set is finite and can be determined from a finite multiplication table and finite matrix data.

Proof. The finite-horizon theorem gives exactly $h_G(s) = 2e + 1$, so every element of $\mathscr{L}_s$ lifts and every integral action reduces into it. If two reductions are conjugate, lift their conjugating matrix to $\mathrm{GL}_d(\mathbb{Z}_p)$ and first conjugate one action by that lift. Their reductions now agree modulo p^s. Since $s > e$, theorem 18.8 makes the integral actions conjugate. The target is finite because it is a quotient of a finite set. □

This is a classification of isomorphism classes, not an equivalence with a finite groupoid. Integral automorphism groups and noninvertible morphisms need not be determined by finite reduction. Likewise the averaging construction is not asserted to be a canonical framed lift. Two outputs agreeing modulo p^{m-e} are conjugate by a matrix congruent to I modulo p^{m-2e} when $m > 2e$, but need not be literally the same matrices.

18.6 A smaller certificate for cyclic groups

Proposition 18.10 (Cyclic relation Jacobian)**.** *Let $G = C_q$ for an integer $q \geq 1$, and let $A \in \mathrm{GL}_d(\mathbb{Z}/p^n)$ satisfy $A^q = I$. Choose a lift $\widetilde{A}$ modulo p^{n+1} and set*

$$c = \frac{\widetilde{A}^q - I}{p^n} \pmod{p}, \qquad L_{\bar{A}}(B) = \sum_{i=0}^{q-1} \bar{A}^i B \bar{A}^{q-1-i}.$$

Then the lifts of the cyclic action to level $n+1$ are exactly the matrices

$$\widetilde{A} + p^n B \quad \text{such that} \quad L_{\bar{A}}(B) = -c.$$

If the affine solution set is nonempty, it has $p^{\dim_k \ker L_{\bar{A}}}$ elements. This criterion uses d^2 variables instead of all group-table cochains.

Proof. Expand the noncommutative product $(\widetilde{A} + p^n B)^q$. Terms with two or more factors $p^n B$ vanish modulo p^{n+1}. The terms with one such factor give exactly $p^n L_{\bar{A}}(B)$. Therefore the cyclic relation is equivalent to the stated linear equation. All candidate matrices remain invertible because their mod-p reductions are fixed. A nonempty fiber of a linear map over k is a translate of its kernel and has the claimed cardinality. □

For $A = I + pJ$ modulo p^2 and $q = p$, the residual is $c = J$ while $L_I(B) = pB = 0$ in k. This recovers the one-step obstruction without exhaustive testing. At other inputs the Jacobian may have nonzero rank, so some nonzero initial relation residuals are removable rather than persistent.

18.7 Why a fixed finite quotient is a real restriction

Proposition 18.11 (A profinite lift need not preserve the finite quotient)**.** *Let p be odd and $J \in M_d(\mathbb{Z}_p)$ be invertible modulo p. The action $C_p \to \mathrm{GL}_d(\mathbb{Z}/p^2)$ given by $I + pJ$ has no integral lift as an action of C_p. Its pullback along $\mathbb{Z}_p \twoheadrightarrow C_p$, however, has a continuous integral lift*

$$\rho(x) = (I + pJ)^x = \sum_{j \geq 0} \binom{x}{j} p^j J^j, \qquad x \in \mathbb{Z}_p.$$

Proof. For each j, the polynomial $\binom{x}{j}$ is continuous on $\mathbb{Z}_p$ and takes values there: approximate x by nonnegative integers, where the binomial coefficient is integral, and use closedness of $\mathbb{Z}_p$ in $\mathbb{Q}_p$. Thus the series converges uniformly because its jth term is divisible by p^j. For nonnegative integer arguments it is the usual matrix power. The nonnegative integers are dense in $\mathbb{Z}_p$, so continuity extends the identity $\rho(x+y) = \rho(x)\rho(y)$ to all pairs. Each value is I modulo p and hence invertible.

Modulo p^2 the series is $I + pxJ$, which depends only on x mod p and gives the specified finite-level action. But $\rho(p) \equiv I + p^2 J \neq I \pmod{p^3}$, so the integral lift does not factor through C_p. Nonexistence of an integral C_p-lift is theorem 17.5. □

The same warning applies to finite quotients used in a Galois problem. A relation imposed by fixing one finite quotient may be absent from the full profinite source. Failure to lift through that fixed quotient therefore does not by itself obstruct a continuous lift with growing image. Conversely, the finite-group horizon depends on $v_p(|G|)$ and gives no uniform bound for arbitrary growing finite quotients.

18.8 What the new block establishes

There are now three distinct tests. A chosen functor's exchange cone is a necessary diagnostic on its detectable inputs, but the central-sign example shows it is not a complete lifting invariant. The class $o_n(\rho_n)$ is a complete one-step obstruction because its vanishing is proved equivalent to solving the actual multiplicativity equations. Finally, the finite horizon plus convergent correction turns a finite witness into a full $\mathbb{Z}_p$-action for a fixed finite group. This gives a positive reconstruction theorem without killing the residual in a Verdier quotient.

In the two-axis vocabulary, the resolution fiber is the actual lift groupoid, the transformation direction includes changes of basis, the relation residual is a normalized cocycle, admissible repairs are cochains, and its class in H^2 is the obstruction. The averaging theorem then supplies a quantitative regime in which repeated repairs converge. These identifications recover and connect classical deformation and ultrametric-stability mathematics. They do not make the abstract obstruction formalism a proof of automorphic existence or Langlands reciprocity.

19 Continuous source resolution and abelian reconstruction

This section keeps the profinite source fixed while allowing its finite images to grow. It separates two independent resolution coordinates: coefficient precision and the finite quotient through which a particular observation factors. Its strongest constructive conclusion is a complete finite-matrix lifting criterion for continuous rank-two representations of $\mathbb{Z}_p^r$. In arbitrary rank the same criterion holds when a specified normalized generator is cyclic modulo p. No prescribed determinant, ramification type, Hodge condition, or automorphic realization is included in these statements.

The continuous cohomology used below is the cohomology of continuous inhomogeneous cochains with a *finite discrete* coefficient module. The finite-quotient description is classical [13, Chapter II, Section 4, Proposition 4.2]; see also [15]. The free pro-p construction and cyclic-centralizer argument are given directly. The claims are written deductions under explicit hypotheses, not assertions of historical novelty.

19.1 The actual continuous obstruction

Fix a profinite group Γ, a prime p, $n, d \geq 1$, and a continuous representation

$$\rho_n : \Gamma \longrightarrow \mathrm{GL}_d(R_n), \qquad R_n = \mathbb{Z}/p^n.$$

The target is finite and discrete. Give $V = M_d(\mathbb{F}_p)$ the adjoint action of $\bar{\rho}$. Choose a section of the reduction map on the finite set $\rho_n(\Gamma)$, taking the identity to the identity. Composing this section with ρ_n gives a continuous matrix-valued lift A_g to R_{n+1}, not necessarily multiplicative. Define

$$c_A(g,h) = \frac{A_g A_h A_{gh}^{-1} - I}{p^n} \pmod{p}.$$

Here division is the identification of $p^n R_{n+1}$ with $\mathbb{F}_p$, as in theorem 18.1.

Theorem 19.1 (Complete continuous one-step obstruction)**.** *The class*

$$o_n^{\mathrm{cts}}(\rho_n) = [c_A] \in H^2_{\mathrm{cts}}(\Gamma, V)$$

is independent of the continuous choices. It vanishes if and only if ρ_n has a continuous *lift to* $\mathrm{GL}_d(R_{n+1})$ *with its specified reduction. More precisely, the repair equation is*

$$\delta b = -c_A, \qquad b : \Gamma \to V \text{ continuous and normalized},$$

and the repaired matrices are $(I + p^n b(g))A_g$. When lifts exist, their framed set is a torsor for $Z^1_{\mathrm{cts}}(\Gamma, V)$. Strict-conjugacy classes and stabilizers are governed by H^1_{cts} and H^0_{cts}, respectively.

Proof. The associativity and square-zero computations in theorem 18.1 hold pointwise without a finiteness assumption on the source. All the resulting cochains are continuous because the targets are finite and the chosen lifts are continuous. Any other continuous lift differs uniquely by $(I + p^n b(g))$ with continuous b, and its residual is $c_A + \delta b$. Hence a zero class is equivalent to an actual continuous repair. Conversely an actual continuous homomorphism is such a repair. The difference of two repairs is a continuous 1-cocycle. Conjugation by $I + p^n X$ changes it by a coboundary, with kernel the invariant elements of V, exactly as in the finite case. □

Theorem 19.2 (Source refinement and inflation)**.** *Choose an open normal $U \subseteq \ker \rho_n$ and compute c_A on $Q = \Gamma/U$. Then*

$$o_n^{\mathrm{cts}}(\rho_n) = 0 \quad \Longleftrightarrow \quad \exists\, U' \lhd_{\mathrm{open}} \Gamma,\ U' \subseteq U : \mathrm{Inf}^{\Gamma/U'}_{\Gamma/U}[c_A] = 0.$$

Thus a continuous next-level lift exists exactly when a repair exists at some finer source quotient*, not necessarily the quotient used initially. For every $j \geq 0$,*

$$H^j_{\mathrm{cts}}(\Gamma, V) \cong \operatorname*{colim}_{U' \subseteq U} H^j(\Gamma/U', V).$$

Proof. Every continuous map $\Gamma^j \to V$ is locally constant. Cover its compact domain by finitely many products of cosets on which the map is constant. Intersect the finitely many open normal subgroups defining these cosets and further intersect with U. The map then factors through $(\Gamma/U')^j$. This applies to cocycles, coboundaries, and cochains witnessing an equality. The finite-quotient cochain complexes therefore have the continuous complex as their filtered union, and the same observation about witnesses proves the asserted cohomology colimit. In particular a class vanishes in that colimit exactly when it vanishes at some subsequent stage. Apply theorem 19.1. This is the finite-module specialization of the cited classical inflation theorem. □

Example 19.3 (An explicit obstruction killed by refining the source)**.** Let p be odd, $J \in M_d(\mathbb{Z})$ nonzero modulo p, and $A = I + pJ$ modulo p^3. Then $A^p = I + p^2 J$ and $A^{p^2} = I$ modulo p^3. The representation of C_p modulo p^2 uses powers of A. For representatives $0 \leq a, b < p$, choose $A_a = A^a$ at the next level. Its cocycle is

$$c(a, b) = \left\lfloor \frac{a+b}{p} \right\rfloor J \pmod{p}.$$

The adjoint residual action is trivial. For any normalized cochain u on C_p,

$$\sum_{t=0}^{p-1} (\delta u)(1, t) = 0, \qquad \sum_{t=0}^{p-1} c(1, t) = J \neq 0.$$

So the finite-quotient class is nonzero. On C_{p^2} put

$$b(x) = \left\lfloor \frac{x}{p} \right\rfloor J \pmod{p} \quad (0 \le x < p^2).$$

Writing each x as its low digit plus p times its high digit shows $\delta b = -\operatorname{Inf}(c)$: the high-digit carry is the negative of the low-digit carry modulo p. The repaired map is exactly $x \mapsto A^x$. Thus inflation kills a genuine finite-quotient obstruction. This calculation does not declare every continuous obstruction removable.

19.2 Reconstructing a continuous tower

Theorem 19.4 (Continuous reconstruction from unobstructed stages)**.** *If $H^2_{\mathrm{cts}}(\Gamma, \operatorname{Ad}\bar{\rho}) = 0$, every given ρ_n with that residual action extends to a continuous representation*

$$\rho : \Gamma \longrightarrow \mathrm{GL}_d(\mathbb{Z}_p).$$

Without the cohomology-vanishing hypothesis, a compatible sequence of continuous lifts still reconstructs uniquely as a framed representation. If Γ is topologically finitely generated, existence of some *extension of the fixed ρ_n to every coefficient level is enough to obtain a compatible sequence and hence an integral lift.*

Proof. The residual adjoint module is unchanged by each coefficient lift. Under the vanishing hypothesis, theorem 19.1 permits recursive choice of compatible lifts ρ_m for all $m \ge n$. Take the inverse limit of their matrix entries. Multiplication is compatible with this limit, and determinants remain units because their reductions modulo p are units. Each finite projection is continuous, so the map into the inverse-limit topology is continuous. The projections determine it uniquely.

For the last assertion, choose r topological generators. A continuous map into a finite group is determined by their images, since the equalizer of two such maps is closed and contains a dense subgroup when the images agree. Thus there are finitely many continuous representations at each level. Form the rooted tree of extensions of ρ_n, with edges given by one-step reduction. It is finitely branching. If every level is nonempty, the root has descendants at arbitrarily large depth. At least one child has the same property, since there are only finitely many children. Repeating gives an infinite compatible path. Apply the first limit argument. This compactness argument supplies no numerical stopping bound and no algorithm for computing H^2_{cts} of an arbitrary source. □

Theorem 19.5 (Unrestricted coefficient lifting for a free pro-p source)**.** *Let $F_r^{(p)}$ be the free pro-p group on finitely many generators. Every continuous*

$$F_r^{(p)} \longrightarrow \mathrm{GL}_d(R_n)$$

has a continuous lift to $\mathrm{GL}_d(\mathbb{Z}_p)$. One may choose arbitrary integral lifts of its generator matrices.

Proof. Construct $F_r^{(p)}$ as the inverse limit of the finite p-group quotients of the abstract free group on these generators. A map from the generators to a pro-p group induces compatible maps to each of its finite p-group quotients, and hence a unique continuous map from this inverse limit. This proves the universal property needed here.

The residual image $P \subseteq \mathrm{GL}_d(\mathbb{F}_p)$ is a p-group, being a finite quotient of a pro-p group. Let H be its inverse image in $\mathrm{GL}_d(\mathbb{Z}_p)$. Its quotient modulo p^m is a finite p-group: its kernel over P has the filtration by congruence subgroups with additive $M_d(\mathbb{F}_p)$ subquotients. Therefore H is pro-p,

including at $p = 2$. Arbitrary integral lifts of the generator matrices lie in H. The just-proved universal property extends them to a continuous map. Reduction agrees with the input on the generators and therefore everywhere. This does not assert that the lift is faithful or has finite image. □

19.3 Continuous powers and the cyclic-centralizer mechanism

Lemma 19.6 (When an integral matrix has continuous procyclic powers)**.** *Let $A \in \mathrm{GL}_d(\mathbb{Z}_p)$. The map $m \mapsto A^m$ on $\mathbb{Z}$ extends uniquely to a continuous homomorphism $\mathbb{Z}_p \to \mathrm{GL}_d(\mathbb{Z}_p)$ if and only if the reduction of A modulo p has p-power order. In particular a commuting tuple of such matrices determines a continuous representation of $\mathbb{Z}_p^r$.*

Proof. Necessity follows because a continuous image of $\mathbb{Z}_p$ in a finite group is a cyclic p-group. Conversely suppose $A^{p^t} \in I + pM_d(\mathbb{Z}_p)$. For $B \in I + p^s M_d(\mathbb{Z}_p)$ with $s \geq 1$, binomial expansion shows $B^p \in I + p^{s+1} M_d(\mathbb{Z}_p)$. For $p = 2$ the quadratic term has depth $2s \geq s + 1$, so this estimate remains valid. It follows that

$$A^{p^{t+j}} \in I + p^{j+1} M_d(\mathbb{Z}_p).$$

Hence exponents congruent modulo p^{t+j} give powers congruent modulo p^{j+1}, also for negative integer exponents by invertibility. Completeness extends the map continuously from the dense subgroup $\mathbb{Z}$ to $\mathbb{Z}_p$. The group law and uniqueness follow by continuity. For commuting matrices the product of these powers defines the final assertion. □

Lemma 19.7 (Cyclic centralizers over a coefficient ring)**.** *Let R be a commutative ring, $B \in M_d(R)$, and suppose a vector v makes*

$$Q = (v, Bv, \ldots, B^{d-1}v) \in \mathrm{GL}_d(R).$$

Then every matrix commuting with B is uniquely $f(B)$ for a polynomial $f \in R[t]$ of degree less than d. The coefficients of f are the entries of $Q^{-1}Mv$. In particular, the centralizer is commutative.

Over $R = \mathbb{Z}/p^s$, it is enough that $\bar{B}$ have a cyclic vector over $\mathbb{F}_p$. Every nonscalar 2×2 matrix over a field has such a vector.

Proof. There is a unique polynomial of degree less than d satisfying $f(B)v = Mv$, since the columns of Q are a basis. If $MB = BM$, then $M - f(B)$ annihilates every $B^j v$, so it is zero. Conversely every polynomial in B commutes with B. This also proves uniqueness and the coefficient formula. A lifted cyclic vector modulo p makes $\det Q$ a unit modulo p^s.

Finally, if a 2×2 matrix has no cyclic vector, it takes each vector into its own span. On a basis e_1, e_2 it is diagonal, and applying this condition to $e_1 + e_2$ makes the two diagonal entries equal. Thus it is scalar. This argument includes the field $\mathbb{F}_2$. □

19.4 A complete normalized-commutator obstruction

Here the source is $\Gamma = \mathbb{Z}_p^r$, where $r \geq 1$ is finite. Write the generator matrices of the given continuous representation ρ_n as $A_1, \ldots, A_r$. They commute in $M_d(R_n)$ and each has p-power order modulo p.

Choose $0 \leq a < n$ and integer scalars λ_i such that

$$A_i \equiv \lambda_i I \pmod{p^a} \quad \text{for every } i.$$

There is no restriction in this display when $a = 0$. Define the *normalized matrices*

$$B_i = \frac{A_i - \lambda_i I}{p^a} \quad \in M_d(R_{n-a}).$$

This is well-defined from the given finite data. It does not invert p inside R_n. Assume that, for some distinguished index $*$, the matrix B_* is cyclic modulo p. Define the finite obstruction record

$$\boxed{\Omega_{a,*}(\rho_n) = \big([B_*, B_i]\big)_{i=1}^{r} \in M_d(R_{n-a})^r,} \qquad [B, C] = BC - CB.$$

Theorem 19.8 (Normalized cyclic lifting criterion)**.** *Under the preceding hypotheses,*

$$\boxed{\rho_n \text{ has a continuous integral lift} \iff \Omega_{a,*}(\rho_n) = 0.}$$

If the obstruction vanishes, an integral lift is constructed from finite polynomial certificates. Express $B_i = f_i(B_*)$ in R_{n-a}, with $\deg f_i < d$ and $f_*(t) = t$ when $d > 1$. Choose an integral lift $\widehat{B}$ of B_* and integer lifts $\widehat{f}_i$ of these polynomials, taking the distinguished generator itself to $\widehat{B}$. Then put

$$\widehat{A}_i = \lambda_i I + p^a \widehat{f}_i(\widehat{B}), \qquad \rho(x_1, \dots, x_r) = \prod_i \widehat{A}_i^{x_i}.$$

For $d = 1$, use the scalar matrices directly. The construction preserves the prescribed reduction modulo p^n and does not pass to an obstruction quotient.

Proof. Suppose first that $\widehat{A}_i$ are any commuting integral lifts. Then $\widehat{B}_i = (\widehat{A}_i - \lambda_i I)/p^a$ are integral. The identities

$$0 = [\widehat{A}_*, \widehat{A}_i] = p^{2a}[\widehat{B}_*, \widehat{B}_i]$$

imply $[\widehat{B}_*, \widehat{B}_i] = 0$, since matrices over $\mathbb{Z}_p$ have no p-torsion. Reducing modulo p^{n-a} proves necessity. This argument permits arbitrary higher digits in all the integral lifts.

Conversely, the cyclic-centralizer lemma supplies the unique polynomials f_i from the vanishing commutators. Their integral polynomial evaluations commute exactly. Multiplication by p^a shows that $\widehat{A}_i$ reduces to A_i modulo p^n. Each is invertible because its reduction modulo p is invertible. That reduction has p-power order by the input's continuity. The procyclic-powers lemma now supplies the continuous representation, whose reduction agrees on the topological generators and hence on all of $\mathbb{Z}_p^r$. In rank one the same argument is simply scalar lifting. Polynomial coefficients and a cyclic vector give a finite positive certificate; a nonzero entry of a displayed commutator gives a finite negative certificate under the verified cyclic hypothesis. □

Remark 19.9 (What normalization reveals). The input relations only force

$$p^{2a}[B_i, B_j] \equiv 0 \pmod{p^n}.$$

When $a > 0$, this can conceal a nonzero commutator in R_{n-a}. The obstruction is therefore not just the original zero commutator re-labelled. It measures the additional commutativity required after removing a common scalar precision. Different permitted scalar choices change B_i by scalar matrices and leave its commutators unchanged. Changing representatives changes them only by zero in R_{n-a}. No preferred normalized matrix is claimed independently of these data.

Corollary 19.10 (A source-fixed finite horizon in the cyclic stratum)**.** *Under the hypotheses of theorem 19.8, the following are equivalent:*

$$\rho_n \text{ lifts over } \mathbb{Z}_p, \qquad \rho_n \text{ extends over } R_{n+a}, \qquad \Omega_{a,*}(\rho_n) = 0.$$

The extension remains a representation of the same $\mathbb{Z}_p^r$. When $a = 0$, the current representation is already a sufficient certificate.

Proof. An integral lift supplies the finite extension. Given a finite extension with commuting matrices A_i' modulo p^{n+a}, subtract the same $\lambda_i I$ and divide by p^a. The resulting matrices B_i' are defined modulo p^n and reduce to B_i modulo p^{n-a}. The equality $p^{2a}[B_*', B_i'] = 0$ modulo p^{n+a} implies $[B_*', B_i'] = 0$ modulo p^{n-a}, so $\Omega_{a,*} = 0$. The normalized criterion supplies an integral lift. This is a bound obtained from the source's actual commutativity relations, not from the orders of its growing finite images. □

19.5 Rank two: a complete decision from the given precision

For $A \in M_2(R_n)$ define its scalar depth by

$$a(A) = \max\{0 \le t \le n : A \text{ is scalar modulo } p^t\}.$$

Writing entries in R_n, this is the minimum of the capped valuations of $A_{12}, A_{21}, A_{11} - A_{22}$. It is unchanged by invertible conjugation. For a tuple put $a = \min_i a(A_i)$.

Theorem 19.11 (Complete rank-two profinite lifting test)**.** *Let p be any prime and let $\rho_n : \mathbb{Z}_p^r \to \mathrm{GL}_2(R_n)$ be continuous, with finitely many topological generators as above.*

1. *If $a = n$, all the A_i are scalar and ρ_n has a continuous integral lift.*
2. *If $a < n$, choose any generator A_* with scalar depth a. Take λ_i to be an integer representative of $(A_i)_{11}$ and form B_i modulo p^{n-a}. Then*

$$\boxed{\rho_n \text{ lifts over } \mathbb{Z}_p \quad \Longleftrightarrow \quad [B_*, B_i] = 0 \text{ in } M_2(R_{n-a}) \ (1 \le i \le r).}$$

The test is finite matrix arithmetic on the existing input*. It does not require enumeration of deeper coefficient lifts. When it passes, the polynomial construction in theorem 19.8 produces exact commuting integral generator matrices.*

Proof. If all generators are scalar, lift their scalar entries integrally. Their residues modulo p have p-power order and therefore equal 1, since $\mathbb{F}_p^\times$ has order prime to p. The scalar lifts are units and define continuous powers.

Otherwise every A_i is scalar modulo p^a, while B_* is nonscalar modulo p by the definition of a. It is cyclic by theorem 19.7. Thus every hypothesis of theorem 19.8 holds. In rank two the polynomials have the form $u_i + v_i t$ and can be recovered from a two-column cyclic basis by solving one invertible 2×2 linear system modulo p^{n-a}. The equivalence with liftability also shows independence of the distinguished generator and of basis choices, although the selected lift itself need not be canonical. □

Corollary 19.12 (A sharp uniform rank-two horizon)**.** *For every p, n, r as above,*

$$\boxed{\rho_n \text{ lifts continuously over } \mathbb{Z}_p \quad \Longleftrightarrow \quad \rho_n \text{ extends continuously to } R_{2n-1}.}$$

For $n \ge 2$ and $r \ge 2$, $2n-1$ is optimal as a uniform coefficient level. For $n = 1$, all continuous rank-two representations of $\mathbb{Z}_p^r$ over $\mathbb{F}_p$ lift.

Proof. For a nonscalar tuple, $a \le n - 1$, so an extension to $2n - 1$ restricts to one at $n + a$. Apply theorem 19.10. Scalar tuples lift directly. This also proves the assertion at $n = 1$.

For sharpness fix $n \ge 2$ and put

$$A = I + p^{n-1}E_{12}, \qquad B = I + p^{n-1}E_{21}.$$

These matrices commute modulo p^{2n-2} and generate a finite abelian p-group there, so they give a continuous extension of their level-n representation to level $2n-2$. However every pair of lifts preserving level n has the form $\widetilde{A} = A + p^n P$, $\widetilde{B} = B + p^n Q$, and

$$\widetilde{A}\widetilde{B} - \widetilde{B}\widetilde{A} \equiv p^{2n-2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \not\equiv 0 \pmod{p^{2n-1}}.$$

Indeed every correction term has valuation at least $2n-1$. This proves nonextension at $2n-1$ for every prime, including 2, and hence nonexistence of an integral lift. Add identity generators for $r > 2$. No smaller uniform coefficient level can distinguish these inputs from extendable ones by mere existence of an extension. □

Example 19.13 (The first hidden commutator)**.** Suppose $n = 2$ and all generator matrices are $A_i = I + pX_i$ modulo p^2. They always commute at that coefficient level. For rank two, theorem 19.11 gives

$$\rho_2 \text{ lifts over } \mathbb{Z}_p \quad \Longleftrightarrow \quad [X_i, X_j] = 0 \text{ in } M_2(\mathbb{F}_p) \text{ for all } i, j.$$

If some X_i is nonscalar, its centralizer is commutative and the displayed test agrees with the distinguished-generator criterion; if all are scalar the lift is immediate. For $X_1 = E_{12}$ and $X_2 = E_{21}$ it fails. Each generator separately has a continuous integral procyclic lift, but the pair does not define an integral action of $\mathbb{Z}_p^2$. On the free pro-p source the same pair *does* lift, by theorem 19.5. The surviving obstruction is therefore an actual source relation, not finite-order torsion imposed by an incidental quotient.

Example 19.14 (A positive repair that changes only unresolved digits)**.** Work at $p = 3$, $n = 3$. Take

$$A_1 = \begin{pmatrix} 1 & 3 \\ 9 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 7 & 12 \\ 9 & 7 \end{pmatrix} \pmod{27}.$$

They commute modulo 27 and both are the identity modulo 3. Choose $a = 1$, $\lambda_1 = \lambda_2 = 1$. Then

$$B_1 = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix} \pmod{9}, \qquad B_2 = 2I + 4B_1 \pmod{9}.$$

The vector e_2 is cyclic for B_1 modulo 3. The displayed integer representatives of A_1, A_2 do not commute exactly: their additive commutator is $\mathrm{diag}(-81, 81)$. The criterion nevertheless passes. Retain B_1 integrally and replace B_2 by the exact polynomial $2I + 4B_1$. This constructs

$$\widehat{A}_1 = \begin{pmatrix} 1 & 3 \\ 9 & 1 \end{pmatrix}, \qquad \widehat{A}_2 = \begin{pmatrix} 7 & 12 \\ 36 & 7 \end{pmatrix}.$$

They commute exactly, remain units over $\mathbb{Z}_3$, and have the original reduction modulo 27. Their continuous powers give the integral representation. This is an explicit non-destructive repair of a tuple, not a claim that its initial arbitrary integer representatives were already an exact action.

19.6 The repaired scope of the two-axis calculus

There is a necessary variance distinction. Let $\mathcal{Q}_\Gamma$ have open normal subgroups as objects and an arrow $U \to U'$ when $U' \subseteq U$. Write $\mathscr{P}_d(n, U)$ for the groupoid of rank-d representations of Γ/U on R_n^d, with invertible intertwining matrices as arrows.

Proposition 19.15 (The source–coefficient diagram has mixed variance)**.** *Reduction of coefficients and inflation of source define a functor*

$$\mathscr{P}_d : \mathbb{N}_{\geq 1}^{op} \times \mathcal{Q}_\Gamma \longrightarrow \mathbf{Gpd}.$$

Each source-inflation functor is fully faithful, and it commutes with coefficient reduction. An object at (m, U') is a compatible refinement of ρ_n at (n, U) precisely when its reduction to level n equals, or is supplied with an isomorphism to, the inflation of ρ_n. There is no total source-coarsening functor on arbitrary representations obtained simply by reversing inflation.

Proof. Coefficient reduction acts on the matrix entries of representations and intertwiners. Source inflation precomposes them with the surjection $\Gamma/U' \to \Gamma/U$. The two operations commute directly on both objects and arrows. An intertwiner of inflated representations commutes with the same matrix image as before inflation, because the source map is onto. Thus inflation is fully faithful. Finally a representation of Γ/U' descends through Γ/U only if the subgroup U/U' acts trivially. This fails for general representations, as the explicit repaired action in theorem 19.3 shows at its higher coefficient level. □

Coefficient refinement and source refinement do different jobs. The former improves arithmetic accuracy. The latter permits more of the source to act, and can kill an inflated finite-quotient obstruction without deleting coefficient information. Neither permits violating a relation of the fixed source itself. In particular these two resolution coordinates cannot both be inserted into the earlier coarsening-only convention without recording their opposite variance or restricting to descending objects.

The constructive abelian theorems avoid an unbounded quotient search altogether on their stated strata. Scalar-depth normalization reveals the missing relation, the cyclic centralizer supplies a finite polynomial certificate, and continuous matrix powers reconstruct the action. In rank two this yields a complete test at every coefficient precision and every finite source rank. In larger rank it requires the stated cyclic normalized generator. Free pro-p sources furnish a contrasting unrestricted coefficient-lifting case because they impose no additional relations on the generators.

These results do not imply a universal scalar-depth test in higher rank, an unconstrained theorem for all profinite groups, or a lift preserving extra arithmetic deformation conditions. The constructions make choices of higher matrix coefficients and may change determinants or other invariants not included among the equations. Continuous Galois representations with local geometric conditions and automorphic existence remain separate targets. The next substantive extension is a specified noncommutative profinite relation system with controlled centralizers or a verified deformation complex, rather than an assumption that every vanishing selected diagnostic gives a global object.

20 Local-to-global reconstruction

20.1 A Čech torsor residual as a nonstable obstruction test bed

The stable Verdier quotient is not the only setting in which the residual/repair/obstruction pattern appears. A classical torsor gives a minimal nonstable model.

Theorem 20.1 (Torsor residual obstruction)**.** *Let $\mathcal{A}$ be a sheaf of abelian groups on a space or a site with the fiber products used below, let P be an $\mathcal{A}$-torsor, and let $\{U_i \to U\}$ be a cover on which P admits local sections $s_i \in P(U_i)$. On overlaps there are unique elements*

$$a_{ij} \in \mathcal{A}(U_{ij})$$

such that

$$s_j = s_i + a_{ij}.$$

Then:

1. $a = (a_{ij})$ *is a Čech* 1*-cocycle;*
2. *changing the local sections by a* 0*-cochain* $b = (b_i)$*,*

   $$s'_i = s_i + b_i,$$

   changes the residual by the Čech coboundary

   $$a'_{ij} = a_{ij} + b_j - b_i;$$

3. *the class*

   $$\Omega(P; \{U_i\}) = [a] \in \check{H}^1(\{U_i\}, \mathcal{A})$$

   is independent of the chosen local sections;
4. $\Omega(P; \{U_i\}) = 0$ *if and only if the local sections can be modified by admissible local repairs so that they glue to a global section of* P *over* U*.*

Proof. On a triple overlap,

$$s_k = s_j + a_{jk} = s_i + a_{ij} + a_{jk},$$

while also $s_k = s_i + a_{ik}$. Freeness of the torsor action gives

$$a_{ij} + a_{jk} = a_{ik},$$

which is the Čech cocycle identity.

If $s'_i = s_i + b_i$, then

$$s'_j = s_j + b_j = s_i + a_{ij} + b_j = s'_i + (a_{ij} + b_j - b_i),$$

so the residual changes by the coboundary δb. Hence its cohomology class is independent of the chosen local sections.

If $[a] = 0$, choose b so that $a + \delta b = 0$. The repaired sections $s'_i = s_i + b_i$ then agree on every overlap, hence glue by the sheaf property to a global section of P. Conversely, if a global section exists, its restrictions have zero transition residual. Comparing those restrictions with the original s_i shows that a differs from zero by a coboundary, so $[a] = 0$. □

Status: proved directly from torsor descent and Čech cohomology.

For the standard torsor interpretation see [5].

In Komorebi language,

overlap residual / local repair = global obstruction class.

Here the residual is a_{ij}, admissible repair is the change $s_i \mapsto s_i + b_i$, repairable residual cocycles are coboundaries, and the obstruction is the cohomology class $[a]$. This local repair relation is not being identified with the monoid of exact endofunctors in the stable quotient construction. This example is important because it realizes the proposed obstruction pattern before stabilization and shows that the abstract vocabulary recovers an established local-to-global obstruction mechanism rather than manufacturing one artificially. It is specifically an obstruction to reconstructing a global *section* (equivalently, to trivializing the torsor), not an obstruction to the existence of the torsor itself.

20.2 Descent data

Let $(\mathcal{C}, J)$ be a site and let $\mathscr{K}$ assign a Komorebi category to each object of the site, with restriction functors on overlaps.

Given a cover $\{U_i \to U\}$, local objects $X_i \in \mathscr{K}(U_i)$ carry descent isomorphisms

$$\phi_{ij} : X_i|_{U_{ij}} \xrightarrow{\sim} X_j|_{U_{ij}}$$

satisfying the cocycle condition on triple overlaps.

Definition 20.2 (Effective descent)**.** Fix a cover and its complete Čech diagram, including all iterated overlaps. Effective descent means that the natural functor from the global category to the category of descent data is an equivalence. Essential surjectivity supplies global objects. Full faithfulness supplies descent of morphisms. Uniqueness of object classes alone is insufficient.

Theorem 20.3 (Descent transfer lemma)**.** *Let $A^\bullet, B^\bullet : \Delta \to \mathbf{Cat}_\infty$ be small Čech descent diagrams for a fixed cover. Suppose $\Lambda^\bullet : A^\bullet \to B^\bullet$ is a natural transformation that is an equivalence at every degree, including every overlap level. Then*

$$\lim_{[n]\in\Delta} A^n \simeq \lim_{[n]\in\Delta} B^n.$$

For ordinary category-valued descent, use the corresponding coherent pseudolimits rather than strict equalizers of object sets.

Proof. Equivalences in a functor category are detected objectwise. Hence $\Lambda^\bullet$ is an equivalence of diagrams. Applying the limit functor preserves this equivalence. The ordinary category formulation is the same argument with coherent descent categories and their quasi-inverses. □

Corollary 20.4 (Global reconstruction transfer)**.** *If the maps from the two global categories to these descent categories are equivalences, then the induced equivalence of descent categories yields an equivalence of global categories.*

This is the abstract form of the principle:

$$\boxed{\text{local equivalence} + \text{restriction compatibility} + \text{effective descent} \Longrightarrow \text{global equivalence.}}$$

21 Seeded local-global reciprocity blueprint

The previous section suggests a precise architecture for Langlands-facing work without claiming that the arithmetic hypotheses have been discharged.

Let

$$\mathbb{K}_{\text{arith}} \qquad \text{and} \qquad \mathbb{K}_{\text{aut}}$$

be arithmetic and automorphic seeded two-axis calculi.

A strong correspondence should include:

$$\begin{aligned} \Lambda(TX) &\simeq T'\Lambda(X), \\ \Lambda(CX) &\simeq C'\Lambda(X), \\ \Lambda(\Delta_{T,C}X) &\simeq \Delta_{T',C'}\Lambda(X), \\ \bar{\Lambda}(\operatorname{Obs} X) &\simeq \operatorname{Obs}'\Lambda(X). \end{aligned}$$

Here $\bar{\Lambda}$ is the induced functor on the designated obstruction quotients. The last two requirements are stronger than matching ordinary local invariants. They demand that the correspondence preserve the structure of failed exchange and failed reconstruction.

Conjecture 21.1 (Seeded reciprocity principle)**.** Suppose arithmetic and automorphic Komorebi systems satisfy:

1. local equivalence at every required place and resolution;
2. compatibility with resolution restriction;
3. preservation of the relevant local invariants;
4. transport of mixed residual and obstruction data;
5. effective descent on both sides;
6. a global separation principle ensuring uniqueness from sufficiently rich local data.

Then the induced local equivalence extends to an equivalence of the corresponding admissible global categories.

Under categorical formulations strong enough to make items (1)–(5) an equivalence of descent systems, the conclusion follows from theorem 20.4. The research content lies in proving that the arithmetic and automorphic systems really instantiate those hypotheses.

Interpretation. This reframes a possible route to local-global reciprocity as:

identify the globalization obstruction and prove that local correspondence transports it.

If

$$\Lambda_*(\Omega_{\mathrm{arith}}) = \Omega_{\mathrm{aut}},$$

and Λ_* preserves and reflects the zero object, then vanishing of one obstruction is equivalent to vanishing of the other. An equivalence, or a conservative exact functor, suffices for this extra detection condition. A general exact functor only preserves vanishing. Effective reconstruction then gives the stated global conclusion. Neither a sheaf/stack model for arbitrary local Langlands data nor automorphic existence follows from the abstract categorical lemmas. Places of a number field are not automatically a covering family for an effective descent problem. Those arithmetic identifications remain obligations.

22 Toy models

22.1 p-adic congruence resolution

Let the resolution level be $n \geq 1$ and take

$$X_n = \mathbb{Z}/p^n\mathbb{Z}.$$

For $m \leq n$, coarsening is reduction

$$C_{n\to m} : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}.$$

For any polynomial $f \in \mathbb{Z}[x]$, the induced transformation satisfies

$$C_{n\to m}(f(x)) = f(C_{n\to m}(x)).$$

Thus the transformation-resolution square commutes exactly. This is a set-level equality, not a cofiber calculation. Regarding the sets as discrete categories gives an identity comparison, while a

stable residual requires a separately specified stable realization. This is a simple instance where arithmetic transformation is resolution-natural.

By contrast, operations involving division by p need not descend uniformly. Their domain of definition and lift multiplicity depend on valuation. This gives a natural source of nontrivial resolution obstruction.

22.2 Dyadic averaging

Let a fine state be

$$x = (x_1, x_2) \in \mathbb{R}^2$$

and let coarse resolution record only the average

$$C(x) = \frac{x_1 + x_2}{2}.$$

Let the fine transformation be

$$A_1(x_1, x_2) = (ax_1, bx_2)$$

and the coarse transformation be multiplication by c:

$$A_0(y) = cy.$$

Then

$$C(A_1 x) - A_0(Cx) = \frac{(a-c)x_1 + (b-c)x_2}{2}.$$

Choosing

$$c = \frac{a+b}{2}$$

gives

$$C(A_1 x) - A_0(Cx) = \frac{a-b}{4}(x_1 - x_2).$$

The defect depends only on the antisymmetric fine mode $x_1 - x_2$, precisely the information erased by averaging. This is a concrete model of a mixed residual exposing structure invisible at coarse resolution.

22.3 Exact symmetry under coarsening

For dyadic vectors, pairwise averaging commutes with reversal up to reversal of the coarse vector. This gives an exact exchange law and hence a zero mixed residual. The contrast with the previous example illustrates that residuals depend jointly on the transformation and the resolution operator.

23 Derived invariants for the zoo

The two-axis formalization suggests new quantities that can be defined without changing the underlying category theory.

23.1 Commutation depth

Fix $X \in \mathbf{T}_s$. First record the exact commutation profile

$$\mathcal{E}_A(X; s) = \{r \leq s : \Delta^A_{s,r}(X) \simeq 0\}.$$

No interval or monotonicity property is assumed. For $R = \mathbb{N}$ and finite s, a well-defined consecutive coarsening depth is

$$\kappa_A(X; s) := \max\{k \in \{0, \ldots, s\} : \Delta^A_{s,r}(X) \simeq 0 \text{ for all } s - k \leq r \leq s\}.$$

The set is nonempty because $\Delta^A_{s,s}(X) \simeq 0$. A larger value means that more consecutive coarsenings preserve this exchange. The preceding version's supremum over exact target levels was uninformative on a fixed finite tower because the identity level $r = s$ is always exact.

23.2 Residual complexity

When a numerical size function $\| \cdot \|$ is available on the fiber, define

$$\mathcal{E}_A(s, r; X) := \|\Delta^A_{s,r}(X)\|.$$

Possible choices include dimension, rank, norm, entropy, categorical amplitude, support size, or proof-complexity measures.

23.3 Resolution branching profile

For a chain $r_0 < r_1 < \cdots < r_n$ and chosen lifts $X_i \in \mathbf{T}_{r_i}$ with $C_{r_{i+1} \to r_i} X_{i+1} \cong X_i$, record

$$B_X = (b_{r_0 \to r_1}(X_0), \ldots, b_{r_{n-1} \to r_n}(X_{n-1})).$$

This profile measures how hidden multiplicity emerges under refinement.

23.4 Zoo growth spectrum

Instead of a single growth function, separate growth by generator type:

$$g_T(n), \qquad g_R(n), \qquad g_\Delta(n).$$

This may distinguish theories in which most complexity comes from representation changes, resolution branching, or residual generation.

24 Candidate axioms for a foundational version

The present note derives its first results from ordinary category and order theory. A stronger foundational program would isolate Komorebi-specific axioms.

Axiom 24.1 (Two-axis presentation)**.** Every admissible structured state is represented in a system

$$\pi : \mathbf{K} \to \mathbf{R}$$

whose fibers encode transformations and whose base encodes resolution.

Axiom 24.2 (Resolution compositionality)**.** Coarsening is functorial:

$$C_{t\to r} = C_{s\to r}C_{t\to s}.$$

Axiom 24.3 (Exchange data)**.** Every designated transformation family carries a coherent comparison

$$A_r C_{s\to r} \Longrightarrow C_{s\to r}A_s.$$

Axiom 24.4 (Residual closure)**.** Every noninvertible exchange comparison generates an admissible residual object that may enter the seed zoo.

Axiom 24.5 (Effective reconstruction, domain-specific)**.** For a specified class of local systems, vanishing global obstruction is equivalent to effective global reconstruction.

These are axioms for a class of structures, not proposed additional axioms of ZFC. Two-axis existence alone imposes no substantive restriction: every category $\mathcal{C}$ has the degenerate presentation over the one-object resolution category with fiber $\mathcal{C}$. A foundational claim needs nontrivial, semantically specified resolutions and a theorem relating them to the intended mathematics.

The last axiom should not be imposed universally without justification. In serious applications it should be a theorem of a sheaf, stack, patching, or descent theory.

25 Machine-native semantics

The seed calculus can be interpreted as a search grammar. A machine state may contain

$$\mathsf{S} = (\Sigma, \mathscr{T}, \mathscr{R}, \Delta, \Omega, \mathsf{Cert}).$$

For a target problem P, search can occur jointly over transformation and resolution:

$$(T^*, R^*) \in \arg\min_{T,R} \mathcal{C}\big(\Omega(T, R; P)\big),$$

where $\mathcal{C}$ is a problem-dependent residual or obstruction cost. This expression describes a search objective. It does not assert that a minimizer exists or that the search is decidable.

This changes the computational question from

"Which operation solves the problem?"

to

"At which resolution and in which representation does the obstruction become simplest?"

A generic discovery loop is:

seed → generate → normalize → measure residual → promote residual → expand zoo → certify.

Formal proof assistants can serve as the certificate layer without requiring the generative search process itself to be trusted.

26 Universal constructions: syntax and stable residual completion

This section addresses the first foundational novelty test. The result is deliberately split into a syntactic theorem, which is standard double-category machinery, and a residual-completion theorem, which is a formal consequence of the free stable envelope of an additive category. The combination gives a rigorous universal construction for the additive/stable version of the seeded two-axis calculus. It does *not* by itself establish that the construction is absent from the literature under another name.

26.1 Free seeded two-axis double syntax

Definition 26.1 (Seeded two-axis double computad)**.** A *seeded two-axis double computad* $\mathbb{S}$ consists of a small double computad together with:

1. a distinguished set $\Sigma \subseteq \mathbb{S}_0$ of seed objects;
2. a designated class of horizontal generating 1-cells, interpreted as transformation generators;
3. a designated class of vertical generating 1-cells, interpreted as resolution generators;
4. designated generating 2-cells, interpreted as exchange or compatibility cells.

Generator boundaries may be well-typed words, but no composition of the generators is identified with another generator at the signature level. An unlabelled realization of $\mathbb{S}$ in a strict double category $\mathbb{D}$ is a map of the underlying double computads. A label-preserving realization additionally requires matching distinguished subsets in the target and preservation of those designations.

The label data are algebraically passive: they distinguish generators but impose no new equations. Hence the ordinary free-double-category construction applies unchanged.

Theorem 26.2 (Free seeded two-axis double category)**.** *For every small seeded two-axis double computad $\mathbb{S}$, there exists a strict double category $\mathbb{D}_{\mathbb{S}}$ and a canonical map of seeded double computads*

$$\iota_{\mathbb{S}} : \mathbb{S} \longrightarrow U\mathbb{D}_{\mathbb{S}}$$

such that for every strict double category $\mathbb{D}$, restriction along $\iota_{\mathbb{S}}$ induces a natural bijection

$$\mathrm{DblCat}(\mathbb{D}_{\mathbb{S}}, \mathbb{D}) \cong \mathrm{DblCptd}(\mathbb{S}, U\mathbb{D}),$$

for the underlying computadic data. If the target carries distinguished subsets for the same labels, restriction gives the analogous bijection with label-preservation imposed on both sides.

Proof. Form horizontal and vertical words in the generating arrows. Form square terms from the generating squares and identity squares by every well-typed horizontal and vertical composition. Quotient the terms by the congruence generated by units, associativity, and interchange. These relations give a strict double category. Every interpretation of the generating data extends by evaluation of terms, and evaluation respects the congruence because the target satisfies the same axioms. Its values on generators force uniqueness. Labels merely restrict the allowed generator assignments when corresponding target labels have been specified. □

Status: proved by translation to standard free double-category machinery.

This establishes a genuine universal property for the *two-axis syntax*. It is not a novelty claim: free double categories and double computads are established constructions; see Fairbanks–Shulman [1].

26.2 Additive indexed systems

The residual layer becomes canonical after passing to stable categories. Let $\mathbf{Cat}^{\mathrm{add}}$ denote the ∞-category of small additive ∞-categories and additive functors, and let $\mathbf{Cat}^{\mathrm{st}}$ denote small stable ∞-categories and exact functors. Ordinary additive categories are included by their nerves. The forgetful functor retains the underlying additive ∞-category, not merely its homotopy category.

Definition 26.3 (Additive two-axis system)**.** An *additive two-axis system* over a small resolution category $\mathbf{R}$ is a functor

$$F : \mathbf{R}^{op} \longrightarrow \mathbf{Cat}^{\mathrm{add}}.$$

Thus every fiber $F(r)$ is additive and every coarsening functor $C_{s\to r}$ is additive.

For a small additive category $\mathcal{A}$, write $\mathrm{St}^{\mathrm{add}}(\mathcal{A})$ for its free stable envelope. We use the established adjunction

$$\mathrm{St}^{\mathrm{add}} : \mathbf{Cat}^{\mathrm{add}} \rightleftarrows \mathbf{Cat}^{\mathrm{st}} : U,$$

characterized by

$$\mathrm{Fun}^{\mathrm{ex}}\big(\mathrm{St}^{\mathrm{add}}(\mathcal{A}), \mathcal{C}\big) \simeq \mathrm{Fun}^{\oplus}(\mathcal{A}, U\mathcal{C})$$

for every stable $\mathcal{C}$; see Saunier–Winges [2].

Theorem 26.4 (Pointwise stable completion)**.** *Let $\mathbf{R}$ be small. Postcomposition with $\mathrm{St}^{\mathrm{add}}$ defines a functor*

$$\mathrm{St}^{\mathrm{add}}_{\mathbf{R}} : \mathrm{Fun}(\mathbf{R}^{op}, \mathbf{Cat}^{\mathrm{add}}) \longrightarrow \mathrm{Fun}(\mathbf{R}^{op}, \mathbf{Cat}^{\mathrm{st}}), \qquad F \longmapsto \mathrm{St}^{\mathrm{add}} \circ F.$$

It is left adjoint to pointwise forgetting

$$U_{\mathbf{R}} : \mathrm{Fun}(\mathbf{R}^{op}, \mathbf{Cat}^{\mathrm{st}}) \longrightarrow \mathrm{Fun}(\mathbf{R}^{op}, \mathbf{Cat}^{\mathrm{add}}).$$

Equivalently, for every additive two-axis system F and stable two-axis system G there is a natural equivalence

$$\mathrm{Map}\big(\mathrm{St}^{\mathrm{add}}_{\mathbf{R}} F, G\big) \simeq \mathrm{Map}\big(F, U_{\mathbf{R}} G\big).$$

Proof. Any adjunction $L \dashv U$ induces an adjunction on functor categories by postcomposition. Concretely, a natural transformation

$$\alpha : L \circ F \Rightarrow G$$

has pointwise adjoint mates

$$\alpha^{\flat}_r : F(r) \longrightarrow U(G(r)).$$

Naturality of the unit and counit implies that the family $\{\alpha^{\flat}_r\}_r$ is natural in r. The mate construction is invertible and respects higher coherence. Applying this to $L = \mathrm{St}^{\mathrm{add}}$ gives the result. □

Status: proved.

This is the precise sense in which the stable version of the two-axis system has a *free residual-complete envelope*: no extra choices are required to create finite cofibers in the fibers, and every additive map into a stable two-axis system extends essentially uniquely to an exact one.

26.3 Exchange data and canonical residuals

Definition 26.5 (Coherent additive transformation family)**.** Let $F : \mathbf{R}^{op} \to \mathbf{Cat}^{\mathrm{add}}$. A coherent additive transformation family A consists of additive endofunctors

$$A_r : F(r) \to F(r)$$

and, for each $r \le s$, a natural transformation

$$\eta^A_{s,r} : A_r C_{s\to r} \Longrightarrow C_{s\to r} A_s,$$

satisfying the identity and transitivity coherences for chains $r \le s \le t$.

The universal property above is an equivalence of functor ∞-categories, not only of their cores. This distinction is essential for noninvertible exchange maps. To extend $\eta : f \Rightarrow g$ between additive functors $\mathcal{A} \to \mathcal{B}$, compose it with $j_{\mathcal{B}} : \mathcal{B} \to \mathrm{St}^{\mathrm{add}}(\mathcal{B})$ and lift the resulting morphism through

$$\mathrm{Fun}^{\mathrm{ex}}(\mathrm{St}^{\mathrm{add}}(\mathcal{A}), \mathrm{St}^{\mathrm{add}}(\mathcal{B})) \simeq \mathrm{Fun}^{\oplus}(\mathcal{A}, \mathrm{St}^{\mathrm{add}}(\mathcal{B})).$$

Full faithfulness transports the supplied composition coherences as well. Accordingly A_r, $C_{s\to r}$, and their coherently supplied exchange cells extend, essentially uniquely relative to the input, to exact functors and natural transformations

$$\widehat{A}_r, \qquad \widehat{C}_{s\to r}, \qquad \widehat{\eta}^A_{s,r} : \widehat{A}_r \widehat{C}_{s\to r} \Longrightarrow \widehat{C}_{s\to r} \widehat{A}_s.$$

Definition 26.6 (Universal stable mixed residual)**.** For $X \in \mathrm{St}^{\mathrm{add}}(F(s))$, define

$$\Delta^A_{s,r}(X) := \mathrm{Cofib}\Big(\widehat{\eta}^A_{s,r,X}\Big) \in \mathrm{St}^{\mathrm{add}}(F(r)).$$

Theorem 26.7 (Universal residual preservation)**.** *Let $G : \mathbf{R}^{op} \to \mathbf{Cat}^{\mathrm{st}}$ be a stable two-axis system and let*

$$\Phi : \mathrm{St}^{\mathrm{add}}_{\mathbf{R}} F \longrightarrow G$$

be an exact natural transformation which intertwines the chosen transformation families and exchange cells. Then for every $r \le s$ and every X there is a canonical equivalence

$$\Phi_r(\Delta^A_{s,r}(X)) \simeq \mathrm{Cofib}\Big(\Phi_r(\widehat{\eta}^A_{s,r,X})\Big).$$

In particular, if the exchange data in G are denoted by η^B, then

$$\Phi_r(\Delta^A_{s,r}(X)) \simeq \Delta^B_{s,r}(\Phi_s X).$$

Proof. Each Φ_r is exact, hence preserves finite colimits and therefore cofiber sequences. Apply Φ_r to the defining cofiber sequence

$$\widehat{A}_r \widehat{C}_{s\to r} X \xrightarrow{\widehat{\eta}^A_{s,r,X}} \widehat{C}_{s\to r} \widehat{A}_s X \longrightarrow \Delta^A_{s,r}(X).$$

The result is a cofiber sequence in $G(r)$. Compatibility of Φ with the transformation and exchange data identifies its first morphism with the target exchange morphism, giving the stated equivalence. □

Status: proved.

Thus residual formation is not merely available in the stable completion: it is *universal under exact realization*.

26.4 Residual promotion is forced by stable closure

Definition 26.8 (Stable seeded zoo)**.** Fix seed sets $\Sigma_r \subseteq \mathrm{Ob}(F(r))$. A *stable seeded sub-zoo* of $\mathrm{St}^{\mathrm{add}}_{\mathbf{R}} F$ is a family of full stable subcategories

$$\mathcal{Z}_r \subseteq \mathrm{St}^{\mathrm{add}}(F(r))$$

such that:

1. $\mathcal{Z}_r$ contains the images of all seeds in Σ_r;
2. $\widehat{A}_r(\mathcal{Z}_r) \subseteq \mathcal{Z}_r$ for every designated transformation generator;
3. $\widehat{C}_{s\to r}(\mathcal{Z}_s) \subseteq \mathcal{Z}_r$ for every $r \leq s$.

The *minimal stable zoo* is the intersection of all such families.

Proposition 26.9 (Existence of the minimal stable zoo)**.** *The minimal stable zoo exists.*

Proof. The family of entire fibers $\mathrm{St}^{\mathrm{add}}(F(r))$ is a stable seeded sub-zoo, so the defining class is nonempty. Arbitrary intersections of full stable subcategories are again full stable subcategories, and invariance under the designated exact functors is preserved under intersection. Hence the intersection is the least stable seeded sub-zoo. □

Theorem 26.10 (Automatic residual promotion)**.** *Let $\mathcal{Z}$ be the minimal stable zoo. If $X \in \mathcal{Z}_s$, then every mixed residual*

$$\Delta^A_{s,r}(X)$$

belongs to $\mathcal{Z}_r$.

Proof. By invariance,

$$\widehat{A}_r \widehat{C}_{s\to r} X \quad \text{and} \quad \widehat{C}_{s\to r} \widehat{A}_s X$$

are objects of $\mathcal{Z}_r$. Since $\mathcal{Z}_r$ is stable, it is closed under cofibers. Therefore the cofiber of the exchange morphism between these two objects, namely $\Delta^A_{s,r}(X)$, also lies in $\mathcal{Z}_r$. □

Status: proved.

Corollary 26.11 (Redundancy of residual promotion in the stable calculus)**.** *In the additive/stable two-axis calculus, "promote every residual to a seed" need not be imposed as an independent closure axiom. Once the zoo is stable and closed under transformation and coarsening, mixed residuals are already forced to lie in it.*

This is a substantive simplification of the initial axiomatization. Residual promotion remains meaningful operationally for search and provenance, but mathematically it is redundant in the stable closure.

26.5 What has and has not been proved

Combining theorems 26.2, 26.4, 26.7 and 26.10 gives the following restricted universal construction.

Theorem 26.12 (Additive/stable seeded two-axis completion)**.** *For a small additive indexed two-axis system with coherent additive transformation/exchange data and designated seeds, there is a canonical stable completion with the following properties:*

1. *the resolution diagram extends universally from additive fibers to stable fibers;*

2. *all coarsening and transformation functors become exact;*

3. *every exchange morphism has a canonical mixed residual given by a cofiber;*

4. *exact realizations preserve these residuals;*

5. *the least stable seed zoo exists and is automatically closed under mixed residual formation.*

The stable completion has the adjunction universal property for stable resolution systems receiving the original additive resolution diagram; with compatible exchange data, residual preservation is automatic for exact realizations.

Proof. Items (1)–(2) are theorem 26.4; item (3) is theorem 26.6; item (4) is theorem 26.7; item (5) is theorems 26.9 and 26.10. Initiality is the adjunction of theorem 26.4. □

Status: proved under the explicit additive/stable hypotheses.

The stronger nonadditive claim remains open. In particular, this theorem does not construct a free *arbitrary* residual-complete double category in which residuals satisfy a universal cofiber property without assuming an additive/stable target. It does, however, close the universal-construction problem for the regime in which the present mixed residual is actually defined as a cofiber.

26.6 Testing residual vanishing on orbit generators

The universal completion makes residuals available, but an application still needs a way to prove they vanish. Exactness gives a reduction to generators, with an important qualification: operation-orbit generators, rather than the original seed list alone, generally control the invariant zoo.

Lemma 26.13 (Residual formation is exact in its input)**.** *Let $P, Q : \mathcal{C} \to \mathcal{D}$ be exact functors between small stable ∞-categories, and let $\eta : P \Rightarrow Q$ be a natural transformation. Then*

$$\Delta_\eta(X) := \mathrm{Cofib}(\eta_X)$$

is an exact functor $\mathcal{C} \to \mathcal{D}$. If $q : \mathcal{D} \to \mathcal{O}$ is exact, the subcategory

$$\{X : q\Delta_\eta(X) \simeq 0\}$$

is thick.

Proof. Cofibers in the functor category are computed pointwise. The full subcategory of exact functors is closed under these finite colimits: a finite colimit of finite-colimit-preserving functors preserves finite colimits, and its value at zero is zero. Hence Δ_η is exact. The composite $q\Delta_\eta$ is exact as well. Its kernel is closed under shifts, cofibers, and retracts, which proves the last assertion. □

Definition 26.14 (Operation orbit of the seeds)**.** In a stable indexed system, let $\mathscr{P}$ be the designated exact transformations and coarsenings. For each r, let $\mathcal{G}_r$ contain every image at level r of a seed under a finite well-typed word in $\mathscr{P}$, including the empty word. Define

$$\mathcal{Z}_r^{\mathrm{thick}} := \mathrm{Thick}(\mathcal{G}_r).$$

Refinement is included here only when an actual exact refinement functor has been separately specified. A multivalued lift relation is not silently treated as an exact functor.

Proposition 26.15 (Orbit generation of the invariant thick zoo)**.** *The family $\{\mathcal{Z}_r^{\mathrm{thick}}\}_r$ is the least family of thick subcategories containing the seeds and preserved by all operations in $\mathscr{P}$.*

Proof. Every operation sends an orbit generator to an orbit generator at the target level. Exactness and preservation of retracts therefore imply that it sends the thick closure of the source orbit into the thick closure of the target orbit. Thus the displayed family is invariant. Any invariant thick family containing the seeds contains all finite operation words applied to them, hence contains all $\mathcal{G}_r$ and their thick closures. This proves minimality. □

Theorem 26.16 (Orbit-generator test for an exchange obstruction)**.** *Fix an exchange $\eta^A_{s,r}$ between exact functors, and an exact quotient q_r of the target fiber. If*

$$q_r\Delta^A_{s,r}(g) \simeq 0 \quad \textit{for every } g \in \mathcal{G}_s,$$

then

$$q_r\Delta^A_{s,r}(X) \simeq 0 \quad \textit{for every } X \in \mathcal{Z}_s^{\mathrm{thick}}.$$

The same assertion holds for $q_r = \mathrm{id}$, in which case the exchange is an equivalence on the entire generated thick zoo.

Proof. By theorem 26.13, the objects on which $q_r\Delta^A_{s,r}$ vanishes form a thick subcategory. By hypothesis it contains $\mathcal{G}_s$, hence it contains $\mathrm{Thick}(\mathcal{G}_s)$. Vanishing of the unquotiented cofiber is equivalent to invertibility of the exchange map. □

Status: proved by exactness and thick generation.

This reduces a universal statement on a thick zoo to a statement on its operation orbit. It is not automatically a finite algorithm. The orbit may be infinite, and checking only the initial seeds does not suffice unless a separate argument propagates vanishing along every operation. In particular, this theorem does not turn finitely many numerical checks into proof of a universal arithmetic comparison.

27 Research program and theorem ledger

ID	Target	Current status
K1	Resolution monotonicity and nested equivalence relations	Proved in theorem 2.5
K2	Grothendieck total category	Proved in theorem 3.2
K3	Two-axis factorization	Proved in theorem 3.3
K4	Resolution ultrametric	Proved with a common state set, universal zeroth relation, nesting, and separation.
K5	Least seed zoo / closure existence	Proved in theorem 5.4
K6	Finitary iterative zoo generation	Proved under continuity hypothesis in theorem 5.6
K7	Lifting coherent transformation families to the total category	Proved in theorem 6.2
K8	Mixed residual vanishing criterion	Proved in stable fibers in theorem 6.4
K9	Exact two-axis normal form	Proved under local confluence in theorem 7.3

ID	Target	Current status
K10	Split residual expansion	Corrected source remainder shift and finite principal-path scope in theorem 7.5.
K11	Stable residual filtration without splitness	Proved for a chosen path. General ordered independence is refuted by K24.
K12	Stable obstruction quotient	Proved for the thick kernel. Constructor descent needs K28 or the directed hypothesis K29.
K13	Free seeded two-axis double syntax	Term-construction proof with typed labels in theorem 26.2.
K14	Universal residual-complete two-axis construction	Proved with additive/stable infinity-category and coherent exchange data. Nonadditive form open.
K15	Effective reconstruction theorem for a nontrivial arithmetic instance	Lattice case proved in K51; the intended reciprocity instantiation remains open.
K16	Descent transfer across a correspondence	Abstract lemma proved in theorem 20.3
K17	Seeded local-global reciprocity instantiation	Program theorem, arithmetic hypotheses open
K18	Lean formalization of K1–K10	Not yet implemented in this note
K19	Pointwise stable completion universal property	Proved in theorem 26.4
K20	Automatic residual promotion in stable zoos	Proved in theorem 26.10
K21	Functorial transport of obstruction quotients	Proved in theorem 11.10
K22	Nonstable residual/repair obstruction test bed	Proved for abelian torsors in theorem 20.1
K23	Localization of total obstruction	Proved pathwise in theorem 11.11.
K24	Ordered filtration independence	General claim refuted in theorems 8.1 and 8.2.
K25	Semantic diamond criterion	Proved in theorem 8.3; total cofibers equivalent, not uniquely identified without more data.
K26	Pure exchange event poset	States are ideals and histories are linear extensions, theorem 8.6.
K27	Ideal-indexed residual diagram	Proved for coherent input; K32 constructs strict chain-natural instances.
K28	Constructor-invariant obstruction quotient	Proved in theorem 11.14; can be zero, theorem 11.12.

ID	Target	Current status
K29	Directed operational quotient	Killability is thick and constructor-invariant, theorem 11.15.
K30	Orbit-generator test for residual vanishing	Proved in theorem 26.16; orbit need not be finite.
K31	Noncollapse certificate	Exact equivariant detector criterion, theorem 11.16.
K32	Strict semantic event realization	Proved for typed pure swaps and strict chain-natural cells, theorem 9.2.
K33	Interaction residual and excision	Total-cofiber criterion, theorem 10.2.
K34	Context-independent event residuals	Equivalent to vanishing square interactions, theorem 10.3.
K35	Principal-ideal reconstruction	Reduced excisive diagrams classified by diagrams on P, theorem 10.5.
K36	Universal excisive quotient	Constructor-compatible minimal thick kernel, theorem 12.1.
K37	Tensor-cube residual formula	Proved by finite-colimit interchange and exact tensor, theorem 13.1.
K38	Integral gcd interaction formula	Explicit Koszul change of basis and cohomology, theorem 13.2.
K39	Prime-overlap kernel	Exact kernel and noncollapse for positive scalar labels, theorem 13.4.
K40	Finite-signature arithmetic certificate	All finite event diagrams in the stated scalar grammar, theorem 13.6.
K41	Integral matrix collapse criterion	Verified Smith data classify free collapse versus torsion support, theorem 13.7.
K42	Nonidentity arithmetic resolution tower	Derived reduction is exact and conservative, not faithful, theorem 14.1.
K43	Coherent weighted exchange cells	Explicit source of residuals and base-change transport, theorem 14.3.
K44	Prime-power interaction cohomology	Full finite-level Koszul cohomology, theorem 14.5.
K45	Nilpotent excision collapse	Nonunit pairs force the prime-power quotient to zero, theorem 14.6.

ID	Target	Current status
K46	Finite prime-component quotient	Exact quotient equivalence and tower coherence, theorem 14.7.
K47	Based tower reconstruction	Uniform bounded ranks and compatible matrices required, theorem 15.1.
K48	Transient negative residual classes	Nonzero finite groups can form a pro-zero system, theorem 15.2.
K49	Completion–repair comparison defect	Explicit generic-fiber formula and vanishing criterion, theorem 15.3.
K50	Arithmetic fracture reconstruction	Perfect integral complexes and their residuals, theorems 15.5 and 15.6.
K51	Effective arithmetic lattice gluing	Existence and uniqueness from specified generic identifications, theorem 15.7.
K52	Compatible operator gluing	Preserves supplied integral polynomial relations, theorem 15.8.
K53	Intrinsic kernel–reduction residual	Natural torsion boundary and transition law, theorem 16.1.
K54	Sharp finite-resolution lifting window	Smith profile, minimal delay, and pro-zero residuals, theorem 16.2.
K55	Finite-group invariant residual	Identified with $H^1[p^n]$, with group-order bound, theorem 16.4.
K56	Equivariant morphism lifting	Obstruction and finite-precision isomorphism test for supplied lattices, theorem 16.5.
K57	Norm-quotient lattice family	Natural non-scalar actions and sharp transient invariants, theorem 16.6.
K58	Perfect-base derived exchange	Exact commutation, not a new residual, theorem 17.1.
K59	Nonperfect-base intrinsic derived defect	Non-scalar action with unbounded cofiber and unit-loss boundary, theorems 17.2 and 17.4.
K60	One-step action-lifting obstruction	The finite cyclic action cannot lift from p^2 to p^3, theorem 17.5.
K61	Exact exchange on coefficient-lifted inputs	Necessary diagnostic; converse refuted by K64, theorem 17.6.
K62	Complete one-step relation obstruction	H^2 class vanishes exactly for a next-level lift, theorem 18.1.

ID	Target	Current status
K63	Repair ambiguity and certificates	H^1, H^0, and exact finite linear witnesses, theorem 18.2.
K64	Blind derived-invariant comparison	Central sign refutes sufficiency of the single exchange cone, theorem 18.3.
K65	Averaged quadratic repair	Exact associativity and explicit division loss, theorem 18.4.
K66	Convergent action repair	Fixed finite group and $m > 2v_p(\|G\|)$, theorem 18.5.
K67	Finite action-lifting horizon	Necessary and sufficient level $\max\{n+e, 2e+1\}$, theorem 18.6.
K68	Controlled integral conjugacy	Agreement beyond group-order valuation suffices, theorem 18.8.
K69	Finite integral action catalogue	Isomorphism classes, not an equivalence of groupoids, theorem 18.9.
K70	Cyclic relation Jacobian	Affine next-level certificate with d^2 variables, theorem 18.10.
K71	Fixed quotient versus profinite lifting	Explicit continuous lift with growing image, theorem 18.11.
K72	Continuous one-step obstruction	Finite discrete adjoint coefficients and continuous repair, theorem 19.1.
K73	Source-inflation criterion	Continuous vanishing occurs on some finer quotient, theorem 19.2.
K74	Continuous tower reconstruction	Vanishing H^2 or a compatible tower; finite-generation compactness, theorem 19.4.
K75	Free pro-p coefficient lifting	Arbitrary integral generator lifts extend continuously, theorem 19.5.
K76	Procyclic matrix powers	Exact residual p-power-order criterion, including $p = 2$, theorem 19.6.
K77	Cyclic-centralizer certificate	Finite polynomial description over a coefficient ring, theorem 19.7.
K78	Normalized cyclic lifting	Complete obstruction and explicit polynomial reconstruction, theorem 19.8.
K79	Cyclic-stratum horizon	Level $n + a$ suffices under the cyclic hypothesis, theorem 19.10.

ID	Target	Current status
K80	Complete rank-two profinite test	All continuous $\mathbb{Z}_p^r$ inputs, no extra deformation conditions, theorem 19.11.
K81	Sharp rank-two horizon	Level $2n-1$, optimal for $n, r \geq 2$, theorem 19.12.
K82	Mixed source–coefficient variance	Source inflation is not arbitrary source coarsening, theorem 19.15.

28 Immediate next formal steps

The general ordered-filtration independence target and the converse of the single derived-invariant test remain refuted. Fixed-finite-group lifting, continuous free pro-p lifting, and rank-two lifting for $\mathbb{Z}_p^r$ now have constructive solutions. Higher-rank abelian lifting is covered on the specified cyclic normalized stratum. Arbitrary profinite relations, geometric local conditions, and automorphic globalization remain separate obligations.

1. **Beyond strict pure exchange.** Extend K32 to a specified non-strict enhanced realization or a rewrite signature with additional critical overlaps. Identify the missing coherent cells explicitly rather than assume that homotopy-category diamonds supply them.
2. **Noncommutative and constrained lifting.** K72–K82 handle continuous source refinement and give complete rank-two $\mathbb{Z}_p^r$ reconstruction. Next specify a noncommutative profinite presentation or a noncyclic higher-rank centralizer. Construct its actual relation obstruction and a sufficient repair criterion. Prescribed determinants and geometric local conditions must be added as equations with their own compatibility checks. No uniform averaging bound over growing finite quotients is assumed.
3. **Arithmetic existence beyond lattices.** K51 closes a concrete lattice case of K15. For the intended K17 application, construct the generic arithmetic or automorphic data and the required identifications, rather than importing them as gluing hypotheses. A globally glued lattice with supplied operators is not an automorphic representation.
4. **Broader coefficient rings.** Determine which parts of the prime-support kernel and matrix criterion persist over a specified Dedekind ring or a higher-dimensional ring. A replacement for the integral Smith decomposition must be proved, not silently imported.
5. **Lean core.** Formalize the finite event-poset correspondence, strict naturality argument, closure statements, and integral certificate checks. This manuscript still has no Lean proof package.
6. **Beyond stability.** Develop another precisely typed nonadditive obstruction example beyond torsor sections. Do not identify a nonabelian obstruction set with a cofiber object without a comparison theorem.

29 Brainstorm: stronger directions

The formalism above suggests several deeper lines of investigation.

29.1 Resolution completion

A resolution tower may admit an inverse limit

$$X_\infty \simeq \varprojlim_r X_r.$$

One can ask whether the seed zoo commutes with completion:

$$\mathcal{Z}(X_\infty) \overset{?}{\simeq} \varprojlim_r \mathcal{Z}(X_r).$$

This is only a proposed comparison. A common category and an actual comparison map must first be constructed before a residual is defined.

29.2 Transformation completion

Repeated transformations may generate an orbit category. Completing this orbit under limits, colimits, idempotent splitting, or derived closure may produce a canonical transformation hull.

29.3 Bi-completion

There are then two completion operations:

$$\widehat{(-)}_T, \qquad \widehat{(-)}_R.$$

After both completions and the comparison have been defined in a common category, the mixed question is whether

$$\widehat{\widehat{X}_T}_R \simeq \widehat{\widehat{X}_R}_T.$$

Given the specified comparison, its defect is a candidate completion-level mixed residual in a setting where such defects are defined. The comparison is now explicit in theorem 15.3: even compatible finite-level repairs can fail to commute with completion, with a nonzero generic-fiber defect. An unconditional bi-completion claim is therefore false.

29.4 Resolution spectrum

For a fixed $X \in \mathbf{T}_s$ and a specified detection predicate P_r at each level $r \le s$, define:

$$\mathrm{Spec}_R(P; X) = \{r \le s : P_r(C_{s\to r}X)\}.$$

This could behave like a support or spectrum for distinguishability.

29.5 Residual genealogy

Residual promotion creates a directed graph

$$s \to \Delta_1 \to \Delta_2 \to \cdots.$$

Classify whether genealogies terminate, become periodic, stabilize, or grow indefinitely. A terminating genealogy would correspond to finite obstruction resolution. Periodic genealogy may encode a persistent but structured defect.

29.6 Seed minimality

Given a generated zoo $\mathcal{Z}$, ask for the smallest seed set Σ with

$$\mathcal{Z}(\Sigma) = \mathcal{Z}.$$

This is a mathematical compression problem. Nonuniqueness of minimal seeds could itself be meaningful.

29.7 Dual zoos

If the calculus carries a duality D, study

$$D\mathcal{Z}(\Sigma) \quad \text{versus} \quad \mathcal{Z}(D\Sigma).$$

The residual of dualization against zoo generation measures whether the generative grammar is duality-complete.

29.8 Proof-resolution

A theorem itself may have resolutions. At coarse resolution one stores only a proposition and certificate. At finer resolution one exposes lemmas, dependencies, computational witnesses, and failed branches. This creates a resolution geometry on proofs and may be useful for machine reasoning and proof compression.

29.9 World-resolution

A foundational extension can assign resolution to models or "worlds" themselves. Morphisms between worlds then have two axes: change of world and change of resolution within a world. This may connect the current calculus to the earlier compatibility/residual/obstruction maps between models of a foundational theory.

30 Provisional philosophical interpretation

The mathematics does not require the following interpretation, but the two-axis formalism sharpens it.

A structure may be characterized simultaneously by:

- what transformations it admits;
- what resolutions distinguish its internal states;
- what survives both kinds of movement;
- what residuals appear when the two directions fail to commute;
- what obstructions prevent refinement, transformation, or global reconstruction.

This suggests the strengthened ontological phrase:

to remain identifiable is to remain coherent under both transformation and resolution.

The mathematical content of the project is not this sentence itself. The mathematical content is the attempt to make "coherent," "transformation," "resolution," "residual," and "reconstruction" precise enough that such a sentence can be tested rather than merely asserted.

31 Conclusion

The first formalization supports a concrete picture:

> seeds generate states,
> transformations generate representations,
> resolutions generate distinguishability,
> exchange maps relate the axes,
> residuals measure failed exchange,
> residuals become new seeds,
> obstructions classify persistent failure,
> descent reconstructs compatible local structure.

The retained categorical core supplies explicit constructions under fixed hypotheses. Stable completion provides functorial cofibers, exact realizations preserve residuals, and the invariant stable zoo contains them automatically. Each normalization path carries a residual filtration, but the filtration is genuinely history-sensitive. The corrected intrinsic object is an ideal-indexed residual diagram for a specified coherent exchange system, whose terminal residual is independent of schedule.

Likewise, a thick obstruction quotient is not automatically a new environment for the original constructors. Their action on the quotient requires an invariant kernel. The invariant saturation gives the universal repair, but its possible collapse is a substantive constraint on admissibility. Directed common postcomposition makes operational killability itself invariant and thick, while exact equivariant detectors certify noncollapse. Finally, thick orbit generation reduces exchange vanishing to a well-defined generator test without promising that the test is finite.

The next layer is now explicit. Strict chain-natural cells construct semantic exchange diagrams. Their second cofibers measure excision, and vanishing of those interaction objects makes event residuals independent of unrelated prior events. Principal-ideal reconstruction retains exactly the poset diagram needed to assemble the whole residual history. In the integral models, common prime support identifies what must be removed, while free cofiber components identify a genuine collapse risk. These are concrete semantic and arithmetic results, not a conclusion drawn solely from the breadth of the vocabulary.

The arithmetic tower sharpens that boundary further. Finite resolution can turn a nonzero nilpotent interaction into a generator of the whole perfect category. Several prime components permit nontrivial excisive quotients, but deleting a component before completion loses a generic fiber. Retaining the localized and completed branches with their overlap instead gives a proved arithmetic fracture square and an effective lattice-gluing theorem. These use established algebraic mechanisms to make the calculus concrete.

The intrinsic exchange layer now distinguishes degree-zero information loss from a genuine failure between fully derived exact functors. Residual torsion gives a sharp finite lifting window for fixed integral constraints. A nonperfect coefficient change can instead expose an unbounded defect and a genuine missing representation lift. Exact comparison is restored on supplied coefficient-lifted inputs under the stated perfectness hypotheses, without taking a quotient that erases the coefficient unit.

The sufficient-lifting layer now adds an explicit distinction between detection and construction. The adjoint relation cocycle is a complete next-level obstruction, whereas derived invariants can be blind to nonliftability. For a fixed finite group, quantitative averaging converts a finite high-precision witness into an integral representation and gives controlled uniqueness up to conjugacy. The stated threshold is known in ultrametric stability, and the direct proof here makes its relation to residual repair explicit. It does not justify substituting fixed finite-group existence for a continuous Galois or automorphic existence problem.

The continuous-source layer now separates coefficient precision from source inflation. A finite-quotient obstruction can disappear when more of the source is allowed to act, but an actual commutativity relation cannot be discarded. Scalar-depth normalization exposes that relation, and a cyclic centralizer constructs compatible integral matrices from finite polynomial data. In rank two the cyclic condition is automatic after normalization, yielding a complete test and the sharp $2n-1$ horizon. This proves reconstruction without eliminating the coefficient category, while the mixed-variance diagram states exactly how the two resolution coordinates interact.

The results do not establish a new foundation or a reciprocity theorem. The remaining arithmetic content requires actual generic parameters, compatible actions, and the specific existence theorem demanded by the intended correspondence. Neither a torsor trivialization, a reconstructed event diagram, nor a glued integral lattice is an automorphic realization by definition.

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A Compact definition sheet

$$\begin{aligned}
\text{Resolution system:} &\quad F : \mathbf{R}^{op} \to \mathbf{Cat}.\\
\text{Fiber:} &\quad \mathbf{T}_r = F(r).\\
\text{Coarsening:} &\quad C_{s\to r} : \mathbf{T}_s \to \mathbf{T}_r.\\
\text{Total category:} &\quad \mathbf{K} = \int_{\mathbf{R}} F.\\
\text{Resolution equivalence:} &\quad X \sim_r Y \iff C_{t\to r}X \cong C_{t\to r}Y.\\
\text{Lift set:} &\quad \mathrm{Lift}_{r\to s}(X) = \{Y : C_{s\to r}Y \cong X\}/\cong.\\
\text{Exchange map:} &\quad \eta^{A}_{s,r} : A_r C_{s\to r} \Rightarrow C_{s\to r} A_s.\\
\text{Stable residual:} &\quad \Delta^{A}_{s,r}(X) = \mathrm{Cofib}(\eta^{A}_{s,r,X}).\\
\text{Stable obstruction quotient:} &\quad \mathcal{O}_{\mathscr{M}} = \mathcal{Z}/\,\mathrm{Thick}(\mathcal{K}_{\mathscr{M}}).
\end{aligned}$$

Invariant obstruction quotient: $\mathcal{O}^{\mathrm{inv}}_{\mathscr{M}} = \mathcal{Z}/\mathcal{J}_{\mathscr{M}}$.

Event residual diagram: $\mathcal{F}_X(I) = \mathrm{Cofib}(X_\varnothing \to X_I)$.

Obstruction class: $\mathrm{Obs}_{\mathscr{M}}(X) = q(X)$.

Seed zoo: $\mathcal{Z}(\Sigma_0) = \mu A.(\Sigma_0 \cup \Phi(A))$.

Resolution ultrametric: $d_R(X,Y) = 2^{-\nu(X,Y)}, \qquad 2^{-\infty} = 0.$

Reconstruction target: $\Omega_G = 0 \iff$ the specified descent datum is effective.

B Preserved audit and verification record for v0.6.0

This section is a historical audit record. The current continuation is recorded in section F; the intervening version records retain their original scopes.

B.1 Coverage and evidence

The v0.5.0 LaTeX source was read in full. This release preserves the running manuscript and revises its statements and proofs rather than replacing it with a shortened summary. The review is an authoring-session self-review, not an independent referee report. The exact source snapshot, change inventory, executable tests, and output are included in the companion package.

The primary audit verdict for the former package is *correct only after stated restrictions and corrections*. In particular, the ordered-filtration ambition fails in general, whereas the retained path-relative theorem survives. Stable quotient existence survives, but it alone does not prove constructor descent or noncollapse. No Lean verification is claimed.

B.2 Mathematical corrections

1. **Resolution:** fixed a common state set and a universal zeroth equivalence relation in the ultrametric construction. Distinguished isomorphism classes of lifts from framed lift groupoids. Excluded $\log 0$ from ordinary branching information.
2. **Exchange:** distinguished invertible comparison maps from exact functors. Required complete local exchange rules for the syntactic normal-form statement. Replaced the split cofiber identification by the correct shift $\Delta \simeq E[1]$. Removed an unsupported recursive termination claim for residual branches.
3. **Coherence:** separated a normal object, its comparison map, and a filtration. Noninvertible exchange cells are extended using the full functor-category stable-envelope universal property, not just a mapping-space adjunction.
4. **Quotients:** distinguished killability, thick killability, and constructor-invariant thick killability. Stated universal uniqueness in the enhanced categorical sense. Corrected the postcomposition orientation in the directed condition.
5. **Descent and scope:** included all overlap degrees and descent of morphisms. Required reflection as well as preservation of obstruction vanishing. Corrected the quotient functor's type. Kept arithmetic globalization and unrestricted reciprocity outside the proved scope.
6. **Derived quantities:** replaced an identity-dominated commutation-depth formula and typed the branching profile along an actual lift chain. Marked zoo growth and search costs as

presentation-dependent. No minimum or decision procedure is asserted by an optimization display.

B.3 Exact finite regression

The companion script `checks/check_models.py` uses integer matrices, rational Gaussian elimination, and finite labelled words. It contains no floating-point tests. The recorded run uses Python 3.13.5 and checks every two-sort word of length 0 through 10, including the empty word and words containing only one sort.

Quantity	Checked count
Initial words	2,047
Reachable word states, summed over initial words	82,499
Elementary rewrite edges, summed over initial words	168,454
Largest event poset	25 events
Products in the three-constructor example	9
Binary supports in the three-coordinate example	8
Exact rank checks for cone models	6

The word test constructs reachable labelled words and order ideals independently, then compares their states and edges. The constructor test confirms the non-thick killability and invariant-collapse examples, together with a nonzero directed projection model. The cone tests compare the ordered dimensions $(0, 1, 3)$ and $(0, 2, 3)$, verify the split projection's shifted cofiber, and distinguish the identity and zero maps with equal endpoints.

All these finite checks passed. They test the stated finite models and implementation paths. The universal statements rest on their written proofs, not on extrapolation from the finite enumeration. The package's result record stores the exact script hash and bounds. Build and visual-review results are recorded separately from the mathematical test status.

B.4 Source verification boundary

The stable-envelope input was checked in Saunier–Winges, arXiv:2502.03408v2, Definition 3.1 and the subsequent functor-category universal property. The stable localization input was checked in Drew, arXiv:1511.08287v1, Definition 1.1 and Theorem 1.3. The quotient kernel and triangulated formulations were checked in the cited Stacks sections. The free strict syntax theorem now has its own term-construction proof. This was an exact-dependency check, not a comprehensive novelty search.

The manuscript does not certify a new axiom of set theory, a classification of all mathematical structures, a physical law, or a Langlands reciprocity theorem. Its new claims concern the explicitly defined exchange grammar and stable constructor systems.

C Audit and verification record for v0.7.0

C.1 Scope, proof status, and provenance

The source base is the supplied v0.6.0 manuscript and its complete package. This release preserves that manuscript, integrates new proof blocks, and updates the abstract, theorem ledger, and next obligations. It does not claim a fresh independent audit of every retained earlier theorem. The new proofs received an authoring-session self-review. No independent referee review or Lean compilation was performed.

The three executed routes were strict semantic realization, interaction/excision reconstruction, and integral quotient classification. They produced K32–K41 under their written hypotheses. The arithmetic test bed uses perfect complexes over $\mathbb{Z}$ and explicit finite integer data. It does not discharge a Langlands-facing existence hypothesis.

C.2 Adversarial boundaries

1. Strict naturality supplies the event diagram only for the specified pure grammar. Weak cells and additional rewrite relations retain separate coherence obligations.
2. A square's commutativity is not its excision. The second cofiber vanishes precisely for a homotopy pushout, not for every commuting square.
3. Context-independent event residuals do not identify ordered filtrations. Nor do they split extensions along dependent events: the $\mathbb{Z}/p^2$ chain is an explicit counterexample.
4. The total normalization residual $C(N)$ and a cube's total cofiber $K(n_1, \ldots, n_m)$ are different objects. The $(6, 10, 15)$ example refutes an inference from highest interaction vanishing to all lower interactions vanishing.
5. The integral noncollapse theorem uses positive scalar labels. Two zero labels give an interaction with a free unit summand and collapse the entire quotient. The matrix criterion identifies the analogous free-cofiber failure.
6. Equality of the kernel with that of localization is proved. No unproved equivalence of complete quotient categories or idempotent completions is substituted for it.

C.3 Exact finite checks

Both regression scripts ran successfully on Python 3.13.5. They use exact integer identities and finite-field elimination, with no random sampling or numerical tolerance. The inherited check was rerun with every two-sort word through length 10: 2,047 initial words, 82,499 states, and 168,454 edges. Its constructor and filtration counterexamples also passed.

The new check covers the following explicitly bounded populations.

Check population	Count
Positive label vectors, arity 1–4, entries 1–6	1,554
Integral exterior-power chain-map identities	5,910
Integral differential-square identities	4,356
Cohomology dimensions over $\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_5, \mathbb{F}_7$	29,856
Initial scalar two-sort words through length 8	511
Ideal states in those scalar diagrams	6,917
Cover arrows and eligible event contexts	10,660
Elementary independent-event squares	4,931
Explicit localized context-map inverses	5,019
Ideal-pair combinatorial excision checks, word length at most 6	5,289
Ordered pairs from an explicit 11-matrix test family	121
Matrix-pair cohomology dimensions over five finite fields	1,815

The matrix family includes identity, zero, rank-deficient, triangular, and torsion examples. Its 121 pairs are exhaustive for that listed family, not for all integer matrices. Nine pairs have two free cone components. Additional signed/zero-vector checks, the excluded all-zero pair, and the named $(6, 10)$, $(6, 10, 15)$, and nonsplit prime-chain examples are recorded separately. No primary case within the stated bounds was skipped.

The integral normal-form checks verify unimodular exterior-power chain isomorphisms. Finite-field ranks provide a distinct check on signs and degree dimensions. These are reproducible implementation checks. They neither instantiate every stable category nor replace the written universal proofs.

C.4 Sources and remaining review

The exact source check used Stacks Tags 014D, 0621, and 05RA, and Drew's arXiv:1511.08287v1, Definition 1.1 and Theorem 1.3. The source record separates the homological Koszul convention from the cohomological convention used here. Direct finite-colimit, lattice, and integer arguments are supplied for the new deductions. This was not a comprehensive novelty search. The companion audit contains the dependency map, script hashes, source record, build logs, and visual-review record.

Independent mathematical review remains outstanding. K15, K17, K18, arbitrary higher-coherent exchange signatures, and the nonadditive universal construction remain open within this project.

D Audit and verification record for v0.8.0

D.1 Scope and proof boundary

The supplied v0.7.0 PDF, source, and full package are preserved unchanged. Their hashes are recorded in the release. This continuation adds K42–K52, updates the restricted lattice status of K15, and retains the earlier open reciprocity obligations. The new proofs have received an authoring-session self-review, not an independent referee review or Lean certification. No fresh complete audit of all inherited theorems is claimed.

The three executed mathematical routes were derived coefficient towers, excision versus completion, and arithmetic reconstruction with retained overlaps. The first produces explicit finite-level interactions. The second refutes unconditional commutation of repair and completion and proves a generic-fiber criterion. The third proves a fracture square and effective lattice gluing, with compatible operator descent.

D.2 Adversarial checks

1. Derived reduction is exact, but naive tensor on arbitrary modules is not. The infinite periodic resolution over $\mathbb{Z}/p^2$ prevents replacing a perfect two-term cone by its degree-zero cohomology.
2. Canonical tensor/base-change exchanges have zero residual. Nonzero scalar cells are specified additional data, not an unexplained consequence of coarsening.
3. Prime-power interactions can generate the whole perfect category even for nonzero labels. A multi-prime quotient removes whole components and can still collapse if every prime is selected.
4. Coherent finite reductions do not imply commutation with an infinite inverse limit. The field factors in K49 exhibit the precise missing information.
5. The based tower theorem requires common bounded ranks and compatible matrices. Arbitrary incompatible level choices are not claimed to lift.
6. The fracture square reconstructs a supplied complex. K51 separately proves effective existence for independent lattice data, but requires the generic identifications explicitly. Operator gluing requires exact intertwining, not only matching characteristic polynomials.
7. Generic-fiber defects are computed in $D(\widehat{B})$, not asserted to be perfect objects over the completed ring. Their cofiber is the [1] shift of the stated fiber.

D.3 Reproducible finite checks

All three check scripts passed in this session with Python 3.13.5. The inherited two scripts reproduce their earlier ranges. The new script additionally uses SymPy 1.14.0 for integer Hermite normal forms; each resulting lattice basis is checked by its exact finite residues and index. There is no random sampling or numerical tolerance.

The new suite checks all 1,080 ordered coefficient pairs in the eight listed rings $\mathbb{Z}/q$ with $q \in \{2, 3, 4, 5, 8, 9, 16, 25\}$. It enumerates kernels and images to check three cohomology degrees and verifies middle-cohomology elementary divisors by torsion counts. It also checks 160 annihilator-transition tuples, 1,728 exchange coherences, 72 prime-component projections, 2,304 positive-label pair instances, all 30 pairs of subspaces of $\mathbb{F}_2^2$ and $\mathbb{F}_3^2$, 2,430 operator-preservation instances, 6,100 rank-one rational lattice membership tests, and 972 principal-part reconstruction witnesses. Exact bounds and nonclaims are in the JSON report.

These are finite regression checks, not computational proofs of universal categorical statements or assertions about every p-adic input. The limiting and gluing theorems have separate written proofs. The arithmetic-square argument and nilpotent-cone argument use classical mechanisms; no historical novelty is inferred from their packaging here.

D.4 Build and source verification

The standalone source is regenerated from the preserved base and the two new proof fragments by `audit/integrate_v080.py`. The source diff, run reports, source checks, build log, and rendered-page audit accompany the release. The first general web search returned irrelevant results and was not used as evidence; the cited primary Stacks Project pages were then opened directly. No external source text or font files are bundled. The lack of a Lean executable in this runtime is recorded as an environment limitation, not a mathematical failure.

E Audit and verification record for v0.9.0

E.1 Authority and evidence status

The supplied v0.8.0 PDF, LaTeX source, and full package are preserved unchanged in the new package. Their hashes identify the base. The continuation adds K53–K61, with written proofs and an authoring-session self-review. It is not a new independent audit of every inherited theorem. There is no Lean proof package and no claim of historical novelty.

Three routes were executed. The kernel and invariants route produces explicit coefficient boundaries and a sharp lifting window. The fully derived regular-base route proves exact commutation, ruling out a putative defect in that regime. The finite nonperfect-base route produces a genuinely nonzero canonical exchange and a separate one-step group-action obstruction. Transitivity of the canonical comparison then gives a positive theorem on coefficient-lifted inputs.

E.2 Proof boundaries checked

1. The boundary uses torsion-freeness of the source lattices and of the target of the linear constraint. The residual at precision n is an annihilator group, with transition multiplication by p^{m-n}, not an arbitrarily identified constant system.
2. Sharp lifting delay concerns solutions for the same fixed integral constraints. It does not grant a lift of constraints or group actions that were supplied only at finite level.
3. Ordinary invariants are not an exact functor. Their cokernel residual is not silently substituted into an exact-functor theorem.
4. Fully derived invariants commute with a coefficient module perfect over the source. At finite singular coefficients that perfectness can fail. The exact counterexample is calculated in enhanced unbounded derived categories, not inside perfect fibers.
5. The $\mathbb{Z}/p^2$ example assumes p odd and J invertible modulo p. Explicit tests outside both hypotheses have nonzero positive cohomology and do not satisfy its asserted quasi-isomorphism.
6. The source of the derived comparison is computed from a bounded-above free resolution of its degree-zero invariant module. Tensoring the original nonnegative invariant complex termwise would give the wrong answer.
7. The vanishing degree-zero comparison follows from naturality of evaluation. The cofiber's cohomology in all integer degrees follows from its long exact sequence, not extrapolation of a truncated numerical calculation.
8. Nonzero derived exchange obstructs a coefficient lift. Its converse is not proved. The separate matrix relation calculation rules out the specified lift from level two to level three.
9. Killing the unbounded target residual kills every perfect k-complex. To conclude that it kills all of $D(k)$, the exact functor must also preserve coproducts.

E.3 Exact regression record

All four check scripts passed in this session. The three inherited scripts retain their source hashes and were rerun at their documented ranges. The new deterministic suite uses exact integers, residue enumeration, and verified integer Smith changes, with no random sampling or tolerance-based acceptance.

New check family	Count
All 2×2 matrices with entries in $[-2, 2]$	625
Matrix/precision cases for kernel and boundary	3,125
Vectors in those kernel enumerations	108,750
Boundary elements checked individually	8,342
Finite lifting-window cases	2,500
Cases witnessing sharpness of the delay	512
Boundary transition elements checked	6,462
Norm-quotient Smith certificates, $2 \le m \le 12$	11
Equivariant-map representation pairs	25
Explicit integral isomorphism lifts	61
Invertible 2×2 action matrices over $\mathbb{F}_p$, $p = 3, 5, 7$	2,544
All next-level lifts of the three selected non-scalar matrices	3,107
Independent full-bar-resolution fixtures	4

The kernel levels are $(p, n) = (2, 1), (2, 2), (2, 3), (3, 1), (3, 2)$. The sharpness and lifting loops use the precisely computed valuation bounds. No matrix in that family is skipped. The full group-action enumeration is exactly $\mathrm{GL}_2(\mathbb{F}_3)$, $\mathrm{GL}_2(\mathbb{F}_5)$, and $\mathrm{GL}_2(\mathbb{F}_7)$. The next-level lifts are all matrices $A + p^2 B$ for the chosen swap J and $B \in M_2(\mathbb{F}_p)$ at those three primes.

As an orthogonal check, the full inhomogeneous cyclic-group cochain differentials, rather than the small periodic resolution, are constructed in degrees zero through two for four fixtures. Twelve verified Smith certificates compute their finite cohomology sizes, and eight chain composites vanish modulo the coefficient modulus. This agrees with the periodic computations. It is an independent implementation check within one authoring session, not independent mathematical review.

The infinite-degree counterexample and the general exactness, lifting, and completion assertions have separate proofs. The test suite does not infer an unbounded derived tensor product from a bounded truncation. The JSON report records exact families, environment, command, counts, and script hash.

E.4 Sources, build, and unresolved obligations

The source check uses Milne's *Class Field Theory*, version 4.03, Chapter II, for finite-group cohomology, boundary maps, and the cyclic resolution, and the official Stacks Project derived-tensor section. The relevant primary PDF pages were also inspected visually. An initial broad search returned irrelevant pages and was discarded in favor of direct primary sources. No external full-text source or font files are bundled. This was a dependency check, not an exhaustive novelty search.

The final source is reproducibly integrated from the exact v0.8.0 source and the new proof fragments. The package includes the source diff, final LaTeX log, static reference check, rendering/layout record, checksum inventory, and run logs. The current build and proofread outcomes are recorded in the companion audit. K17, K18, arbitrary higher-coherent rewriting, and a sufficient intrinsic arithmetic-lifting criterion remain open. None is closed by the number of new ledger entries.

F Audit and verification record for v0.10.0

F.1 Source authority and evidence status

The source base is the supplied v0.9.0 manuscript and full package. Their SHA-256 hashes are recorded in `audit/base_sha256.json`, and the three original files are preserved in `previous/`. This continuation adds K62–K71. All inherited theorem, lemma, proposition, corollary, and proof blocks are preserved verbatim. Current prose and the ledger now distinguish the refuted exchange-converse claim from the proved fixed-finite-group lifting criterion. Historical version and audit records remain historical.

The new block has written proofs and an authoring-session self-review, not independent review or Lean certification. It does not assert that all earlier mathematics received a fresh audit. The Lean and Lake executables were not available on the runtime path. This is a tool limitation, not a mathematical obstruction. No historical novelty is claimed for the deformation-theoretic or finite-group stability results.

F.2 Executed routes and decisive outcomes

Three routes were pursued. First, the normalized multiplicativity defect gives a complete next-level obstruction in $H^2(G, \operatorname{Ad}\bar{\rho})$, with explicit linear repair and nonexistence certificates. Second, a central sign factor makes the derived-invariant test blind to an action that still cannot lift. This refutes sufficiency of that one diagnostic. Third, finite averaging gives a convergent correction whenever defect depth exceeds twice the valuation of the group order. Its finite-horizon corollary supplies a necessary-and-sufficient existence test for a fixed finite group.

The corresponding ultrametric stability threshold is established in [14, Lemma 7.3]. The source was checked at the exact v5 version, with the displayed hypotheses and proof, not from its abstract. The direct matrix proof here is independent of that citation and supplies the implementation contract. The normalized group-cohomology conventions were checked against [13, Chapter II, 1.17–1.19]. The new source record states the translation and the limits of the overlap check. An initial broad search returned irrelevant material, which was discarded. No inference of novelty is drawn from that search.

F.3 Proof boundaries examined

1. The ideal $p^n R_{n+1}$ is square-zero even at $n = 1$. Matrix corrections use left multiplication, which fixes the signs in the cocycle equation and strict-conjugacy action.

2. The next-level obstruction is a class, not the chosen raw cocycle. The affine equation is necessary and sufficient for the entire fixed group multiplication table. A failed choice of representative matrices is not by itself an obstruction.

3. The approximate conjugations α_g are not an action. The averaging proof starts with an exact associativity identity and controls its quadratic remainder before averaging. Division by $|G|$ loses $e = v_p(|G|)$ digits, explicitly accounted for at each step.

4. The convergence threshold is strictly $m > 2e$. The finite witness must extend the requested ρ_n, but the reconstructed integral action need only preserve that level, not the witness's top e digits. The example $3 \pmod 8$ for C_2 verifies that this distinction is real.

5. The central-sign example uses odd p, so its idempotents are defined. It is a counterexample for a fixed finite group; its restriction to C_p supplies the impossible relation. No identification of

its entire exchange cone with the adjoint H^2 class is made.

6. Finite catalogue means a finite set of isomorphism classes. It does not claim finite automorphism groups or a category equivalence under reduction. Strict framed matrices need not be unique.

7. The profinite example changes the source from C_p to $\mathbb{Z}_p$. The latter has an integral lift because it does not impose the same finite-order relation. This prevents applying a fixed-finite-quotient nonlifting verdict indiscriminately to a Galois representation.

F.4 Finite computation contract

All five exact regression scripts passed, including reruns of the four inherited suites. The new script uses integer and modular arithmetic only in mathematical checks. It examines eight specified cochain fixtures, including the nonabelian group S_3, with 593 cocycle triples and 56 changes of section. Positive solutions and negative left-nullspace witnesses are checked directly against their equations.

The cyclic suite enumerates all 367 valid rank-two actions and all 40,987 next-level generator matrices in its six explicitly listed (p, n, q) cases. The averaging suite performs 330 corrections in 110 deterministic fixtures, including primes dividing and not dividing the group order. Every use of division tracks finite working precision, and the exact associativity identity is checked before the predicted new defect depth. There are also 36 rank-one horizon comparisons, 98 conjugacy-average checks, and all 3,107 next-level lifts of three selected non-scalar nonliftable matrices. The sign and profinite power calculations are supplementary finite checks.

These are bounded tests of the implementation. They do not establish p-adic convergence, exhaustive classification for arbitrary groups, or historical novelty. Those conclusions either have their written proofs above or are not claimed. Full iterators, exclusions, counts, interpreter version, wall times, command lines, hashes before and after execution, and report hashes are stored in the package. Each suite has a 40-second subprocess timeout. No memory cap was set. Numerical-library thread limits are cooperative.

F.5 Manuscript and continuation

The standalone source is reproduced by `audit/integrate_v0100.py`. The source diff, inheritance checks, LaTeX log, rendering record, and checksum manifest are included. No external source text or font files are bundled. Final proofreading checks the new formulas, hypotheses, labels, references, and surrounding integration prose.

The sufficient existence question is closed here only for a fixed finite source group over $\mathbb{Z}_p$. Continuous lifting with growing finite images, geometric local conditions, and automorphic realization remain separate obligations. A useful continuation must address those actual hypotheses rather than merely rename the already-classical finite-group obstruction or averaging argument.

G Audit and verification record for v0.11.0

G.1 Source authority and evidence status

The supplied v0.10.0 PDF, source, and full package are preserved unchanged in `previous/`. Their SHA-256 hashes are recorded in `audit/base_sha256.json`. The new continuation adds K72–K82 and the proof fragment `proofs/profinite_lifting.tex`. All inherited theorem, lemma, proposition, corollary, and proof blocks remain verbatim and in their original relative order. Current summary prose, the theorem ledger, and next-step statements are updated. Historical audit and version entries remain historical.

The new results have written proofs and an authoring-session self-review. They have not received an independent mathematical review or Lean certification. Lean and Lake are not available on the runtime path. No claim of historical novelty is made for continuous cohomology, free pro-p groups, or cyclic centralizers, nor is novelty inferred for their combination from a limited search.

G.2 Executed routes and mathematical boundaries

The continuous-obstruction route retains the actual profinite source and proves the finite-quotient inflation criterion, including an explicit carry cocycle killed by source refinement. It does not infer that all finite-quotient obstructions eventually vanish. The free-source route constructs lifts without source relations beyond those of a free pro-p group. The abelian route retains commutativity, exposes its normalized residual, and gives a complete criterion on the cyclic normalized stratum, including every rank-two input. The example with E_{12}, E_{21} proves the uniform $2n-1$ horizon sharp for $n, r \geq 2$.

The self-review checked the following distinctions.

1. Continuous H^2 here has a finite discrete coefficient module. No statement about arbitrary topological coefficient theories is substituted for that definition.
2. A continuous repair cochain may require a finer finite source quotient. It is not required to factor through the initial quotient.
3. The cohomology-vanishing theorem gives a sufficient condition and the finitely branching tree gives a compactness criterion. Neither supplies a general numerical horizon for all profinite groups.
4. In the cyclic test, the normalized generator must be cyclic modulo p. Rank two makes that condition automatic after minimal scalar-depth normalization. No such assertion is made for arbitrary higher rank.
5. Cancellation of p^{2a} takes place in torsion-free integral matrices for necessity, or is translated into the stated lower-modulus congruence for a finite witness. It is not division by a unit in R_n.
6. The positive construction preserves the given reduction, but not extra determinant, ramification, or geometric conditions. The constructed matrices are units over $\mathbb{Z}_p$, not necessarily unimodular over $\mathbb{Z}$.
7. The continuous-powers argument includes $p = 2$. The explicit carry example retains its separate odd-prime hypothesis.
8. Source inflation and coefficient reduction have opposite variance. A preliminary draft description treating both as unrestricted coarsenings was replaced by theorem 19.15; reversing inflation would require descent through the smaller quotient.

G.3 Exact computations and usable certificates

All six regression suites passed with Python 3.13.5. The inherited five suites were rerun at their previously recorded bounds. The new standard-library suite uses exact integer and modular arithmetic with no random sampling or floating tolerance.

The source-inflation checks cover 495 cocycle triples, all 3,107 refined-quotient pairs for $p = 3, 5, 7$, and 83 repaired matrix values. The centralizer checks cover 1,017 cyclic matrices and 982,827 candidate centralizer pairs in the explicitly listed finite rings and companion-matrix family. Complete rank-two tables contain all 10, 33, 832, and 29,889 valid continuous two-generator actions over $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{Z}/4$, and $\mathbb{Z}/9$, respectively. Each positive case is checked against an exactly commuting integral polynomial certificate.

An orthogonal enumeration of all 1,048,576 ordered pairs of admissible generator matrices modulo 8 gives 59,392 commuting pairs. Their 664 distinct reductions modulo 4 agree exactly with the positive set from the normalized criterion. All 4,096 rank-two tangent triples over $\mathbb{F}_2$ are also checked. The sharpness suite includes 24 prime/precision families and all 72,353 correction pairs in three stated finite fixtures. Higher-rank checks use 54 constructed three-generator fixtures, not an exhaustive higher-rank population, with 162 recovered polynomial certificates and 648 finite continuity checks. Exact ranges and nonclaims are in `checks/profinite_results.json`.

The included command-line tool `tools/lift_abelian_rank2.py` validates the matrix input and returns either exact integral generators or a nonzero normalized-commutator certificate. It applies only to the unconstrained rank-two $\mathbb{Z}_p^r$ problem. Its positive and negative examples are preserved in `examples/`. The universal lifting and convergence statements follow from the written proofs, not from these bounded computations.

G.4 Sources, build, and reproducibility

The direct-limit cohomology input was checked in Milne's *Class Field Theory*, version 4.03, Chapter II, Section 4, Proposition 4.2, printed page 87. That PDF page was inspected visually. The continuous-cochain differential was checked on the official Stacks Project Tag 0DVG. Broad searches returning unrelated material were not treated as evidence. This is a dependency check, not an exhaustive novelty survey. No external full-text paper or font file is bundled.

The source is regenerated from the exact preserved base and the new proof and audit fragments by `audit/integrate_v0110.py`. The package includes the source diff, final LaTeX log, source-preservation checks, run manifests, visual-review record, and checksums. The runner removes stale result files before execution and records each script's before/after hash. A shell inspection command initially rejected a leading-hyphen `printf` argument; it was rerun without that argument and had not modified source. The companion audit records final build outcomes and the precise review scope.

H Version ledger

v0.11.0 Proves continuous one-step obstruction and source-inflation criteria, continuous tower reconstruction, and free pro-p coefficient lifting. Gives a normalized cyclic-centralizer lifting criterion for $\mathbb{Z}_p^r$, a complete rank-two test, and the sharp uniform horizon $2n-1$. Correctly separates source inflation from coefficient reduction by mixed variance. Adds exact positive and negative examples, a certificate CLI, and a sixth regression suite. Preserves the entire v0.10.0 package. Written proofs and self-review only.

v0.10.0 Adds a complete H^2 next-level obstruction, cochain repair certificates, and a central-sign counterexample to the sufficiency of derived-invariant exchange. Gives a direct averaging proof of known finite-group ultrametric stability, an exact finite lifting horizon, controlled conjugacy, a finite class catalogue, a cyclic Jacobian certificate, and an explicit profinite scope boundary. Five exact suites pass. Preserves every inherited theorem/proof block and the full v0.9.0 package. Written proofs and self-review only.

v0.9.0 Adds intrinsic kernel and invariant comparisons, explicit cohomological residuals, sharp precision-window lifting, equivariant morphism correction, and norm-quotient examples. Proves perfect-base derived exchange and exhibits a non-scalar cyclic action with an unbounded canonical exchange defect over singular finite coefficients. Proves the one-step failure to lift that action and exact exchange for supplied coefficient-lifted inputs. Preserves the full v0.8.0 package. All four finite suites pass, including independent full-bar checks. Written proofs and self-review only.

v0.8.0 Adds nonidentity derived arithmetic coarsenings, coherent weighted exchanges, the prime-power Koszul formula, and nilpotent excision collapse. Classifies finite prime-component quotients and their coherence. Proves uniform based-tower reconstruction, computes transient negative classes, and identifies the generic-fiber defect of completion versus repair. Proves arithmetic fracture reconstruction, effective lattice gluing, and compatible operator descent. Preserves the v0.7.0 package and adds exact finite tests. Written proofs and self-review only.

v0.7.0 Constructs strict chain-natural event diagrams. Defines interaction residuals and proves the total-cofiber and context-independence criteria. Proves principal-ideal reconstruction for reduced excisive event diagrams. Constructs the universal constructor-compatible excisive quotient. Gives tensor-cube, integral gcd, prime-overlap kernel, finite-signature, and integral matrix collapse theorems. Adds explicit nonsplitting, higher/lower interaction, and zero-label collapse boundaries, together with exact finite regression and a preserved full v0.6.0 package. Written proofs and self-review only.

v0.6.0 Audits size, variance, categorical regimes, lift multiplicity, cofiber shifts, exactness, descent, and quotient interpretation. Refutes general ordered filtration independence, including inside the typed two-axis grammar. Proves the semantic diamond criterion and the order-ideal description of pure exchange histories. Constructs the coherent ideal-indexed residual diagram. Proves constructor-invariant quotient and directed invariance theorems, records a quotient-collapse counterexample, and adds exact orbit-generator tests and a noncollapse certificate. Includes finite exact regression checks. Written proofs only, not Lean verification.

v0.5.0 Combines the stable residual filtration with the Verdier obstruction quotient to prove obstruction localization along any chosen normalization path: if all elementary exchange residuals vanish in the obstruction quotient, so does the total normalization residual; contrapositively, a persistent total obstruction forces at least one persistent elementary exchange obstruction. Under directed annihilators the statement upgrades from quotient-vanishing to operational killability.

v0.4.0 Adds a direct nonstable obstruction test bed: local sections of an abelian torsor produce overlap residuals, local changes of trivialization act as repairs, and the resulting Čech H^1 class vanishes exactly when the local sections can be repaired to reconstruct a global section. This closes the first descent test-bed obligation and shows that the residual/repair/obstruction pattern recovers a classical cohomological mechanism.

v0.3.0 Proves the stable residual filtration along a chosen normalization path by iterated octahedral cofiber sequences. Constructs the universal stable obstruction quotient by the thick closure of operationally killable residuals, proves exact detection under a directed common-annihilator hypothesis, and proves functorial transport of obstruction quotients under exact correspondences. K11 and K12 are closed in these stable forms; path-independent filtration and nonstable obstruction remain open.

v0.2.0 Adds the free seeded two-axis double-computad universal property and proves the pointwise free stable completion theorem for additive indexed systems. Mixed residuals become canonical cofibers, exact realizations preserve them, and residual promotion is shown to be automatic in the minimal stable zoo. K13 is closed at the strict syntactic level and K14 is closed under additive/stable hypotheses; the nonadditive residual universal construction remains open.

v0.1.0 Initial continuous formalization. Introduces indexed resolution categories, total Grothendieck category, factorization, resolution ultrametric, seed zoo closure, exchange transformations, stable residuals, exact normal forms, split residual expansion, obstruction taxonomy, descent transfer, and a seeded local-global reciprocity blueprint.