

CORE SERIES XII

Patching Support and Hecke Packets

Conditional automorphic completion

Monograph

Komorebi P.

Reading release 1.0.0 / 10 October 2026

REALIZATION CALCULUS

A reading map

This volume develops the algebraic steps in conditional automorphic completion. Read its support and packet theorems as deductions for the rings, modules, complexes, and arithmetic inputs that are actually supplied. They do not create missing residual occupancy or missing classicality.

The companion visual guide gives 27 worked lessons. The Lean and evidence guide exposes the production sources, the 49-claim crosswalk, and the hypotheses of the formal interfaces. The links below are reading routes, not additional theorem certificates.

Topic	Main statement	Visual	Lean section
Local conditions	3.1	V3	4
Support and quotients	6.9	V10	8
Derived support	6.16	V11	9
Hecke packets	8.2	V12	10
Stable cores	8.11	V15	11
Companion completion	10.5	V23	11
Conditional realization	12.6	V17	12

What the verification record means

The inherited formal record covers all 49 revised internal claims. The original-wording ledger remains separate: 41 complete, seven partial, and one unchecked. This release preserves those historical statuses rather than promoting amended statements into proofs of their stronger predecessors.

No fresh Lean compilation is claimed here. Current checks concern source preservation, inherited evidence authenticity, bounded computations, references, and publication quality. Representability, applicable patching, classicality, attached representations, Frobenius density, and the comparison inputs retain their stated external status.

The final theorem requires a compatible Hecke point over the stated field. The propagation step compares two already existing families associated with the same automorphic object. Paper XI's general realization conjecture is not an input to this paper.

Contents

1 Introduction . . . 3
2 Global deformation data and exact boundaries . . . 5
3 Local conditions are quotient packets . . . 6
4 Adequacy and the residual input . . . 8
5 Auxiliary group algebras and augmentation . . . 10
6 Ordinary and derived patching support . . . 12
7 Transport through an isomorphism $R \cong \mathbb{T}$ . . . 18
8 Finite Hecke packets and infinite occupancy . . . 19
9 Pointwise automorphy from support . . . 23
10 Compatible systems and companion completion . . . 26
11 The coefficient prime two . . . 30
12 Conditional automorphic completion . . . 31
13 Exact boundary of the article . . . 35

ABSTRACT. Let $\bar{\rho}$ be a residual Galois representation and let R be a global deformation ring with declared local conditions. We separate the global automorphy problem into residual occupancy, deformation support, Hecke occupancy, and arithmetic realization. Local conditions are encoded by quotient rings rather than by idempotents, since irreducible components of a local deformation ring need not be clopen. For an ordinary or derived patching datum, full support descends through the augmentation quotient. Near faithfulness is shown to be equivalent to full support for finite modules over Noetherian rings. An isomorphism $R \cong \mathbb{T}$ transports support, Fitting ideals, and point fibers, but a quotient map can erase an entire component. Finite Hecke algebras are resolved into canonical Artin packets, so simultaneous generalized eigenspaces are available without assuming semisimplicity. Nested packets in one finite carrier stabilize; for changing finite carriers, nonzero stable cores give compatible inverse limit vectors. These results give a pointwise criterion: a characteristic zero deformation point in automorphic support produces a Hecke eigensystem. With attached Galois representations, local global compatibility, and classicality, the point is automorphic. A fixed compatible system then propagates the conclusion to every coefficient place. For finite coefficient quotients, compatible conjugacy classes admit compatible continuous representatives. Declared polynomial quotient conditions survive their adic reconstruction. These completion statements require explicit finite occupancy, continuity, and integral Frobenius data. Universal companion existence and unrestricted reciprocity are not proved.

Publication and supporting files

This reading edition, dated 10 October 2026, retains the complete revised mathematical proofs. Fourteen inherited vector figures are unchanged. Four diagrams have been redrawn to separate their arrows, expose the supplied inputs, and synchronize the final architecture with the five gates of Theorem 12.6. Supporting Markdown (`.md`) files are included with the source: the publication note, formal-evidence boundary, original-to-revised claim crosswalk, current status, source-preservation ledger, and build instructions. They record the statement hypotheses, evidence scope, and editorial changes alongside the preserved original and revised source snapshots.

The saved release reports complete mapped formal evidence for all forty-nine revised internal claims, while the original-wording census is forty-one complete, seven partial, and one unchecked. These are different contracts. The completed tensor and point-factorization results are algebraic deductions for supplied deformation data; representability, residual occupancy, applicable patching and classicality theorems, attached Galois representations, Frobenius density, semisimplification, and the general Brauer–Nesbitt theorem remain external arithmetic or algebraic inputs. Neither Paper XI's general Conjecture 66 nor universal companion existence is assumed.

1 Introduction

Automorphy lifting joins several kinds of mathematics that are easy to conflate. A residual representation must first occur in an automorphic carrier. The chosen characteristic zero representation must lie on a deformation problem controlled by local conditions. A patching argument must show that the automorphic module reaches the relevant part of the deformation space. A comparison such as $R \cong \mathbb{T}$ must identify the arithmetic and Hecke coordinates. Finally, the resulting Hecke point must be classical and must carry the expected Galois representation with the required local compatibility.

None of these steps is merely a change of notation. A deeper lift above one residual representation cannot create a missing residual companion. Freeness over an auxiliary group algebra does not imply that every component of a deformation ring is occupied. An isomorphism of rings transports support that already exists, but it does not create a nonzero automorphic vector. A nonzero class in completed or torsion cohomology need not be classical without a theorem that says so.

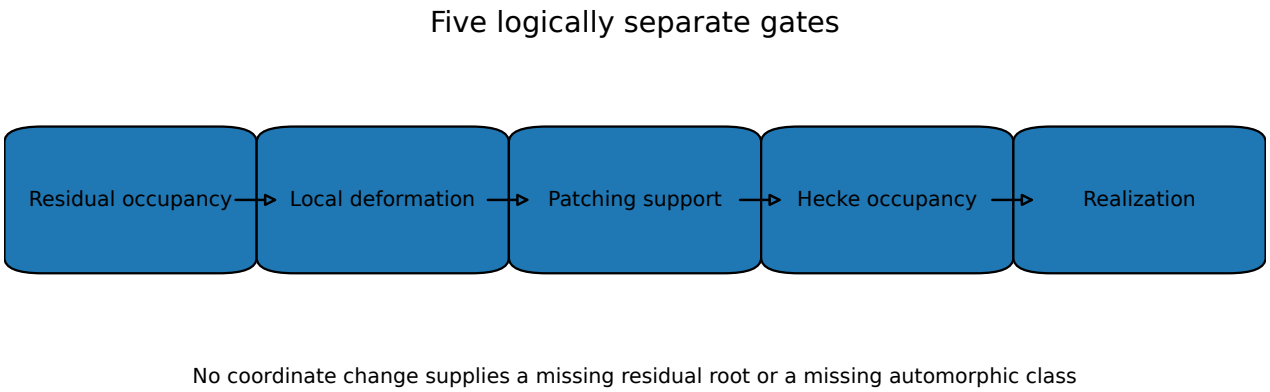


FIGURE 1. The global argument has several logically separate gates.

The Taylor Wiles method and its later forms solve these problems in many important settings [27, 24, 9, 14, 12]. Higher dimensional and derived versions use a broader range of local deformation rings, residual image conditions, torsion classes, and cohomological carriers [8, 3, 6, 25, 17, 22, 7, 19]. The present article does not replace those theorems. It isolates the algebraic support statements that explain exactly what a patching package supplies once its arithmetic hypotheses have been verified.

The first correction concerns local packets. A local deformation ring is usually complete and local. Its irreducible components are rarely open and closed. They cannot generally be represented by central idempotents. The correct object is a quotient ring defining a local condition, together with the support of a module or complex on the resulting spectrum. Central idempotents reappear later, where they belong, inside finite Artin Hecke algebras and finite dimensional characteristic zero carriers.

The second correction concerns patching. The useful global conclusion is full support, often expressed as near faithfulness. For a finite module over a Noetherian ring, near faithfulness is equivalent to full support. Full support descends through the quotient that removes the auxiliary variables. This statement is elementary, but it is the exact bridge from a patched module to componentwise occupancy at finite level. The same conclusion holds for perfect complexes through derived support.

The third correction concerns Hecke packets. Semisimplicity is not required to isolate a maximal Hecke packet. Every finite commutative algebra over a field is a product of Artin local algebras. Its canonical central idempotents project onto simultaneous generalized eigenspaces. Trace is only a secondary criterion in characteristic zero. Support and localization are the primary notions.

The last part assembles these statements into a conditional theorem. A characteristic zero point of R is automorphic when it lies in the support of the automorphic carrier, the comparison with the Hecke algebra is available, and the corresponding Hecke fiber satisfies the declared realization theorem. If the point belongs to a fixed strictly compatible system, equality of Frobenius polynomials propagates the automorphic identification to every coefficient place by Chebotarev density and Brauer Nesbitt [5, 18, 12].

The organization is as follows. Local quotient conditions are developed in section 3. Adequacy and residual occupancy appear in section 4. Auxiliary group algebras are treated in section 5. Ordinary and derived patching support appear in section 6. The effect of $R \cong \mathbb{T}$ is isolated in section 7. Finite

Hecke packets and infinite occupancy are developed in section 8. Pointwise automorphy and compatible systems appear in sections 9 and 10. The coefficient prime two is discussed in section 11. The final conditional theorem is stated in section 12.

2 Global deformation data and exact boundaries

Throughout, S is a finite set of places. Frobenius elements are geometric Frobenius elements; all Hecke polynomials, attached representations, and common algebraic spines use that same convention and the normalization specified by the declared realization theorem.

Let F be a number field. Fix a prime p, a finite extension $E/\mathbb{Q}_p$, its valuation ring $\mathcal{O}$, uniformizer ϖ, and residue field k. Let

$$\bar{\rho} : G_{F,S} \longrightarrow \mathrm{GL}_n(k)$$

be a continuous residual representation, where S is a finite set of places containing every place at which the deformation problem is allowed to ramify.

Definition 2.1 (Global deformation datum)**.** A global deformation datum consists of the following objects.

(1) A residual representation $\bar{\rho}$.
(2) For every $v \in S$, a framed local deformation ring $R_v^{\square}$.
(3) A quotient $R_v^{\tau_v}$ representing the chosen local condition whenever such a quotient is known to exist.
(4) A global deformation ring $R^{\square}$ with a continuous local map
$$R^{\mathrm{loc}} := \widehat{\otimes}_{v\in S,\mathcal{O}} R_v^{\square} \longrightarrow R^{\square}.$$
(5) The induced global ring with local conditions
$$R^{\tau} := R^{\square} \,\widehat{\otimes}_{R^{\mathrm{loc}}}\, R^{\mathrm{loc},\tau}, \qquad R^{\mathrm{loc},\tau} := \widehat{\otimes}_{v\in S,\mathcal{O}} R_v^{\tau_v}.$$

Representability of deformation functors and the construction of local condition quotients require hypotheses. The standard foundations are due to Mazur and Schlessinger, with many later refinements for local conditions and pseudorepresentations [16, 21, 4].

Standard Fact 2.2 (Represented deformation problem)**.** Under the usual finiteness and Schur type hypotheses, framed lifts of $\bar{\rho}$ are represented by complete Noetherian local $\mathcal{O}$ algebras. A declared local deformation condition is represented by a quotient when the corresponding subfunctor is representable. Tangent and obstruction spaces are described by the appropriate first and second cohomology groups.

Standard Fact 2.3 (Automorphy lifting package)**.** Published automorphy lifting theorems begin with residual automorphic occupancy and impose hypotheses on the residual image, local deformation conditions, regularity, and the global deformation problem. They then use ordinary or derived patching to compare a deformation ring with a Hecke algebra and to prove automorphy for characteristic zero points in the supported locus. The precise assumptions depend on the theorem being used [8, 3, 6, 25, 17, 1].

The article treats standard facts 2.2 and 2.3 as external inputs. Its results begin after the relevant rings, modules, complexes, and comparison maps have been supplied.

Definition 2.4 (Residual automorphic occupancy)**.** The residual datum $(\bar{\rho}, \tau)$ is automorphically occupied when a nonzero localized automorphic or cohomological Hecke carrier realizes the residual representation and the declared local conditions.

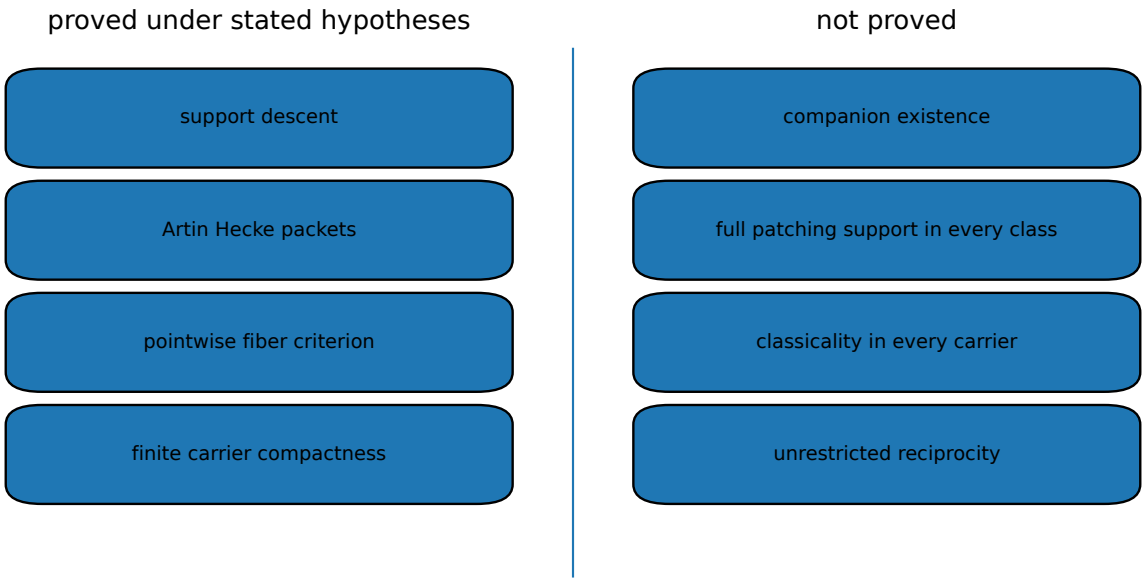


FIGURE 2. The article proves formal support and occupancy theorems. It does not prove the universal existence statements shown on the right.

This is a depth zero hypothesis. A positive depth deformation argument above another residual root cannot produce it.

3 Local conditions are quotient packets

The completed tensor product combines the local deformation conditions. The first theorem records the exact quotient law.

Theorem 3.1 (Completed tensor quotient)**.** *Let $R_1, \dots, R_s$ be complete Noetherian local $\mathcal{O}$ algebras with common residue field k. Let $I_i \subseteq R_i$ be closed ideals. Put*

$$R^{\mathrm{loc}} = R_1 \mathbin{\widehat{\otimes}}_{\mathcal{O}} \cdots \mathbin{\widehat{\otimes}}_{\mathcal{O}} R_s$$

and let J be the closed ideal generated by the images of the I_i. Then there is a canonical isomorphism

$$\boxed{(R_1/I_1) \mathbin{\widehat{\otimes}}_{\mathcal{O}} \cdots \mathbin{\widehat{\otimes}}_{\mathcal{O}} (R_s/I_s) \cong R^{\mathrm{loc}}/J.}$$

Proof. Write $S = \bigotimes_{i=1}^{s} R_i$ for the ordinary tensor product. Its ideal of definition is

$$\mathfrak{a} = \mathfrak{m}_{\mathcal{O}} S + \sum_i \mathfrak{m}_{R_i} S, \qquad L = \sum_i I_i S.$$

The scalar term retains the base adic topology when the family is empty. The ordinary quotient theorem gives a canonical algebra isomorphism

$$S/L \cong \bigotimes_{i=1}^{s} (R_i/I_i),$$

carrying the image of $\mathfrak{a}$ to the defining ideal on the right. The quotient topology on S/L is precisely this image ideal topology.

Here completion means the actual Hausdorff completion for the additive uniformity of the indicated adic topology. For every adic ring S and ideal L, there is a canonical bicontinuous ring isomorphism

$$\widehat{S}/\overline{L\widehat{S}} \cong \widehat{S/L}.$$

Indeed, the completed quotient map kills the closed extended ideal. In the other direction, the quotient map from S/L to $\widehat{S}/\overline{L\widehat{S}}$ extends to its completion. The latter quotient is complete: the adic uniformity has a countable base, and the quotient of a complete countably based additive group is complete. Its closed defining subgroup makes it Hausdorff. The two composites agree with the identity on dense quotient representatives, so continuity proves both inverse laws.

Applying this comparison to the displayed ordinary tensor isomorphism proves the claim. The closed extended ideal is exactly J, since extension commutes with the finite sum of the factor ideals. The maps preserve coefficients and each canonical factor map. The common residue field identifies $S/\mathfrak{a}$ with k; finite generation of $\mathfrak{a}$, supplied by the Noetherian factor hypotheses, identifies the completed topology with its maximal ideal topology. Proper quotients retain locality. If a condition is the unit ideal, the ring comparison remains valid with the zero ring, without treating it as a local ring with residue field k. □

Corollary 3.2 (Simultaneous local conditions)**.** *A continuous point of R^{loc} with Hausdorff target factors continuously through R^{loc}/J if and only if its restriction to every R_i factors continuously through R_i/I_i. Each factor is unique. If the original point is local, its quotient factor is local.*

Proof. Let $f : R^{\mathrm{loc}} \to A$ be the point. Since A is Hausdorff and f is continuous, $\ker f$ is closed. Thus it contains the closed ideal J if and only if it contains the algebraic ideal generated by the images of the I_i. This is equivalent to every local restriction killing I_i. The universal property of the actual ideal quotient gives the factors, and the quotient topology gives their continuity. Surjectivity of the quotient map gives uniqueness. If f reflects units, lift an element of the quotient to the original ring to see that its factor also reflects units. □

Proposition 3.3 (Global base change for local conditions)**.** *In the global deformation datum of definition 2.1, let $R^{\square}$ be a complete Noetherian local $\mathcal{O}$ algebra receiving a continuous local $\mathcal{O}$ algebra map from R^{loc}. Then*

$$R^{\tau} = R^{\square} \mathbin{\widehat{\otimes}}_{R^{\mathrm{loc}}} R^{\mathrm{loc},\tau}$$

represents the base change of the global deformation problem to the simultaneous local conditions. A continuous local $\mathcal{O}$ algebra point of $R^{\square}$ with complete Hausdorff coefficient target factors uniquely and continuously through R^{τ} exactly when all of its local restrictions satisfy the declared quotient conditions.

Proof. Put $A = R^{\mathrm{loc}}$, $B = R^{\square}$, and let $J \subseteq A$ be the closed local condition ideal. By theorem 3.1, the independently constructed ring $R^{\mathrm{loc},\tau}$ is canonically A/J. The ordinary tensor identity

$$B \otimes_A (A/J) \cong B/JB$$

carries the defining ideal of the tensor to the image of $\mathfrak{m}_B$. Here the structural map is local by the stated global deformation datum, so it carries $\mathfrak{m}_A$ into $\mathfrak{m}_B$. Completing, and applying the completion and quotient comparison from the preceding proof, gives a bicontinuous isomorphism

$$R^{\tau} \cong B/\overline{JB}.$$

The construction commutes with the canonical map from B, which is therefore surjective. It is also unchanged by transporting the tensor factor from A/J to the actual completed tensor of the local quotients. The transported ideal is exactly its natural defining ideal.

For a continuous algebra point $x : B \to C$ with complete Hausdorff target, the kernel of x is closed. Consequently x has a unique continuous factor through R^{τ} if and only if it kills JB, equivalently if its restriction to A kills J. By corollary 3.2, this is exactly the requirement that all local restrictions satisfy

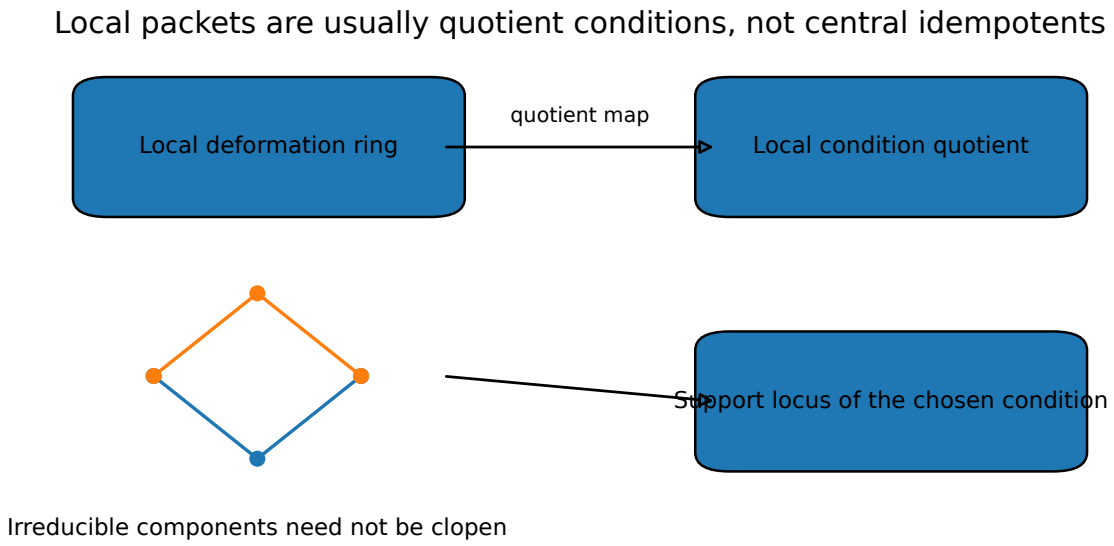


FIGURE 3. Local conditions are imposed by quotient rings. Components are support loci unless a genuine product decomposition exists.

their quotient conditions. The factor is an algebra map because its composite with the canonical global map is x. If x is local, surjectivity of that map shows that its factor reflects units and is local. Uniqueness also proves naturality under coefficient preserving maps of targets. This identifies the represented base change; it does not supply representability of a new arithmetic deformation problem. □

The local coefficient points in proposition 3.3 are test rings in the represented deformation category. Later field valued points $x : R \to E'$ are algebraic or generic specializations, and continuity of ρ_x is imposed separately. Such a field specialization need not be a local ring map: under $\mathcal{O} \to E$, a uniformizer becomes a unit.

Proposition 3.4 (Idempotents in a local ring)**.** *If R is a commutative local ring, then its only idempotents are zero and one.*

Proof. Let $e^2 = e$. Since $e + (1 - e) = 1$, at least one of e and $1 - e$ is a unit. If e is a unit, then $e = 1$. If $1 - e$ is a unit, then $e = 0$. □

Remark 3.5. An irreducible component of $\operatorname{Spec} R$ for a local deformation ring need not be open and closed. It therefore need not be selected by an idempotent. Components and local conditions should be described by quotient ideals and module support. Idempotent packet formulas are reserved for finite Artin algebras or other settings with an actual product decomposition.

4 Adequacy and the residual input

Adequacy is one of the residual image hypotheses used in several automorphy lifting theorems. The definition and its role are external to the support arguments below [13, 25, 17].

Standard Fact 4.1 (Adequate image)**.** Let $G \subseteq \mathrm{GL}(V)$ act absolutely irreducibly over a finite field k of characteristic p. Under the standard definition, adequacy imposes vanishing conditions on $H^1(G, k)$ and

$H^1(G, \mathrm{ad}^0 V)$, a scalar condition, and a generalized eigenspace detection condition for every irreducible submodule of $\mathrm{ad}^0 V$.

The following small example makes the projector detection condition explicit. It is a certificate for one finite representation, not a new adequacy criterion.

Let $G = S_3$, let $k = \mathbb{F}_7$, and let V be the standard two dimensional representation. In the basis

$$v_1 = (1, -1, 0), \qquad v_2 = (0, 1, -1),$$

a transposition and a three cycle act by

$$s = \begin{pmatrix} 6 & 1 \\ 0 & 1 \end{pmatrix}, \qquad t = \begin{pmatrix} 0 & 6 \\ 1 & 6 \end{pmatrix}.$$

Theorem 4.2 (Finite adequacy certificate)**.** *The standard representation of S_3 over $\mathbb{F}_7$ is adequate. More precisely,*

$$H^0(S_3, \mathrm{ad}^0 V) = 0, \qquad H^1(S_3, \mathbb{F}_7) = 0, \qquad H^1(S_3, \mathrm{ad}^0 V) = 0,$$

and

$$\mathrm{ad}^0 V \cong \mathrm{sgn} \oplus V.$$

The sign summand is detected by an eigenprojector of a three cycle, and the standard summand is detected by an eigenprojector of a transposition.

Proof. The coordinate map

$$(a, b) \longmapsto (a, -a + b, -b)$$

identifies the displayed matrices with the permutation action on the sum zero subspace of $\mathbb{F}_7^3$. This action is faithful. The four matrices $1, s, t, st$ span M_2 over every field of characteristic seven: the determinant of their coordinate matrix is $4 \neq 0$. A subspace stable under the group is therefore stable under every endomorphism, so the representation is absolutely irreducible. The characteristic polynomials of the identity, transpositions, and three cycles are respectively

$$(X - 1)^2, \qquad (X - 1)(X + 1), \qquad (X - 2)(X - 4).$$

Thus all eigenvalues already lie in $\mathbb{F}_7$.

A trace zero matrix fixed by conjugation by both s and t is zero, as follows by comparing its four entries. This proves $H^0(S_3, \mathrm{ad}^0 V) = 0$. For any $\mathbb{F}_7[S_3]$ module M and one cocycle f, put

$$u = -6^{-1} \sum_{h \in S_3} f(h).$$

Summing $f(gh) = gf(h) + f(g)$ over h gives $gu - u = f(g)$. Hence every one cocycle is a coboundary and $H^1(S_3, M) = 0$, giving both asserted first cohomology vanishings.

Write

$$W(a, b, c) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \qquad Z = W(2, 3, 4), \quad A = W(1, 1, 2), \quad B = W(1, 5, 6).$$

The three coordinate columns have determinant $3 \neq 0$. Under conjugation,

$$sZ = -Z, \quad tZ = Z, \qquad sA = -A, \quad sB = A + B, \quad tA = B, \quad tB = -A - B.$$

Consequently $(z, (a, b)) \mapsto zZ + aA + bB$ is an equivariant isomorphism from $\mathrm{sgn} \oplus V$ to $\mathrm{ad}^0 V$.

The sign witness is $W_{\mathrm{sgn}} = Z$, while $W_{\mathrm{std}} = W(6, 4, 3) = 3A + 3B$ belongs to the standard summand. The corresponding projectors and trace pairings are

$$P_{t,2} = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}, \qquad \mathrm{Tr}(P_{t,2} W_{\mathrm{sgn}}) = 3 \neq 0,$$

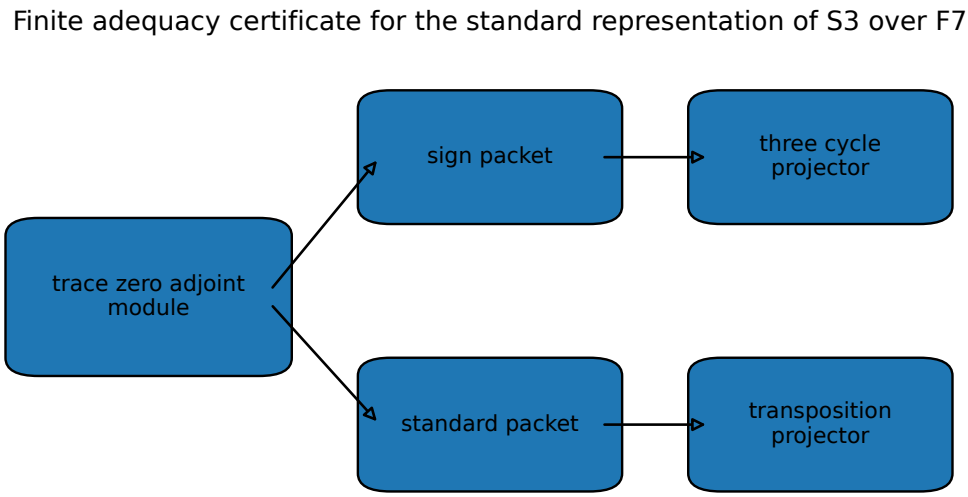


FIGURE 4. The two irreducible adjoint packets are detected by different generalized eigenspace projectors.

$$P_{s,1} = \begin{pmatrix} 0 & 4 \\ 0 & 1 \end{pmatrix}, \qquad \operatorname{Tr}(P_{s,1} W_{\mathrm{std}}) = 6 \neq 0.$$

For an operator D with $(D - \alpha I)(D - \beta I) = 0$ and $\alpha \neq \beta$, the canonical projector is

$$P_{D,\alpha} = (\alpha - \beta)^{-1}(D - \beta I).$$

Its image equals $\ker((D - \alpha I)^n)$ for every $n \geq 1$, and it kills $\ker((D - \beta I)^n)$. This identifies the displayed matrices with the generalized eigenspace projectors required in the adequacy definition.

For detection on every irreducible submodule, one can check the stronger assertion that canonical projectors detect every nonzero matrix. In addition to the two projectors above, use

$$P_{s,-1} = \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}, \qquad P_{tst^{-1},1} = \begin{pmatrix} 1 & 0 \\ 4 & 0 \end{pmatrix}.$$

The four trace pairings with these matrices have trivial common kernel in $M_2(\mathbb{F}_7)$, by row reduction. Every irreducible adjoint submodule contains a nonzero matrix, so some canonical projector has nonzero trace pairing on that submodule. This verifies the last adequacy clause and completes the certificate. □

Remark 4.3. The theorem verifies adequacy for one finite image. A global automorphy lifting argument must verify the published residual image hypothesis for its actual representation. The calculation does not replace that step, and it does not provide a uniform coefficient prime two theorem.

5 Auxiliary group algebras and augmentation

Let Δ be a finite abelian p group and let

$$\Lambda = \mathcal{O}[\Delta].$$

The augmentation map is

$$\epsilon : \Lambda \longrightarrow \mathcal{O}, \qquad \sum_{g \in \Delta} a_g g \longmapsto \sum_{g \in \Delta} a_g.$$

Its kernel is the augmentation ideal $\mathfrak{a}$.

Proposition 5.1 (Functorial group algebra transfer)**.** *A group homomorphism $q : \Delta' \to \Delta$ induces an $\mathcal{O}$ algebra homomorphism*

$$q_* : \mathcal{O}[\Delta'] \longrightarrow \mathcal{O}[\Delta].$$

If q is surjective, then q_* is surjective. Augmentation commutes with the transfer:

$$\epsilon_\Delta q_* = \epsilon_{\Delta'}.$$

Proof. Extend q linearly from the group basis. Multiplicativity follows from $q(gh) = q(g)q(h)$. Surjectivity follows because every basis element of $\mathcal{O}[\Delta]$ has a preimage. Both augmentation maps send every group element to one, which gives the final identity. □

Theorem 5.2 (Residual augmentation algebra)**.** *Suppose*

$$\Delta \cong \prod_{i=1}^{r} C_{p^{n_i}}.$$

Choose generators γ_i, and let $X_i = \gamma_i - 1$. Then

$$\boxed{k[\Delta] \cong k[X_1, \dots, X_r]/(X_1^{p^{n_1}}, \dots, X_r^{p^{n_r}}).}$$

The augmentation ideal is generated by the X_i, and $k[\Delta]$ is local.

Proof. In characteristic p,

$$(1 + X_i)^{p^{n_i}} - 1 = X_i^{p^{n_i}}.$$

Thus the assignment $X_i \mapsto \gamma_i - 1$ gives a surjective homomorphism from the displayed truncated polynomial ring to $k[\Delta]$. Both sides have dimension $\prod_i p^{n_i} = |\Delta|$ over k, so the map is an isomorphism. The quotient by the ideal generated by the X_i is k. Since each X_i is nilpotent, that ideal is the unique maximal ideal. □

Corollary 5.3 (No residual character splitting)**.** *The residual group algebra $k[\Delta]$ has no nontrivial idempotents. In particular, character projectors do not split a finite p group algebra in residue characteristic p.*

Proof. The first assertion follows from locality in theorem 5.2. For the general finite p group assertion, let $e \in k[G]$ be idempotent and write $\epsilon : k[G] \to k$ for augmentation. Suppose $\epsilon(e) = 0$ and $e \neq 0$. The $\mathbb{F}_p$ span W of the left translates ge is a nonzero finite dimensional $\mathbb{F}_p$ vector space, stable under G. Its cardinality is divisible by p. The orbit counting formula implies that its fixed point set also has cardinality divisible by p. Since zero is fixed, there is a nonzero fixed vector $v \in W$.

A vector in the regular representation fixed by all left translations has constant coefficients. Consequently $ve = \epsilon(e)v = 0$. On the other hand, $e^2 = e$ implies $we = w$ for every $w \in W$, hence $ve = v$, a contradiction. Thus every idempotent of augmentation zero is zero. Since $\epsilon(e)^2 = \epsilon(e)$ in the field k, either $\epsilon(e) = 0$ or $\epsilon(e) = 1$; in the latter case apply the preceding argument to $1 - e$. □

For $M \otimes_{\Lambda,\epsilon} \mathcal{O}$, we use a right Λ module M. The left module convention is $\mathcal{O} \otimes_{\Lambda,\epsilon} M$. Throughout, $\mathfrak{a}M$ denotes the augmentation relation submodule: it is generated by ma for right modules and by am for left modules, with $a \in \mathfrak{a}$. This convention also applies to nonabelian groups in corollary 11.2. For abelian groups, both tensor orders agree over the commutative group algebra.

Theorem 5.4 (Coinvariant recovery)**.** *Let M be a free Λ module of rank d. Then*

$$\boxed{M/\mathfrak{a}M \cong M \otimes_{\Lambda,\epsilon} \mathcal{O} \cong \mathcal{O}^d.}$$

The statement is valid for every coefficient prime, including two.

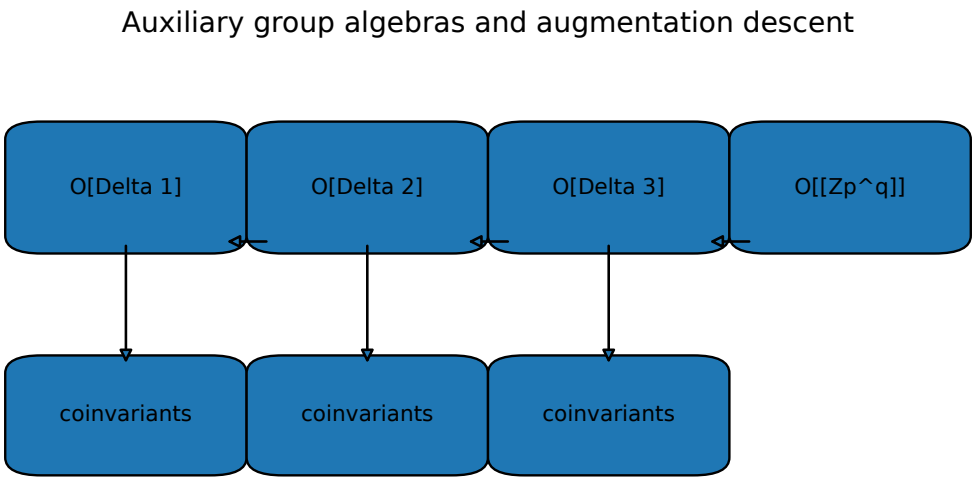


FIGURE 5. Auxiliary group algebras refine through quotient maps, while augmentation recovers the coarse level.

Proof. Choose a Λ basis of M. Tensoring the resulting isomorphism $M \cong \Lambda^d$ along augmentation gives $\mathcal{O}^d$. Tensoring by $\Lambda/\mathfrak{a}$ is the same as quotienting by $\mathfrak{a}M$. □

Standard Fact 5.5 (Iwasawa limit)**.** For $\Delta_m = (\mathbb{Z}/p^m\mathbb{Z})^r$, the inverse limit of $\mathcal{O}[\Delta_m]$ is the Iwasawa algebra

$$\mathcal{O}[[\mathbb{Z}_p^r]] \cong \mathcal{O}[[X_1, \dots, X_r]],$$

where X_i corresponds to a chosen topological generator minus one [20].

The augmentation formalism is uniform in p. The difficult coefficient two questions lie elsewhere. Residual image hypotheses, local deformation geometry, and classicality may change at two even though coinvariant recovery remains exact.

6 Ordinary and derived patching support

The algebraic conclusion supplied by a patching argument is most naturally stated through support. This section isolates the commutative algebra needed to descend that conclusion from a patched object to finite level.

Definition 6.1 (Support and near faithfulness)**.** Let R be a commutative ring and let M be an R module. Its support is

$$\operatorname{Supp}_R(M) = \{\mathfrak{p} \in \operatorname{Spec} R : M_{\mathfrak{p}} \neq 0\}.$$

If R is Noetherian and M is finite, then M is *nearly faithful* when $\operatorname{Ann}_R(M)$ is a nilpotent ideal.

Theorem 6.2 (Finite module support)**.** *Let R be a commutative ring and let M be a finite R module. Then*

$$\boxed{\operatorname{Supp}_R(M) = V\big(\operatorname{Ann}_R(M)\big).}$$

Proof. Let $\mathfrak{p} \in \operatorname{Spec} R$. If $\operatorname{Ann}_R(M) \not\subseteq \mathfrak{p}$, choose $a \in \operatorname{Ann}_R(M) \setminus \mathfrak{p}$. The element a becomes a unit in $R_{\mathfrak{p}}$, so $M_{\mathfrak{p}} = 0$.

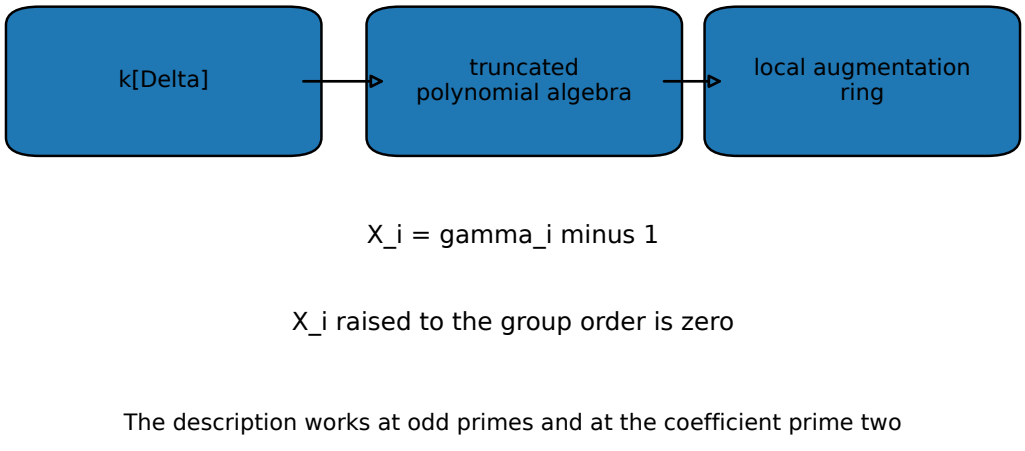


FIGURE 6. The residual auxiliary algebra is a local truncated polynomial algebra at every coefficient prime.

Conversely, suppose $M_{\mathfrak{p}} = 0$. Choose generators $m_1, \dots, m_d$ of M. For each i, there is an element $s_i \notin \mathfrak{p}$ such that $s_i m_i = 0$. The product $s = s_1 \cdots s_d$ lies outside $\mathfrak{p}$ and annihilates every generator. Thus $s \in \operatorname{Ann}_R(M) \setminus \mathfrak{p}$. This proves the equality. □

Theorem 6.3 (Fitting support criterion)**.** *Let R be Noetherian and let M be a finite R module. Then*

$$\boxed{\operatorname{Supp}_R(M) = V\big(\operatorname{Fitt}_0^R(M)\big).}$$

Proof. If $M = 0$, the statement follows from the usual convention $\operatorname{Fitt}_0(0) = R$. Assume $M \neq 0$, and choose a finite presentation

$$R^m \xrightarrow{A} R^n \longrightarrow M \longrightarrow 0$$

with $n \geq 1$. The zeroth Fitting ideal is generated by the n by n minors of A. After localization at a prime $\mathfrak{p}$, the module $M_{\mathfrak{p}}$ vanishes exactly when $A_{\mathfrak{p}}$ is surjective. A map onto the free module $R_{\mathfrak{p}}^n$ is surjective exactly when one maximal minor is a unit. Thus $M_{\mathfrak{p}} = 0$ exactly when $\operatorname{Fitt}_0^R(M) \not\subseteq \mathfrak{p}$. □

Theorem 6.4 (Higher Fitting strata)**.** *Let R be Noetherian and let M be finite. For $r \geq 1$, put*

$$\mu_{\mathfrak{p}}(M) = \dim_{\kappa(\mathfrak{p})} \left(M \otimes_R \kappa(\mathfrak{p}) \right).$$

Then

$$\boxed{\{\mathfrak{p} \in \operatorname{Spec} R : \mu_{\mathfrak{p}}(M) \geq r\} = V\big(\operatorname{Fitt}_{r-1}^R(M)\big).}$$

Proof. Use a finite presentation with target rank n. If $r > n$, every fiber has dimension at most n, so the left hand locus is empty. The convention $\operatorname{Fitt}_{r-1}(M) = R$ for $r - 1 \geq n$ makes the right hand locus empty as well. It remains to consider $1 \leq r \leq n$. At $\mathfrak{p}$, the fiber dimension is

$$n - \operatorname{rank} A(\mathfrak{p}).$$

It is at least r exactly when the rank is at most $n - r$, which is equivalent to the vanishing of every minor of size $n - r + 1$. Those minors generate $\operatorname{Fitt}_{r-1}^R(M)$. □

Proposition 6.5 (Fitting ideals and base change)**.** *Let $R \to S$ be a ring map and let M be a finitely presented R module. Then for every $j \geq 0$,*

$$\boxed{\mathrm{Fitt}_j^S(M \otimes_R S) = \mathrm{Fitt}_j^R(M)S.}$$

Proof. Base change carries a finite presentation matrix for M to a presentation matrix for $M \otimes_R S$. Every defining minor is carried to the corresponding minor over S. □

Remark 6.6. Support records nonvanishing. The higher Fitting strata record lower bounds on the number of generators in each fiber. They should not be confused with automorphic multiplicity formulas, which require additional representation theoretic input.

Theorem 6.7 (Near faithfulness is full support)**.** *Let R be Noetherian and let M be a finite R module. The following conditions are equivalent.*

(1) M is nearly faithful.
(2) $\mathrm{Supp}_R(M) = \operatorname{Spec} R$.
(3) $\mathrm{Ann}_R(M) \subseteq \mathrm{Nil}(R)$.

If R is reduced, these conditions are equivalent to faithfulness.

Proof. By theorem 6.2, full support is equivalent to

$$V\big(\mathrm{Ann}_R(M)\big) = \operatorname{Spec} R.$$

This holds exactly when $\mathrm{Ann}_R(M)$ is contained in every prime ideal, hence in the nilradical. Because R is Noetherian, the ideal $\mathrm{Ann}_R(M)$ is finitely generated. An ideal generated by finitely many nilpotent elements is nilpotent. This proves the equivalence of the first three conditions. If R is reduced, its nilradical is zero. □

Corollary 6.8 (Support on every irreducible component)**.** *Under the hypotheses of theorem 6.7, a nearly faithful module is nonzero at the generic point of every irreducible component of* $\operatorname{Spec} R$.

Proof. Every generic point is a prime of R, and full support contains every prime. □

Theorem 6.9 (Support after a quotient)**.** *Let R be Noetherian, let $I \subseteq R$ be an ideal, and let M be a finite R module. Under the canonical identification*

$$\operatorname{Spec}(R/I) \cong V(I) \subseteq \operatorname{Spec} R,$$

one has

$$\boxed{\mathrm{Supp}_{R/I}(M/IM) = \mathrm{Supp}_R(M) \cap V(I).}$$

Proof. Let $\mathfrak{p} \supseteq I$. Localizing gives

$$(M/IM)_{\mathfrak{p}} \cong M_{\mathfrak{p}}/I_{\mathfrak{p}}M_{\mathfrak{p}}.$$

If $M_{\mathfrak{p}} = 0$, the quotient is zero. If $M_{\mathfrak{p}} \neq 0$, then $I_{\mathfrak{p}}$ lies in the maximal ideal of the local ring $R_{\mathfrak{p}}$. Nakayama's lemma shows that $M_{\mathfrak{p}}/I_{\mathfrak{p}}M_{\mathfrak{p}} \neq 0$. The stated equality follows. □

Definition 6.10 (Ordinary patching datum)**.** An ordinary patching datum consists of a Noetherian ring R_∞, an ideal $\mathfrak{a} \subseteq R_\infty$, a finite R_∞ module M_∞, and identifications

$$R \cong R_\infty/\mathfrak{a}, \qquad M \cong M_\infty/\mathfrak{a}M_\infty.$$

In automorphy lifting applications, R_∞ is a patched deformation ring and M_∞ is a patched automorphic module. The definition itself contains no automorphy assertion.

Theorem 6.11 (Ordinary patching support descent)**.** *Let $(R_\infty, \mathfrak{a}, M_\infty, R, M)$ be an ordinary patching datum. If*

$$\operatorname{Supp}_{R_\infty}(M_\infty) = \operatorname{Spec} R_\infty,$$

then

$$\boxed{\operatorname{Supp}_R(M) = \operatorname{Spec} R.}$$

Equivalently, if M_∞ is nearly faithful, then M is nearly faithful. If R is reduced, then M is faithful.

Proof. Apply theorem 6.9 with $I = \mathfrak{a}$. The support of the quotient is

$$\operatorname{Spec} R_\infty \cap V(\mathfrak{a}) = V(\mathfrak{a}),$$

which identifies with $\operatorname{Spec} R$. The remaining statements follow from theorem 6.7. □

Corollary 6.12 (Componentwise finite level occupancy)**.** *In the setting of theorem 6.11, every irreducible component of* $\operatorname{Spec} R$ *is contained in the support of M. In particular, no component of the finite level deformation ring is erased by the quotient from the patched level.*

Proof. The module M has full support by theorem 6.11. □

Example 6.13 (Freeness over auxiliary variables is not enough)**.** Let k be a field, let $S = k[[t]]$, and put

$$R = S[[x]]/(x(x - t)).$$

The ring R is reduced and has two irreducible components. Let

$$M = R/(x) \cong S.$$

Then M is free of rank one over S, but

$$\operatorname{Supp}_R(M) = V(x)$$

is only one component of $\operatorname{Spec} R$. Thus freeness over the auxiliary power series ring does not imply full support over the deformation ring.

Proof. The two nonassociate prime factors x and $x - t$ occur with multiplicity one, so the principal quotient is reduced. The displayed isomorphism makes M free over S. Its annihilator as an R module is (x), hence

$$\operatorname{Supp}_R(M) = V(x)$$

by theorem 6.2. The generic point $(x - t)$ of the other component does not contain x, so it is not in the support. □

Derived patching replaces one module by a perfect complex. The correct support is detected after derived specialization.

Definition 6.14 (Derived support)**.** Let R be a commutative ring and let C be a perfect complex of R modules. Its derived support is

$$\operatorname{Supp}_R(C) = \left\{ \mathfrak{p} \in \operatorname{Spec} R : C \overset{\mathbf{L}}{\underset{R}{\otimes}} \kappa(\mathfrak{p}) \not\simeq 0 \right\}.$$

Standard Fact 6.15 (Cohomological description of derived support)**.** If R is Noetherian and C is perfect, then

$$\operatorname{Supp}_R(C) = \bigcup_i \operatorname{Supp}_R H^i(C).$$

In particular, derived support is closed and is empty exactly when $C \simeq 0$ [26, 11].

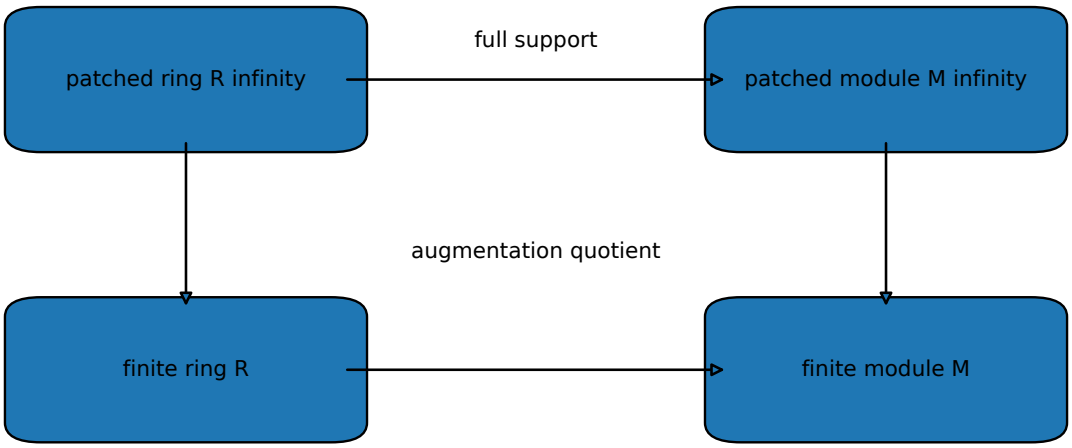


FIGURE 7. Full support at the patched level descends through the augmentation quotient.

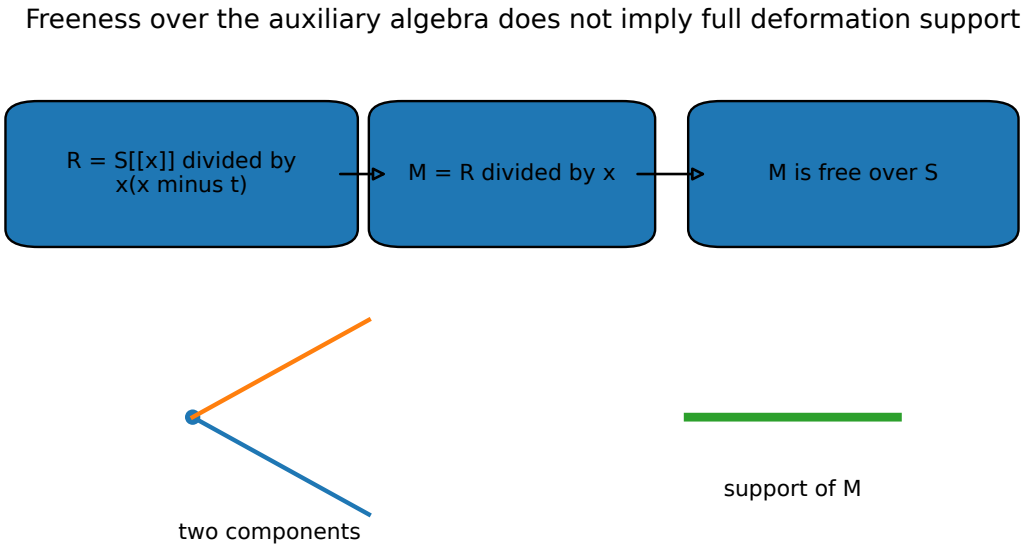


FIGURE 8. A module can be free over the auxiliary ring while occupying only one deformation component.

Theorem 6.16 (Derived quotient support)**.** *Let R be a commutative ring, let $I \subseteq R$ be an ideal, and let C be a perfect complex over R. Under* $\operatorname{Spec}(R/I) \cong V(I)$*, one has*

$$\boxed{\operatorname{Supp}_{R/I}\left(C \overset{\mathbf{L}}{\otimes}_{R} R/I\right) = \operatorname{Supp}_{R}(C) \cap V(I).}$$

Proof. Choose a bounded complex P of finite projective R modules representing C. Scalar extension to a commutative R algebra S is computed degree by degree as $S \otimes_R P$; the canonical symmetry of tensor products gives the displayed order $C \otimes_R^{\mathbf{L}} S$. Scalar extension preserves the bounds, finite generation, and projectivity of the terms. A derived isomorphism between two such projective models lifts to a homotopy equivalence, which scalar extension preserves. The resulting derived object and its residue field fibers are therefore independent of the chosen model.

Let $q \in \operatorname{Spec}(R/I)$, and let $\mathfrak{p}$ be its inverse image in R. The composite point $R \to R/I \to \kappa(q)$ has kernel $\mathfrak{p}$, so it factors through a field embedding $\kappa(\mathfrak{p}) \to \kappa(q)$. The ordinary tensor associator in every degree gives an isomorphism of complexes

$$\kappa(q) \otimes_{R/I} \big((R/I) \otimes_R P\big) \cong \kappa(q) \otimes_R P \cong \kappa(q) \otimes_{\kappa(\mathfrak{p})} \big(\kappa(\mathfrak{p}) \otimes_R P\big).$$

Extension of fields is faithfully flat. It preserves and reflects exactness in each degree, hence also acyclicity. The left hand complex is thus nonzero in the derived category exactly when the residue field fiber of C at $\mathfrak{p}$ is nonzero. This proves the support identity on corresponding primes. The image of $\operatorname{Spec}(R/I) \to \operatorname{Spec} R$ is exactly $V(I)$, giving the stated ambient spectrum equality, including the case $I = R$. □

Definition 6.17 (Derived patching datum)**.** A derived patching datum consists of a Noetherian ring R_∞, an ideal $\mathfrak{a}$, a perfect complex C_∞ over R_∞, and identifications

$$R \cong R_\infty/\mathfrak{a}, \qquad C \simeq C_\infty \overset{\mathbf{L}}{\underset{R_\infty}{\otimes}} R.$$

Theorem 6.18 (Derived patching support descent)**.** *Let $(R_\infty, \mathfrak{a}, C_\infty, R, C)$ be a derived patching datum. If*

$$\operatorname{Supp}_{R_\infty}(C_\infty) = \operatorname{Spec} R_\infty,$$

then

$$\boxed{\operatorname{Supp}_R(C) = \operatorname{Spec} R.}$$

For every point $x : R \to E'$ to a field, the derived fiber

$$C \overset{\mathbf{L}}{\underset{R,x}{\otimes}} E'$$

is nonzero.

Proof. Apply theorem 6.16 to the patched complex and use the actual ring identification $R \cong R_\infty/\mathfrak{a}$. Full patched support restricts to $V(\mathfrak{a})$, which corresponds to all of $\operatorname{Spec} R$. The supplied derived isomorphism with C preserves this support by the projective model argument in that theorem.

For a field valued point $x : R \to E'$, put $\mathfrak{p} = \ker x$. The point factors as $R \to \kappa(\mathfrak{p}) \to E'$, where the second map is a field embedding. Tensor association on a projective model identifies the fiber at x with the extension to E' of the fiber at $\mathfrak{p}$. The latter is nonzero by full support. Faithful flatness of the field extension reflects exactness in every degree, so the fiber at x is nonzero as well. This applies to every field valued point, with no closed point restriction. □

Corollary 6.19 (A nonzero cohomological degree survives)**.** *Under the hypotheses of theorem 6.18, for every field valued point $x : R \to E'$, there is an integer i such that*

$$H^i\left(C \overset{\mathbf{L}}{\underset{R,x}{\otimes}} E'\right) \neq 0.$$

Proof. The derived fiber is nonzero by theorem 6.18. If every cohomology object vanished, a cochain model of that fiber would be acyclic. Its image in the derived category would then be zero, a contradiction. Hence some integer degree has nonzero cohomology. □

Remark 6.20. The support theorem does not decide the cohomological degree in which the point occurs. It also does not prove that the surviving class is classical. Those are separate arithmetic inputs.

7 Transport through an isomorphism $R \cong \mathbb{T}$

An isomorphism between a deformation ring and a Hecke algebra is an exact coordinate change. It transports support and finite presentation data, but it does not create a supported module.

Theorem 7.1 (Support and Fitting transport)**.** *Let $\varphi : R \to \mathbb{T}$ be an isomorphism of Noetherian rings. Let M be a finite $\mathbb{T}$ module, viewed as an R module through φ. Then the homeomorphism*

$$\varphi^* : \operatorname{Spec} \mathbb{T} \longrightarrow \operatorname{Spec} R$$

satisfies

$$\boxed{\varphi^*\big(\operatorname{Supp}_{\mathbb{T}}(M)\big) = \operatorname{Supp}_R(M).}$$

For every $j \geq 0$,

$$\boxed{\varphi\left(\operatorname{Fitt}_j^R(M)\right) = \operatorname{Fitt}_j^{\mathbb{T}}(M).}$$

Proof. Localization commutes with transport along an isomorphism, so a localization of M vanishes on one side exactly when the corresponding localization vanishes on the other. This proves the support identity. A finite presentation matrix over R is carried entry by entry to a presentation matrix over $\mathbb{T}$. The minors generating each Fitting ideal are therefore carried to the corresponding minors. □

Corollary 7.2 (Full support is invariant under $R \cong \mathbb{T}$)**.** *In the setting of theorem 7.1, M has full R support if and only if it has full $\mathbb{T}$ support.*

Proof. Apply the support homeomorphism in theorem 7.1. □

Proposition 7.3 (A quotient can erase a component)**.** *Let*

$$R = k[[x, y]]/(xy)$$

and let

$$\mathbb{T} = R/(x) \cong k[[y]].$$

Viewed as an R module, $\mathbb{T}$ has support $V(x)$, which misses the generic point (y) of the other irreducible component. Thus a surjection $R \twoheadrightarrow \mathbb{T}$ does not by itself imply full component occupancy.

Proof. The annihilator of the R module $\mathbb{T}$ is (x). Hence its support is $V(x)$ by theorem 6.2. The prime (y) does not contain x, so it is not in this support. □

Theorem 7.4 (Field fiber criterion)**.** *Let R be a commutative ring, let M be a finite R module, and let $x : R \to E'$ be a homomorphism to a field. Put $\mathfrak{p} = \ker x$. Then*

$$\boxed{M \otimes_{R,x} E' \neq 0 \quad \Longleftrightarrow \quad \mathfrak{p} \in \operatorname{Supp}_R(M).}$$

Proof. The map x factors through the local ring $R_{\mathfrak{p}}$ and then through an embedding of the residue field $\kappa(\mathfrak{p})$ into E'. Since M is finite, Nakayama's lemma gives

$$M_{\mathfrak{p}} \neq 0 \quad \Longleftrightarrow \quad M \otimes_R \kappa(\mathfrak{p}) \neq 0.$$

Extension of scalars from $\kappa(\mathfrak{p})$ to E' preserves nonzero vector spaces. □

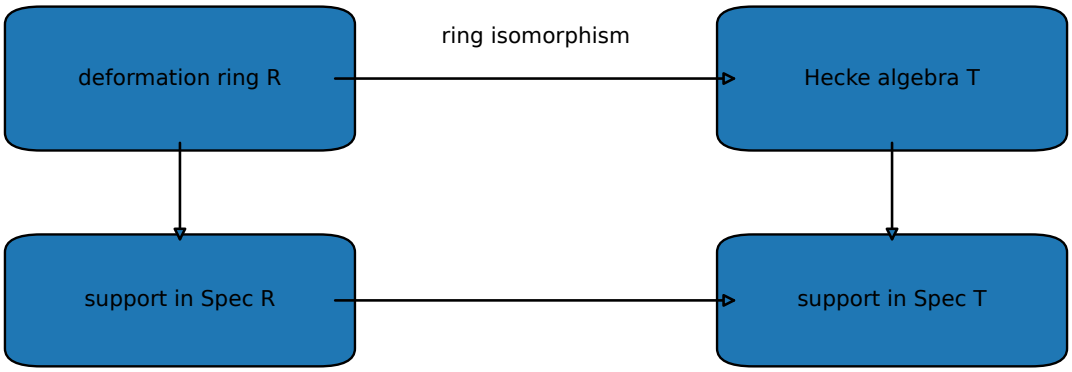


FIGURE 9. An isomorphism transports occupied support exactly. A quotient can remove a component.

Corollary 7.5 (Supported deformation points give nonzero Hecke fibers)**.** *Let $R \cong \mathbb{T}$, and let M be a finite Hecke module. If a point $x : R \to E'$ lies in $\mathrm{Supp}_R(M)$, then the corresponding Hecke fiber*

$$M \otimes_{\mathbb{T},x} E'$$

is nonzero.

Proof. Transport the support through theorem 7.1, then apply theorem 7.4. □

8 Finite Hecke packets and infinite occupancy

Finite Hecke carriers admit canonical packets even when the Hecke action is not semisimple. The packets come from the Artin local factors of the finite commutative Hecke algebra.

Standard Fact 8.1 (Artin decomposition)**.** Let E' be a field and let $\mathbb{T}$ be a finite dimensional commutative E' algebra. Then

$$\boxed{\mathbb{T} \cong \prod_{\mathfrak{m} \in \mathrm{Max}\,\mathbb{T}} \mathbb{T}_{\mathfrak{m}}.}$$

Each factor is an Artin local algebra. Equivalently, there is a unique complete family of central idempotents $(e_{\mathfrak{m}})$ such that

$$e_{\mathfrak{m}} e_{\mathfrak{n}} = 0 \quad (\mathfrak{m} \neq \mathfrak{n}), \qquad \sum_{\mathfrak{m}} e_{\mathfrak{m}} = 1,$$

and $e_{\mathfrak{m}}\mathbb{T} \cong \mathbb{T}_{\mathfrak{m}}$ [2, 15].

Theorem 8.2 (Canonical Hecke packet decomposition)**.** *Let $\mathbb{T}$ be a finite dimensional commutative algebra over a field E', and let V be a finite $\mathbb{T}$ module. Then*

$$\boxed{V = \bigoplus_{\mathfrak{m} \in \mathrm{Max}\,\mathbb{T}} e_{\mathfrak{m}} V.}$$

For every maximal ideal $\mathfrak{m}$, the canonical map gives

$$\boxed{e_{\mathfrak{m}} V \cong V_{\mathfrak{m}}.}$$

The packet $e_{\mathfrak{m}} V$ is nonzero exactly when $\mathfrak{m} \in \mathrm{Supp}_{\mathbb{T}}(V)$.

Proof. The idempotent relations in standard fact 8.1 give the direct sum decomposition. Localization at $\mathfrak{m}$ kills every factor except the $\mathfrak{m}$ factor, where $e_{\mathfrak{m}}$ becomes one. Thus the localization map identifies $e_{\mathfrak{m}} V$ with $V_{\mathfrak{m}}$. The final statement is the definition of support. □

Corollary 8.3 (Simultaneous generalized eigenspaces)**.** *Assume that E' is algebraically closed and that $\mathbb{T}$ is generated by commuting operators $T_1, \dots, T_d$ on V. For a maximal ideal*

$$\mathfrak{m} = (T_1 - \lambda_1, \dots, T_d - \lambda_d),$$

there is an integer $N \geq 1$ such that

$$\boxed{e_{\mathfrak{m}} V = \left\{ v \in V : (T_i - \lambda_i)^N v = 0 \text{ for every } i \right\}.}$$

Thus the canonical Artin packet is the simultaneous generalized eigenspace.

Proof. The maximal ideal of the Artin local algebra $\mathbb{T}_{\mathfrak{m}}$ is nilpotent. Hence a common power of every $T_i - \lambda_i$ kills the local packet. On every other local factor, at least one $T_i - \lambda_i$ is a unit, so no nonzero vector can satisfy all the displayed nilpotence conditions.

Conversely, suppose every $(T_i - \lambda_i)^N$ kills v. Every monomial of total degree greater than $d(N-1)$ contains one factor to exponent at least N. Therefore a power of $\mathfrak{m}$ kills v, which places v in the $\mathfrak{m}$ primary packet. □

Corollary 8.4 (Rank and trace detection in characteristic zero)**.** *Assume that E' has characteristic zero. For every maximal Hecke packet,*

$$\boxed{\mathrm{Tr}_{E'}(e_{\mathfrak{m}} | V) = \mathrm{rank}_{E'}(e_{\mathfrak{m}} | V) = \dim_{E'} e_{\mathfrak{m}} V.}$$

In particular, the packet is occupied exactly when its projector has positive rank, equivalently nonzero trace.

Proof. An idempotent linear map has minimal polynomial dividing $X(X-1)$, which has distinct roots. It is diagonalizable with eigenvalues zero and one. Its trace and rank both equal the multiplicity of the eigenvalue one. A positive integer is nonzero in characteristic zero. □

Remark 8.5. The trace criterion can fail in positive characteristic when the packet dimension is divisible by the characteristic. The rank and localization criteria remain valid.

Example 8.6 (A nonsemisimple Hecke packet)**.** Let

$$\mathbb{T} = E'[u]/(u^2) \times E'[v]/(v^3)$$

and let $V = \mathbb{T}$ be the regular module. The two canonical central idempotents split

$$V = V_u \oplus V_v$$

with dimensions two and three. Multiplication by u is nilpotent on V_u, and multiplication by v is nilpotent on V_v. Neither local factor is semisimple, but the two maximal Hecke packets are isolated exactly by the Artin idempotents.

Proof. The product decomposition gives the two central idempotents $(1,0)$ and $(0,1)$. Their ranges are the two local factors. The classes of u and v are nilpotent of orders two and three, respectively. □

The next two theorems distinguish a fixed finite carrier from a sequence of changing carriers.

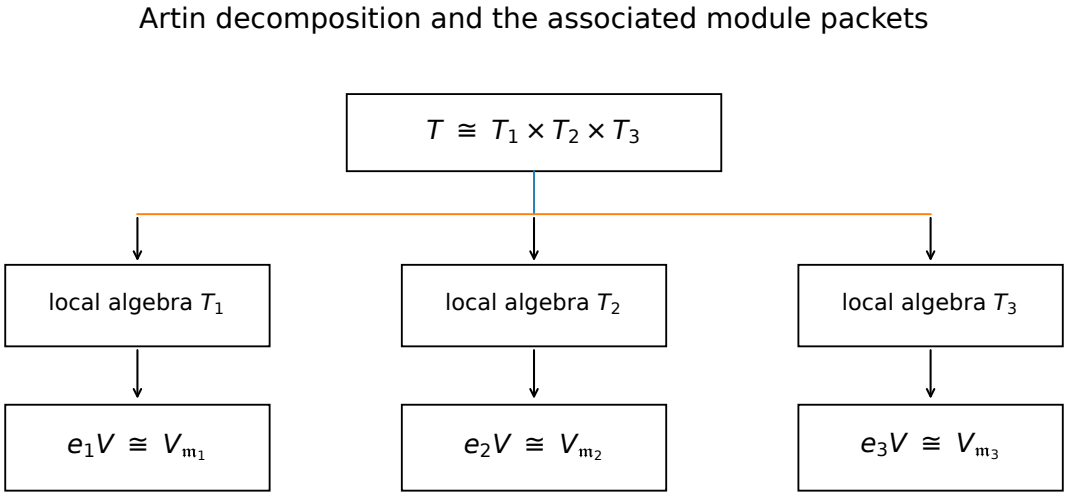


FIGURE 10. A finite commutative Hecke algebra splits into canonical Artin local packets.

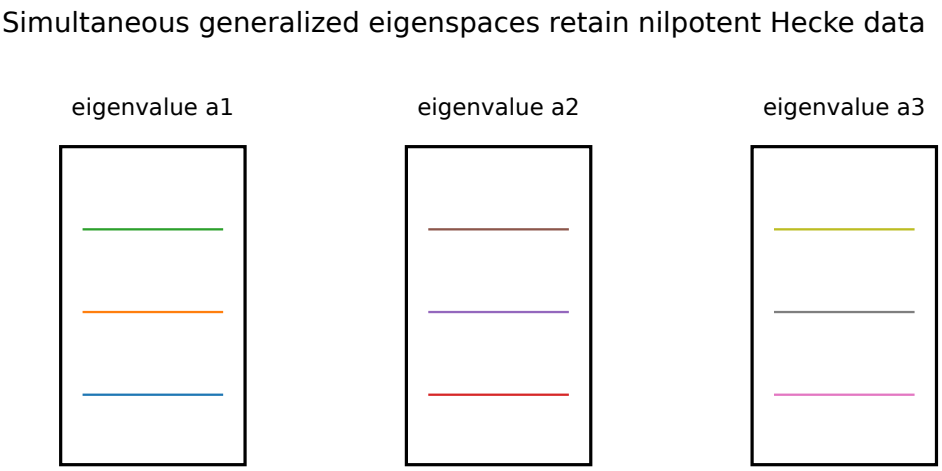


FIGURE 11. An Artin packet is a simultaneous generalized eigenspace, not necessarily an ordinary eigenspace.

Theorem 8.7 (Nested packets in one finite carrier)**.** *Let V be a finite dimensional vector space. Let $e_1, e_2, \dots$ be idempotents such that*

$$e_{m+1}e_m = e_m e_{m+1} = e_{m+1}$$

for every m. If $e_m V \neq 0$ for every m, then the descending chain stabilizes and

$$\boxed{\bigcap_{m\geq 1} e_m V \neq 0.}$$

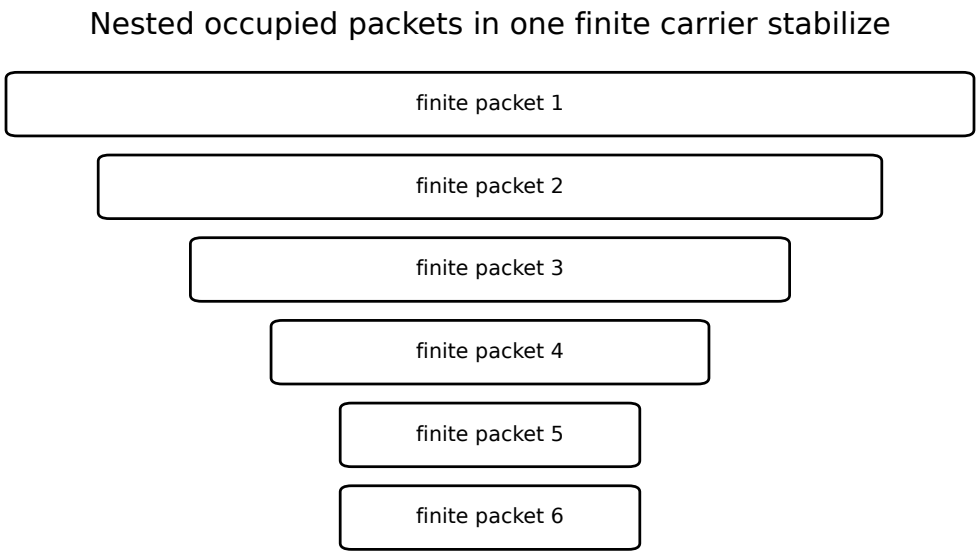


FIGURE 12. Nested nonzero packets in one finite carrier stabilize after finitely many steps.

Proof. The displayed relations give

$$e_{m+1}V \subseteq e_mV.$$

The nonnegative dimensions form a nonincreasing sequence of positive integers, so they eventually become constant. Once two consecutive dimensions agree, the corresponding inclusion is equality. Thus the ranges stabilize at a nonzero subspace, which is their intersection. □

Corollary 8.8 (Finite witness of incompatibility)**.** *Under the hypotheses of theorem 8.7, if*

$$\bigcap_{m\geq 1} e_mV = 0,$$

then $e_mV = 0$ for some finite m.

Proof. This is the contrapositive of theorem 8.7. □

Definition 8.9 (Stable core)**.** Let

$$V_1 \longleftarrow V_2 \longleftarrow V_3 \longleftarrow \cdots$$

be an inverse system of finite dimensional vector spaces. For $n \geq m$, write $f_{n,m} : V_n \to V_m$. The stable core at level m is

$$Y_m = \bigcap_{n\geq m} \operatorname{im}(f_{n,m}).$$

Lemma 8.10 (Surjectivity on stable cores)**.** *For every m, the transition map $V_{m+1} \to V_m$ induces a surjection*

$$\boxed{Y_{m+1} \twoheadrightarrow Y_m.}$$

Proof. Fix $y \in Y_m$. For $n \geq m+1$, let

$$A_n = \{z \in \operatorname{im}(f_{n,m+1}) : f_{m+1,m}(z) = y\}.$$

Each A_n is a nonempty affine subspace of V_{m+1}, and the sequence (A_n) is descending. A descending sequence of affine subspaces in a finite dimensional vector space stabilizes whenever every term is nonempty. Choose an element in its stable value. It lies in Y_{m+1} and maps to y. □

Theorem 8.11 (Nonzero stable core criterion)**.** *Let (V_m) be an inverse system of finite dimensional vector spaces. If*

$$Y_m \neq 0$$

for every m, then

$$\boxed{\varprojlim_m V_m \neq 0.}$$

More precisely, the inverse limit contains a compatible sequence whose first coordinate is any prescribed nonzero element of Y_1.

Proof. Choose $0 \neq y_1 \in Y_1$. By lemma 8.10, choose recursively $y_{m+1} \in Y_{m+1}$ mapping to y_m. Every y_m is nonzero because it maps to the nonzero vector y_1. The sequence (y_m) is a nonzero element of the inverse limit. □

Corollary 8.12 (Surjective systems)**.** *If every transition map $V_{m+1} \to V_m$ is surjective and every V_m is nonzero, then the inverse limit is nonzero.*

Proof. Surjectivity gives $Y_m = V_m$. Apply theorem 8.11. □

Example 8.13 (Tail escape in an infinite carrier)**.** Let $V = \ell^2(\mathbb{N})$, and let P_m be the orthogonal projection onto the closed span of

$$e_m, e_{m+1}, e_{m+2}, \dots.$$

Every range $P_m V$ is nonzero and the ranges are nested, but

$$\boxed{\bigcap_{m \geq 1} P_m V = 0.}$$

Thus the finite carrier hypothesis in theorem 8.7 cannot be removed.

Proof. A vector in every tail range has its first $m-1$ coordinates equal to zero for every m. All of its coordinates vanish. □

9 Pointwise automorphy from support

Support produces a nonzero Hecke fiber. A realization theorem is still required to turn that fiber into a classical automorphic representation and to identify its Galois representation.

Definition 9.1 (Automorphic Hecke datum)**.** An automorphic Hecke datum consists of the following objects.

(1) A commutative $\mathcal{O}$ algebra $\mathbb{T}$.
(2) A finite $\mathbb{T}$ module H, or a perfect $\mathbb{T}$ complex C, obtained from an automorphic or cohomological carrier.
(3) For every unramified place v outside a finite set, a polynomial
$$P_v(X) \in \mathbb{T}[X].$$
(4) A declared characteristic zero locus on which a nonzero field fiber is known to be classical and is known to carry an attached Galois representation whose Frobenius polynomial at v is the specialization of $P_v(X)$.

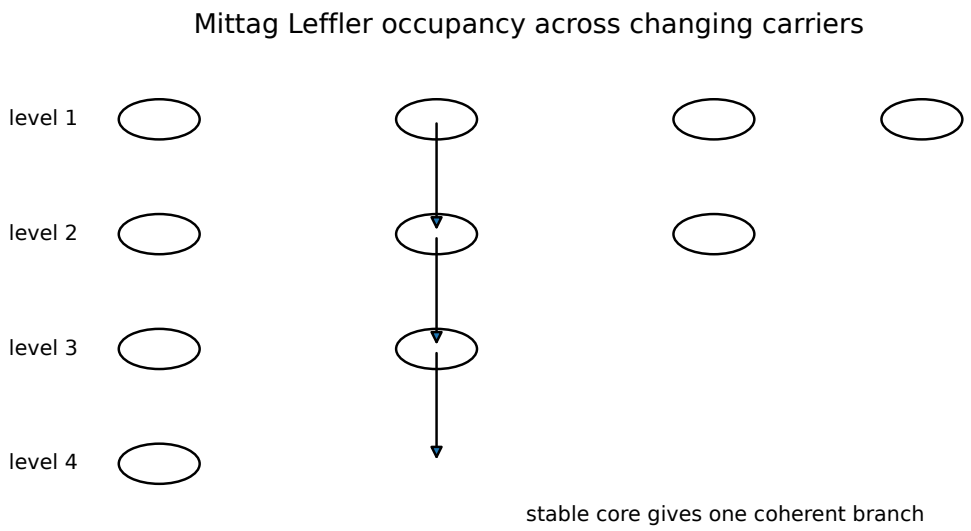


FIGURE 13. Changing finite carriers require a nonzero stable core, not only nonzero individual levels.

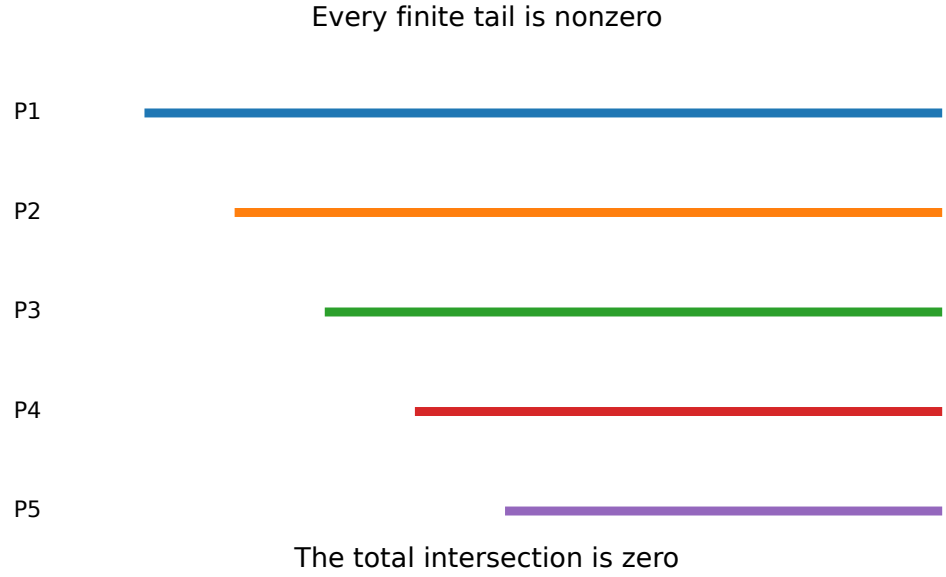


FIGURE 14. Nested nonzero ranges can escape to infinity in an infinite dimensional carrier.

Standard Fact 9.2 (Chebotarev and Brauer Nesbitt identification)**.** Let E' be a Hausdorff topological field of characteristic zero, with continuous ring operations. Let

$$\rho_1, \rho_2 : G_{F,S} \longrightarrow \mathrm{GL}_n(E')$$

be continuous semisimple representations. If their characteristic polynomials agree at every Frobenius element outside a density zero exceptional set, then

$$\boxed{\rho_1 \cong \rho_2.}$$

This follows from Chebotarev density and the Brauer Nesbitt theorem [5, 18, 12].

Theorem 9.3 (Support gives a Hecke eigensystem)**.** *Let $R \cong \mathbb{T}$, let H be a finite $\mathbb{T}$ module, and let $x : R \to E'$ be a field valued point. If*

$$\ker x \in \mathrm{Supp}_R(H),$$

then

$$\boxed{H_x := H \otimes_{\mathbb{T},x} E' \neq 0.}$$

Every element of $\mathbb{T}$ acts on H_x through the scalar prescribed by x.

Proof. Nonvanishing follows from corollary 7.5. The tensor product relation

$$th \otimes 1 = h \otimes x(t)$$

shows that $t \in \mathbb{T}$ acts through $x(t)$. □

Theorem 9.4 (Pointwise Galois realization)**.** *Let E' be a Hausdorff topological field of characteristic zero, with continuous ring operations. Require both $\rho_x = x \circ \rho_R$ and the attached representation below to be continuous. Assume the following data.*

(1) A represented deformation ring R with universal representation

$$\rho_R : G_{F,S} \longrightarrow \mathrm{GL}_n(R).$$

(2) An isomorphism $R \cong \mathbb{T}$.

(3) An automorphic Hecke datum $(\mathbb{T}, H, (P_v))$.

(4) A characteristic zero point $x : R \to E'$ lying in $\mathrm{Supp}_R(H)$.

(5) A classical automorphic representation π_x occurring in the fiber H_x, together with a semisimple attached representation

$$\rho_{\pi_x} : G_{F,S} \longrightarrow \mathrm{GL}_n(E')$$

whose unramified characteristic polynomial is $x(P_v(X))$.

(6) Equality

$$\det\left(X - \rho_x(\mathrm{Frob}_v)\right) = x(P_v(X))$$

for $\rho_x = x \circ \rho_R$ at every unramified place in a Chebotarev dense set.

Then

$$\boxed{\rho_x^{\mathrm{ss}} \cong \rho_{\pi_x}.}$$

If ρ_x is irreducible, then $\rho_x \cong \rho_{\pi_x}$.

Proof. The coefficients of the characteristic polynomials of ρ_x and ρ_{π_x} are continuous and constant on conjugacy classes. Their equality on the declared dense union of Frobenius conjugacy classes therefore extends to every element of $G_{F,S}$, by Hausdorffness. A semisimplification has the same characteristic polynomials as the original representation. The algebraic Brauer Nesbitt input in standard fact 9.2 thus gives $\rho_x^{\mathrm{ss}} \cong \rho_{\pi_x}$. If ρ_x is irreducible, it is already semisimple. □

Corollary 9.5 (Full support gives pointwise automorphy on the realized locus)**.** *In the setting of theorem 9.4, suppose*

$$\mathrm{Supp}_R(H) = \mathrm{Spec}\, R.$$

For every characteristic zero point $x : R \to E'$ with $\ker x \in U \subseteq \mathrm{Spec}\, R$, assume the coefficient topology and continuity hypotheses of theorem 9.4 and the declared classicality and Galois realization input. Then every such point is automorphic in the precise sense of theorem 9.4.

Proof. Every point of U lies in the support, so theorem 9.3 gives a nonzero Hecke fiber. Apply the declared realization input and then theorem 9.4. □

Theorem 9.6 (Derived pointwise realization)**.** *Let $e : R \cong \mathbb{T}$, and let C be a perfect automorphic Hecke complex with full derived support. Choose a bounded finite projective model for C, supported in degrees $[a, b]$. For every characteristic zero point $x : R \to E'$, write $y = x \circ e^{-1}$. Then some integer $i \in [a, b]$ has*

$$H^i\left(C \overset{\mathbf{L}}{\underset{\mathbb{T},y}{\otimes}} E'\right) \neq 0.$$

Every $t \in \mathbb{T}$ acts on this cohomology through $y(t)$.

For the realization conclusion, give E' a Hausdorff topological field structure. Supply a universal representation ρ_R, a continuous specialization $\rho_x = x \circ \rho_R$, and a semisimplification of ρ_x. Suppose its characteristic polynomials on a Chebotarev dense set are $y(P_v(X))$. Let the declared classicality range contain $[a, b]$. Assume that every nonzero Hecke eigensystem in that range is realized by a classical automorphic representation occurring in that degree, with a continuous semisimple attached representation whose characteristic polynomials on the same dense set are $y(P_v(X))$. With the standard Chebotarev density and Brauer Nesbitt inputs of standard fact 9.2, the specialization ρ_x is automorphic after semisimplification. If ρ_x is irreducible, it is itself isomorphic to the attached representation.

Proof. Full derived support gives a nonzero fiber at y. Scalar extension of the chosen projective model preserves its degree bounds. Its cohomology therefore vanishes below a and above b. Since vanishing in every degree is equivalent to being zero in the derived category, some degree $i \in [a, b]$ has nonzero cohomology. Choose a nonzero vector in that cohomology module.

Multiplication by $t \in \mathbb{T}$ is a cochain endomorphism of the original model. In the scalar extension the tensor relation identifies this endomorphism with $y(t)$ times the identity. Passing to the derived category and then to cohomology preserves this equality. Thus the chosen vector is a common Hecke eigenvector with character y.

The interval inclusion puts this eigensystem in the declared classicality range. Apply the realization input to obtain an occurring automorphic representation and its continuous semisimple attached representation. Its Frobenius characteristic polynomials agree with those of ρ_x on the prescribed dense set. Continuity extends the equality to every group element. The supplied semisimplification preserves these polynomials, so the general Brauer Nesbitt input gives an invertible linear intertwiner. If ρ_x is irreducible, it is already semisimple and the same comparison applies directly. □

Remark 9.7. Derived support does not identify the cohomological degree, Hodge type, or classical weight of the surviving eigensystem. Those properties must come from the external local and global realization theorem.

10 Compatible systems and companion completion

Pointwise automorphy at one coefficient place and existence of companions at other coefficient places are different statements. They meet only after a common algebraic Frobenius spine has been fixed.

Definition 10.1 (Algebraic Frobenius spine)**.** An algebraic Frobenius spine consists of a number field L, a finite set of places S of F, and polynomials

$$P_v(X) \in L[X]$$

for every finite place $v \notin S$, together with any additional local labels required by the declared compatibility problem.

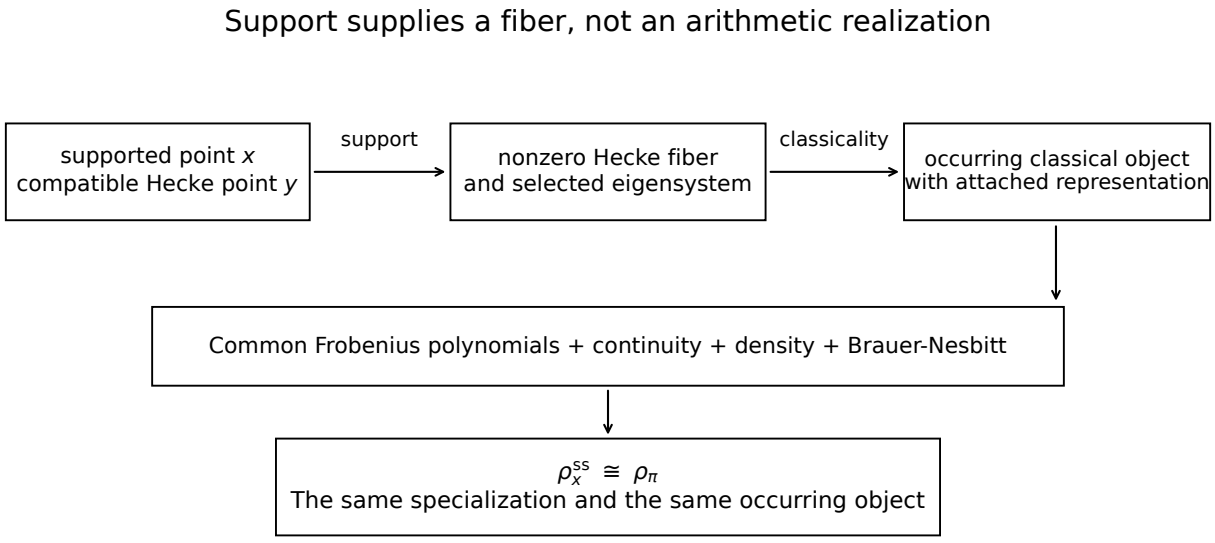


FIGURE 15. Support gives a nonzero Hecke fiber. Classicality and Galois realization complete the pointwise argument.

Definition 10.2 (Finite companion forest)**.** Fix a coefficient place λ of L. A finite companion forest for the spine is an inverse system of finite sets

$$X_{\lambda,1} \longleftarrow X_{\lambda,2} \longleftarrow X_{\lambda,3} \longleftarrow \cdots$$

whose points are admissible realizations modulo λ^m, with edges given by reduction. A complete realization is admissible for this forest when all its finite reductions belong to the displayed levels; the empty level obstruction below concerns this declared admissibility problem.

Theorem 10.3 (Companion branch criterion)**.** *Suppose every level $X_{\lambda,m}$ is finite and nonempty, and every point reduces along the displayed forest to the fixed residual root. Then*

$$\boxed{\varprojlim_m X_{\lambda,m} \neq \varnothing.}$$

Proof. The disjoint union of the levels is an infinite rooted tree. It is locally finite because every level is finite. It contains a vertex at every depth. König's infinity lemma gives an infinite branch [10]. That branch is a point of the inverse limit. □

Corollary 10.4 (Empty finite companion obstruction)**.** *If $X_{\lambda,m} = \varnothing$ for some m, then no complete λ adic realization admissible for this forest exists.*

Proof. A complete realization would reduce to a point at every finite level. □

Finite forests alone give compatible classes. To obtain actual continuous representations, we use the following explicit coefficient and admissibility data. They also delimit the meaning of completion in the next theorem.

Fix a topological group Γ, a finite matrix index set D, and a set Λ of coefficient indices. For each λ, let $\mathcal{O}_\lambda$ be a commutative ring, complete and separated for a finitely generated ideal I_λ, with finite residue ring $\mathcal{O}_\lambda / I_\lambda$. Give $\mathcal{O}_\lambda$ its ideal adic topology and every quotient its discrete topology. Let

$$i_\lambda : \mathcal{O}_\lambda \longrightarrow K_\lambda, \qquad e_\lambda : L \longrightarrow K_\lambda$$

be supplied ring homomorphisms, where L and K_λ are fields, K_λ has a declared topology, and i_λ is continuous.

For every positive integer n, let $X_{\lambda,n}$ be a finite nonempty set of conjugacy classes of actual continuous representations

$$\Gamma \longrightarrow \mathrm{GL}_D(\mathcal{O}_\lambda / I_\lambda^n).$$

Conjugacy means change of basis by an invertible matrix over that quotient. Assume reduction preserves these sets and that $X_{\lambda,1}$ consists of the declared residual class. Each admissible class must have a continuous representative. The auxiliary level zero is the unique class over the trivial quotient and imposes no extra occupancy hypothesis.

Choose elements $\gamma_v \in \Gamma$ on a sample set Ω, with declared good subsets $\Omega_\lambda \subseteq \Omega$. In the arithmetic application these are the chosen Frobenius elements at the permitted places. For each $v \in \Omega_\lambda$, supply an integral polynomial $P_{\lambda,v} \in \mathcal{O}_\lambda[X]$ such that

$$i_\lambda(P_{\lambda,v}) = e_\lambda(P_v) \quad \text{in } K_\lambda[X].$$

Here and below a ring homomorphism applied to a polynomial acts on its coefficients. Assume every representative of every class in $X_{\lambda,n}$ has characteristic polynomial $P_{\lambda,v}$ mod I_λ^n at γ_v. Also supply a subgroup $H_\lambda \leq \Gamma$ on which all these finite representations are trivial. This is an inertia condition when H_λ is the specified inertia subgroup.

Finally, put

$$B_\lambda = \mathcal{O}_\lambda[x_{g,a,b} : g \in \Gamma,\ a, b \in D].$$

For each declared local condition index u, supply an ideal $J_{\lambda,u} \subseteq B_\lambda$. Evaluation at a representation sends $x_{g,a,b}$ to its (a, b) matrix entry at g. Assume that evaluation at every admissible finite representative kills $J_{\lambda,u}$, with polynomial coefficients reduced modulo I_λ^n. For this class based theorem, the equations must hold in every admissible frame. A condition imposed only in one chosen frame needs a separate compatible frame argument. No arbitrary arithmetic condition is identified with such an ideal without a separate argument.

Theorem 10.5 (Conditional simultaneous companion completion)**.** *Under the coefficient and finite admissibility hypotheses above, one can choose continuous representations*

$$\rho_\lambda : \Gamma \longrightarrow \mathrm{GL}_D(\mathcal{O}_\lambda) \qquad (\lambda \in \Lambda)$$

whose reduction class belongs to $X_{\lambda,n}$ for every positive n. They retain the declared residual class. After applying i_λ, they are continuous, trivial on H_λ, and satisfy

$$\det(X - i_\lambda(\rho_\lambda(\gamma_v))) = e_\lambda(P_v), \qquad v \in \Omega_\lambda.$$

Their integral evaluation points factor through every actual quotient $B_\lambda / J_{\lambda,u}$.

Proof. The finitely generated ideal and finite residue ring give finite coefficient quotients at every precision. An individual conjugacy class has finitely many frames because the group of invertible basis matrices over each quotient is finite. Thus the finitely many admissible classes give a finite nonempty set of actual continuous representations at each positive level. Continuity is invariant under conjugation. Reduction gives total maps between these sets. Compactness, as in theorem 10.3, selects strictly compatible representatives. This step neither assumes transition surjectivity nor lifts a prescribed conjugating matrix.

Completeness and separatedness identify $\mathrm{GL}_D(\mathcal{O}_\lambda)$ with the compatible inverse limit of the quotient matrix groups. Lift entries and inverse entries to construct this identification. The two reduction and reconstruction identities hold entrywise. Finite quotient continuity implies continuity for the ideal adic topology, including the matrix inverse, so the reconstructed representation is continuous.

Each coefficient of its characteristic polynomial agrees with the prescribed integral coefficient modulo every positive power of I_λ. Separatedness gives equality of the full polynomials. The same argument applied to the matrix entries of $\rho_\lambda(g) - 1$ gives triviality on H_λ. Continuous coefficient change and the matching equations for $P_{\lambda,v}$ give the asserted common spine over the local fields.

For $p \in J_{\lambda,u}$, evaluation commutes with reduction. Its value is zero modulo every positive power of I_λ, hence is zero integrally by separatedness. Therefore the evaluation homomorphism kills the whole ideal and factors uniquely through $B_\lambda / J_{\lambda,u}$. Choose a realization for each λ. □

The quotient conclusion is a proved effectivity statement for the declared polynomial relations. For local conditions presented by continuous maps from represented local rings, corollary 3.2 also accounts for the closed ideal generated by their defining ideals. Neither statement constructs a deformation ring for a condition that has not been shown representable. Hodge conditions, Weil–Deligne conditions, integral lifts of arbitrary Frobenius polynomials, and the finite occupancy hypotheses still require their own arithmetic inputs. The theorem retains a residual conjugacy class, not a prescribed residual matrix frame, and asserts no uniqueness up to integral conjugacy.

Standard Fact 10.6 (Strict compatibility and one member determination)**.** A strictly compatible semisimple system is determined at every coefficient place by its common unramified characteristic polynomials. If an automorphic representation gives a compatible system with the same polynomials, then Chebotarev density and Brauer Nesbitt identify the two systems member by member after semisimplification [23, 3].

Theorem 10.7 (One member propagation in a fixed system)**.** *Let*

$$\{\rho_\lambda\}_{\lambda \in \Lambda}$$

be a semisimple strictly compatible system with Frobenius spine (P_v). Suppose that for one coefficient place λ_0, the representation ρ_{λ_0} is identified with the Galois representation attached to an automorphic representation π. Assume that the compatible system attached to π has the same polynomials $P_v(X)$. Then for every $\lambda \in \Lambda$,

$$\boxed{\rho_\lambda \cong \rho_{\pi,\lambda}^{\mathrm{ss}}.}$$

Proof. For every λ, the two semisimple representations have the same unramified characteristic polynomials at all places outside one finite set. Apply standard fact 9.2. □

Remark 10.8. The theorem propagates automorphy inside a compatible system that already exists. It does not construct a missing coefficient place member and does not prove universal companion existence.

Proposition 10.9 (Finite Frobenius separator)**.** *Let $t_1, \dots, t_r$ be linearly independent functions on a set Ω with values in a field E'. Then there are points $\omega_1, \dots, \omega_r \in \Omega$ such that the matrix*

$$\big(t_i(\omega_j)\big)_{j,i}$$

is invertible. In particular, when the trace functions of finitely many compatible constituents are linearly independent on the unramified Frobenius set, finitely many Frobenius traces recover their multiplicities in any known semisimple carrier.

Proof. Let

$$W = \operatorname{span}\{(t_1(\omega), \dots, t_r(\omega)) : \omega \in \Omega\} \subseteq E'^r.$$

If $W \neq E'^r$, a nonzero linear functional on E'^r would vanish on every evaluation vector, giving a nontrivial linear relation among the functions. Thus $W = E'^r$. Choose r evaluation vectors forming a basis. □

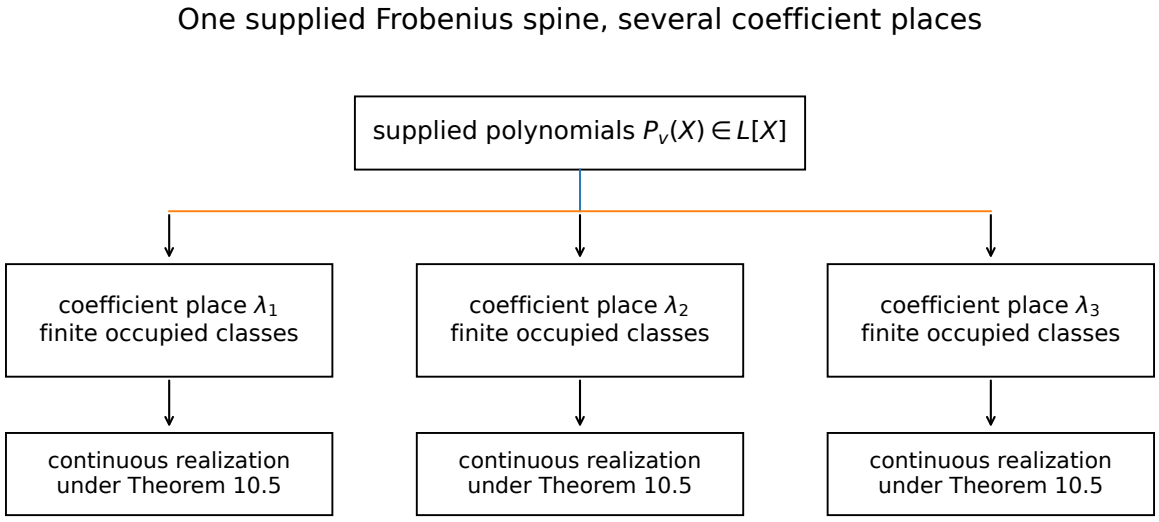


FIGURE 16. Under the coefficient and effectivity hypotheses of theorem 10.5, local companion branches realize the Frobenius spine fixed in advance.

11 The coefficient prime two

The auxiliary augmentation algebra works at every coefficient prime. What changes at two is the availability of integral character projectors and, in many applications, the residual image and local deformation hypotheses.

Proposition 11.1 (Failure of integral sign projectors). *Let $C_2 = \langle g \rangle$. Over $\mathbb{Q}_2$, the two sign projectors are*

$$\boxed{e_+ = \frac{1+g}{2}, \qquad e_- = \frac{1-g}{2}.}$$

Neither belongs to $\mathbb{Z}_2[C_2]$. After reduction modulo two,

$$\boxed{\mathbb{F}_2[C_2] \cong \mathbb{F}_2[X]/(X^2), \qquad X = g - 1.}$$

This ring is local and has no nontrivial idempotents.

Proof. The displayed rational elements are the unique primitive idempotents because they project to the two characters of C_2. Their coefficients contain $1/2$, so they are not integral. In characteristic two,

$$(g-1)^2 = g^2 - 2g + 1 = 0.$$

The map $X \mapsto g - 1$ gives the displayed isomorphism. The quotient by (X) is $\mathbb{F}_2$, and (X) is nilpotent. Thus the ring is local. □

Corollary 11.2 (Dyadic coinvariant recovery). *Let Δ be a finite two group, let $\Lambda = \mathcal{O}[\Delta]$ with $\mathcal{O}$ a finite extension of $\mathbb{Z}_2$, and let M be finite free over Λ. Then*

$$\boxed{M \otimes_{\Lambda,\epsilon} \mathcal{O} \cong M/\mathfrak{a}M}$$

is free over $\mathcal{O}$ of the same Λ rank. No semisimple character decomposition is required.

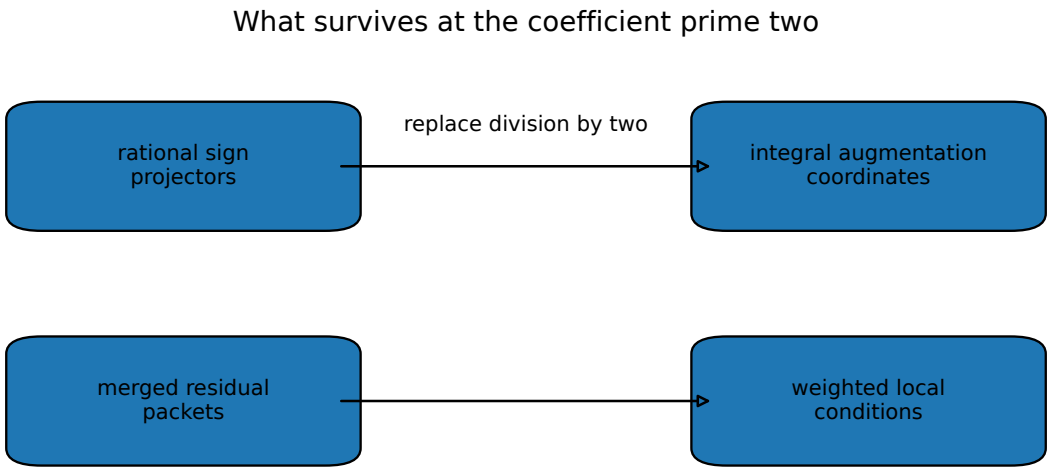


FIGURE 17. Augmentation survives at two, while semisimple character projectors and arithmetic hypotheses require separate treatment.

Proof. Use the right module convention fixed in section 5. Quotienting by all balancing relations identifies $M \otimes_{\Lambda,\epsilon} \mathcal{O}$ with $M/\mathfrak{a}M$, without any commutativity assumption on Δ. Applying ϵ to the coordinates of a finite Λ basis gives an $\mathcal{O}$ basis with the same index set. □

Remark 11.3. The corollary is an algebraic statement about augmentation and coinvariants. It does not supply a coefficient two adequacy theorem, a local deformation ring with the desired geometry, or a classicality theorem. Those inputs must be checked in the automorphy lifting theorem being used.

12 Conditional automorphic completion

We now assemble the preceding statements. The theorem is deliberately written as a chain of gates. Each gate has a distinct mathematical source.

Definition 12.1 (Ordinary completion datum at a point)**.** An ordinary completion datum at a characteristic zero point $x : R \to E'$ consists of the following objects.

(1) A represented global deformation ring R with declared local quotient conditions and universal representation ρ_R.

(2) An ordinary patching datum
$$(R_\infty, \mathfrak{a}, M_\infty, R, H)$$
whose finite level module H is an automorphic Hecke carrier.

(3) Full patched support
$$\operatorname{Supp}_{R_\infty}(M_\infty) = \operatorname{Spec} R_\infty.$$

(4) An isomorphism
$$R \cong \mathbb{T}$$
with the Hecke algebra acting on H. The R action on H supplied by the patched control isomorphism is the restriction of this $\mathbb{T}$ action along the displayed isomorphism.

(5) An automorphic Hecke datum $(\mathbb{T}, H, (P_v))$ and a pointwise classicality and Galois realization theorem satisfying all the inputs of theorem 9.4 for the nonzero fiber at x: an occurring classical automorphic representation with a continuous semisimple attached representation; equality of the characteristic polynomials of both that representation and $\rho_x = x \circ \rho_R$ with $x(P_v(X))$ on the same declared Chebotarev dense union of Frobenius conjugacy classes; and the standard semisimplification, density, and general Brauer Nesbitt inputs. The coefficient topology and continuity requirements are retained in theorem 12.2.

Theorem 12.2 (Ordinary conditional automorphic completion)**.** *Let $x : R \to E'$ admit an ordinary completion datum. Assume E' is a Hausdorff topological field of characteristic zero with continuous ring operations, and that both $\rho_x = x \circ \rho_R$ and its attached comparison representation are continuous. Then*

$$\boxed{H \otimes_{\mathbb{T},x} E' \neq 0.}$$

The specialization

$$\rho_x = x \circ \rho_R$$

is identified after semisimplification with the Galois representation attached to a classical automorphic representation in that fiber. If ρ_x is irreducible, it is itself automorphic.

Proof. Full patched support descends to full support of H over R by theorem 6.11. The isomorphism $R \cong \mathbb{T}$ transports that support by theorem 7.1. The field fiber is nonzero by theorem 9.3. The declared classicality and realization theorem supplies an automorphic representation in the fiber and an attached Galois representation. The comparison with ρ_x follows from theorem 9.4. □

Definition 12.3 (Derived completion datum at a point)**.** A derived completion datum is defined in the same way, with a perfect patched complex C_∞ of full derived support and a finite level automorphic complex

$$C \simeq C_\infty \overset{\mathbf{L}}{\underset{R_\infty}{\otimes}} R$$

in place of the ordinary modules. Under the Hecke ring isomorphism $e : R \cong \mathbb{T}$, require an actual derived carrier identification between the scalar transport of this finite level complex and the automorphic Hecke complex. This identification retains the R action induced by patched control as the restriction of the $\mathbb{T}$ action along e.

Theorem 12.4 (Derived conditional automorphic completion)**.** *Let $x : R \to E'$ admit a derived completion datum, with Hecke identification $e : R \cong \mathbb{T}$, and put $y = x \circ e^{-1}$. Then*

$$\boxed{C \overset{\mathbf{L}}{\underset{\mathbb{T},y}{\otimes}} E' \not\simeq 0.}$$

Some integer cohomological degree contains a nonzero Hecke eigensystem with character y. Its degree lies between the bounds of any chosen finite projective model for C.

For the conditional conclusion, retain the Hausdorff coefficient topology, continuous specialization, supplied semisimplification, and common Frobenius polynomial hypotheses in theorem 9.6. If the declared classicality and Galois realization theorem applies to this eigensystem and supplies an occurring classical automorphic representation with a continuous semisimple attached representation and those Frobenius polynomials, then ρ_x is automorphic after semisimplification. If ρ_x is irreducible, it is itself isomorphic to the attached representation. This pointwise applicability condition does not require a uniform classicality assertion in other degrees.

Proof. Full support descends from the patched perfect complex by theorem 6.18. Scalar transport through e, followed by the actual derived isomorphism with the automorphic Hecke complex, gives full support

for C. Its fiber at y is therefore nonzero. The bounded model and induced Hecke action argument in theorem 9.6 supplies a nonzero cohomology eigenvector in an integer degree within the model's bounds.

If realization applies to this eigensystem, it supplies the occurring automorphic object and a continuous semisimple attached representation with the declared Frobenius polynomials. The representation comparison in theorem 9.6 then identifies the supplied semisimplification of ρ_x with that attached representation. Only applicability to this eigensystem is used; a classicality theorem for every degree in a prescribed interval is unnecessary. □

Theorem 12.5 (Compatible system completion from one automorphic member)**.** *Let*

$$\{\rho_\lambda\}_{\lambda\in\Lambda}$$

be a semisimple strictly compatible system with common Frobenius spine. Suppose the member at λ_0 admits an ordinary or derived completion datum and is identified with the Galois representation attached to an automorphic representation π. Assume that the compatible system attached to π has the same Frobenius spine. Then

$$\boxed{\rho_\lambda \cong \rho_{\pi,\lambda}^{\mathrm{ss}} \quad \text{for every } \lambda \in \Lambda.}$$

Proof. The completion theorem at λ_0 gives the automorphic representation π. Apply theorem 10.7 to the two compatible systems. □

Theorem 12.6 (Five gate formulation)**.** *Fix a characteristic zero deformation point $x : R \to E'$. Give E' a Hausdorff topological field structure. The conditional argument uses the following five gates.*

(1) ***Residual gate.*** *Residual occupancy in the declared automorphic carrier and representability of the chosen deformation problem are supplied arithmetic inputs. In particular, the deformation ring R and its universal representation ρ_R are available.*

(2) ***Patching gate.*** *An applicable patching theorem supplies an ordinary finite module or a perfect complex with full patched support, together with the coefficient quotient and the module or derived control isomorphism. Write M for the resulting finite level carrier over R.*

(3) ***Comparison gate.*** *Let H be a finite ordinary Hecke module or a perfect Hecke complex over $\mathbb{T}$. Supply a ring map $f : R \to \mathbb{T}$ satisfying*

$$(f^*)^{-1}\big(\mathrm{Supp}_R(M)\big) \subseteq \mathrm{Supp}_{\mathbb{T}}(H).$$

Use ordinary or derived support according to the carrier. Actual scalar extension satisfies this inclusion, as does transport through a ring isomorphism with the corresponding carrier identification.

(4) ***Fiber gate.*** *Supply a Hecke point $y : \mathbb{T} \to E'$ with $y \circ f = x$. Then the fiber of H at y is nonzero. It contains a nonzero Hecke eigenvector, in some integer cohomological degree in the derived case, with character y.*

(5) ***Realization gate.*** *The declared classicality theorem applies to the resulting eigensystem and supplies an occurring classical automorphic representation π, with a continuous semisimple attached representation. Require the specialization $\rho_x = x \circ \rho_R$ to be continuous and supply its semisimplification. On the same Chebotarev dense set, assume that the characteristic polynomials of both ρ_x and the attached representation are $y(P_v(X))$. Retain the standard density and general Brauer Nesbitt inputs of standard fact 9.2.*

Then ρ_x^{ss} is isomorphic to the representation attached to π. If ρ_x is irreducible, the same conclusion holds for ρ_x itself.

For propagation, suppose both a semisimple compatible family $\{\rho_\lambda\}$ containing ρ_x and a family $\{\rho_{\pi,\lambda}\}$ attached to this same π already exist. Require the distinguished members to be ρ_x and its attached representation, respectively. Assume the coefficient, continuity, density, semisimplification and common algebraic Frobenius spine hypotheses

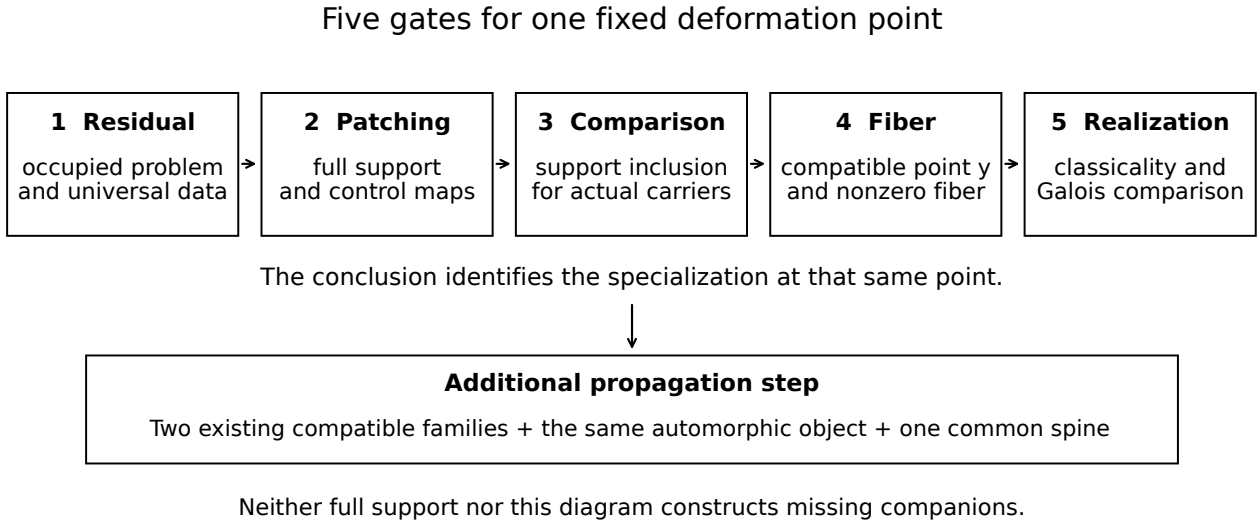


FIGURE 18. The final theorem is a composition of separate residual, support, comparison, fiber, and realization gates.

of theorem 10.7, with that spine agreeing with the specialized Hecke polynomials at the distinguished place. Then

$$\rho_\lambda \cong \rho_{\pi,\lambda}^{\mathrm{ss}} \qquad \text{for every } \lambda.$$

Proof. The residual and representability inputs specify the setting in which the supplied patching theorem applies. The formal deduction begins with its actual rings, carriers and comparison maps. In the ordinary branch, finite generation and full support at finite level follow from the coefficient quotient and module control isomorphism. In the derived branch, the derived control isomorphism gives a perfect finite level model, and theorem 6.18 gives full support.

The comparison inclusion transports this full support to H. More specifically, $y \circ f = x$ identifies the contraction of $\ker y$ with $\ker x$, so support at the source point gives support at the Hecke point. In the ordinary case, theorem 7.4 gives a nonzero actual tensor fiber. In the derived case, the support criterion gives a nonzero actual derived fiber and hence a nonzero cohomology vector. Its integer degree lies within the bounds of the chosen finite projective model. In both cases, multiplication on the original carrier induces the scalar $y(t)$ on the specialized vector for every $t \in \mathbb{T}$.

Choose this nonzero eigensystem and apply the declared realization theorem to it. No realization statement in other degrees is needed. The two continuous representations have the same characteristic polynomials on the declared dense Frobenius conjugacy classes. Continuity extends their equality to the whole group. The supplied semisimplification preserves those polynomials, and the general Brauer Nesbitt input gives the required invertible linear intertwiner. The irreducible conclusion follows as in theorem 9.4.

Finally, retain the same π in its attached family. The distinguished member identities connect both families to the pointwise construction, while the common algebraic spine gives the required Frobenius polynomials at every coefficient place. Apply theorem 10.7 to these two existing families. The construction does not produce a missing companion or an attached compatible family. □

TABLE 1. Proved formal statements and arithmetic inputs that remain external.

Proved in this article	Not proved in this article
Local conditions combine through completed quotient rings	Representability of every desired local deformation condition
Full ordinary and derived support descends through the patching quotient	Near faithfulness or full support for every global deformation problem
An isomorphism $R \cong \mathbb{T}$ transports support and Fitting data	An $R \cong \mathbb{T}$ theorem outside published hypotheses
Finite Hecke algebras split into Artin packets	Positive automorphic occupancy of every residual or characteristic zero packet
Nonzero stable cores produce compatible inverse limit points	Universal residual companion existence
One automorphic member determines a fixed compatible system	Existence of a compatible system for every geometric representation
Augmentation and coinvariant recovery work at coefficient prime two	A universal coefficient two replacement for residual image, local geometry, and classicality hypotheses

13 Exact boundary of the article

The formal results above sharpen the logical role of patching support, Hecke packets, and compatible systems. They do not replace the arithmetic theorems that supply the gates.

Remark 13.1. The strongest global conclusion is conditional automorphic realization inside a declared class for which the five gates have been verified. The article does not prove unrestricted global Langlands reciprocity.

Remark 13.2. A numerical certificate can check the finite algebra appearing in an example. It cannot verify a published automorphy lifting theorem, a classicality theorem, or a universal existence statement.

Lean verification and revision scope

The preserved Lean release 0.17.0, reverified in evidence release 0.17.1, reports 748 named theorem declarations written for Paper XII, including supporting lemmas and interface results. A further 81 theorem declarations are reused unchanged from Paper X and counted separately. The cumulative verification rebuilds all 96 registered project modules with Lean 4.34.0-rc2 and warnings treated as errors. The transitive axiom audit covers 1,229 named declarations and permits only `propext`, `Classical.choice`, and `Quot.sound`. Three deliberately invalid controls test rejection of a false proof, a placeholder, and an incorrect empty family normalization. The pinned Mathlib dependency cache is not itself rebuilt. Exact compiler results, source hashes, and the verification scope are recorded with the archive.

The preserved release records complete mapped Lean statements for all 49 internal numbered claims of its revised manuscript. The present traditional edition retains those proof bodies, exposes the local-map and coefficient-target hypotheses in proposition 3.3, makes the $r > n$ convention explicit in the proof of theorem 6.4, states all the pointwise comparison inputs in definition 12.1, and declares a finite ramification set and coherent geometric Frobenius normalization. These editorial clarifications are recorded separately from the inherited eight amendments; they are not counted as newly compiled Lean certificates. No new statement amendment was needed for the completion and tensor results. The eight

inherited amendments remain explicitly separate from the original wording, and the cited arithmetic inputs remain outside the internal claim census. Formal proof coverage is recorded claim by claim in the roadmap. These checks certify the formal statements under their explicit hypotheses, not every prose claim, novelty assertion, or external arithmetic result.

The completion block proves theorem 3.1 and proposition 3.3 using actual algebraic tensor products, actual Hausdorff completions, and the actual closed generated ideals. Both inverse laws, continuity in both directions, and the canonical factor maps are checked. An independent comparison with the algebraic inverse limit completion identifies the topology and local ring structure. The global theorem retains continuous local algebra points and their naturality, rather than only an unstructured point existence statement. The empty family retains the scalar adic topology. The zero condition and unit ideal boundaries are checked separately.

The preceding Fitting block proves theorems 6.3, 6.4 and 7.1 and proposition 6.5. Its intrinsic Fitting ideals are independent of the chosen finite presentation, including the number of generators and relations. Arbitrary base change, all higher fiber dimension strata, and the complete support and Fitting transport statement are checked. The Noetherian and finite module hypotheses of the manuscript provide the finite presentation needed by the general construction.

The new checks for proposition 5.1, theorem 5.4, and corollary 11.2 include actual group algebra transfer and augmentation. Coinvariant recovery uses the quotient by all balancing relations, with its universal property proved. In the commutative case, this construction is identified with the ordinary tensor product. The augmentation relations generate exactly the augmentation ideal acting on the module. Applying augmentation to the coordinates of a supplied free basis gives a coefficient basis with the same index set. The right module argument applies to arbitrary groups, including nonabelian two groups, and includes rank zero.

The new checks for theorem 5.2, corollaries 5.3 and 8.4, and proposition 11.1 construct the full multivariable truncated polynomial algebra isomorphism and identify its augmentation generators. The residual group algebra is local for abelian p groups in characteristic p. For each supplied Hecke idempotent, the trace is the image dimension cast into the coefficient field; occupancy is equivalent to positive rank and, in characteristic zero, nonzero trace. The dyadic calculation gives the two rational sign projectors, proves that neither lies in the integral group algebra, and identifies the residual algebra with $\mathbb{F}_2[X]/(X^2)$.

Corollary 5.3 is now checked for every finite p group, including nonabelian groups, over every field of characteristic p. The proof uses a finite prime field orbit span and a nonzero fixed vector. The original unrestricted statement is restored, and the earlier explicit abelian restriction is no longer needed.

The checks for proposition 7.3, theorem 8.2, and corollary 8.3 now include the actual formal power series quotient and the canonical Hecke packet maps. In the node $k[[x,y]]/(xy)$, quotienting by (x) gives $k[[y]]$, with support $V(x)$; Lean verifies both irreducible components and the exclusion of the generic point (y). The components can still meet, so this is not a claim that their supports are disjoint.

For the Hecke results, the canonical ring decomposition in standard fact 8.1 remains a declared external input. Its coordinates are the actual localization maps. The module direct sum, the packet identification with the localized module, and the formula $v \mapsto v/1$ are constructed in Lean. Nilpotence in the selected Artin local factor, together with invertibility of a generator on each other factor, gives a single positive exponent for the simultaneous generalized eigenspace. The result includes the explicit generators $T_i - \lambda_i$, agrees with the packet used in the trace theorem, and does not assume semisimplicity.

The revised five gate statement in theorem 12.6 is now checked for both ordinary and derived carriers and for a general support comparison. The comparison is an actual inclusion of subsets of prime spectra, and scalar extension is proved to satisfy it. The coefficient quotient and ordinary or derived control

maps yield finite level support; that conclusion is not supplied as a separate patching assumption. The specialized tensor module or derived cohomology supplies an actual nonzero eigenvector. The declared realization theorem is applied to this particular eigensystem, and the resulting representation comparison is an actual invertible linear intertwiner. The system conclusion uses the family attached to the very same automorphic object, with distinguished member identities and a common algebraic polynomial spine.

The original comparison wording did not explicitly supply a compatible Hecke point over the fixed coefficient field. Support preservation alone is insufficient: extension from $\mathbb{Q}$ to $\mathbb{Q}(i)$ preserves full support, but there is no ring map $\mathbb{Q}(i) \to \mathbb{Q}$ extending the identity. The revision therefore states $y \circ f = x$ and exposes the arithmetic comparison and existing-family inputs. Residual occupancy, representability and the applicable patching theorem remain supplied arithmetic provenance. The original full statement remains partial. The completed tensor quotients and intrinsic Fitting ideals are now checked as described above, including the full Fitting transport clause.

The derived realization checks cover the revised theorems 9.6 and 12.4. They prove vanishing outside the bounds of an actual finite projective model and construct a nonzero cohomology vector inside those bounds. The Hecke action is induced from multiplication on the original cochain model, carried through scalar extension and derived localization, and proved to act by the specialized character on actual cohomology. The conditional arithmetic step gives an actual invertible linear intertwiner from the declared realization input and common Frobenius polynomials.

The original formulation of theorem 9.6 did not ensure that the surviving degree lies in the classicality range. A complex concentrated in degree one has full support over a field, while a classicality condition restricted to degree zero says nothing about its nonzero fiber. The revision requires the declared range to contain the model interval. The conditional conclusion of theorem 12.4 retains applicability at the selected eigensystem itself. Both statements now expose the topology, continuity, semisimplification, and common polynomial inputs needed for the representation comparison. Their original unconditional cohomological conclusions are checked, but their original complete statements remain partial. The general support comparison and compatible-point formulation is now checked in the revised theorem 12.6, as described above.

The derived support checks prove theorems 6.16 and 6.18 and corollary 6.19 as originally stated. Perfect complexes are actual derived category objects equipped with bounded finite projective models and derived isomorphisms to those models. Scalar extension is constructed on cochain complexes. Its associativity, preservation of homotopy equivalences, and independence under actual derived isomorphisms are checked. The quotient support formula uses the actual map of spectra and its image $V(I)$. Every field valued point factors through the residue field at its kernel, and faithful flatness of field extensions preserves and reflects acyclicity. The surviving integer degree is expressed using the actual derived cohomology functor, with a corresponding witness in the specialized cochain model. These proofs do not require the external cohomological support description in standard fact 6.15. The external classicality theorems remain supplied arithmetic inputs; the conditional derived realization argument is checked as described above.

The adequacy checks prove theorem 4.2 as originally stated. They construct the faithful standard representation from the actual permutation module, prove irreducibility after extension to the algebraic closure, and split every group element's characteristic polynomial over $\mathbb{F}_7$. The trace zero adjoint is the kernel of matrix trace with its conjugation action, and its sign plus standard decomposition is an actual equivariant linear isomorphism. The averaging proof is connected to Mathlib's actual first group cohomology, while the invariant calculation gives actual degree zero vanishing. Canonical projectors are identified with generalized eigenspace projectors for every positive exponent. Their trace pairings detect every nonzero endomorphism and hence every irreducible adjoint submodule. The two printed

constituent witnesses and their traces are checked exactly. The adequacy definition remains a cited convention; no external adequacy theorem is assumed in this finite certificate.

The realization checks cover theorems 9.4, 10.7, 12.2 and 12.5 and corollary 9.5. Characteristic polynomial coefficients are continuous by the principal minor formula. Equality on the declared Frobenius conjugacy classes therefore extends to the whole Galois group. The finite dimensional algebra input is explicit: a supplied semisimple representative preserves every characteristic polynomial, and a general Brauer Nesbitt theorem identifies semisimple representations with equal polynomials. Lean proves independence of this representative and the irreducible conclusion. Construction of semisimplification, the general Brauer Nesbitt theorem, and arithmetic Chebotarev density remain external inputs.

The ordinary completion argument derives finite generation, full finite level support, and an actual nonzero Hecke tensor fiber from the patched support and control isomorphisms. The declared classicality input then supplies the occurring automorphic object and its attached representation. Both existing compatible systems, with their common polynomial spine, are retained as hypotheses in the propagation results. These results do not construct missing companions or an attached compatible system.

The pointwise realization and ordinary completion statements now explicitly require Hausdorff coefficient topology and continuity of the compared representations, including the universal specialization. The corresponding locus statement is restricted to such points. These three revised claims are counted as complete; their original wording remains partial. This clarification also makes the coefficient topology in standard fact 9.2 explicit without claiming a new proof of that external arithmetic fact.

The earlier revision made two further substantive corrections. In corollary 6.12, every irreducible component is contained in the full support. The previous sentence reversed that containment. In theorem 10.5, the coefficient, continuity, integrality, class finiteness, and quotient condition hypotheses are now explicit. Finite forests by themselves establish branches, not all arithmetic effectivity assertions. Corollary 3.2 now gives the closed kernel argument and the continuity and uniqueness of its factors.

The original manuscript and its literal claim records are preserved with the release. In that original census, 41 claims are fully checked, seven are partial, and one is not checked as written. The corrected corollary 6.12 and restricted theorem 10.5 are counted as proved only for their revised statements. The original unrestricted completion wording is not retrospectively marked proved. Source hashes, the exact amendments, compiler logs, and a mathematical self-review accompany the code.

Compiler and axiom checks are distinguished from semantic proof review and external arithmetic results.

References

1. Patrick B. Allen, Frank Calegari, Ana Caraiani, Toby Gee, David Helm, Bao V. Le Hung, James Newton, Peter Scholze, Richard Taylor, and Jack A. Thorne, *Potential automorphy over CM fields*, Annals of Mathematics **197** (2023), no. 3, 897–1113.
2. Michael F. Atiyah and Ian G. Macdonald, *Introduction to commutative algebra*, Addison Wesley, Reading, Massachusetts, 1969.
3. Thomas Barnet Lamb, Toby Gee, David Geraghty, and Richard Taylor, *Potential automorphy and change of weight*, Annals of Mathematics **179** (2014), no. 2, 501–609.
4. Joël Bellaïche and Gaëtan Chenevier, *Families of Galois representations and Selmer groups*, Astérisque, vol. 324, Société Mathématique de France, Paris, 2009.
5. Richard Brauer and Cecil Nesbitt, *On the modular characters of groups*, Annals of Mathematics **42** (1941), no. 2, 556–590.
6. Frank Calegari and David Geraghty, *Modularity lifting beyond the Taylor Wiles method*, Inventiones Mathematicae **211** (2018), no. 1, 297–433.
7. Ana Caraiani and Peter Scholze, *On the generic part of the cohomology of compact unitary Shimura varieties*, Annals of Mathematics **186** (2017), no. 3, 649–766.
8. Laurent Clozel, Michael Harris, and Richard Taylor, *Automorphy for some l adic lifts of automorphic mod l Galois representations*, Publications Mathématiques de l'IHÉS **108** (2008), 1–181.

9. Fred Diamond, *The Taylor Wiles construction and multiplicity one*, Inventiones Mathematicae **128** (1997), no. 2, 379–391.
10. Reinhard Diestel, *Graph theory*, fifth ed., Graduate Texts in Mathematics, vol. 173, Springer, Berlin, 2017.
11. David Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics, vol. 150, Springer, New York, 1995.
12. Toby Gee, *Modularity lifting theorems*, Essential Number Theory **1** (2022), no. 1, 73–126.
13. Robert Guralnick, Florian Herzig, Richard Taylor, and Jack A. Thorne, *Adequate subgroups*, Journal of the Institute of Mathematics of Jussieu **11** (2012), no. 4, 907–920, Appendix to Jack A. Thorne, On the automorphy of l-adic Galois representations with small residual image.
14. Mark Kisin, *Moduli of finite flat group schemes, and modularity*, Annals of Mathematics **170** (2009), no. 3, 1085–1180.
15. Hideyuki Matsumura, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989.
16. Barry Mazur, *Deforming Galois representations*, Galois Groups over $\mathbb{Q}$ (Yasutaka Ihara, Kenneth Ribet, and Jean Pierre Serre, eds.), Mathematical Sciences Research Institute Publications, vol. 16, Springer, New York, 1989, p. 385–437.
17. Konstantin Miagkov and Jack A. Thorne, *Automorphy lifting with adequate image*, Forum of Mathematics, Sigma **11** (2023), e8, 31 pages.
18. Jürgen Neukirch, *Algebraic number theory*, Grundlehren der Mathematischen Wissenschaften, vol. 322, Springer, Berlin, 1999.
19. James Newton and Jack A. Thorne, *Torsion Galois representations over CM fields and Hecke algebras in the derived category*, Forum of Mathematics, Sigma **4** (2016), e21, 88 pages.
20. Luis Ribes and Pavel Zalesskii, *Profinite groups*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 40, Springer, Berlin, 2010.
21. Michael Schlessinger, *Functors of Artin rings*, Transactions of the American Mathematical Society **130** (1968), no. 2, 208–222.
22. Peter Scholze, *On torsion in the cohomology of locally symmetric varieties*, Annals of Mathematics **182** (2015), no. 3, 945–1066.
23. Jean Pierre Serre, *Linear representations of finite groups*, Graduate Texts in Mathematics, vol. 42, Springer, New York, 1977.
24. Richard Taylor and Andrew Wiles, *Ring theoretic properties of certain Hecke algebras*, Annals of Mathematics **141** (1995), no. 3, 553–572.
25. Jack A. Thorne, *On the automorphy of l adic Galois representations with small residual image*, Journal of the Institute of Mathematics of Jussieu **11** (2012), no. 4, 855–920.
26. Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.
27. Andrew Wiles, *Modular elliptic curves and Fermat's last theorem*, Annals of Mathematics **141** (1995), no. 3, 443–551.