OPERATION KOMOREBI

REALIZATION CALCULUS



CORE SERIES X

Universal Residual Seeds

Deformation forests, transport categories, and conditional system completion

---

Traditional monograph

Komorebi P.

Verification edition / 20 October 2026

REALIZATION CALCULUS

Reading scope

This reading edition joins the preserved Block 27 manuscript, the subsequent residue-counting continuation, and the final checked conditional occupied-system completion. The live internal ledger is 20/20 at the explicitly amended conditional scope. Historical checkpoint totals remain dated evidence of their own source revisions. Universal companion existence, carrier existence, and nonzero occupancy are not inferred. The final assembly and its exact inputs are in Section D.

Abstract

A residual representation is a genuine zeroth-order coordinate. All positive-depth perturbations above it may fill its formal deformation neighborhood, but they cannot move into a different residual class. This paper develops the corresponding universal object: a residual forest whose roots are residual representations, whose finite levels are lifts modulo powers of a uniformizer, and whose infinite branches are complete characteristic-zero objects. We prove the residual partition, no-crossing, inverse-limit reconstruction, finite-branch compactness, exact smooth and singular branch counts, packet decomposition of tangent and obstruction cohomology, a directed minimum-seed theorem, rational seed-zeta formulas, and finite companion-forest completion. The coefficient prime two is represented by a weighted forest in which characteristic-zero packets may merge modulo two and become visible only at later depth. We then define a prime-independent system seed through a common Frobenius spine, prove a compatible-carrier summand theorem under finite Frobenius separation, and reduce finite occupancy in a fixed semisimple carrier to the nonnegative integer trace of an exact joint projector. The paper does not prove that every admissible residual companion exists or that every finite Hecke packet is occupied. Those two existence statements remain the exact horizontal boundary beyond vertical deformation theory.

A route through the complete reading edition

The central distinction is between a residual root, a coherent branch, and an occupied packet in a carrier. Sections 2–7 develop the finite and formal machinery. Sections 8–11 assemble its arithmetic uses under stated hypotheses. The residue-counting continuation is retained in Appendix C. Appendix D states the later checked conditional endpoint, while Appendix E supplies the worked visual atlas.

Read for	Follow this route
The mathematical core	Residual partition, reconstruction, smooth and singular counts, cohomology packets, directed seed rank.
The arithmetic assembly	Common coefficient data, finite Frobenius separation, explicit carrier summands, and inverse-limit-closed profiles.
The formal contract	The separate Lean guide prints the terminal source and all three direct dependency modules, with a statement register and hypothesis-by-hypothesis explanation.
The finite examples	Atlas pages V1–V30 show the actual residue tables, projector tests, countermodels, and norm-character examples.

The live ledger is conditional 20/20. This is the recorded closure of the amended formal contract, not a certification of every historical prose sentence at its broadest possible interpretation. The final source is over the rational absolute Galois group. Its carrier data, constituent comparisons, and completed source family are explicit inputs. The 17/20 passages attached to Block 27 and the scalar/Mellin continuation

remain historical records. A later stronger residual-system attempt records 19/20 and no promoted endpoint. These records concern different source contracts.

Two outputs require two sets of inputs. The terminal theorem proves that a transported target family exists and that the declared summand can be recovered inside a supplied compatible carrier. It does not identify those two objects. An application that needs such an identification must supply its comparison. Likewise, zero multiplicities are permitted by the formal assembly, so a nonzero conclusion retains a separate nonzero premise.

Proofs, checks, and publication. The exact mathematical arguments, computational illustrations, and inherited Lean records have different evidential roles. The October 10 reading pass corrected the digit-extraction display, reconciled checkpoint wording, and rebuilt the reader set without a fresh Lean kernel check. The separately dated October 20 verification records identify the newly selected source, production execution, owned-declaration audit, and bounded checks. The unrecovered October 16–17 packages are not substituted into that evidence. The complete historical frontmatter and its original limits are preserved in Appendix B rather than placed in front of the current table of contents.

Contents

Reading route and current contract **1**

1 The universal seed at residual depth **5**

1.1 What the paper is designed to separate 6

2 Residual partition and no crossing **6**

3 Lift forests and inverse-limit reconstruction **8**

4 Tangent and obstruction packets **11**

5 The weighted dyadic residual forest **13**

6 Seed zeta functions and branching dimensions **15**

7 Residual transport categories **18**

8 Prime-independent system seeds **20**

9 Compatible carriers and finite Frobenius separation **26**

10 Finite spectral occupancy **35**

11 A conditional system-completion criterion **37**

12 Worked finite models **39**

12.1 Smooth branch 39

12.2 Weighted dyadic chart 39

12.3 Singular branch 40

12.4 Directed transport 40

13 Computational certificates **40**

14 Claim ledger **41**

15 Conclusion **42**

A Algorithms implemented in the companion package **42**

A.1 Residual towers 42

A.2 Transport condensation 42

A.3 Companion forests 43
A.4 Cohomology packets 43
A.5 Finite occupancy 43

B Reproducibility boundary **43**

C Retained residue-counting continuation **53**

D Conditional occupied-system completion: final checked scope **57**
D.1 Two conclusions with separate inputs 57
D.2 Why the assembly closes 57
D.3 Occupancy and inherited support 58
D.4 Evidence tied to this exact source 58
D.5 A later attempt with a different scope 58

References **59**

E Worked visual atlas **61**

1 The universal seed at residual depth

The earlier papers treat a seed as a primitive object together with an operator grammar, a packet system, a topology, and an exact reconstruction law. In arithmetic deformation problems, the primitive coordinate has two layers:

$$\boxed{\text{residual root} \quad + \quad \text{positive-depth deformation coordinates.}}$$

The distinction is structural. A correction divisible by the uniformizer changes higher digits but leaves the residual digit fixed. Therefore the correct universal arithmetic seed cannot be one residual representation unless the admissible residual space has only one point. It must contain every residual root and the complete formal neighborhood above each root.

Let $\mathcal{O}$ be a complete discrete valuation ring with uniformizer ϖ, maximal ideal (ϖ), residue field

$$k = \mathcal{O}/(\varpi), \qquad |k| = q,$$

and fraction field K. When a characteristic-zero conclusion is intended, assume additionally that $\operatorname{char} K = 0$; the general counting statements do not require that restriction. Let $\mathcal{M}$ be a moduli functor on complete local Noetherian $\mathcal{O}$-algebras with residue field k, equipped with a well-defined reduction map

$$\operatorname{red}_A : \mathcal{M}(A) \longrightarrow \mathcal{M}(k).$$

For a Galois representation this means either that a framed integral model has been fixed or that the residual semisimplified class is used.

Definition 1.1 (Residual seed system)**.** For $\bar{x} \in \mathcal{M}(k)$, define the residual fiber

$$\mathcal{M}_{\bar{x}}(A) = \{x \in \mathcal{M}(A) : \operatorname{red}_A(x) = \bar{x}\}.$$

The point $\bar{x}$ is the residual root. The family of all nonempty fibers is the residual special fiber, and the formal deformation problems above them form the residual forest.

For a Galois problem the principal example is

$$\bar{\rho} : G_{F,S} \longrightarrow \mathrm{GL}_n(k),$$

with a deformation functor $\mathrm{Def}_{\bar{\rho}}$ cut out by a determinant and selected local conditions.

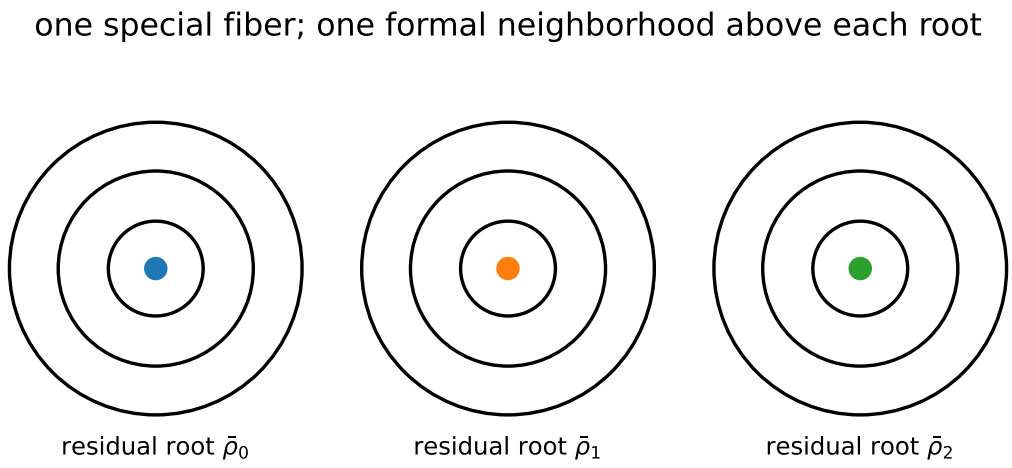


Figure 1: The universal residual seed is not one point. It is the special fiber together with one complete formal neighborhood above each admissible root.

1.1 What the paper is designed to separate

Three statements are often compressed into one sentence even though they have different logical content.

1. **Vertical completeness:** every compatible perturbation stream above a fixed root reconstructs a point of its complete deformation space.
2. **Horizontal companion existence:** every required residual root, including roots at other coefficient primes, actually exists.
3. **Occupancy:** the resulting arithmetic packet occurs in a nonzero global carrier of the required type.

The first belongs to deformation and inverse-limit theory. The second is a residual companion problem. The third is an arithmetic or automorphic nonvanishing problem. Neither of the last two follows from arbitrarily deep perturbation above one already occupied root.

2 Residual partition and no crossing

Theorem 2.1 (Residual partition)**.** *For every coefficient algebra A,*

$$\boxed{\mathcal{M}(A) = \bigsqcup_{\bar{x} \in \mathcal{M}(k)} \mathcal{M}_{\bar{x}}(A).}$$

Proof. Every A-valued object has one reduced object $\mathrm{red}_A(x) \in \mathcal{M}(k)$, so the union is exhaustive. Two distinct fibers cannot intersect because an object cannot have two different reductions. Hence the union is disjoint. □

Theorem 2.2 (No crossing under positive-depth perturbation)**.** *Suppose $x, y \in \mathcal{M}(\mathcal{O})$ are related by a deformation operation that is congruent to the identity modulo ϖ. Then*

$$\mathrm{red}(x) = \mathrm{red}(y).$$

Every finite composition remains in one residual fiber. If an infinite perturbation stream converges in the ϖ-adic topology, its limit remains in the same fiber. The existence of that limit is an explicit hypothesis, not a consequence of positive depth alone.

Proof. A correction in $\varpi\mathcal{O}$ vanishes after reduction modulo ϖ. Thus one operation preserves the residual root, and induction proves the finite statement. For a convergent stream, convergence at the first finite precision makes its reduction eventually equal to that of its limit. Every term already has the initial reduction, so the limit has that reduction too. Completeness applies to Cauchy streams; it does not make every root-preserving stream Cauchy. For example, $0, 3, 0, 3, \ldots$ has constant reduction modulo 3 but no limit modulo 9. □

Corollary 2.3 (Zeroth-order obstruction)**.** *If two admissible objects have different residual roots, no sequence of genuinely positive-depth perturbations beginning at one can reach the other. A horizontal transport between their roots is a zeroth-order operation, not a deeper lift.*

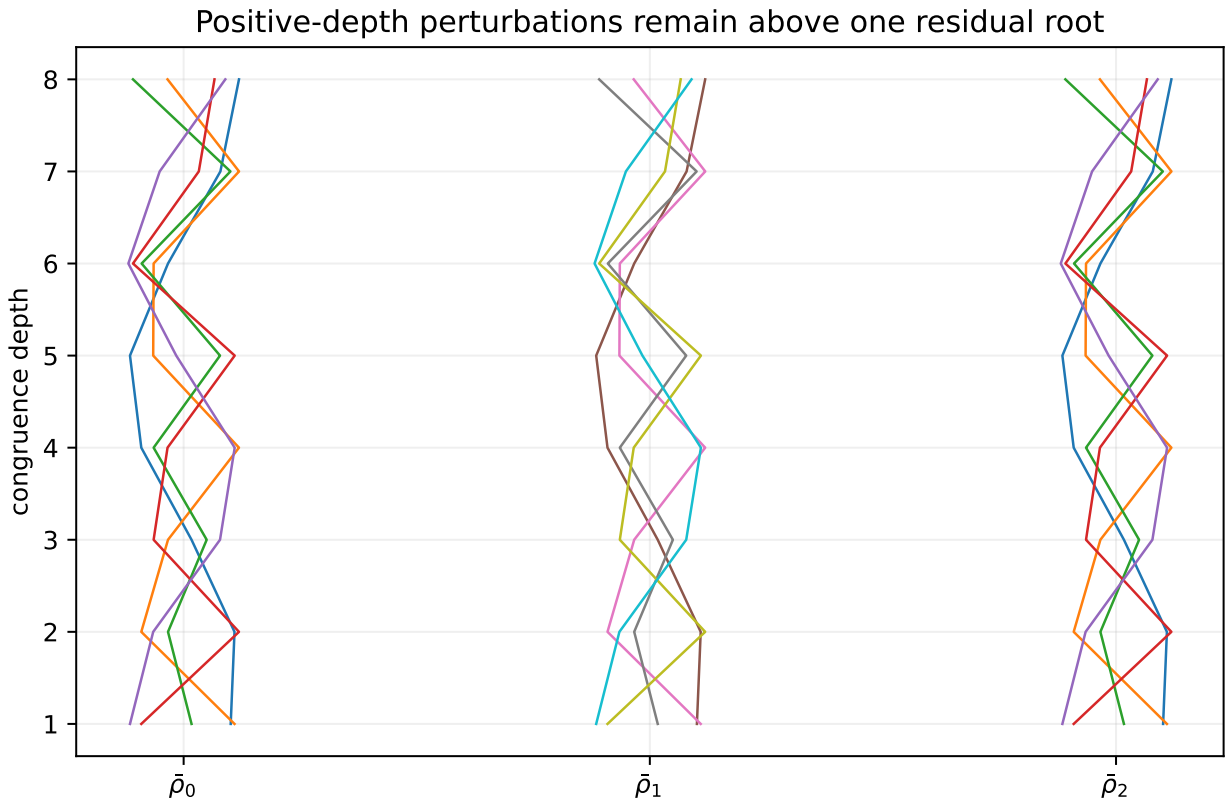


Figure 2: Each perturbation stream remains above its initial residual root. Refinement adds digits vertically; it does not move a branch horizontally.

Numerical Certificate 2.4 (Exact residual partition)**.** The finite implementation partitions the integers $0, \ldots, 8$ into the three classes modulo 3:

$$\{0, 3, 6\}, \qquad \{1, 4, 7\}, \qquad \{2, 5, 8\}.$$

Every tested correction of the form $x \mapsto x + ph$ preserves the residual class exactly.

3 Lift forests and inverse-limit reconstruction

Fix $\bar{x} \in \mathcal{M}(k)$. For $m \geq 1$, set

$$X_m(\bar{x}) = \mathcal{M}_{\bar{x}}(\mathcal{O}/\varpi^m).$$

Reduction gives transition maps

$$r_{m+1,m} : X_{m+1}(\bar{x}) \longrightarrow X_m(\bar{x}).$$

A compatible branch is a sequence $(x_m)_{m\geq 1}$ satisfying

$$r_{m+1,m}(x_{m+1}) = x_m.$$

Classical Input 3.1 (Represented complete deformation problems)**.** When a deformation functor is represented by a complete local ring $R_{\bar{x}}$, its $\mathcal{O}$-valued points are continuous local maps $R_{\bar{x}} \to \mathcal{O}$. Completeness identifies such maps with compatible maps to every quotient $\mathcal{O}/\varpi^m$.

Theorem 3.2 (Formal inverse-limit reconstruction)**.** *Assume $\mathcal{M}_{\bar{x}}$ is represented by a complete local $\mathcal{O}$-algebra $R_{\bar{x}}$. Then reduction induces a natural bijection*

$$\boxed{\mathcal{M}_{\bar{x}}(\mathcal{O}) \cong \varprojlim_m \mathcal{M}_{\bar{x}}(\mathcal{O}/\varpi^m).}$$

Proof. A continuous local homomorphism $R_{\bar{x}} \to \mathcal{O}$ reduces to a compatible family of homomorphisms into $\mathcal{O}/\varpi^m$. Conversely, compatible finite-level maps determine a map

$$R_{\bar{x}} \longrightarrow \varprojlim_m \mathcal{O}/\varpi^m \cong \mathcal{O}.$$

The two constructions are inverse because equality in a separated complete ring is detected modulo every power of its maximal ideal. □

Theorem 3.3 (Finite-branch compactness)**.** *Suppose every $X_m(\bar{x})$ is finite and nonempty and every node has finitely many children. Then the lift tree above $\bar{x}$ contains an infinite compatible branch.*

Proof. The tree is infinite because it has a nonempty level at every depth, and it is finitely branching by hypothesis. König's lemma gives an infinite path. The edge relation is reduction, so that path is a compatible branch. □

Surjectivity of every transition map is sufficient but not necessary. Finite branching and nonempty arbitrarily deep levels are the compactness mechanism.

Proposition 3.4 (Smooth formal branch count)**.** *If*

$$R_{\bar{x}} \cong \mathcal{O}[[t_1, \ldots, t_d]] \quad \text{as complete local } \mathcal{O}\text{-algebras},$$

then

$$\boxed{\#X_m(\bar{x}) = q^{d(m-1)}.}$$

Fix a uniformizer and a section $s : k \to \mathcal{O}$ of reduction, with $s(0) = 0$. Every $\mathcal{O}$-valued branch is encoded by d positive-depth digit streams

$$t_i = s(a_{i,1})\varpi + s(a_{i,2})\varpi^2 + \cdots, \qquad a_{i,j} \in k,$$

Proof. A local map sends each parameter into the image of (ϖ) in $\mathcal{O}/\varpi^m$. Conversely, every tuple in that image is nilpotent and defines a coefficient-preserving evaluation homomorphism. This homomorphism is local. To prove uniqueness without assuming continuity of an arbitrary competing homomorphism, observe in one variable that a homomorphism killing X^N only sees the finite polynomial truncation modulo X^N. Induction over the finite number of variables gives uniqueness in all dimensions. The actual local maps are therefore in bijection with the full parameter tuple set. Its cardinality is $q^{d(m-1)}$, including the zero-dimensional chart.

For integral maps, reduce an arbitrary tuple in $(\varpi)\mathcal{O}$ at every precision, evaluate there, and use the represented-map reconstruction theorem. The finite evaluations are compatible, the reconstructed algebra map is local, and every competing local map has the same finite reductions. Separatedness gives uniqueness. Local maps are continuous in the maximal-ideal adic topologies. This establishes the integral parameter classification rather than assuming it from the finite count.

For the digit claim, subtract the chosen representative of the residue and divide the remainder by the uniformizer. Iterating gives unique digits. Conversely, the partial sums of any digit stream form a compatible Cauchy family and reconstruct uniquely by completeness and separatedness. More precisely, for each parameter there is a unique stream $(a_{i,j})_{j\geq 1}$ such that

$$\varpi^{n+1} \mid t_i - \sum_{j=1}^{n} s(a_{i,j})\varpi^j \qquad (n \geq 0).$$

The section is a set-theoretic section of reduction, not an asserted embedding of residue fields into the coefficient ring. Finally, a supplied local $\mathcal{O}$-algebra isomorphism from $R_{\bar{x}}$ transports both the map classification and the count. □

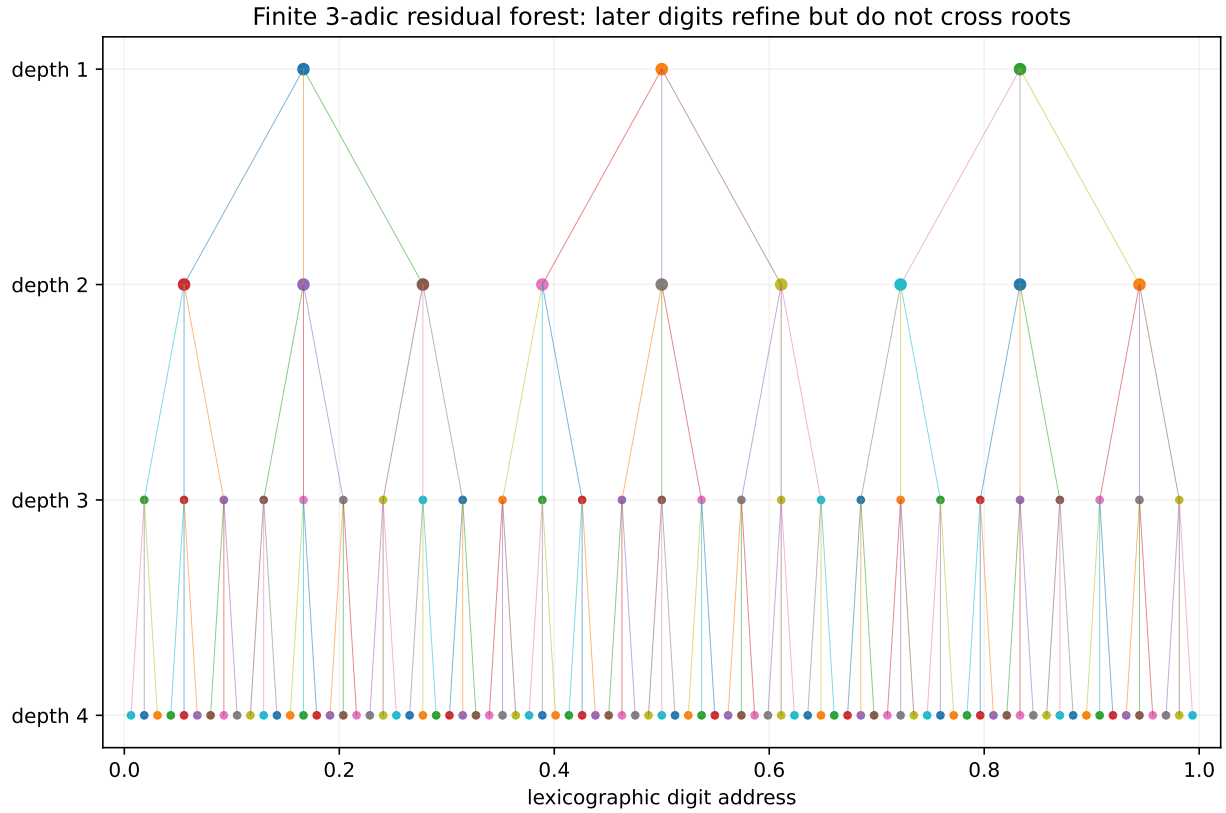


Figure 3: The first four levels of a full 3-adic digit forest. A path is a compatible finite-depth perturbation stream.

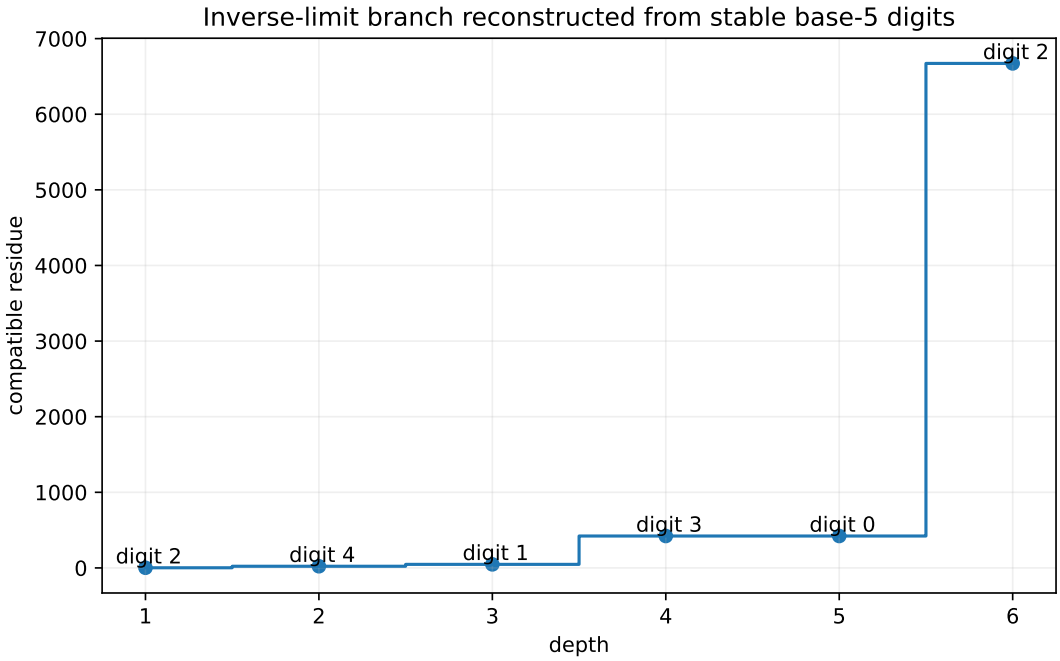


Figure 4: A compatible base-five residue tower reconstructs the digit stream $(2, 4, 1, 3, 0, 2)$.

4 Tangent and obstruction packets

For a residual representation $\bar{\rho}$, first-order deformations and lifting obstructions are classically controlled by cohomology of the adjoint module. The projector calculus does not replace that theory. It resolves the controlling complex into invariant packets when suitable projectors exist.

Classical Input 4.1 (Tangent and obstruction spaces)**.** For a framed representation problem, first-order lifts are described by $Z^1(G, \mathrm{ad}\,\bar{\rho})$. For an unframed Schur problem, tangent directions are described by $H^1(G, \mathrm{ad}\,\bar{\rho})$. Relations in a presentation of a deformation ring map into $H^2(G, \mathrm{ad}\,\bar{\rho})$, which contains the obstruction classes.

Let

$$C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \to \cdots$$

be a cochain complex over a field. Suppose a finite family of projectors $P_j^r : C^r \to C^r$ satisfies

$$\sum_j P_j^r = I, \qquad P_i^r P_j^r = \delta_{ij} P_j^r, \qquad d_r P_j^r = P_j^{r+1} d_r.$$

Theorem 4.2 (Cohomology packet decomposition)**.** *Under these hypotheses, each packet is a subcomplex and*

$$\boxed{H^r(C^\bullet) \cong \bigoplus_j H^r(P_j C^\bullet).}$$

In particular,

$$H^1 = \bigoplus_j H_j^1, \qquad H^2 = \bigoplus_j H_j^2,$$

so tangent and obstruction spaces reconstruct exactly from their packet components.

Proof. For each packet, use the actual image submodules $C_j^r = \mathrm{im}\, P_j^r$. Commutation with the differential makes these a subcomplex. Write

$$Z^r = \ker d_r, \qquad B^r = \mathrm{im}\, d_{r-1} \subseteq Z^r, \qquad H^r = Z^r / B^r,$$

and define Z_j^r, B_j^r, H_j^r using the restricted differentials of $C_j^\bullet$. The inclusion $i_j : C_j^\bullet \to C^\bullet$ and the image-valued projection $p_j : C^\bullet \to C_j^\bullet$ are cochain maps, so they induce maps on these genuine quotients.

For $i \neq j$, orthogonality gives $H(p_i)H(i_j) = 0$. Idempotence gives $H(p_j)H(i_j) = I$, and completeness gives $\sum_j H(i_j)H(p_j) = I$. Thus the maps

$$[z] \longmapsto ([P_j^r z])_j, \qquad ([z_j])_j \longmapsto \left[\sum_j i_j z_j\right]$$

are inverse linear maps. The finite product equals the direct sum, proving the assertion. In the formalization, both sides are also identified with Mathlib's standard homology objects of the genuine graded complexes. □

Scope of the splitting. The algebraic proof works over any commutative ring and does not divide by 2, so it applies in characteristic two when the stated integral chain projectors exist. It does not construct those projectors. The identification of a particular Galois tangent or obstruction problem with this cohomology remains the preceding classical input. The splitting detects vanishing:

$$H^r(C^\bullet) = 0 \quad \Longleftrightarrow \quad H^r(C_j^\bullet) = 0 \text{ for every } j.$$

It does not force either side to vanish or decouple nonlinear deformation equations. Moreover, commutation with the differential is essential: complementary idempotents on the underlying vector spaces need not preserve cocycles or coboundaries.

Naturality. If a cochain map $F : C^\bullet \to D^\bullet$ intertwines the supplied projectors, it restricts to maps $F_j : C_j^\bullet \to D_j^\bullet$. The direct-sum decomposition commutes with $H^r(F)$ and the family $H^r(F_j)$. This is checked using representatives in the actual quotients, independently of any dimension computation.

Remark 4.3 (Odd and dyadic nonlinear equations). At odd residue characteristic, a differential graded Lie model may be written in Maurer–Cartan form

$$d\eta + \frac{1}{2}[\eta, \eta] = 0,$$

and a cyclic grading converts this into exact packet convolutions. At residue characteristic two, division by 2 is not integral. A quadratic or divided-square refinement must be retained. This is one reason the dyadic residual forest is weighted rather than obtained by formal substitution into the odd-prime formulas.

Numerical Certificate 4.4 (Finite packet complex)**.** The exact modular test uses two invariant subcomplexes over $\mathbf{F}_5$. Their cohomology dimensions are

$$(0, 0, 0), \qquad (1, 2, 1),$$

and the reconstructed total is

$$\dim(H^0, H^1, H^2) = (1, 2, 1).$$

The packet-sum residual is zero.

New checked realization of the finite pattern. The Block 04 implementation separately uses

$$\mathbf{F}_5^2 \xrightarrow{d_0} \mathbf{F}_5^4 \xrightarrow{d_1} \mathbf{F}_5^2, \quad d_0(a, b) = (a, 0, 0, 0), \quad d_1(a, b, c, d) = (b, 0).$$

The first packet is the exact complex with dimensions $(1, 2, 1)$, and the second packet has zero differentials and dimensions $(1, 2, 1)$. The actual middle cohomology is linearly isomorphic to $\mathbf{F}_5^2$, which is checked in Lean. Independent exhaustive coset tests check all three cohomology degrees after changes of basis. This is a new specified realization of the displayed dimension pattern, not a claimed rerun of the original companion implementation.

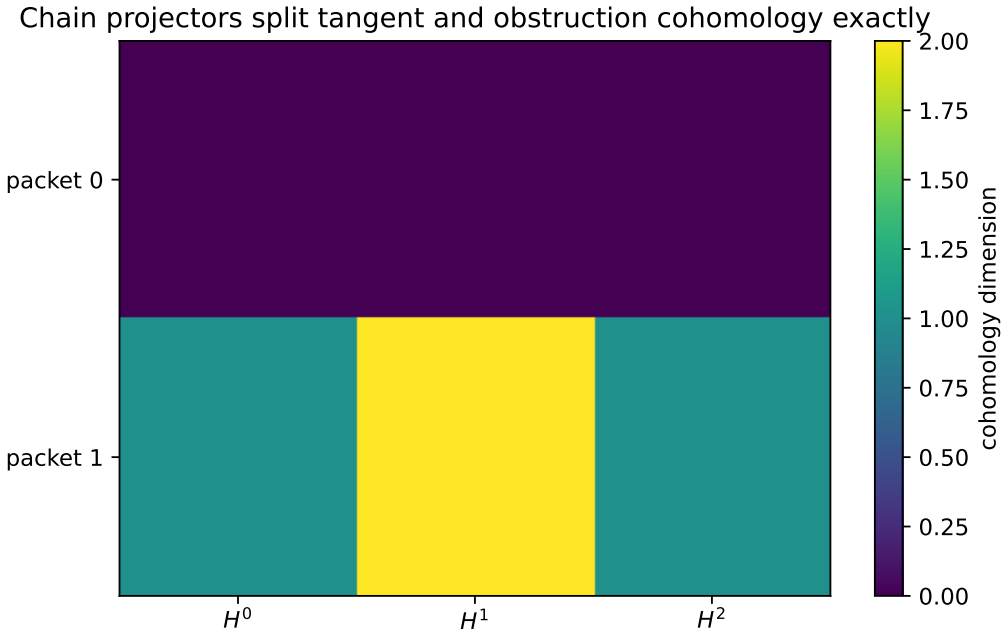


Figure 5: A finite chain-projector model. Tangent and obstruction dimensions are carried by one packet and reconstruct exactly.

5 The weighted dyadic residual forest

At an odd prime, distinct sign characters remain distinct in the residue field. At the coefficient prime two,

$$1 \equiv -1 \pmod 2.$$

Thus distinct characteristic-zero packets can share one residual root and separate only after more precision is retained.

Definition 5.1 (Visibility weight)**.** For two integral packet coordinates $a, b \in \mathcal{O}$, define their separation depth

$$w(a,b) = \min\{m \geq 1 : a \not\equiv b \pmod{\varpi^m}\},$$

when such an m exists. A dyadic packet decomposition is weighted by these first visibility depths and by the valuations of any denominators required by its rational projectors.

Example 5.2 (The two sign branches)**.** The characters $\chi_+(g) = 1$ and $\chi_-(g) = -1$ have the same reduction modulo 2, but

$$1 \not\equiv -1 \pmod 4.$$

Their first separation depth is therefore 2. One mod-two root contains two branches that become distinct at the next level.

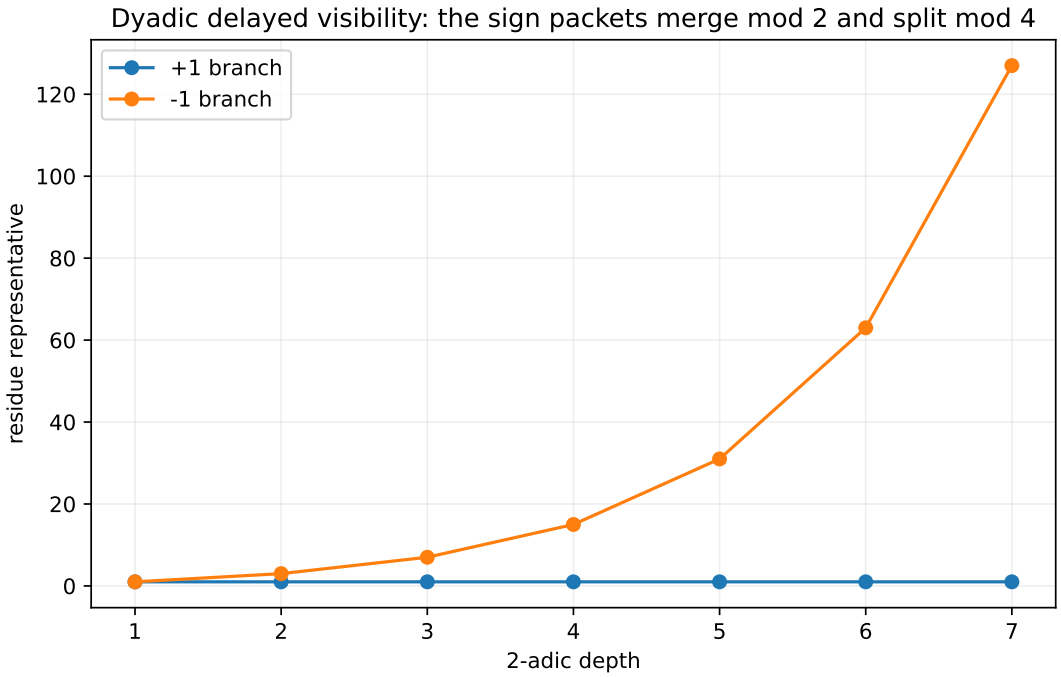


Figure 6: The sign packets merge at residual depth and separate at depth two. This is a genuine change in integral coordinates, not a numerical plotting effect.

A simple weighted formal model already captures delayed visibility. Let $w_i \geq 1$ and constrain

$$t_i \in (\varpi^{w_i}\mathcal{O} + \varpi^m\mathcal{O})/\varpi^m\mathcal{O}.$$

Then coordinate i is invisible until $m > w_i$.

Proposition 5.3 (Weighted smooth-chart count)**.** *For weights* $\boldsymbol{w} = (w_1, \ldots, w_d)$,

$$\boxed{N_{\boldsymbol{w}}(m) = q^{\sum_{i=1}^{d} \max(m-w_i,0)}.}$$

Proof. The image of $\varpi^{w_i}\mathcal{O}$ in $\mathcal{O}/\varpi^m$ has one element when $m \leq w_i$, and q^{m-w_i} elements when $m > w_i$. Multiplying the independent coordinate counts gives the formula. □

This model does not assert that every dyadic Galois deformation ring is a weighted formal ball. It gives the exact local counting law once such weighted coordinates or a Smith-normal-form chart have been proved.

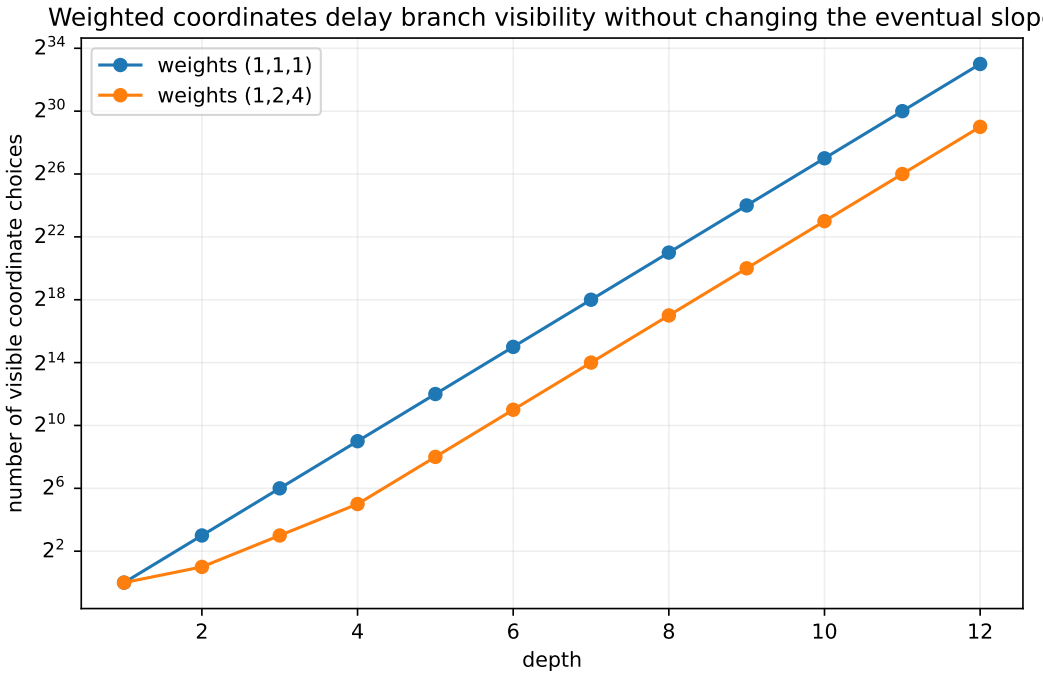


Figure 7: Weights delay finite-depth visibility. Once all coordinates are active, the asymptotic slope equals the ordinary formal dimension.

6 Seed zeta functions and branching dimensions

Finite-depth counts record more than the existence of a branch. They distinguish smooth, weighted, and singular local geometry.

Definition 6.1 (Seed zeta function)**.** For a residual root $\bar{x}$ with finite levels, define

$$\boxed{Z_{\bar{x}}(T) = \sum_{m \geq 1} \#X_m(\bar{x}) T^m.}$$

When a weighted chart is specified, write $Z_{\boldsymbol{w}}(T)$.

Theorem 6.2 (Smooth seed zeta function)**.** *For a smooth formal component of dimension d,*

$$\boxed{Z_d(T) = \frac{T}{1 - q^d T}.}$$

Proof. By the proved formal-model map-to-parameter equivalence, $\#X_m = q^{d(m-1)}$ counts all the actual local coefficient-algebra maps. Summing the geometric series gives the identity. The checked statement is an equality of infinite formal power series, with zero constant coefficient. □

Theorem 6.3 (Weighted seed zeta function)**.** *Let $1 \leq w_1 \leq \cdots \leq w_d$, put $W = \sum_i w_i$, and let*

$$a_m = q^{\sum_i \max(m - w_i, 0)}.$$

Then

$$\boxed{Z_{\boldsymbol{w}}(T) = \sum_{m=1}^{w_d} a_m T^m + q^{-W} \frac{(q^d T)^{w_d + 1}}{1 - q^d T}.}$$

In particular, every weighted smooth chart has a rational seed zeta function.

Proof. The finite sum records the depths before all coordinates are active. For $m \geq w_d + 1$,

$$a_m = q^{dm-W},$$

so the remaining terms form one geometric series. □

Theorem 6.4 (Singular power seed)**.** *For integers $e \geq 1$ and $m \geq 1$, in $\mathcal{O}/\varpi^m$ the equation*

$$x^e = 0$$

has exactly

$$\boxed{q^{m-\lceil m/e \rceil}}$$

solutions. Its seed zeta function is

$$\boxed{Z_e(T) = \frac{\sum_{r=1}^{e} q^{r-1} T^r}{1 - q^{e-1} T^e}.}$$

Proof. The congruence is equivalent to

$$e\, v_\varpi(x) \geq m,$$

so x must lie in $\varpi^{\lceil m/e \rceil}\mathcal{O}/\varpi^m$, which has the stated cardinality. Write $m = ek + r$, $1 \leq r \leq e$. The exponent becomes $(e-1)k + r - 1$. Summing over $k \geq 0$ and $1 \leq r \leq e$ gives the rational expression. □

Definition 6.5 (Branching dimension)**.** When the limit exists, define the finite-level counting exponent

$$\dim_{\text{seed}}(\bar{x}) = \lim_{m \to \infty} \frac{\log_q \#X_m(\bar{x})}{m}.$$

A limsup may be used in general.

Corollary 6.6 (Fractional singular dimension)**.** *For $x^e = 0$,*

$$\boxed{\dim_{\text{seed}} = 1 - \frac{1}{e}.}$$

In particular, $x^2 = 0$ has branching dimension $1/2$.

Counting exponent versus compatible branches. *The preceding exponent counts all finite-level solutions, including nodes which cannot lift indefinitely. It is not the cardinality or dimension of the space of infinite branches. In fact every compatible tower of solutions to $x^e = 0$, for $e \geq 1$, is the zero tower. For each m, a solution at depth em lies in $\varpi^m\mathcal{O}/\varpi^{em}$; its reduction at depth m is therefore zero. Compatibility proves the assertion at every depth. This observation and the real logarithmic limit are both formalized in Block 02.*

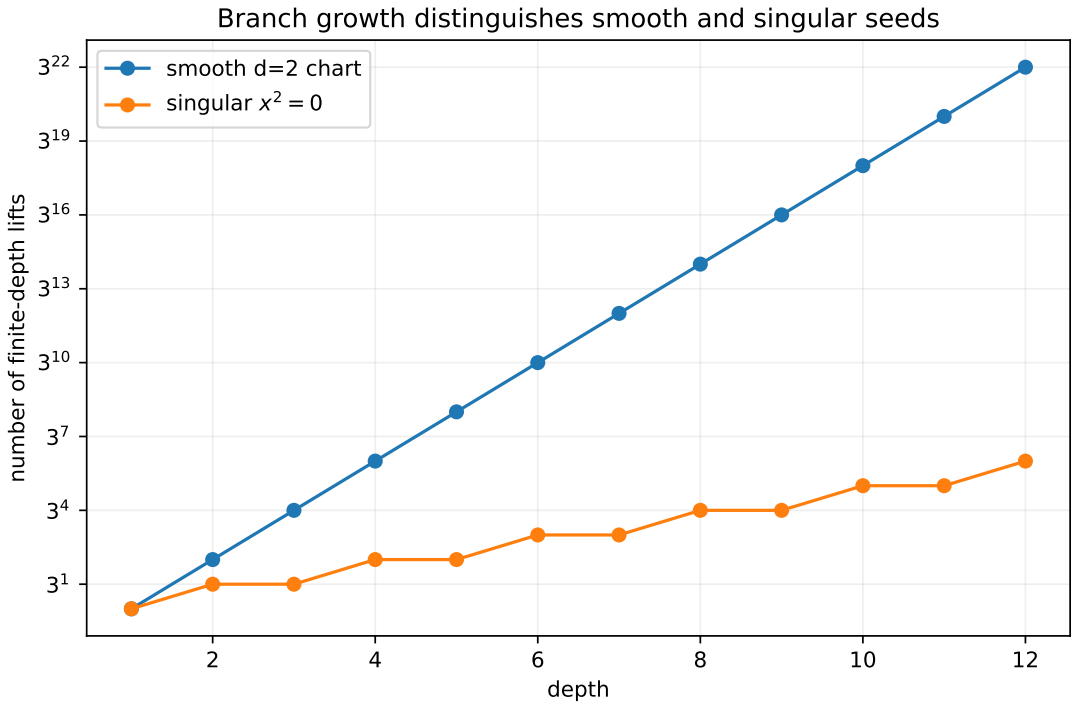


Figure 8: A smooth two-dimensional formal chart and the singular equation $x^2 = 0$ have visibly different exponential growth.

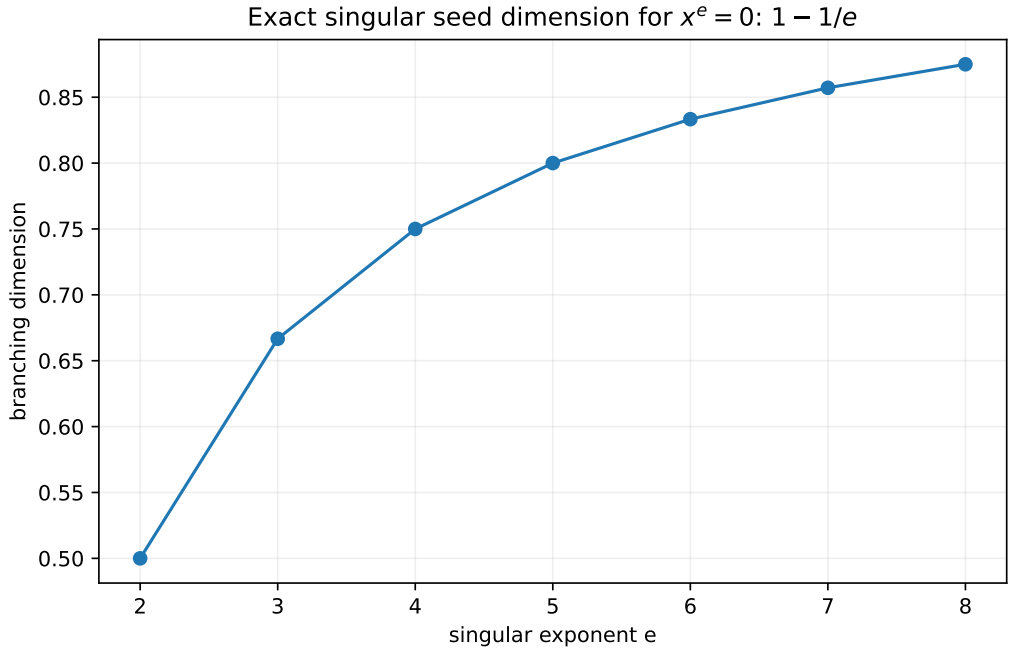


Figure 9: The finite-depth branching dimension of $x^e = 0$ is $1 - 1/e$, giving a family of fractional local growth exponents.

7 Residual transport categories

Vertical lifting never changes a residual root. Horizontal coverage requires operations whose residual effect is not congruent to the identity. Examples that may define such transports include twisting, duality, restriction, induction, tensor product, symmetric powers, exterior powers, and functorial transfer. The word "may" is essential: each operation must preserve the deformation conditions and the realization statement being transported.

Definition 7.1 (Admissible residual transport)**.** An admissible transport from $\bar{\rho}$ to $\bar{\rho}'$ consists of

$$\mathcal{T} : \bar{\rho} \longmapsto \bar{\rho}'$$

together with natural maps

$$\mathcal{T}_A : \mathrm{Def}_{\bar{\rho}}(A) \longrightarrow \mathrm{Def}_{\bar{\rho}'}(A)$$

for every coefficient algebra A, compatible with base change and reduction. When the transport is reversible, it is an isomorphism in the residual transport groupoid. One-way operations give a directed category.

For a finite collection of roots, record a directed edge $u \to v$ whenever occupancy or the relevant property is known to propagate from u to v.

Definition 7.2 (Directed seed rank)**.** For a directed transport graph G, define

$$\mathrm{srank}_G = \min\{|S| : \mathrm{Cl}_G(S) = V(G)\},$$

where $\mathrm{Cl}_G(S)$ is the set of vertices reachable from S.

Theorem 7.3 (Directed minimum-seed theorem)**.** *Let G be a finite directed graph. Collapse every strongly connected component to one vertex, obtaining the condensation DAG. Then*

$$\boxed{\mathrm{srank}_G = \#\{\textit{source strongly connected components of } G\}.}$$

One representative from each source component is a minimum seed set.

Proof. Every vertex of a finite DAG is reachable from a source, so one seed in each source component reaches the entire condensation graph. Within a strongly connected component, one vertex reaches every other. Hence the source representatives are sufficient.

Conversely, a source component has no incoming edge from another component. It cannot be reached from a seed outside itself, so every generating seed set must meet it. Therefore the number of source components is also a lower bound. □

For reversible transports the condensation has one vertex per connected component, recovering the undirected minimum-seed theorem.

Formal scope and propagation. Reachability is the reflexive-transitive closure of the supplied directed edge relation, and a strongly connected component is the actual quotient by mutual reachability. The minimum seed count is defined independently as the least cardinality of a set that reaches all vertices. Every covering set must meet every source component; conversely, meeting every source component suffices in a finite graph. Thus any choice of one representative from each source is optimal. Empty

graphs have rank zero, and self-loops are permitted. Adding admissible edges cannot increase the minimum seed count.

If a property P propagates along every supplied edge and holds at all selected seeds, it holds at every reached vertex. These are explicit hypotheses. The minimum-seed theorem does not construct valid arithmetic edges or supply initial occupancy.

Why the finite hypothesis cannot be dropped. Take vertices $\{*\} \sqcup \{v_n : n \in \mathbf{N}\}$, with $*$ isolated and edges $v_{n+1} \to v_n$. Every strongly connected component is a singleton, and $\{*\}$ is the unique source component. No finite seed set covers all vertices: it can reach only indices no larger than the largest chosen index. Both the unique-source statement and the failure of every finite cover are checked in Lean. Hence one source component in an infinite condensation does not imply that one seed reaches all objects.

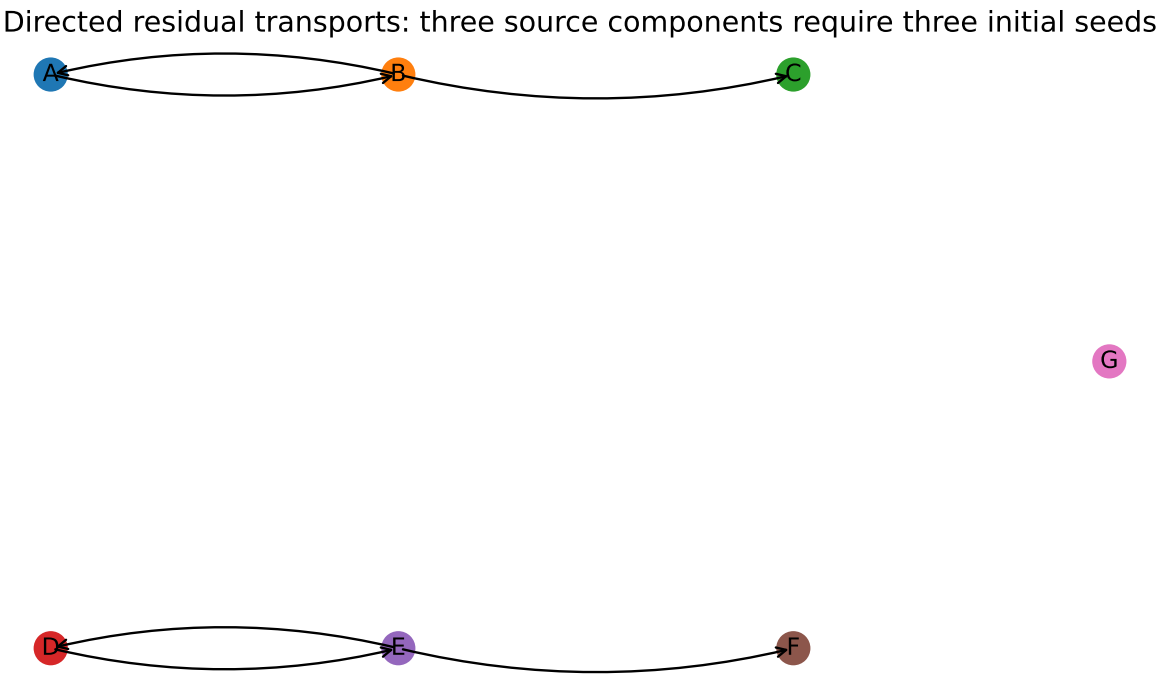


Figure 10: A finite directed transport model. The components $\{A, B\}$, $\{D, E\}$, and $\{G\}$ are sources; C and F are reached downstream.

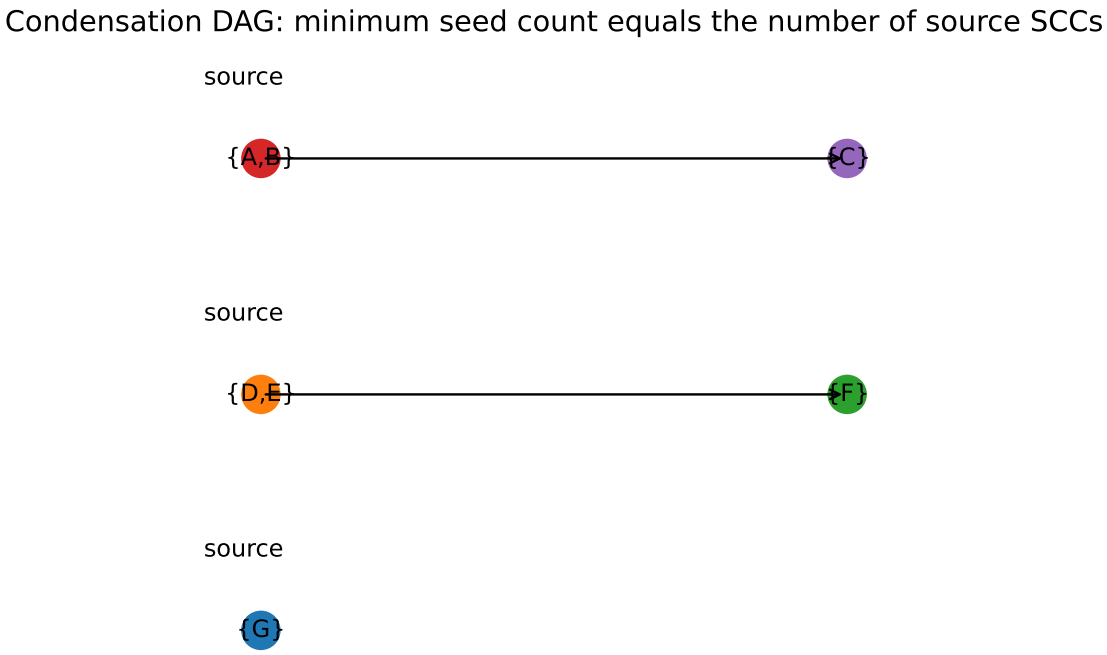


Figure 11: The condensation DAG makes the minimum seed count visible without searching every subset of roots.

Numerical Certificate 7.4 (Directed seed count)**.** The test graph has five strongly connected components and three source components. The theorem gives seed rank three. Exhaustive search over every subset of vertices independently returns three, and the selected representatives reach all seven roots.

New explicit checked realization. Separately from the unretrieved original companion implementation, Block 05 specifies vertices A, B, C, D, E, F, G and the eight edges

$$A \to B,\ B \to A,\ B \to C,\ C \to F,\ D \to E,\ E \to D,\ E \to F,\ G \to F.$$

The source components are $\{A, B\}$, $\{D, E\}$, and $\{G\}$; the other components are $\{C\}$ and $\{F\}$. A bounded path search to length three is proved closed under every original edge, which certifies equality with unbounded reachability. The resulting label certificate is identified with the actual quotient by mutual reachability. Lean checks seed rank three and exactly the four minimum sets

$$\{A, D, G\}, \quad \{A, E, G\}, \quad \{B, D, G\}, \quad \{B, E, G\}.$$

Independent exact tests enumerate every directed graph on zero through four vertices, including loops, and every seed subset. These finite tests audit the implementation; the universal theorem is a separate formal proof.

8 Prime-independent system seeds

A compatible system should not be treated as an unrelated collection of λ-adic roots. Its members share algebraic Frobenius polynomials, Hodge–Tate data, determinant information, and local Weil–Deligne packets.

Definition 8.1 (Common arithmetic spine)**.** Let M be a number field. A common spine $\mathfrak{C}$ consists of:

- characteristic polynomials $P_v(X) \in M[X]$ at almost all places;
- coefficient-prime independent weight and determinant data;
- prescribed local Weil–Deligne or inertial packet labels at the ramified places.

For each finite place λ of M, let

$$X_{\lambda,m}(\mathfrak{C})$$

be the finite set of admissible objects modulo ϖ_λ^m realizing the reduction of this spine.

The system seed is the fibered object

$$\boxed{\mathscr{S}_{\mathrm{sys}} = \mathscr{S}_2^{\mathrm{wt}} \times_{\mathfrak{C}} \prod_{\lambda \nmid 2}{}' \mathscr{S}_\lambda.}$$

The fiber product notation states a compatibility requirement. It does not prove that every fiber is nonempty.

Theorem 8.2 (Adelic finite-depth completion)**.** *Assume:*

1. *an algebraic common spine $\mathfrak{C}$ over a number field M is fixed;*
2. *for every finite place λ of M and every $m \geq 1$, the companion level $X_{\lambda,m}(\mathfrak{C})$ is finite and nonempty;*
3. *the reduction forests are finitely branching;*
4. *the selected local conditions are closed under inverse limits.*

Then for every λ there is an infinite compatible branch

$$\rho_\lambda \in \varprojlim_m X_{\lambda,m}(\mathfrak{C}),$$

and the collection $\{\rho_\lambda\}_\lambda$ realizes the prescribed strictly compatible spine.

Proof. Finite-branch compactness gives an infinite branch in every coefficient-prime forest. Inverse-limit closure gives an admissible λ-adic object. By definition each finite level has the prescribed reductions of the common Frobenius and local packets. Equality at every finite precision implies equality in the complete coefficient ring. Thus all members realize the same spine. □

Corollary 8.3 (Residual companion obstruction)**.** *If*

$$X_{\lambda,1}(\mathfrak{C}) = \varnothing$$

for one coefficient prime λ, no compatible completion with spine $\mathfrak{C}$ exists. Deeper perturbations at other primes cannot create the missing residual companion.

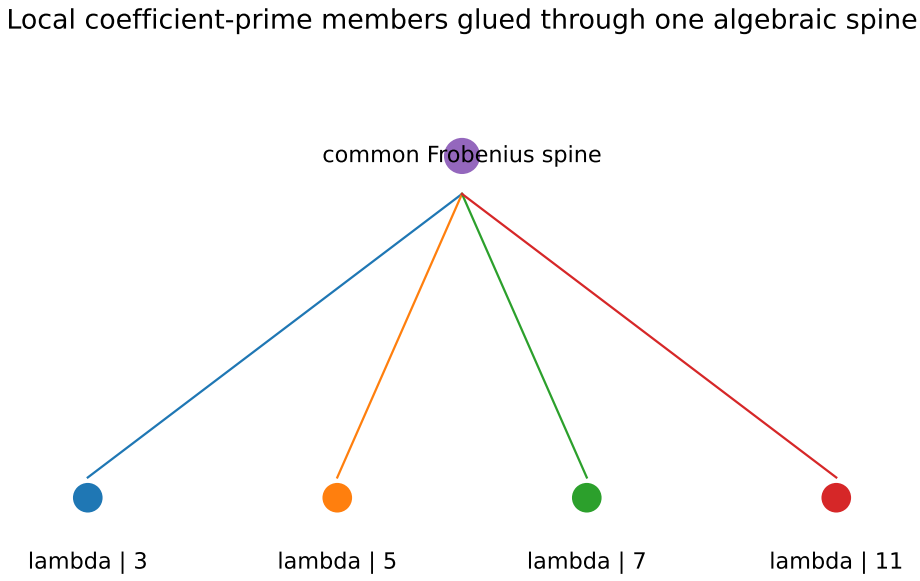


Figure 12: The local members are coordinates of one system only after their finite data descend to a common algebraic spine.

Example 8.4 (A scalar companion failure)**.** For

$$f(x) = x^2 + 1,$$

there are two simple branches at $p = 5$ and $p = 13$, no residual roots at $p = 3$ or $p = 7$, and one root modulo 2 that has no child modulo 4. A complete 5-adic branch therefore does not imply an adelic solution.

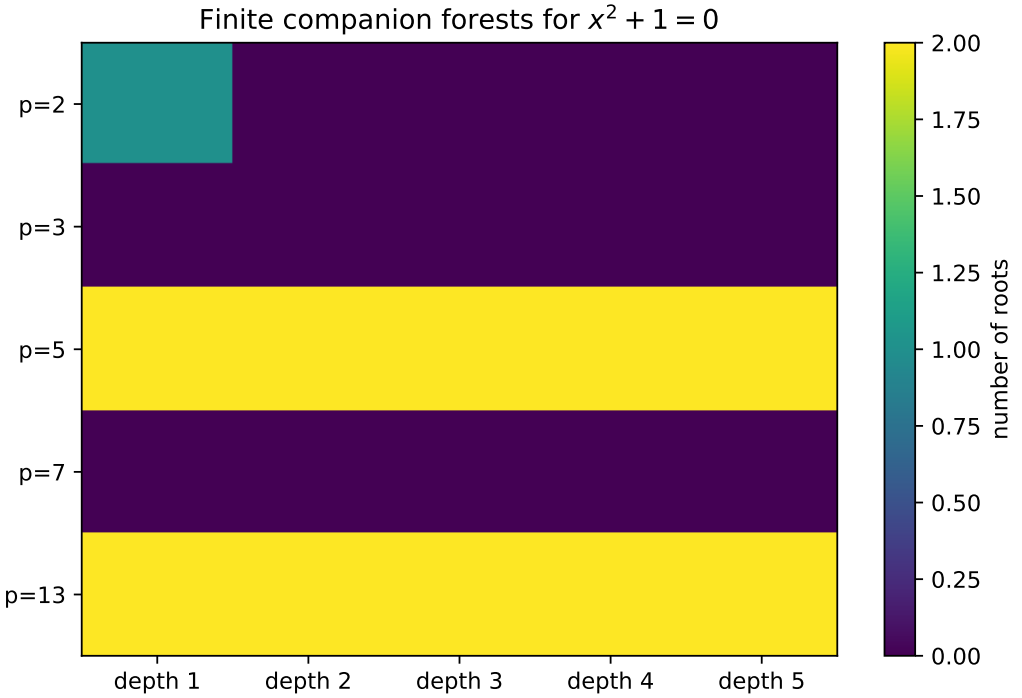


Figure 13: Finite companion levels for $x^2 + 1 = 0$. The obstruction at $2, 3, 7$ is visible before any deep lifting argument begins.

Conditional scope and complete assembly (Block 23). The theorem above is conditional on the chosen spine and the stated local inverse-limit closure. In the fixed-rank integral representation setting, "closed under inverse limits" means that every compatible family of admissible finite representation classes, after coherent matrix framing and reconstruction, satisfies the full selected local realization relations. It is not merely closedness of an unrelated label space. Weight and packet relations retain their supplied arithmetic meaning; they are parameters of the universal theorem. For a general moduli functor, effectivity requires its own reconstruction hypothesis or the corresponding full meaning of admissible inverse-limit closure.

The complete Lean assembly now includes the common polynomials, good-place sets with finite complements, coefficient-specific prescribed determinants, and every selected weight and packet relation. Determinants are recovered from their actual finite-class determinants. A single common field-valued character on the whole Galois group is not assumed. The returned representations are continuous, and their reductions form the actual compatible branches. Their common polynomial identities hold at every declared good place, hence almost everywhere for each coefficient prime.

This checks Claim 8.2 as the universal conditional source theorem. The previous partial status also demanded proofs that particular intended arithmetic packets satisfy its premises; that stronger application target is not a conclusion of the theorem. The Block 23 source-claim count was therefore sixteen checked and four partial. Concrete Hodge–Tate and Weil–Deligne attachments, torsion interpretations, closure instances, and companion nonemptiness remain obligations of arithmetic applications. Inertial-type labels are not upgraded to full Weil–Deligne data.

Checked representation-level completion and its arithmetic boundary. The eighth formalization block works with actual matrix representations of a fixed group, not only with underlying sets. For a

complete separated ring R and ideal I, compatible maps $\rho_m : G \to \mathrm{GL}_d(R/I^m)$ reconstruct a unique integral representation for that fixed compatible family. Multiplication and inverses are verified after every reduction. If all finite representations are continuous for the discrete quotient topology, the integral representation is continuous for the ideal-adic topology. This does not assert uniqueness among all possible branches of the allowed forest.

The forest has as its level the actual subtype of representations satisfying the declared finite conditions. Those conditions must be preserved by reduction. Finiteness and nonemptiness at every level give an infinite branch and hence an actual continuous representation. Finiteness of the coefficient quotient alone is not a substitute for finiteness of the allowed representation set for an arbitrary source group. Surjectivity of the reductions is not required.

All coefficients of the characteristic polynomial commute with reduction. Consequently, prescribed integral characteristic polynomials at every precision hold exactly in the reconstructed representation. For a common coefficient field E, local integral polynomials lie in $\mathcal{O}_\lambda[X]$ and are compared with $P_v \in E[X]$ only after mapping both into the local coefficient field. No field embedding $E \to \mathcal{O}_\lambda$ is assumed. The allowed-place predicate is explicit and may depend on λ. Trivial action on a supplied subgroup is likewise detected at every precision; identifying that subgroup with arithmetic inertia is a separate input.

The eleventh block assembles continuous coefficient-field representations, exact common polynomials and the prescribed local predicate from positive finite depths. Depth-zero finiteness and nonemptiness are derived. Claims 8.2 and 11.1 already assume inverse-limit closure of the selected local conditions; an unconditional closure theorem is not a missing hypothesis of their conditional proofs. Their conditional matrix core is checked. The remaining full arithmetic interpretation must identify the admissible finite objects and the actual local invariants with these data. The coherent-framing obligation is resolved by Block 14 below. An arbitrary local predicate is not being renamed Hodge–Tate or Weil–Deligne compatibility.

Completion from conjugacy classes (Block 14). Over an I-adically complete separated commutative coefficient ring R, for a fixed finite rank, the formalization now takes finite-level objects to be actual matrix representations modulo conjugation by an invertible matrix. Assume the coefficient quotients R/I^k are finite. Conjugation commutes with coefficient reduction, and each conjugacy class has finitely many frames over a finite coefficient ring. Thus finite allowed class levels give finite full preimages of allowed matrix representations. Downward admissibility and positive-depth nonemptiness yield an exactly compatible matrix branch and an actual integral representation. More strongly, every prescribed compatible class branch has such representatives, using finite nonempty fibers over that branch. No surjectivity of reductions or lifting of prescribed change-of-basis matrices is assumed. This does not assert uniqueness up to integral conjugacy.

For finitely generated I with finite R/I, finiteness of R/I^k is derived. One continuous representative per allowed class suffices, because continuity is proved invariant under conjugation. Characteristic polynomials, determinants and trivial action on a supplied subgroup are likewise actual invariants. The unframed common-spine theorem constructs continuous coefficient-field representations with the prescribed whole polynomials and the explicitly supplied inverse-limit-closed local predicate.

An actual $\mathbb{Z}_3$ example applies the class-level completion theorem for the cyclic group of order two and polynomial $X^2 - 1$, producing a nonidentity generator. Another checked example has compatible classes represented by $\mathrm{diag}(1, -1)$ modulo 3 and $\mathrm{diag}(-1, 1)$ modulo 9: these chosen matrices disagree under reduction, and an actual swap of basis repairs them. Arbitrary independent representatives therefore do

not replace the framing argument.

The framing obligation is finished. Identifying the manuscript's arithmetic admissible objects, actual local invariants and finite-level packet reductions with this interface remains separate. An uninterpreted local predicate is not a construction of Hodge–Tate or Weil–Deligne compatibility. The source's local closure hypothesis remains explicit. Block 23 completes the universal conditional implication; these particular arithmetic applications remain unproved.

Actual local comparisons from fixed-depth certificates (Block 15). Let R be an I-adically complete separated commutative ring with finite coefficient quotients and an injective map into a field K. Fix a finite matrix rank. For finitely many local group maps $\phi_s : H_s \to G$ and integral template representations τ_s, fix detecting depths d_s. At precision m retain matrices $P_{s,m}$ as part of the admissible object and require

$$\rho_m(\phi_s(h))P_{s,m} = P_{s,m}\tau_{s,m}(h).$$

For $m \geq d_s$ require their determinants to remain nonzero after reduction to R/I^{d_s}. Earlier levels omit that test. Finite occupied augmented levels give a compatible branch of both representations and comparisons. The integral limits satisfy the actual equations and have nonzero determinants, so the comparisons are units over K. A converse supplies all finite nodes from any integral realization with these detectors. Finite-level invertibility and compatible prior choices of comparisons are not assumed.

The unframed positive-depth version also retains continuity and whole common polynomials. Its actual local isomorphisms are derived from the certificates; there is no uninterpreted local-stability predicate for this particular problem. The fixed depths and augmented-level occupancy remain explicit inputs.

The formalization specializes to the actual maps $G_{\mathbb{Q}_p} \to G_{\mathbb{Q}}$ induced by the field embeddings and choices of algebraic closures. A nontrivial continuous quadratic character of $G_{\mathbb{Q}}$, constructed from its action on $i^2 = -1$, gives the integral 2-adic actions

$$\tau(\sigma) = \operatorname{diag}(1,-1), \quad \rho(\sigma) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

at a nonidentity quadratic element. The determinant 2 is detected modulo 4. The representations are conjugate over $\mathbb{Q}_2$ but not integrally conjugate. A separate $\mathbb{Z}_3$ example invokes simultaneous completion and reconstructs a comparison which is an integral nonunit and a field unit.

The bound is substantive: for the unipotent actions $\rho_a(g) = \begin{pmatrix} 1 & ag \\ 0 & 1 \end{pmatrix}$, every integral intertwiner with ρ_1 has determinant aP_{22}^2. Taking $a = \pi^n$ for a nonzero $\pi \in I$ rules out every fixed detecting depth, even though all the field-valued approximants are isomorphic and their integral limit is trivial. This does not refute Claim 8.2, which already assumes inverse-limit closure of its chosen local conditions.

Exact prescribed local Galois action is stronger than an inertial-type label. The common arithmetic packet interpretation, its coefficient comparisons and the actual Hodge–Tate/Weil–Deligne attachment remain separate. The new theorem does not supply a uniform bound for all intended arithmetic packets. Those arithmetic applications remain unfinished; the universal conditional implication is checked in Block 23.

Derived comparison bounds (Block 16). For a finite group Γ whose order is a unit in a commutative local coefficient ring R, any prescribed equivariant residual isomorphism lifts to an integral isomorphism. Lift its matrix to A and define

$$P = |\Gamma|^{-1} \sum_{g \in \Gamma} \rho(g) A \tau(g^{-1}).$$

Reindexing proves equivariance, and reduction recovers the prescribed matrix. Its determinant is a unit. Complete local coefficients with finite quotients, occupied compatible finite conditions and this residual type therefore give actual integral completion with detector depth one. A fixed continuous map $G \to \Gamma$ permits pullback to infinite source groups. The finite-level representations must factor through the same map, and the residual type is imposed on their Γ-actions. No uniform determinant bound or augmented certificate occupancy is assumed in this sector.

For a second sector let R be a DVR with uniformizer π, fraction field K and matrix rank n. If a fixed template τ has full K-linear action span, finite generation and denominator clearing give a fixed c such that

$$\pi^c M_n(R) \subseteq \operatorname{span}_R\{\tau(g) : g \in G\}.$$

Normalize a field comparison to an integral P with a unit entry. The displayed span constructs an integral B with $BP = \pi^c I$, so

$$\det P \mid \pi^{cn}, \qquad \det P \not\equiv 0 \pmod{\pi^{cn+1}}.$$

One c works for all integral representations in the fixed field-isomorphism class. Jacobson density derives full span from actual irreducibility and a scalar commutant. The formalization states these hypotheses directly and does not assume the DVR fraction field is algebraically closed.

If at every precision there is an integral representation of this rational type satisfying the finite conditions, the derived bound supplies the simultaneous certificates of Block 15 and preserves the type after completion. The integral approximants are not required to be coherent, but their existence is a substantive premise. Arbitrary torsion nodes are not asserted to lift.

Over $\mathbb{Z}_3$, the actual average corrects $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ to $P = \left(\begin{smallmatrix} 1 & -1/2 \\ 0 & 1 \end{smallmatrix}\right)$ for the involutions $\rho(g) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$ and $\tau(g) = \operatorname{diag}(1, -1)$. It preserves the residual comparison and has determinant one. Pullback through the quadratic character of the actual $G_{\mathbb{Q}}$ gives continuous arithmetic examples. In contrast, the identity and $\operatorname{diag}(1, \operatorname{sign})$ over $\mathbb{Z}/4$ agree modulo two but are not integrally conjugate. Their group order is not a unit. These restricted results do not supply general arithmetic packet attachment, all finite-level existence statements, or cross-coefficient compatibility.

9 Compatible carriers and finite Frobenius separation

One useful completion mechanism is available when the desired packet already occurs as a semisimple summand of a known compatible carrier.

Theorem 9.1 (Finite Frobenius separation)**.** *Let $\mathscr{W}_1, \dots, \mathscr{W}_r$ be pairwise nonisomorphic strictly compatible systems whose members are absolutely irreducible, with common algebraic unramified trace values $t_i(v)$ in a number field E. There are r distinct permitted unramified places $v_1, \dots, v_r$ for which*

$$C = (t_i(v_j))_{j,i} \in M_r(E), \qquad \det C \neq 0.$$

Every coefficient-field embedding $\sigma_\lambda : E \hookrightarrow E_\lambda$ therefore sends C to an invertible matrix. Its entries agree with the ordinary representation traces $\operatorname{Tr}(W_{i,\lambda}(\operatorname{Frob}_{v_j}))$ when the selected places satisfy the unramified compatibility convention for that coefficient prime. In particular, places above the residue characteristic of λ are excluded from this ordinary trace comparison.

Proof. Fix one coefficient prime. The classical independence of the characters of pairwise nonisomorphic absolutely irreducible representations and the density of unramified Frobenius conjugacy classes give

independent restricted trace functions. Density here concerns the union of the conjugacy classes, not an unsupported assertion about a set of chosen representatives.

For r independent functions, their evaluation linear forms span the r-dimensional coefficient dual: a proper span would leave a nonzero coefficient vector annihilated by every evaluation, contradicting independence. Choose a basis from the evaluations. Its r distinct sample points give an invertible square matrix. The common algebraic traces put that matrix over E; an injective field embedding reflects the nonvanishing of its determinant. All further field embeddings preserve the same nonvanishing. □

Formalization boundary. Characteristic-zero independence is now Lean-checked for actual finite-dimensional irreducible representations of arbitrary monoids. Finiteness of the group and algebraic closedness of the field are not needed for this result. The proof uses the actual monoid algebra and Jacobson density to select constituent actions. Jointly continuous coordinate actions have continuous characters. Coverage in every open normal quotient of a compact totally disconnected group implies density of the conjugacy-saturated samples and hence the finite separator. General arithmetic Frobenius coverage remains unformalized. Block 25 now proves the bridge from componentwise nonisomorphic systems to nonisomorphic members at the selected coefficient prime, under density on the required good-place overlaps. It also proves the complete restricted separator for a fixed finite cyclotomic Galois group over $\mathbf{Q}$, with arithmetic coverage derived from the proved Dirichlet theorem. The full arithmetic theorem remains partial. Selection is existential and gives no effective upper bound on the selected primes. A common algebraic matrix is still distinct from an ordinary unramified trace comparison at all coefficient primes simultaneously.

System isomorphism and good-place overlaps (Block 25). The system equivalence used here is componentwise representation isomorphism:

$$\mathscr{W}_i \simeq \mathscr{W}_j \quad \Longleftrightarrow \quad \forall \lambda,\ W_{i,\lambda} \simeq W_{j,\lambda}.$$

For finite-dimensional irreducible representations of any monoid over a characteristic-zero field, the new character-rigidity theorem proves that equality of characters is equivalent to an actual equivariant linear isomorphism. Neither a finite group nor an algebraically closed coefficient field is needed.

Suppose two members are isomorphic at λ_0. Injectivity of the common coefficient embedding gives $t_i(v) = t_j(v)$ at every place good for λ_0. At another coefficient prime λ, this equality compares actual member traces on the intersection of the two good-place sets. Conjugacy density of that intersection and continuity extend the equality to the whole group; character rigidity then gives an actual isomorphism at λ. Thus the nonisomorphism of systems supplies the required nonisomorphism of members. Equality of optional labels or record fields is not used as a substitute for representation equivalence.

The generic separator now derives this bridge and permits any additional finite set of excluded places. Its coverage premise retains every finite deletion: for each finite excluded set and each open normal quotient, every quotient class contains a conjugate of an allowed sample. This implies density on each pair of cofinite good-place sets; no all-coefficient-prime common ordinary-trace domain is assumed.

Arithmetic coverage for a fixed cyclotomic extension (Block 25). Let $n > 0$ and let $L = \mathbf{Q}(\zeta_n)$ be a cyclotomic number field. The canonical isomorphism

$$\mathrm{Gal}(L/\mathbf{Q}) \simeq (\mathbb{Z}/n\mathbb{Z})^\times$$

defines σ_p by $\sigma_p(\zeta_n) = \zeta_n^p$ when $p \nmid n$. For every automorphism σ, finite set $B \subset \mathbb{N}$ and bound b, Lean proves that there is a rational prime $p > b$, with $p \notin B$ and $p \nmid n$, such that $\sigma_p = \sigma$. The prime-existence step uses the proved Dirichlet theorem in the pinned Mathlib dependency.

The arithmetic attachment is checked in this finite cyclotomic case. There is an actual maximal prime $\mathfrak{P}$ of $\mathcal{O}_L$ above (p), its ramification index is one, and σ_p belongs to its decomposition group. For every integral $x \in \mathcal{O}_L$,

$$\overline{\sigma_p(x)} = \overline{x}^{\,p} \quad \text{in } \mathcal{O}_L/\mathfrak{P}.$$

The residue identity is proved on all integral elements using $\mathcal{O}_L = \mathbb{Z}[\zeta_n]$. Existence of $\mathfrak{P}$ is obtained by lying over, rather than assumed. This specifies arithmetic Frobenius, with p-power residue action. The action and unramifiedness statements also hold for every maximal prime above the selected p.

Consequently, for pairwise componentwise nonisomorphic compatible irreducible representation families of this fixed finite Galois group, the common algebraic trace matrix can be selected at distinct rational primes outside any prescribed finite set, with nonzero determinant. Every coefficient embedding preserves its nonvanishing; actual member trace comparisons retain that member's good-prime condition. No coverage or character-continuity hypothesis is added in this cyclotomic specialization: the prime coverage is proved, and the finite extension's Krull topology is discrete. A concrete conductor-four instance uses the actual field $\mathbf{Q}(\zeta_4)$.

This is a complete restricted arithmetic case of Claim 9.1. It does not establish coverage in arbitrary finite quotients of the absolute Galois group, nor the Hodge–Tate and Weil–Deligne comparisons of Claim 9.2. The source-claim ledger therefore remains seventeen checked and three partial.

Absolute Galois restriction and arithmetic witnesses (Block 26). Fix a cyclotomic number field $L = \mathbf{Q}(\zeta_n)$ with $n > 0$ and an embedding $j : L \hookrightarrow \overline{\mathbf{Q}}$. The new restriction map

$$q_L : G_{\mathbf{Q}} \longrightarrow \mathrm{Gal}(L/\mathbf{Q}), \qquad j(q_L(\gamma)x) = \gamma(j(x)),$$

is an actual continuous surjective homomorphism with open kernel. Surjectivity is derived by extending automorphisms to the algebraic closure. Pullback along a surjective homomorphism preserves and reflects equivariant linear isomorphisms, invariant subspaces, and irreducibility. These assertions keep the underlying linear maps and submodules and do not assume a finite acting group. In particular, the equivalence relation for systems is preserved and reflected componentwise.

The arithmetic witnesses use the integral closure $\mathcal{A}$ of $\mathbb{Z}$ in $\overline{\mathbf{Q}}$. For each rational prime p, Lean constructs a maximal ideal $\mathfrak{Q}$ of $\mathcal{A}$ above (p) and an actual element $\gamma_p \in G_{\mathbf{Q}}$ such that

$$\gamma_p \in \mathrm{Stab}_{G_{\mathbf{Q}}}(\mathfrak{Q}), \qquad \overline{\gamma_p(a)} = \overline{a}^{\,p} \quad (a \in \mathcal{A}).$$

The existence argument applies Mathlib's proved profinite stabilizer-surjectivity theorem to this actual invariant ring. The residue p-power map is proved bijective by taking p-th roots in $\overline{\mathbf{Q}}$ and showing that they remain integral when their p-th powers are integral. This constructs an arithmetic Frobenius witness in the prime-ideal decomposition group; it does not assert that an arbitrary global lift belongs to that group.

When $p \nmid n$, the comparison with the finite cyclotomic sample is proved:

$$q_L(\gamma_p) = \sigma_p, \qquad \sigma_p(\zeta_n) = \zeta_n^p.$$

Reduction distinguishes n-th roots of unity at such a prime. Indeed, if $x^n = y^n = 1$ and their reductions agree, the factorization of $x^n - y^n$ gives

$$(x - y) \sum_{i=0}^{n-1} x^i y^{n-1-i} = 0.$$

The second factor reduces to $n\overline{y}^{n-1} \neq 0$, so the domain property gives $x = y$. This proves both Frobenius recognition on the cyclotomic generator and triviality of its inertia action. Consequently, every representation pulled back through q_L is unramified there.

The valuation formulation is checked separately. For any valuation subring $A \subset \overline{\mathbf{Q}}$ with residue characteristic $p \nmid n$, its actual residue-action kernel fixes every n-th root of unity. A supplied element of its decomposition subgroup with p-power residue action restricts to σ_p. These statements use Mathlib's valuation inertia and decomposition subgroups. No identification of the valuation and integral-closure prime models is assumed in the formal comparison.

For a compatible family $\rho_{i,\lambda}$ of representations of the fixed finite group, set $V_{i,\lambda} = \rho_{i,\lambda} \circ q_L$. Suppose the actual absolute-Galois members are irreducible and the systems are pairwise componentwise nonisomorphic. The finite prime separator now yields distinct permitted rational primes p_j, actual prime ideals $\mathfrak{Q}_j$ and arithmetic Frobenius witnesses γ_j as above, outside any prescribed finite set, such that the common algebraic trace matrix has nonzero determinant. Every coefficient embedding preserves this nonvanishing, and

$$\iota_\lambda(t_i(p_j)) = \operatorname{tr}(V_{i,\lambda}(\gamma_j))$$

whenever p_j is good for that coefficient member. The operator and trace values agree for every global lift with the same cyclotomic restriction. Neither Frobenius-witness existence nor quotient coverage is a supplied hypothesis in this restricted arithmetic theorem. The common cyclotomic factorization and the compatible trace data remain explicit hypotheses.

This completes the passage to actual absolute-Galois Frobenius witnesses for the fixed cyclotomic class. It does not establish prime coverage for arbitrary finite Galois extensions. Claim 9.1 therefore remains partial, and the source ledger remains seventeen checked and three partial.

The boundaries are formalized as well. Pullback along a nonsurjective homomorphism can make inequivalent irreducible representations isomorphic. Samples can cover one quotient while missing an open quotient of the full group. Finally, in $\mathbb{Z}_2$, the distinct square roots 1 and -1 have the same residue, so the condition that the root order be a unit cannot be discarded.

General finite-field Frobenius comparison (Block 27). Let $L/\mathbf{Q}$ now be any finite normal number field, with a chosen embedding $j : L \hookrightarrow \overline{\mathbf{Q}}$. Write $\mathcal{A}$ for the integral closure of $\mathbb{Z}$ in $\overline{\mathbf{Q}}$. The embedding restricts to an injective map $\mathcal{O}_L \to \mathcal{A}$ which intertwines the actual Galois actions. If $\mathfrak{Q} \subset \mathcal{A}$ is maximal above (p), its contraction $\mathfrak{P} \subset \mathcal{O}_L$ is a nonzero maximal ideal above (p), and the induced residue map is injective.

Restriction carries the actual decomposition group at $\mathfrak{Q}$ into that at $\mathfrak{P}$, carries ideal inertia into ideal inertia, and preserves the arithmetic p-power residue action. Thus the absolute Frobenius constructed in Block 26 gives a simultaneous finite Frobenius at the contracted prime. This assertion applies to arbitrary finite normal number fields and does not require a cyclotomic generator.

The exceptional primes are controlled by the actual field discriminant:

$$S_L = \{p \text{ rational prime} : p \mid \operatorname{disc}(L)\}.$$

This set is finite. For $p \notin S_L$, every prime of $\mathcal{O}_L$ above (p) is unramified. Lean proves that the actual inertia cardinality is the ramification index. It follows that inertia is trivial and the arithmetic Frobenius at a fixed finite prime is unique as a field automorphism. Neither trivial inertia nor uniqueness is assumed in place of the arithmetic unramifiedness condition.

The choice of absolute prime and Frobenius lift is also controlled. Let $(\mathfrak{Q}_u, \gamma_u)$ and $(\mathfrak{Q}_v, \gamma_v)$ be any two actual arithmetic witnesses at the same rational prime. Transitivity on primes of $\mathcal{A}$ above (p) gives $a \in G_{\mathbf{Q}}$ with

$$\mathfrak{Q}_v = a\mathfrak{Q}_u, \qquad \gamma_v^{-1} a \gamma_u a^{-1} \in I_{\mathfrak{Q}_v}.$$

Consequently, any representation trivial on $I_{\mathfrak{Q}_v}$ has the same trace on the two witnesses. For representations factoring through L, triviality on this actual absolute inertia is derived whenever $p \notin S_L$. The restrictions of all absolute witnesses at such a prime therefore form a single conjugacy class. This conclusion requires only the indicated finite factorization and the discriminant exclusion.

There is a final compatibility between embeddings. For a normal subfield $L \subset \overline{\mathbf{Q}}$, restriction defined through a chosen embedding is conjugate to canonical restriction by one fixed automorphism of L. This identifies their conjugacy classes and lets the arithmetic comparison apply directly to the fixed fields of open normal subgroups of $G_{\mathbf{Q}}$. Those fixed fields are proved to be finite Galois number fields, with the original open normal subgroup as the kernel of restriction.

The remaining prime-existence obligation now has a precise formulation. For every finite Galois subfield L, every finite set B of rational primes, and every $\sigma \in \mathrm{Gal}(L/\mathbf{Q})$, require a prime $p \notin B \cup S_L$ and an actual absolute arithmetic witness γ_p such that

$$q_L(\gamma_p) \sim \sigma.$$

Lean proves that this assertion is equivalent to the cofinite quotient coverage required by the general separator, for any fixed choice of actual Frobenius witnesses. The equivalence derives independence from the witness choices; it does not supply the displayed prime-existence assertion. Arbitrary finite exclusions absorb S_L, and the resulting conditional density theorem preserves the separate good-prime sets of coefficient members. General prime coverage remains unproved, so Claim 9.1 remains partial. The source ledger is still seventeen checked and three partial. This new arithmetic interface has base $\mathbf{Q}$. For the source's general-base instances involving $G_{F,S}$, the analogous actual-prime and norm-power Frobenius attachment and coverage over F, or a proved transfer to the rational-base result, also remain necessary.

Finite deletion can destroy density. In the actual discrete group $C_2 = \{1, a\}$, set $s(0) = a$ and $s(m+1) = 1$. The original samples are surjective and satisfy quotient coverage. Removing index zero leaves only the identity, so both density and quotient coverage fail. The actual rational sign and trivial characters agree on the remaining samples but differ globally. This checked example explains why ordinary coverage alone cannot justify deleting the bad places in the system comparison.

Theorem 9.2 (Compatible-carrier summand completion)**.** *Suppose a semisimple strictly compatible carrier decomposes as*

$$\mathscr{W} \cong \bigoplus_{i=1}^{r} \mathscr{W}_i^{\oplus c_i},$$

with the $\mathscr{W}_i$ as above. At one coefficient prime λ_0, suppose a semisimple summand has the form

$$V_{\lambda_0} \cong \bigoplus_{i=1}^{r} W_{i,\lambda_0}^{\oplus m_i}, \qquad 0 \le m_i \le c_i.$$

Then

$$\boxed{V_\lambda = \bigoplus_{i=1}^{r} W_{i,\lambda}^{\oplus m_i}}$$

defines a strictly compatible summand at every coefficient prime. The vector (m_i) is uniquely recoverable from finitely many Frobenius traces supplied by the separation theorem.

Proof. Use the same multiplicities in every member of the compatible carrier. Direct sums preserve strict compatibility, and $m_i \le c_i$ makes V_λ a summand of W_λ. At the separating places, the target trace vector is $C\boldsymbol{m}$. Full column rank makes $\boldsymbol{m}$ the unique solution. □

Recovery and realization are different steps. For a certified square separator the coefficient vector is $C^{-1}t$. This is a valid carrier multiplicity vector only when every entry is a nonnegative integer and satisfies its capacity bound $m_i \le c_i$. Those tests and uniqueness of natural multiplicities in characteristic zero are separately Lean-checked, as is transport through a field embedding. Reduction modulo a residue characteristic is not a field embedding and may lose rank. The representation construction and common-polynomial part are now Lean-checked as described below. The full strictly compatible arithmetic statement, including its local conditions, remains partially formalized.

Scope of the supplied-carrier hypothesis. Claim 9.2 supplies the compatible carrier, its decomposition and the compatible constituent systems. Constructing unknown such objects is not required by that conditional theorem. The remaining formal obligation is that finite sums preserve the intended actual local arithmetic structures, together with the arithmetic separator input. The algebraic monodromy results below do not by themselves identify attached Weil–Deligne representations or Hodge–Tate comparison data.

Explicit summands inside a supplied decomposition. Let K be a field, let G be a monoid, and let $\rho_i : G \to \mathrm{End}_K(W_i)$ be actual representations on finite-dimensional spaces, indexed by a finite set I. For natural numbers $m_i \le c_i$, set

$$U_m = \prod_{i\in I}\prod_{k<m_i} W_i, \qquad U_c = \prod_{i\in I}\prod_{k<c_i} W_i.$$

These finite products realize the corresponding finite direct sums. The representation on U_m acts componentwise. Define $j : U_m \to U_c$ by retaining the first m_i copies and filling every remaining coordinate with zero. Define $r : U_c \to U_m$ by restriction to the selected copies. Lean proves

$$rj = 1_{U_m}, \qquad j\rho_m(g) = \rho_c(g)j, \qquad r\rho_c(g) = \rho_m(g)r.$$

The image of j is exactly the subspace whose unused coordinates vanish. Consequently jr is an equivariant idempotent. A supplied equivariant isomorphism $D : W \xrightarrow{\sim} U_c$ gives the split maps $D^{-1}j$ and rD inside the original carrier. The decomposition D is an input; these statements do not construct a decomposition of an unknown arithmetic carrier.

The same formalization proves, for each $g \in G$,

$$\det\big(XI - \rho_m(g)\big) = \prod_{i\in I}\det\big(XI - \rho_i(g)\big)^{m_i}, \qquad \mathrm{Tr}\big(\rho_m(g)\big) = \sum_{i\in I}(m_i : K)\,\mathrm{Tr}\big(\rho_i(g)\big).$$

Constituents may have different dimensions. Zero multiplicities, zero-dimensional constituents, and the empty index set are included. If at a permitted sample g the constituent polynomials are the images of $P_{i,g} \in E[X]$ under a specified map $E \to K$, the summand polynomial is the image of $\prod_i P_{i,g}^{m_i}$. Applying this independently at each coefficient field preserves a supplied common polynomial family. This does not establish the Hodge–Tate or Weil–Deligne conditions in strict compatibility.

Joint continuity of the constituent actions implies joint continuity of the constructed action. The inclusion and retraction are continuous for the product topologies, and triviality on a supplied subgroup is preserved. Neither continuity nor triviality is inferred for a carrier whose corresponding hypothesis has not been supplied. The maps transported by D are proved algebraically; their topological interpretation additionally requires the appropriate continuity of D and its inverse.

For the three-character example below, the construction produces an actual three-dimensional summand inside the carrier of dimension seven. At the sample $(0, 1)$ its characteristic polynomial is $(X-1)^2(X+1)$. A separate checked counterexample uses the coordinate inclusion $\mathbf{Q} \to \mathbf{Q}^2$, its coordinate retraction, and the swap involution: the split law holds, but compression sends the swap to zero and therefore fails to preserve its square. Thus selecting whole constituent copies, with the proved equivariance, is essential to this construction.

Algebraic monodromy data on the selected copies. Fix a field K and a nonzero scalar $q \in K^\times$. On each constituent W_i, supply an automorphism F_i and a nilpotent endomorphism N_i with $F_i N_i F_i^{-1} = qN_i$. The scalar q is common to this finite family. The componentwise operators F_m, N_m on U_m satisfy the same relation; F_m is invertible and N_m is nilpotent. The split maps constructed above intertwine both operators. The proof first identifies the independently defined kernel through an equivariant linear isomorphism

$$\ker N_m \simeq \prod_{i \in I} \prod_{k < m_i} \ker N_i.$$

The relation, with $q \neq 0$, proves that both F_i and its inverse preserve $\ker N_i$. Thus $F_i|_{\ker N_i}$ is an actual automorphism, and Lean proves

$$\det(XI - F_m|_{\ker N_m}) = \prod_{i \in I} \det(XI - F_i|_{\ker N_i})^{m_i}.$$

The corresponding kernel dimensions and traces are additive with multiplicity. Common constituent kernel polynomials, when supplied through a coefficient-ring map, give the mapped product polynomial for the summand. Whole-space Frobenius polynomials are not substituted for these kernel polynomials.

The exact special chain has basis $e_0, \ldots, e_{n-1}$ and operators

$$Ne_0 = 0, \qquad Ne_j = e_{j-1}\ (j > 0), \qquad Fe_j = aq^{-j}e_j, \qquad a, q \in K^\times.$$

Lean proves $N^n = 0$, and for $n > 0$ also $N^{n-1} \neq 0$ and $\ker N = Ke_0$. Consequently its kernel polynomial is $X - a$, whereas its whole-space polynomial is $\prod_{j=0}^{n-1}(X - aq^{-j})$. The zero-length chain is included as a zero-dimensional space; the assertion $\ker N = Ke_0$ is made only for positive length.

For $n > 1$, replacing N by zero retains exactly the same F and the same whole-space characteristic polynomial, while changing the degree of the kernel polynomial from one to n. Both pairs still satisfy the algebraic monodromy relation. This Lean-checked example shows that whole Frobenius data alone do not determine the monodromy kernel polynomial. It is not asserted to be a counterexample among the intended strictly compatible arithmetic systems.

These statements concern supplied algebraic pairs. No Weil-group or inertia action, Hodge–Tate object, arithmetic comparison, or realization of the pair from a Galois representation is constructed here. The polynomial convention is $\det(XI - F|_{\ker N})$; no identification with an inertia-dependent local Euler factor is asserted. Claim 9.2 remains partial at its full arithmetic scope.

Checked group, inertia and monodromy finite sums (Block 12). Let K be a field, G a group, and $\nu : G \to K^\times$ a character. The new datum consists of an actual representation r and nilpotent N with $r(g)N = \nu(g)Nr(g)$ for every g. For a supplied subgroup H, define

$$E(D, H) = \ker N \cap V^H.$$

This is the intersection of independently defined kernel and actual invariant subspaces. For normal H, the restricted whole-group representation and its inverse automorphisms are constructed. A change from φ to φh, with $h \in H$, leaves the restricted operator itself unchanged.

For a finite family of finite-dimensional constituents and natural multiplicities, the actual local space of the constructed sum is linearly equivalent to the sum of the constituent local spaces, equivariantly for every group element. Its characteristic polynomials multiply, its dimensions and traces add, and maps intertwining both the group and N restrict to the local spaces. The explicit bounded-copy inclusion and retraction still split there. Finite inertia image and an open inertia kernel are preserved.

Writing $Q_D(T)$ for the reversed local characteristic polynomial, Lean proves

$$Q_D(T) = \det(1 - Tr(\varphi) \mid E(D, H)), \qquad Q_D(0) = 1, \qquad Q_{\bigoplus_i D_i^{\oplus m_i}}(T) = \prod_i Q_{D_i}(T)^{m_i}.$$

The abstract formulas do not need characteristic zero. Normality and finite dimension appear explicitly where used. Arithmetic identification of G, H, ν and φ remains separate. In the geometric-Frobenius convention, $\nu(\varphi)$ is the inverse residue cardinality; it is this unit, rather than the residue cardinality, that enters the earlier special-chain parameter.

An explicit action of $C_2 \times \mathbb{Z}$ has $\nu(\varphi) = 1/3$, $F = \operatorname{diag}(2, 6, 18)$ and $Ne_1 = e_0$, $Ne_2 = e_1$, $Ne_0 = 0$. Trivial inertia gives local dimension one and denominator $1 - 2T$; scalar sign inertia gives dimension zero and denominator one. The entire F and N are equal in these two examples, and $N^2 \neq 0$. Thus inertia cannot be omitted. Conversely, unramified chains of lengths one and three have equal Euler denominators while only the first has $N = 0$. Euler factors therefore do not identify complete monodromy data.

The remaining arithmetic comparison is $\mathrm{WD}(\bigoplus_i V_i^{\oplus m_i}) \cong \bigoplus_i \mathrm{WD}(V_i)^{\oplus m_i}$ for the actual attached functor, intertwining the Weil action and monodromy, together with the intended coefficient and Hodge–Tate comparisons. These are not supplied by the abstract datum. Claim 9.2 remains partial at its full arithmetic scope.

Recovered multiplicities determine the semisimple isomorphism type relative to the specified constituents, not a canonical embedding or projector inside a repeated isotypic component. For example, the two projectors $\operatorname{diag}(1, 0)$ and $\operatorname{diag}(0, 1)$ on a two-dimensional trivial representation are distinct equivariant idempotents with trace one.

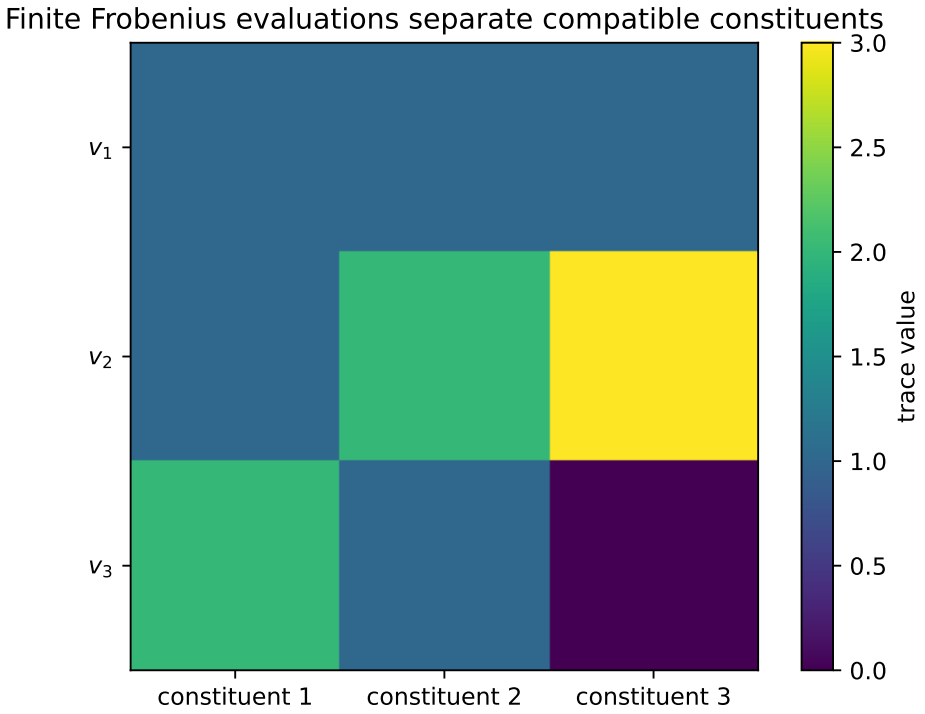


Figure 14: A finite set of trace evaluations separates three compatible constituents and recovers their multiplicities.

Numerical Certificate 9.3 (Multiplicity recovery)**.** The original companion matrix was not retrieved or rerun. A new exact realization uses the trivial character and two independent sign characters of $(\mathbf{Z}/2\mathbf{Z})^2$, evaluated at $(0,0),(1,0),(0,1)$:

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \qquad \det C = 4, \qquad C \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}.$$

Lean checks that these are the traces of actual one-dimensional representations, recovers $(2,0,1)$ uniquely, rejects a target whose unique solution is $(1/2,0,0)$, and rejects $(4,0,1)$ under capacities $(3,2,2)$. The separator becomes singular modulo two. These are finite-image representation checks, not fabricated Frobenius data for an arithmetic carrier.

The theorem is substantial only when a suitable compatible carrier and its irreducible compatible constituents are already known. It does not assert that every geometric representation embeds into such a carrier.

Finite-exponential reconstruction and a cubic inverse (Block 13). Let K be a characteristic-zero field and let (r,N) be supplied group data with nilpotent N and $r(g)N = \nu(g)Nr(g)$. For a supplied scalar map satisfying

$$t(gh) = t(g) + \nu(g)t(h),$$

the formalization constructs

$$\rho(g) = \exp(t(g)N)r(g), \qquad \exp(aN) = \sum_{j=0}^{d-1} \frac{a^j}{j!} N^j \quad (N^d = 0).$$

Lean proves the representation laws, actual finite-copy and split-map compatibility, and joint continuity from continuity of N, r and t with the stated continuous vector-space operations. The monodromy fixed kernel intersected with subgroup invariants is unchanged by this twist. For normal inertia, its restricted action and finite-dimensional Euler denominator agree. Twisting again by $-t$ recovers the datum with N still supplied.

Block 13 also established an extracted inverse on the explicit class $N^3 = 0$:

$$\log_2 U = (U - I) - \tfrac{1}{2}(U - I)^2, \qquad a^{-1} \log_2(\exp(aN)) = N \quad (a \neq 0).$$

Both exponential/logarithmic inverse identities are checked on the cubic nilpotent and unipotent loci. If a supplied common sample u has $r_D(u) = r_E(u) = I$ and $t(u) \neq 0$, an intertwiner of the reconstructed cubic actions recovers both original group equivariance and monodromy equivariance. The general nilpotence exponential theorem is not restricted to this cubic class, but the extracted inverse in Block 13 was. Block 20 below removes that restriction by using the full finite logarithm.

The rational affine-group example has $u = (1, 1)$, $d = (0, 1/3)$ and

$$\rho(u) = \begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho(d) = \operatorname{diag}(1, 3, 9).$$

Reversing the factor order or dropping the quadratic term gives checked failures. For the length-four chain, the quadratic logarithm has erroneous e_3-to-e_0 coefficient $-1/3$, so its cubic restriction is substantive. A zero-parameter sample also fails to recover N.

No arithmetic group, tame cocycle or required sample is constructed by these statements. The attached comparison for every intended representation, coefficient comparisons and Hodge–Tate/period data remain separate. Finite inertia image and an open inertia kernel of r are not asserted to persist under this twist to ρ. Claim 9.2 remains partial.

10 Finite spectral occupancy

System completion produces arithmetic members. It does not prove that a desired packet occurs in a global carrier. In a fixed finite-dimensional semisimple carrier, however, occupancy is an exact projector question.

Let E be a field of characteristic zero and let $T_1, \dots, T_N$ be commuting operators on a finite-dimensional E-vector space A, each diagonalizable over E. Thus the selected spectra split over the coefficient field. Let $a_i \in E$ be a selected target scalar for T_i, possibly absent from its spectrum. Define

$$e_i = \prod_{\mu \in \operatorname{Spec}(T_i) \setminus \{a_i\}} \frac{T_i - \mu I}{a_i - \mu}, \qquad e_T = \prod_{i=1}^{N} e_i,$$

with $e_i = 0$ if a_i is absent.

Theorem 10.1 (Exact finite occupancy trace)**.** *The operator e_T is the projector onto the joint eigenspace*

$$A_T = \{v : T_i v = a_i v \text{ for every } i\}.$$

Moreover,

$$\boxed{\operatorname{Tr}(e_T) = \operatorname{rank}(e_T) = \dim A_T \in \mathbf{Z}_{\geq 0}.}$$

The trace is identified with its unique nonnegative integer preimage under $\mathbf{Z} \hookrightarrow E$. Thus $\mathrm{Tr}(e_T) > 0$ refers to that integer, not to an ordering of an arbitrary field E. The finite packet is occupied exactly when this integer is positive. In positive characteristic the field-valued trace can vanish for a nonzero projector; the rank test, not a positive field trace, must then be used.

Proof. For each operator, its finite set of actual eigenvalues is used without multiplicity. The polynomial

$$p_{i,a}(X) = \prod_{\mu \in \mathrm{Spec}(T_i) \setminus \{a\}} (a - \mu)^{-1}(X - \mu)$$

is used when a occurs, and the zero polynomial otherwise. For an eigenvector $T_i v = bv$, evaluation gives

$$p_{i,a}(T_i)v = \begin{cases} v & b = a, \\ 0 & b \neq a. \end{cases}$$

Because the eigenspaces span A, this proves that $e_i = p_{i,a_i}(T_i)$ maps all of A into the selected eigenspace and fixes that eigenspace pointwise. It therefore proves the individual projector and image assertions, including an absent target.

Polynomials in commuting operators commute. If projections P, Q onto U, W commute, then PQ maps into both U and W, and fixes every vector in $U \cap W$. Induction proves that the actual ordered product e_T is a projection onto A_T; commutation also makes the result independent of the ordering. An empty list of conditions gives the identity.

For a projection onto a subspace, the direct sum of its image and kernel identifies it with $I \oplus 0$. Thus its trace in E is the natural dimension of its image cast into E. In characteristic zero this cast is injective, so there is a unique natural integer represented by the trace. That integer is positive exactly when the joint eigenspace contains a nonzero vector. □

Naturality and finite hypotheses. If $FT_i = U_i F$ and the source operators split, then $Fe_{T,a} = e_{U,a}F$. This follows first on source eigenvectors and then on their span. In particular, a change of basis conjugates the constructed projector and preserves its trace. Adding conditions decreases the actual joint eigenspace and cannot increase its dimension. These statements do not assume an orthonormal basis or a positive definite inner product.

A specified finite realization. On $\mathbf{Q}^4$, take the diagonal operators with diagonal lists

$$(1,1,1,0), \quad (2,2,0,0), \quad (3,0,0,0), \quad (0,4,4,4)$$

and target scalars $(1,2,3,4)$. All four targets occur individually. Imposing these conditions successively gives actual projector ranks $3,2,1,0$. The last packet is empty despite individual feasibility. The formal code checks these operators, projector formulas, traces and dimensions. This is a newly specified realization, not a claim to have recovered the original companion implementation.

Why the hypotheses cannot be omitted. For

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

each operator is idempotent, but PQ is not idempotent and has trace $1/2$. For the Jordan block $J = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, the singleton-spectrum Lagrange polynomial is 1, so the formula produces the identity although the eigenspace at 1 is only a line. Both failures are exact formal counterexamples to dropping the corresponding hypotheses, not counterexamples to the theorem with those hypotheses retained.

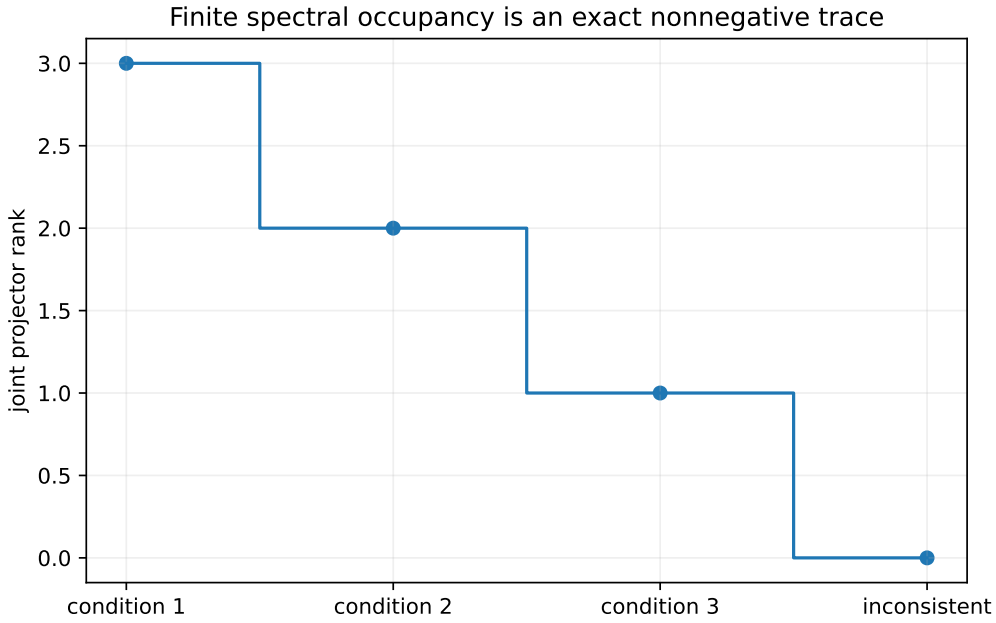


Figure 15: A consistent sequence of finite conditions narrows the carrier from ranks 3 to 2 to 1. One inconsistent condition forces the joint projector to zero.

Conjecture 10.2 (Universal occupancy)**.** For every admissible finite arithmetic packet in the intended global class, there exists a finite-dimensional carrier in which the associated joint projector has positive trace.

This conjecture is not a consequence of the projector formula. The formula measures occupancy exactly once the carrier is fixed. It cannot force the trace to be positive.

11 A conditional system-completion criterion

The preceding results can be assembled without promoting their missing hypotheses to theorems.

Theorem 11.1 (Residual-system completion criterion)**.** *Fix an admissible arithmetic class. Suppose:*

1. *every object has a unique residual root in an exhaustive residual moduli space;*
2. *a common algebraic spine is defined for the desired system;*
3. *every coefficient-prime residual companion exists;*
4. *every companion forest has nonempty finite levels, finite branching, and inverse-limit closed local conditions;*
5. *the required horizontal transports preserve the common spine and the selected local deformation conditions.*

Then the desired strictly compatible arithmetic system exists. If, in addition, the system is a summand of a known compatible carrier, the summand is recovered at every coefficient prime by finitely many Frobenius traces.

Proof. The first four hypotheses give the system by adelic finite-depth completion. The transport hypothesis identifies any roots that are meant to be generated horizontally without changing the prescribed spine. The final assertion is the compatible-carrier summand theorem. □

Corollary 11.2 (Exact location of the remaining existence problem)**.** *Inverse-limit reconstruction alone is not a nonemptiness theorem. The existence requirements must retain both residual companions and liftability through every finite depth:*

companion existence: $X_{\lambda,1}(\mathfrak{C}) \neq \varnothing$ *for every* λ*;*
finite-depth liftability: $X_{\lambda,m}(\mathfrak{C}) \neq \varnothing$ *for every* $\lambda, m \geq 2$*;*
finite occupancy: $\operatorname{Tr}(e_T) > 0$ *for every required global packet.*

Formal reconstruction does not create a missing residual root, repair an empty higher level, or force global occupancy. Nonempty finite levels with finite branching supply a branch, after which inverse-limit closure reconstructs an admissible object.

An occupied node can have no child. The actual order-two representation with generator

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \pmod 4$$

satisfies $A^2 = I$ and $\det(XI - A) = X^2 - 1$. It has no lift modulo 8 with the same characteristic polynomial. Indeed, every lift has entries congruent to $1, 2, 2, 3$ modulo 4. Trace zero then forces its determinant to be 3 modulo 8, whereas $X^2 - 1$ requires determinant $-1 = 7$ modulo 8. Both the occupied node and the absence of any next-level matrix or representation with the prescribed polynomial are Lean-checked. Other branches of the same unrestricted involution forest survive, so this example does not contradict finite-branch compactness.

Local properties need their own limit argument. Over a domain, for a nonzero $\pi \in I$, the representations of the additive integer group given by

$$\rho_n(g) = \begin{pmatrix} 1 & \pi^n g \\ 0 & 1 \end{pmatrix}$$

are all nontrivial and converge pointwise in the ideal-adic topology to the trivial representation. The convergence and the failure of limit preservation for nontriviality are formally checked. Thus not every property of approximating integral representations can be asserted for a limit. This example is not claimed to disprove stability for any particular intended Galois local condition.

Complete conditional existence boundary (Block 24). The eleventh block established the corresponding conditional matrix criterion. Block 24 now uses the complete common-spine conclusion of Claim 8.2: the returned coefficient-field representations are continuous, have the prescribed common polynomials and coefficient-specific determinants, and satisfy every selected local realization relation under the stated inverse-limit closure hypothesis. All finite reductions belong to the admissible classes.

For each supplied required packet p, let A_p be its finite-dimensional carrier over a characteristic-zero field E_p, and let e_p be the actual joint spectral projector for its finite list of split commuting conditions.

No order on E_p is assumed. The source notation $\mathrm{Tr}(e_p) > 0$ means that the unique natural number n_p with $\mathrm{Tr}(e_p) = (n_p : E_p)$ is positive. The checked projector formula identifies n_p with the dimension of the actual joint eigenspace.

An admissible realization here includes continuous coefficient-field members and membership of every finite reduction in its designated admissible class. Under the completion hypotheses and these supplied carrier hypotheses, Lean proves

$$\exists \rho,\ (x_p)_p : \quad \rho \text{ is an admissible realization of the full chosen spine}, \quad 0 \neq x_p \in \mathrm{im}(e_p) \text{ for every } p$$
$$\iff \quad \big(\forall \lambda,\ X_{\lambda,1}(\mathfrak{C}) \neq \varnothing\big) \ \wedge\ \big(\forall \lambda,\ \forall m \geq 2, X_{\lambda,m}(\mathfrak{C}) \neq \varnothing\big) \ \wedge\ \big(\forall p,\ n_p > 0\big).$$

The packet index may be infinite. Each x_p belongs to its own supplied carrier; no common vector across distinct carriers or embedding of the reconstructed system into those carriers is asserted. Neither an unknown carrier nor positive trace is constructed without the corresponding hypotheses.

This completes Claim 11.2 at its conditional source scope, on the same counting basis used for Claim 8.2 in Block 23. The full arithmetic application premises remain separate. In particular, the criterion does not prove universal occupancy, Hodge–Tate or Weil–Deligne attachment, or the Frobenius recovery clause of Claim 11.1. The latter still depends on the partial Claims 9.1 and 9.2.

Every required packet is necessary. For each omitted index $N \in \mathbb{N}$, consider one-dimensional rational carriers with the identity operator, requiring eigenvalue 0 at N and eigenvalue 1 at every other index. The actual spectral projector has trace 0 at N and trace 1 elsewhere. Lean proves that any predetermined finite sample of packet indices can miss an empty required packet: choose N outside that sample. The family depends on the omitted index. This does not assert that one fixed family passes every finite prefix while failing global occupancy.

Conjecture 11.3 (Natural finite seed rank)**.** For the admissible residual transport category attached to a fixed arithmetic class, the directed seed rank is finite. A one-seed theorem additionally requires a source component from which every required object is reachable and transports preserving deformation and occupancy data. For a finite condensation DAG, having exactly one source implies that reachability condition. For an infinite category, the existence of a unique source alone does not establish it.

12 Worked finite models

12.1 Smooth branch

For $q = 3$ and formal dimension $d = 2$, the finite levels have cardinalities

$$1, 9, 81, 729, 6561, \ldots.$$

The seed zeta function is

$$Z(T) = \frac{T}{1 - 9T}.$$

12.2 Weighted dyadic chart

For weights $(1, 2, 4)$, the counts are

$$1, 2, 8, 32, 256, 2048, \ldots.$$

The first coordinate is active immediately, the second one level later, and the third at depth five. The exact rational function generated by the code is

$$Z_{(1,2,4)}(T) = \frac{T(1 - 6T - 8T^2 - 32T^3)}{1 - 8T}.$$

The negative signs in this presentation arise from moving the denominator to $1 - 8T$; every power-series coefficient is positive.

12.3 Singular branch

For $x^2 = 0$ over a residue field of size three, the counts are

$$1, 3, 3, 9, 9, 27, 27, 81, \ldots,$$

and

$$Z(T) = \frac{T + 3T^2}{1 - 3T^2}.$$

The exact branching dimension is $1/2$.

12.4 Directed transport

The finite graph in figure 10 has three source components. The source-component theorem and brute-force enumeration both return the minimum seed count three.

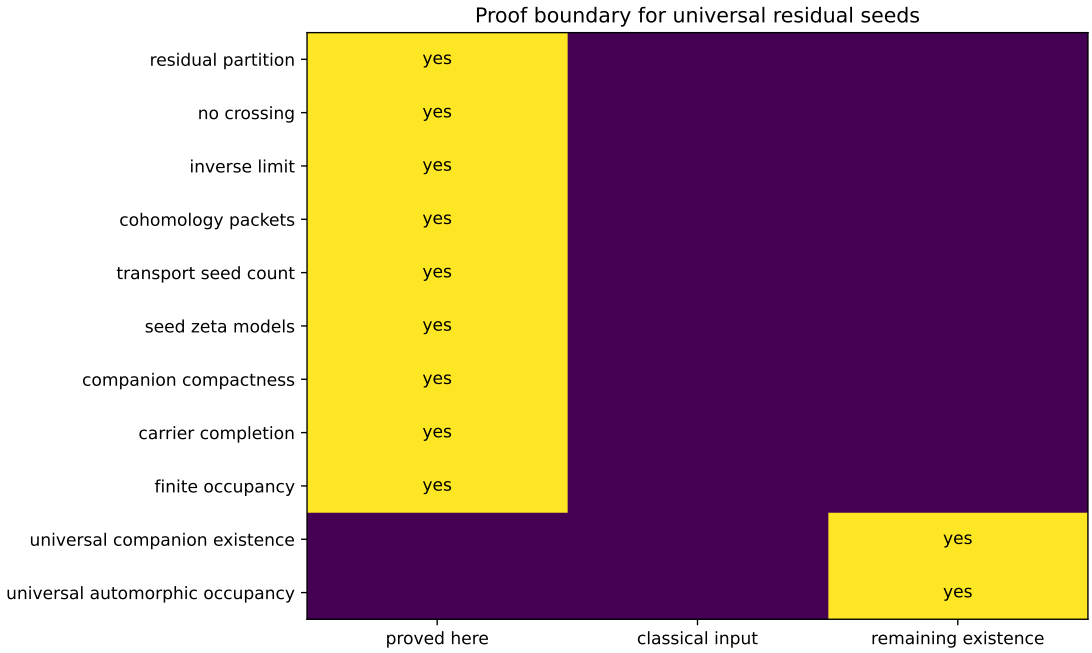


Figure 16: The finite and formal statements proved in this paper are separated from the two remaining universal existence assertions.

13 Computational certificates

The code uses exact integer, rational, symbolic, and modular arithmetic for all mathematical identities. Floating-point arithmetic is used only to render the figures and to report a regression estimate alongside an exact branching dimension.

Table 1: Selected exact certificates.

Certificate	Result
Residual packet count in the mod-three model	3
Compatible base-five inverse-limit branch	yes
First dyadic sign separation depth	2
Source strongly connected components	3
Brute-force minimum directed seed count	3
Cohomology packet reconstruction residual	0
Compatible common-spine test	passed
Inconsistent spine detected	yes
Recovered carrier multiplicities	$(2, 0, 1)$
Finite occupancy ranks	$(3, 2, 1, 0)$

The complete automated suite contains 28 tests. They cover residual partition, no crossing, inverse-limit digits, weighted and singular zeta coefficients, directed source components, cohomology packets, companion forests, common-spine consistency, finite Frobenius separation, multiplicity recovery, and exact occupancy projectors.

14 Claim ledger

Statement	Status	Scope or boundary
Residual partition	proved	any moduli functor with well-defined reduction
No crossing	proved	operations trivial at residual depth
Inverse-limit reconstruction	proved	represented complete deformation problem
Finite-branch compactness	proved	finite nonempty levels and finite branching
Smooth and weighted seed zeta formulas	proved	stated formal coordinate models
Singular power zeta and dimension	proved	scalar equation $x^e = 0$
Cohomology packet decomposition	proved	chain projectors commuting with the differential
Directed minimum-seed theorem	proved	finite directed transport graph
Adelic finite-depth completion	proved	common spine, all finite levels occupied, closed local conditions
Compatible-carrier summand completion	proved	known semisimple compatible carrier and finite separation
Finite occupancy trace	Lean checked	supplied commuting split operators; characteristic-zero trace criterion
Universal residual companions	not proved	requires nonempty root at every coefficient prime

Statement	Status	Scope or boundary
Universal compatible-system completion	not proved	algebraic spine and companion occupancy remain necessary
Universal automorphic occupancy	not proved	requires positive global packet traces
Unrestricted reciprocity	not claimed	belongs to later local and global papers

15 Conclusion

The residual seed theory becomes precise once vertical and horizontal generation are separated. Positive-depth perturbations can be complete within one formal component. They cannot change the residual root. The universal arithmetic seed is therefore a forest rather than one branch:

$$\boxed{\text{residual roots} \longrightarrow \text{finite deformation levels} \longrightarrow \text{inverse-limit branches.}}$$

Projectors add three useful structures to this forest. They split tangent and obstruction complexes into exact subproblems, they measure delayed dyadic visibility through weights, and they turn finite carrier occupancy into an integer trace. Directed transport adds a fourth: it converts the informal question "how many seeds are needed?" into the source-component count of a condensation graph.

The system-level boundary is now narrow enough to state without ambiguity. One must construct a common algebraic spine, populate every residual companion forest, lift those companions through all depths, and prove positive occupancy of the required global packets. The present paper proves the formal reconstruction, counting, transport, compactness, carrier, and finite-projector theorems used in that reduction. It does not assert the missing universal existence statements.

A Algorithms implemented in the companion package

A.1 Residual towers

The residual module partitions coordinates modulo p, verifies compatibility

$$x_{m+1} \equiv x_m \pmod{p^m},$$

and reconstructs the digit a_{m-1} by

$$a_{m-1} = \left\lfloor \frac{x_m}{p^{m-1}} \right\rfloor \pmod{p}.$$

For canonical representatives $0 \le x_m < p^m$, set $x_0 = 0$. Compatibility also gives $a_{m-1} = (x_m - x_{m-1})/p^{m-1} \in \{0, \dots, p-1\}$. The floor in the first formula is essential. For example, the compatible base-five residues $4, 19$ give the second digit 3, not the rational number $19/5$. The executable companion already uses integer floor division. This edition corrects the displayed algorithm, not the implementation.

A.2 Transport condensation

The transport module computes strongly connected components by two depth-first passes, forms the condensation graph, finds all source components, and selects one representative from each. A brute-force subset search is included only as an independent finite certificate.

A.3 Companion forests

For an integral polynomial f, the finite root forest is obtained by exact enumeration of

$$f(x) \equiv 0 \pmod{p^m}$$

and by connecting a child x_{m+1} to its reduction modulo p^m. This is intended for small exact examples, not large deformation rings.

A.4 Cohomology packets

Modular Gaussian elimination computes the ranks of the differentials. For

$$C^{i-1} \xrightarrow{d_{i-1}} C^i \xrightarrow{d_i} C^{i+1},$$

the finite dimension is evaluated as

$$\dim H^i = \dim C^i - \operatorname{rank} d_i - \operatorname{rank} d_{i-1}.$$

Packet subcomplexes are computed separately and checked against the total.

A.5 Finite occupancy

For each commuting semisimple operator, a Lagrange polynomial constructs the exact spectral idempotent. Their product gives the joint packet projector. Symbolic rational arithmetic verifies idempotence and the equality of trace and rank.

B Reproducibility boundary

The companion code is a finite certificate library. It does not construct universal Galois deformation rings, compute general continuous group cohomology, prove representability, discover compatible systems, or evaluate automorphic trace formulas. Those are mathematical inputs or later existence problems. The release is designed to make every finite example and every displayed count independently reproducible without confusing computation with proof.

Historical Block 27 formalization boundary

The following checkpoint discussion belongs to the Block 27 source. Its statements of partial status are not the live ledger of this edition. The later conditional contract is stated in Appendix D. The original frontmatter for that checkpoint is preserved here before the detailed development.

Block 27 formal-verification status. Seventeen source proof claims are checked at their declared conditional scope and three remain partial. Block 27 proves actual prime and Frobenius restriction to arbitrary finite normal number fields, derives unramifiedness outside the discriminant, and proves independence from Frobenius choices. Its general arithmetic coverage equivalence retains the prime-existence assertion as a premise. Block 25 proves the system-to-member isomorphism bridge under the required overlap density and proves actual arithmetic Frobenius separation for representations of a fixed cyclotomic Galois group over $\mathbf{Q}$. General arithmetic Frobenius coverage remains unproved here, so Claim 9.1 remains partial. Claim 11.2 is checked as the full-spine existence boundary with a nonzero joint eigenvector in every supplied required packet. The residual, higher-depth, and positive-trace requirements are retained; universal arithmetic occupancy is not proved. Claim 8.2 is checked as a universally quantified conditional completion theorem, retaining its explicit local-closure and finite-level-existence hypotheses. Earlier accounting included verification of particular arithmetic application premises in this claim; those applications remain separate and unfinished. For the fixed quadratic packet of actual $G_{\mathbf{Q}}$ characters, local triviality after the field-induced map from $G_{\mathbf{Q}_p}$ is proved equivalent to $p \equiv 1 \pmod 4$ or global triviality. Its preservation under coefficient completion and simultaneous finite-level existence criterion are proved. The packet characters are trivial on actual valuation inertia at every odd prime, with a locally dominating valuation constructed. The equality of the valuation decomposition group with the entire local Galois group is not established, and ramification at two is not claimed. The full packet identifications, Galois-to-Weil–Deligne attachment, and coefficient and Hodge–Tate/period comparisons remain unfinished.

Cumulative formalization status after Block 27. The cumulative formal ledger records 17 source proof claims checked at their stated conditional scope, 3 partially formalized, and no untouched proof claims. Claims 8.2 and 11.2 are checked with their stated hypotheses retained. Claims 9.1, 9.2, and 11.1 remain partial. The finite cyclotomic case of Claim 9.1 is now checked with arithmetic coverage proved. Arithmetic applications of the conditional completion and occupancy criterion still require verification of their premises. This status does not prove any universal existence conjecture. No PDF has been typeset or visually certified for this code-focused checkpoint.

Retained checkpoint discussion. The smooth formal-model classification, its finite point count, its integral digit streams, and its rational seed zeta identity are now checked in Lean. The actual representing-ring algebra isomorphism remains a supplied hypothesis; neither smoothness nor Galois representability is inferred from numerical counts. All local maps are included in the classification, and locality and actual maximal-ideal-adic continuity are separately proved. Arbitrary choices of parameter representatives do not add phantom maps.

The cumulative ledger contains seventeen checked source proof claims and three partial claims, with no untouched proof claims. Block 27 removes the finite-number-field restriction and Frobenius-choice attachment gaps, while general prescribed-conjugacy prime coverage remains unproved. Exact finite occupancy is checked. Claims 8.2 and 11.2 are complete at their conditional source scope, with arithmetic application premises still unproved in general. Claim 11.2 retains every residual level, every higher depth, and every required supplied packet; it does not prove their nonemptiness or positive trace unconditionally. Claims 9.1, 9.2, and 11.1 retain the gaps recorded in the current ledger. Block 25 proves the general system-to-member bridge under overlap density and the full finite cyclotomic arithmetic separation case, with actual unramified Frobenius primes supplied. General absolute-Galois Frobenius

coverage remains unfinished. Blocks 21–22 identify one actual quadratic arithmetic packet with its coordinate-ring points and prove its conditional finite-depth completion with actual simultaneous local-triviality conditions. The p-adic criterion is proved at every arithmetic prime. The packet characters are trivial on actual valuation inertia at odd primes; identification of its decomposition group with the full local Galois group remains open here, and ramification at two is not proved. These results do not identify every packet in the common arithmetic spine or attach Hodge–Tate and Weil–Deligne invariants. General nilpotent logarithmic recovery remains checked under its stated sample hypotheses. Universal occupancy and natural finite seed rank remain conjectures. The original companion package has not been rerun. The release provides separate exact checks and cumulative compiler evidence. No new PDF was typeset.

Finite-group comparison with precision loss (Block 17). Let R be a commutative domain, Γ a finite group, and ρ, τ actual integral matrix representations. Suppose $a = |\Gamma| \neq 0$ in R and $a \mid \pi^e$. An equivariant matrix modulo $(\pi)^{e+k}$ lifts to an integral equivariant matrix with the prescribed reduction modulo $(\pi)^k$. Indeed, lift its entries to A and form

$$S = \sum_{g \in \Gamma} \rho(g) A \tau(g^{-1}).$$

The finite equations imply $S - aA = \pi^{e+k}E$ with E integral. Choose b with $\pi^e = ab$ and put $P = A + b\pi^k E$. Then $aP = S$, so cancellation of $a \neq 0$ proves equivariance. Over a local ring, if π is a nonunit, $k > 0$ and the supplied comparison is invertible, P is an integral unit. Neither completeness nor a finite residue field is needed for this step.

For a DVR, $e = v_\pi(|\Gamma|)$ is derived from the actual group order. The single precision $e + 1$, independent of rank, detects integral isomorphism. It makes reduction injective on integral isomorphism classes, and finite coefficient quotients give finitely many integral classes in fixed rank. This is a formalization of classical finite-group rigidity (Maranda, 1953, Theorem 2), with an explicit proof of the comparison's precision loss.

For complete local coefficients with finite quotients, downward-compatible allowed representations occupied at every precision, and the required finite-group type at depth $e + 1$, reconstruction followed by this comparison produces the required integral representation. Continuity is proved for a finite discrete source and for pullback through a fixed continuous map to that source. Occupancy at only one precision is insufficient. The result lifts comparisons between actual integral representations; it does not lift arbitrary quotient representations or supply the missing arithmetic packet interpretation.

For C_2 over $\mathbb{Z}_2$, take

$$\tau(g) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \rho(g) = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The matrix A intertwines modulo 4 and its average is $P = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$. No exact intertwiner can retain A modulo 4, although P retains it modulo 2. The actual trivial and sign representations also agree modulo 2 but are not integrally isomorphic. These checked examples show why residual-only detection and full preservation of the input precision cannot be asserted here.

Lifting a summand inside a supplied finite-group carrier (Block 18). Let R be a commutative integral domain complete and separated for the (π)-adic filtration. Let Γ be a finite group, and let $\rho : \Gamma \to \mathrm{GL}_n(R)$

be an actual integral representation. Assume that $a = |\Gamma| \neq 0$ in R and $a \mid \pi^e$. For $k > 0$, suppose

$$\bar{E} \in M_n\big(R/(\pi)^{e+k}\big), \qquad \bar{E}^2 = \bar{E}, \qquad \bar{\rho}(g)\bar{E} = \bar{E}\bar{\rho}(g) \quad (g \in \Gamma).$$

There is an actual integral matrix E satisfying

$$E^2 = E, \qquad \rho(g)E = E\rho(g), \qquad E \bmod (\pi)^k = \bar{E} \bmod (\pi)^k.$$

No integral summand or coherent family of quotient projectors is assumed. No local-ring assumption or finiteness of the residue field is needed.

To prove this, apply the preceding averaging correction with both representations equal to ρ. This produces an exact equivariant matrix A with the prescribed reduction at depth k, so that $A^2 - A \in (\pi)^k M_n(R)$. Averaging alone need not preserve idempotence. Use the integral polynomial

$$f(A) = 3A^2 - 2A^3.$$

The identities

$$f(A) - A = (1 - 2A)(A^2 - A), \qquad f(A)^2 - f(A) = (A^2 - A)^2(4A^2 - 4A - 3)$$

show that successive corrections preserve the retained reduction and improve the idempotence defect. In these formulas scalar constants denote scalar matrices. The Lean proof establishes at least one additional ideal digit at each step, constructs the resulting compatible family of finite-level matrices, and uses completeness and separatedness to recover E. Every matrix commuting with A commutes with E. There is no division by 2, so the correction also works in residue characteristic 2.

The actual submodules $U = \operatorname{im} E$ and $W = \ker E$ are invariant, and addition gives an equivariant isomorphism

$$U \oplus W \simeq R^n.$$

The inclusion $i : U \hookrightarrow R^n$ and retraction $r(v) = Ev$ are constructed as linear maps and satisfy $ri = 1_U$ and $ir = E$. Their equivariance is proved. Coordinatewise reduction maps U onto $\operatorname{im}(\bar{E} \bmod (\pi)^k)$ and maps W onto $\ker(\bar{E} \bmod (\pi)^k)$. Both equalities are surjectivity statements onto the actual submodules over the finite-level quotient, not merely containment. The same projector defines a summand after pullback through any group homomorphism to Γ.

For $R = \mathbb{Z}_2$ and the sign carrier $\rho(s) = \operatorname{diag}(1, -1)$, the projector

$$\bar{E} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \pmod 4$$

is equivariant and the theorem constructs a nonzero proper equivariant projector retaining its reduction modulo 2. No exact equivariant endomorphism retains this prescribed matrix modulo 4. At the insufficient input depth 1, the projector $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right) \pmod 2$ has no exact equivariant lift retaining its reduction. These are statements about the supplied carrier and prescribed projectors, not about nonexistence of all integral summands.

This block constructs one summand and its complement inside a supplied finite-group carrier. It advances the carrier step in Claim 9.2. It does not identify the intended arithmetic packets, construct the arithmetic Weil–Deligne attachment, establish the Hodge–Tate comparisons, or prove universal occupancy. At the Block 18 checkpoint, the ledger recorded fifteen checked claims and five partially formalized claims; see the Block 23 conditional-scope correction above.

Simultaneous projector-family lifting and rigidity (Block 19). Keep the complete domain, supplied finite-group carrier and group-order hypotheses of Block 18. Let J be any finite label set and $k > 0$. Suppose $(\bar{E}_i)_{i\in J}$ is a complete orthogonal family of equivariant idempotents modulo $(\pi)^{e+k}$. There exists an integral equivariant family $(E_i)_{i\in J}$ satisfying

$$E_i^2 = E_i, \qquad E_iE_j = 0\ (i \neq j), \qquad \sum_{i\in J} E_i = 1, \qquad E_i \bmod (\pi)^k = \bar{E}_i \bmod (\pi)^k.$$

The loss is at most e, independent of $|J|$ and the ambient matrix rank. There is no local-ring or finite-residue-field hypothesis for existence. The formal theorem also treats empty label sets whenever their input equations can hold.

Average every supplied matrix once. Inductively let C be the complement of the previously lifted projectors and correct the next averaged matrix inside the corner defined by C. The integral polynomial correction preserves the retained reduction and commutation with C and the carrier. Multiplication by C gives the required supported projector. This consumes no further precision. Once every label has been lifted, the complementary idempotent reduces to zero at positive depth. Adic separatedness forces it to vanish, proving the sum-to-identity equation.

The actual invariant images $U_i = \operatorname{im} E_i$ give an equivariant linear isomorphism

$$\bigoplus_{i\in J} U_i \simeq R^n, \qquad (v_i)_i \longmapsto \sum_i v_i, \qquad v \longmapsto (E_iv)_i.$$

Lean represents the finite direct sum by the finite product of the image submodules. The existing coordinatewise reduction theorem maps every U_i onto the corresponding reduced projector image. No chosen bases or unproved freeness of the images are used.

For rigidity alone, let R be a commutative local ring, let π be a nonunit, and let E, F be complete equivariant families on the same carrier with $E_i \equiv F_i \pmod{(\pi)^k}$ for every label and $k > 0$. Then

$$U = \sum_{i\in J} F_iE_i, \qquad U \equiv 1 \pmod{(\pi)^k}, \qquad UE_i = F_iU.$$

Reduction reflects invertibility in this setting, so U is an integral unit and $UE_iU^{-1} = F_i$. It commutes with the carrier and induces actual equivariant isomorphisms $\operatorname{im} E_i \simeq \operatorname{im} F_i$. This rigidity theorem requires neither a finite acting group, a domain, nor completeness. More generally a unit-reflecting coefficient map suffices. The combined result proves existence and uniqueness up to this labeled conjugacy; exact uniqueness of the matrices is false.

For the continuous $\mathbb{Z}_2$ carrier $\rho(s) = \operatorname{diag}(1, -1, 1)$, conjugate the three coordinate projectors by $1 + 2e_{01}$ and reduce modulo 4. They form a complete equivariant family. The general theorem constructs three nonzero integral summands retaining this family modulo 2. No exact equivariant family retains the whole prescribed family modulo 4. Separate checked examples show that independent equivariant idempotent lifts can lose orthogonality, and that distinct complete families with the same reduction can be related by the asserted identity-congruent conjugacy.

Block 19 completes this finite-group projector-family step. The actual arithmetic packets, Galois-to-Weil–Deligne attachment, coefficient and Hodge–Tate/period comparisons, and arithmetic Frobenius coverage remain outside this result. At the Block 19 checkpoint, the ledger recorded fifteen checked claims and five partial claims; see the Block 23 conditional-scope correction above. Universal occupancy remains conjectural.

General finite logarithmic recovery (Block 20). Let A be a rational algebra, without any commutativity assumption on A. For a unipotent element U and a positive integer d satisfying $(U-1)^d = 0$, put

$$L_d(U) = \sum_{j=1}^{d-1} \frac{(-1)^{j+1}}{j} (U-1)^j.$$

The value is independent of the chosen annihilating exponent. The finite logarithm and the nilpotent exponential satisfy

$$L(\exp N) = N, \qquad \exp(L(U)) = U.$$

Both maps preserve every annihilating exponent, and hence preserve the exact nilpotency class of N and $\exp(N) - 1$, or of $U - 1$ and $L(U)$. The Lean finite sum includes its zero-index term with coefficient zero; the displayed mathematical sum starts at one. The formal statements also handle the zero algebra and the exponent-zero case.

The proof evaluates Mathlib's formal power-series inverse identities at constant nilpotent series. Such evaluation is exactly a finite polynomial. For an arbitrary noncommutative ambient algebra, perform the calculation inside the commutative subalgebra generated by the element and then apply its inclusion. No analytic convergence or topology is assumed. The finite logarithm commutes with ring homomorphisms between rational algebras.

Let K be a characteristic-zero field and N an endomorphism with $N^d = 0$. For $t \neq 0$ the actual recovery formula is

$$N = \frac{1}{t} \sum_{j=1}^{d-1} \frac{(-1)^{j+1}}{j} \big(\exp(tN) - 1\big)^j.$$

All coefficients lie in K. The vector spaces need not be finite dimensional, but nilpotence still means that some finite power of the whole operator is zero. A linear map intertwines two nilpotent operators exactly when it intertwines their exponential samples at a common nonzero parameter. Also

$$\ker\big(\exp(tN) - 1\big) = \ker N.$$

For supplied covariance data and the twisted cocycle of Block 13, let $\rho_D(g) = \exp(t(g)N_D) r_D(g)$. If a common element u satisfies $r_D(u) = r_E(u) = 1$ and $t(u) \neq 0$, every intertwiner of ρ_D and ρ_E recovers both the original r-intertwiner and the N-intertwiner. There is no cubic nilpotence restriction. This recovers equivariant projectors for both original components. Reconstructed-action isomorphisms induce actual isomorphisms of the inertia-fixed monodromy kernels. Their local characteristic polynomials agree under the existing finite-dimensional and normal-subgroup hypotheses.

For actual finite original inertia images the common kernel sample is derived. Suppose $u \in H$ satisfies $v(u) = 1$ and $t(u) \neq 0$ and both $r_D(H)$ and $r_E(H)$ are finite. Their sampled actions have positive finite orders a, b. At u^{ab} both original actions are trivial, and

$$t(u^{ab}) = (ab)t(u) \neq 0$$

by the cocycle identity and characteristic zero. Thus the same full intertwiner equivalence holds using the actual finite-image hypotheses, without a separately supplied common trivial-action sample. The initial nonzero cocycle sample and the arithmetic interpretation of the data remain requirements.

Actual affine representations with monodromy chains of every finite length invoke these theorems. In particular a length-six example has $N^5 \neq 0$ and $N^6 = 0$ and is recovered from its actual reconstructed

action. The length-four example still defeats the old quadratic logarithm, while the full finite logarithm recovers it.

The sample hypotheses retain substantive content. A zero parameter loses N. For unrestricted supplied algebraic data, one can also absorb $\exp(tN)$ into r and set the new monodromy to zero; the reconstructed representation is unchanged. The absorbed original action fails the required trivial-action sample condition. This is a counterexample in the algebraic datum class, not a failure of uniqueness of arithmetic Weil–Deligne attachment with finite original inertia action.

Block 20 removes the general logarithmic-extraction gap from the local algebraic comparison. It does not construct the actual arithmetic attachment, identify every intended packet, prove the coefficient and Hodge–Tate/period comparisons, or prove arithmetic Frobenius coverage. At the Block 20 checkpoint, the ledger recorded fifteen checked claims and five partial claims; see the Block 23 scope correction above.

Interpreted quadratic arithmetic completion (Block 21). Fix the continuous quotient

$$q : G_{\mathbf{Q}} \longrightarrow C_2$$

obtained by the action on a chosen square root i of -1 in the algebraic closure. Its surjectivity and the equivalence

$$q(\sigma) = 1 \quad \Longleftrightarrow \quad \sigma(i) = i$$

are proved using the existing actual quadratic Galois construction. For a commutative ring B, define the arithmetic packet

$$\mathcal{P}_q(B) = \{\chi : G_{\mathbf{Q}} \longrightarrow B^\times \mid \ker q \subseteq \ker \chi\},$$

where the maps are group homomorphisms. Unique descent through the actual surjective map q is derived, rather than supplied. Every member has finite image and is continuous for the coefficient topology, since its descent has the finite discrete domain C_2.

Let R be any commutative ring and B any commutative R-algebra. The coordinate ring $A_R = R[X]/(X^2 - 1)$ represents precisely this specified packet:

$$\mathrm{Hom}_{R\text{-alg}}(A_R, B) \simeq \{x \in B : x^2 = 1\} \simeq \mathrm{Hom}(C_2, B^\times) \simeq \mathcal{P}_q(B).$$

The first map evaluates the coordinate. Its inverse is the quotient universal property. A root is a unit with itself as inverse and determines a character by its value on the generator of C_2. Finally pull back along q. All inverse identities and naturality under coefficient homomorphisms are proved. In particular, the point reductions and actual arithmetic character reductions coincide. There is no invertibility-of-two, reducedness or nontriviality hypothesis on B. No smoothness or étaleness statement at two is used or asserted.

Let $I \subseteq R$ and suppose that R is I-adically complete. Using the checked ring-map reconstruction on the quadratic coordinate ring, Lean proves

$$\mathcal{P}_q(R) \simeq \varprojlim_{m \geq 0} \mathcal{P}_q(R/I^m).$$

The zero-precision quotient is included only for the formal indexing. The inverse constructs an actual arithmetic character, with exactly the prescribed reductions. It is unique, and its continuity is derived from its factorization through the same finite quotient.

There is also a completion theorem without chosen compatible characters. Suppose that R/I^m is finite for every $m \geq 1$, and that $C_m \subseteq \mathcal{P}_q(R/I^m)$ are allowed subsets such that every reduction of an allowed member is allowed. Then

$$\begin{aligned} &\exists \chi \in \mathcal{P}_q(R), \quad \chi \text{ is continuous and } \chi \bmod I^m \in C_m \text{ for all } m \geq 1 \\ &\qquad \Longleftrightarrow \quad C_m \neq \varnothing \text{ for all } m \geq 1. \end{aligned}$$

The root interpretation makes the allowed finite levels genuinely finite. Finite-branch compactness chooses the compatible roots, and the ring reconstruction gives their integral root. Naturality turns this into the required arithmetic character. Surjectivity of transitions and individual liftability of every finite node are not assumed. The root-level theorem needs only the allowed root subsets to be finite.

For $R = \mathbf{Z}_2$, Lean derives finiteness of all positive maximal-ideal quotients and applies this completion theorem. Since $\mathbf{Z}_2$ is a domain, its only roots of $X^2 - 1$ are 1 and -1. Thus the integral packet has exactly two distinct characters: the trivial character and the sign character describing the action on i. They have the same reduction modulo two and distinct reductions modulo four.

The distinction between finite occupancy and continuation is witnessed by an actual finite arithmetic character. The roots modulo eight are $1, 3, 5, 7$. The root 3 defines a member of $\mathcal{P}_q(\mathbf{Z}/8\mathbf{Z})$, but no member of $\mathcal{P}_q(\mathbf{Z}/16\mathbf{Z})$ reduces to it. Indeed the only scalar lifts of 3 are 3 and 11, and both square to 9 modulo sixteen. The formal proof also classifies all roots modulo sixteen and the reductions with children. Consequently a nonempty finite node need not lie on an infinite branch, even in this explicitly interpreted arithmetic packet.

Block 21 completes this restricted arithmetic interpretation and its conditional completion. The packet is defined by one fixed quotient; it is not the collection of all quadratic characters of $G_{\mathbf{Q}}$. No local ramification criterion, Hodge–Tate weight, Weil–Deligne attachment, or cross-coefficient local invariant was established by Block 21. The full manuscript still requires its remaining packet identifications and arithmetic comparisons. At the Block 21 checkpoint, the ledger recorded fifteen checked claims and five partial claims; see the Block 23 conditional-scope correction above. Universal positive occupancy remains conjectural.

Actual quadratic local conditions and completion (Block 22). Continue with the fixed packet $\mathcal{P}_q$ above. For a characteristic-zero field K, let

$$j_K : G_K \longrightarrow G_{\mathbf{Q}}$$

be the actual field-induced homomorphism between absolute Galois groups. It includes a chosen embedding of algebraic closures. Let i_K be the image of the chosen square root i under that embedding. Lean proves compatibility with the Galois actions and hence

$$(q \circ j_K)(\sigma) = 1 \quad \Longleftrightarrow \quad \sigma(i_K) = i_K.$$

The fixed-field theorem for the algebraic closure then gives

$$q \circ j_K = 1 \quad \Longleftrightarrow \quad \exists x \in K,\ x^2 = -1.$$

If no such root exists, the local quotient is surjective onto C_2. These are statements about the actual local Galois map, not a supplied arithmetic interpretation of a scalar label.

For every commutative coefficient ring B and $\chi \in \mathcal{P}_q(B)$, define $\chi_K = \chi \circ j_K$. The local character is continuous and commutes with coefficient change. The precise local admissibility condition is

$$\mathrm{LocallyTrivial}_K(\chi) \quad \Longleftrightarrow \quad \chi_K = 1.$$

Descent through the fixed quadratic quotient proves

$$\mathrm{LocallyTrivial}_K(\chi) \iff \big(\exists x \in K,\ x^2 = -1\big) \ \vee\ \chi = 1.$$

No reducedness, characteristic-zero or domain hypothesis is imposed on B. In particular, the formula applies to the finite coefficient quotients used in completion.

The p-adic square-root criterion is proved for every prime:

$$\big(\exists x \in \mathbf{Q}_p,\ x^2 = -1\big) \iff p \equiv 1 \pmod 4.$$

A root satisfies $|x|_p^2 = 1$ and is therefore integral. Reduction modulo three rules out a root in $\mathbf{Q}_3$, and reduction modulo four rules out a root in $\mathbf{Q}_2$. At an odd prime, any residue root of $X^2 + 1$ has nonzero derivative and lifts by the pinned Mathlib Hensel theorem. The finite-field square criterion then gives the stated congruence. An explicit Hensel application at the residue root 2 modulo 5 supplies a root in $\mathbf{Z}_5$. Consequently

$$\boxed{\mathrm{LocallyTrivial}_{\mathbf{Q}_p}(\chi) \iff p \equiv 1 \pmod 4 \ \vee\ \chi = 1.}$$

The arithmetic prime p is independent of the coefficient prime. For the two integral $\mathbf{Z}_2$-valued characters, the sign character is locally trivial exactly at primes congruent to one modulo four. Both distinct global characters are locally trivial at five, while the sign character is nontrivial on the full local Galois groups at three and two.

The inertia theorem uses actual residue actions. For a valuation subring A of an extension field L/K, let D_A be its decomposition subgroup in $\mathrm{Aut}_K(L)$ and put

$$I_A = \ker\big(D_A \longrightarrow \mathrm{Aut}(k_A)\big),$$

where k_A is the residue field. If 2 is a unit in A, every member of I_A fixes every $y \in L$ with $y^2 = -1$. Indeed $y^{-1} = -y$, so y belongs to A and is a unit there. An automorphism sends y to y or $-y$. Their residues are distinct because 2 and y are units, whereas inertia fixes residues. The second possibility is excluded.

For every prime p, Lean constructs a valuation subring $A \subseteq \overline{\mathbf{Q}_p}$ containing the integral image of $\mathbf{Z}_p$ such that the induced map $\mathbf{Z}_p \to A$ is local. For odd p, two is a unit. The preceding argument applies to every valuation subring containing that integral image, and hence every member of $\mathcal{P}_q(B)$ is trivial on its inertia subgroup after actual local restriction. The existence theorem supplies a locally dominating choice, so this assertion is not vacuous.

This inertia is the actual residue-action kernel inside D_A. No equality $D_A = G_{\mathbf{Q}_p}$ is established in this block. Nor is ramification at two proved: the nontrivial full local character there is a separate result. At three the $\mathbf{Z}_2$-valued sign character is nontrivial on the full local Galois group but trivial on every inertia subgroup covered by the theorem. Thus local triviality and the stated inertia triviality are distinguished by checked arithmetic examples.

The local conditions survive coefficient completion by a proved theorem. For an I-adically separated commutative ring R,

$$\mathrm{LocallyTrivial}_K(\chi) \iff \mathrm{LocallyTrivial}_K(\chi \bmod I^m) \text{ for every } m \geq 1.$$

If -1 has a square root in K, every packet member satisfies the condition. Otherwise the condition says that the character is globally trivial, and separatedness detects that equality through all reductions. This proves the closure statement for the independently defined arithmetic property.

Suppose now that R is I-adically complete and that every positive quotient R/I^m is finite. Let $C_m \subseteq \mathcal{P}_q(R/I^m)$ be allowed subsets preserved by reduction, and let $(p_s)_{s\in S}$ be any indexed family of arithmetic primes. The index set S need not be finite. Lean proves the simultaneous arithmetic completion criterion

$$\begin{aligned} &\exists \chi \in \mathcal{P}_q(R), \quad \chi \text{ continuous}, \quad \chi \bmod I^m \in C_m \ (m \geq 1), \quad \chi_{\mathbf{Q}_{p_s}} = 1 \ (s \in S) \\ &\iff \quad \forall m \geq 1, \ \exists \psi \in C_m, \quad \forall s \in S, \ p_s \equiv 1 \pmod 4 \ \vee \ \psi = 1. \end{aligned}$$

The constructed character also satisfies every odd-prime valuation-inertia condition stated above. Finite-branch compactness selects all finite characters together; no coherent branch or surjective transition maps are assumed. The new local-closure implication is proved, not supplied. Every-level occupancy remains a substantive hypothesis. One nonsplit selected prime forces the completed global character to be trivial.

For the integral dyadic packet, a single detecting precision is also proved: reduction modulo four detects local triviality uniformly in the arithmetic prime. Modulo two does not suffice. The sign character is nontrivial locally at three, while its reduction modulo two is trivial. This is a coefficient-depth failure, distinct from splitting of the arithmetic field.

Block 22 finishes this actual local admissibility and completion step for the fixed quadratic packet. It does not identify all packets in the manuscript's common arithmetic spine, attach Weil–Deligne objects, or prove coefficient and Hodge–Tate/period comparisons. The decomposition and full local Galois group identification remains a separate inertia obligation, and ramification at two is not claimed. The full arithmetic ledger at the Block 22 checkpoint recorded fifteen checked source claims and five partial claims; the Block 23 conditional-scope correction gives sixteen checked and four partial. Universal occupancy and natural finite seed rank remain conjectures.

C Retained residue-counting continuation

The following continuation is retained at its original checkpoint scope. Its 17/20 status and the statement that Claims 9.1, 9.2 and 11.1 remain partial are historical descriptions of that checkpoint. Section D supplies the later amended conditional completion. The counting formulas, hypotheses and remaining norm-image obligations below are preserved.

Scope and notation. Let K be a number field of degree $d \geq 1$, with ring of integers $\mathcal{O} = \mathcal{O}_K$. Write $\mathrm{N}\, I = \#(\mathcal{O}/I)$ for the absolute norm of a nonzero integral ideal and $n(x) = \mathrm{N}_{K/\mathbb{Q}}(x) \in \mathbb{Z}$ for $x \in \mathcal{O}$. Thus $\mathrm{N}(x) = |n(x)|$. Let ι be the mixed embedding in its Euclidean coordinates, $L(J) = \iota(J)$, and D the actual unit-norm fundamental cone used in the preceding counting block. Its norm function scales by t^d under a positive real dilation t. The full-paper ledger remains **17 checked claims and 3 partial claims**.

Actual additive residues. Fix nonzero integral ideals $M \subseteq J$, and put $q = [J : M] = \#(J/M)$. The formal quotient of ideal lattices is identified with the additive congruence relation $x - a \in M$ on actual elements of J. In particular, the residue label is not an independent parameter attached to a count.

Proposition C.1. *There is a constant $B \geq 0$, depending on K, M, J, such that, for every $a \in J/M$ and every real $t \geq 1$,*

$$\left| \#\{x \in J : x \bmod M = a,\ \iota(x) \in tD\} - \frac{\mathrm{vol}(D)}{q\ \mathrm{covol}(L(J))} t^d \right| \leq B(t^{d-1} + 1).$$

The same constant works for all residues a.

The estimate is proved from the actual boundary charts and the fixed-lattice discrepancy theorem. The index factor comes from the covolume identity $\mathrm{covol}(L(M)) = q\ \mathrm{covol}(L(J))$. For a complex weight $w : J/M \to \mathbb{C}$, summing the estimates gives the leading term

$$\frac{\mathrm{vol}(D) t^d}{q\ \mathrm{covol}(L(J))} \sum_{a \in J/M} w(a),$$

with error at most $B \sum_a |w(a)|(t^{d-1} + 1)$. If the finite sum of weights is zero, only that error remains. Writing $Y = t^d$ gives an error $O(Y^{1-1/d} + 1)$ for the actual norm threshold. These assertions include $d = 1$.

Lean attachment: `ResidueLatticeIdeal`, `ResidueLatticeIdealPoints`, `ResidueLatticeIdealWeights`, and `ResidueLatticeNormCounts`.

The residue norm and its sign. For $m \in \mathbb{N}$, reducing the multiplication matrices of $\mathcal{O}$ modulo m and taking their determinants constructs a monoid homomorphism

$$\overline{n} : \mathcal{O}/m\mathcal{O} \longrightarrow \mathbb{Z}/m\mathbb{Z}, \qquad \overline{n}([x]) = n(x) \bmod m.$$

The construction is proved independent of the integral basis. For $m \neq 0$, its source is finite of cardinality m^d. Consequently $x - y \in m\mathcal{O}$ implies $n(x) \equiv n(y) \pmod{m}$. No surjectivity of $\overline{n}$ is asserted.

The relation to the geometric sign is also proved. If σ ranges over real embeddings and τ over one embedding from each complex pair, then

$$n(x) = \prod_{\sigma} \sigma(x) \prod_{\tau} |\tau(x)|^2.$$

For $x \neq 0$, the second factor is positive. Thus the sign of $n(x)$ is exactly the sign of the real-coordinate product. Cutting D into its positive and negative portions D_+, D_- adds no boundary outside the existing frontier, and both portions retain the proved boundary atlas. They are disjoint and their union is D.

For $\varepsilon \in \{1, -1\}$ define $\overline{n}_\varepsilon(a) = \varepsilon \overline{n}(a)$. On the corresponding sign portion,

$$\mathrm{N}(x) \bmod m = \overline{n}_\varepsilon([x]).$$

The sign cannot be dropped: $-1 \equiv 2 \pmod{3}$, whereas $\mathrm{N}(-1) = 1$ and $\mathrm{N}(2) = 2$ for principal ideals of $\mathbb{Z}$.

The exact ideal filter. Suppose $IJ = (x)$, $n(x)$ has the indicated sign, and $\gcd(\mathrm{N}\, J, m) = 1$. For $c \in \mathbb{Z}/m\mathbb{Z}$,

$$\mathrm{N}\, I \equiv c \pmod{m} \quad \Longleftrightarrow \quad \overline{n}_\varepsilon([x]) = c(\mathrm{N}\, J \bmod m).$$

This gives an actual finite residue target, which is allowed to be empty. The product identity before cancellation does not require coprimality.

Let $[J] = C^{-1}$ in $\mathrm{Cl}(\mathcal{O})$, and let $\mathcal{G}(J)$ denote the actual fundamental-cone generator set. For every predicate $P : \mathbb{R} \to \mathrm{Prop}$ and every real X, the norm-preserving correspondence gives

$$\begin{aligned} &\#\{x \in \mathcal{G}(J) : \mathrm{N}(x) \leq X\, \mathrm{N}\, J,\ P(\mathrm{N}(x))\} \\ &\qquad = \#\mu(K)\, \#\{I \neq 0 : [I] = C,\ \mathrm{N}\, I \leq X,\ P(\mathrm{N}\, I\ \mathrm{N}\, J)\}. \end{aligned}$$

The same norm-preserving equivalence carries arbitrary complex weights. In particular, write $G_{J,\chi}(Y)$ for the sum of $\chi(\mathrm{N}(x) \bmod m)$ over these generators with $\mathrm{N}(x) \leq Y$, and $T_{C,\chi}(X)$ for the corresponding sum over ideals in class C. For multiplicative χ preserving 0 and 1, the exact identity is

$$G_{J,\chi}(X\, \mathrm{N}\, J) = \#\mu(K)\, \chi(\mathrm{N}\, J \bmod m)\, T_{C,\chi}(X). \tag{1}$$

Both signs are included before applying this full roots-of-unity factor. A root of unity can interchange the sign portions, so that factor is not automatically valid for either portion separately.

Lean attachment: `NormResidueArithmetic`, `NormResidueIdealFilter`, `NormSignArithmetic`, `IdealNormFilterAttachment`, and `IdealNormWeightedAttachment`.

Ordinary class characters: an unconditional power saving. The preceding block proved, for every ordinary ideal class C,

$$A_C(X) = c_K X + O_{K,C}(X^{1-1/d}), \qquad c_K = \frac{2^{r_1}(2\pi)^{r_2} R_K}{\#\mu(K)\sqrt{|\Delta_K|}}.$$

Here A_C counts actual nonzero integral ideals, r_1, r_2 count the real and complex places, R_K is the regulator, and Δ_K is the discriminant. For a weight $w : \mathrm{Cl}(\mathcal{O}) \to \mathbb{C}$, define the actual finite ideal sum

$$S_w(X) = \sum_{\substack{I \neq 0 \\ \mathrm{N}\, I \le X}} w([I]).$$

Exact grouping by ideal class gives $S_w(X) = \sum_C w(C) A_C(X)$. If $\sum_C w(C) = 0$, the common leading coefficient cancels, so there is $B_w \ge 0$ such that

$$|S_w(X)| \le B_w X^{1-1/d} \qquad (X \ge 1).$$

For every nontrivial character $\chi : \mathrm{Cl}(\mathcal{O}) \to \mathbb{C}^\times$, the finite character sum is zero. Thus the displayed estimate applies without an additional arithmetic cancellation hypothesis. This conclusion concerns ordinary class characters; it does not identify them with the further norm-residue characters needed for Frobenius separation.

An actual Mellin endpoint. The counting functions are constant on the integer norm steps. Their measurability, and hence that of S_w, is proved from this fact. For zero-sum w, define

$$F_w(h) = (1+h) \int_1^\infty x^{-2-h} S_w(x)\, dx \qquad (h \ge 0).$$

Since $1 - 1/d < 1$, the logarithmic first moment

$$M_w = \int_1^\infty (1 + \log x)\, |x^{-2} S_w(x)|\, dx$$

is finite. Lean proves the explicit estimate $|F_w(h) - F_w(0)| \le h M_w$ for every $h \ge 0$, and therefore $F_w(h) = F_w(0) + O(h)$ as $h \downarrow 0$. The function here is the displayed integral of the actual ideal sum. Its identification with the intended arithmetic Euler product is a separate step.

For a finite family of nontrivial ordinary class characters, suppose functions p, Z satisfy $h\, p(h) \to P \neq 0$, $h\, Z(h) \to Q \neq 0$, and, for all sufficiently small $h > 0$,

$$Z(h) = \left(p(h) \prod_i F_{\chi_i}(h) \right)^e, \qquad e \in \mathbb{N}.$$

The checked real-axis criterion then gives $e = 1$ and $F_{\chi_i}(0) \neq 0$ for every i. The pole limits and product identity remain hypotheses of this application.

Lean attachment: `IdealClassCharacterBound`, `IdealCharacterMellinMeasurable`, `IdealCharacterMellin`, and `IdealCharacterMellinPole`.

An actual unit action on ideal residues. Fix $m > 0$ and a nonzero integral ideal J. Multiplication by $u \in (\mathcal{O}/m\mathcal{O})^\times$ gives a proved permutation of J/mJ. Its inverse follows from $(ab-1)x \in mJ$ when $ab - 1 \in m\mathcal{O}$ and $x \in J$. No surjectivity of $J/mJ \to \mathcal{O}/m\mathcal{O}$ or coprimality condition on J is used. The well-defined residue norm on this quotient satisfies

$$\overline{n}_J(u \cdot a) = \overline{n}(u)\,\overline{n}_J(a).$$

For a multiplicative $\chi : \mathbb{Z}/m\mathbb{Z} \to \mathbb{C}$ preserving 1, an exhibited unit with

$$u \in (\mathcal{O}/m\mathcal{O})^\times, \qquad \chi(\overline{n}(u)) \neq 1 \tag{2}$$

therefore forces

$$\sum_{a \in J/mJ} \chi\big(s\,\overline{n}_J(a)\big) = 0 \qquad (s \in \mathbb{Z}/m\mathbb{Z}).$$

Taking $s = 1, -1$ gives the two finite means needed by the sign-corrected norm weights. Their vanishing follows from the actual permutation.

The assembled ideal norm-character estimate.

Proposition C.2. *Let $C \in \mathrm{Cl}(\mathcal{O})$, let $J \neq 0$ be an integral ideal with $[J] = C^{-1}$, and let $m > 0$. Let $\chi : \mathbb{Z}/m\mathbb{Z} \to \mathbb{C}$ preserve multiplication, 0, and 1. Assume a residue-ring unit satisfying (2), and assume $\chi(\mathrm{N}\,J \bmod m) \neq 0$. Then*

$$T_{C,\chi}(X) = \sum_{\substack{I \neq 0,\ [I]=C \\ \mathrm{N}\,I \leq X}} \chi(\mathrm{N}\,I \bmod m) = O_{K,J,m,\chi}(X^{1-1/d}).$$

More precisely, some $B \geq 0$ bounds its absolute value by $BX^{1-1/d}$ for every real $X \geq 1$.

The assembly identifies the residue sum with the sum over actual generators, recombines the sign portions, and uses $Y = t^d$ to obtain $|G_{J,\chi}(Y)| \leq B_0(Y^{1-1/d} + 1)$. Equation (1) then permits division by the nonzero factor $\#\mu(K)\chi(\mathrm{N}\,J \bmod m)$. The final theorem supplies its finite means and counting estimates internally. For $d = 1$ the exponent is zero and the conclusion is a bounded sum, so the argument does not require $d > 1$.

What remains open. Nontriviality of χ on the larger residue ring does not supply (2): its restriction to the image of the norm on units may be trivial. Witness existence for the intended characters, their remaining norm-image cases, and suitable inverse-class representatives with $\chi(\mathrm{N}\,J) \neq 0$ are still arithmetic obligations. Measurability and the Mellin attachment for these norm-character sums remain to be checked. The compiled Mellin results above concern ordinary class characters. The resulting estimates must also be attached to the required Euler-product identities and prime-occurrence argument. The conditional pole criterion does not supply those identities. Claims 9.1, 9.2, and 11.1 remain partial, including the separate local compatibility obligations. The ledger stays **17 checked / 3 partial**.

Lean attachment: `IdealResidueUnitActionMean`, `NormResidueUnitCounting`, `NormResidueGeneratorAttachment`, `IdealNormCharacterBound`, and `NormResidueCharacterCounting`.

D Conditional occupied-system completion: final checked scope

The current endpoint is `PaperX.ConditionalOccupiedSystemCompletion.conditional_occupied_system_completion`. The retained release records 20/20 internal claims checked at their amended and conditional source scopes. This is an assembly theorem with explicit inputs, not unrestricted companion existence or carrier existence.

D.1 Two conclusions with separate inputs

The first conclusion starts with actual completed source members at every coefficient place, admissibility at every positive depth, reduction-natural horizontal transport, and an inverse-limit-closed target profile. The predecessor theorem `exists_transported_system_of_inverseLimitClosed` constructs a coefficient-indexed target family. Each target member satisfies every declared finite-depth admissibility condition, is related to the source by the exact all-depth relation `TransportedAlong`, and has the declared complete target profile.

The second conclusion starts with a supplied finite compatible family of pairwise nonisomorphic irreducible constituents, a supplied carrier and its explicit constituent-copy decomposition, and a supplied initial subrepresentation with its constituent-copy presentation. Claim 9.1's arithmetic occurrence result supplies a finite set of distinct good primes outside any prescribed finite forbidden set. Its actual Frobenius trace matrix has nonzero determinant. Claim 9.2 then uses this separator to recover the selected multiplicity vector and its summand inside the carrier.

Write ρ_i for the supplied constituents, c_i for the carrier multiplicities and m_i for the selected multiplicities, with $m_i \le c_i$. At a distinguished coefficient place λ, the input presentations have the form

$$V_{\mathrm{car},\lambda} \simeq \bigoplus_i \rho_{i,\lambda}^{\oplus c_i}, \qquad U_{0,\lambda} \simeq \bigoplus_i \rho_{i,\lambda}^{\oplus m_i}.$$

These are actual equivariant presentations. They are not inferred from an unexplained equality of numerical traces. The selected family $\bigoplus_i \rho_{i,\lambda}^{\oplus m_i}$ has a split equivariant inclusion in the carrier. The finite trace sample uniquely determines the multiplicity vector in the declared constituent packet.

The package `CarrierRecoveryAt` also supplies continuity of the selected copies, the common Frobenius characteristic-polynomial product, finite-copy Weil–Deligne and smooth-inertia data, local characteristic and Euler polynomial products, monodromy-kernel dimension and inertia-fixed trace formulas, and genuine semilinear period comparisons with weight-multiplicity sums. The constituent local packets and comparisons are semantic inputs. The conclusion proves their stated finite-copy closure.

D.2 Why the assembly closes

The final proof is a conjunction. The first branch applies the checked inverse-limit-closed horizontal completion theorem to the completed source family. The second branch constructs the finite arithmetic separator and applies the checked compatible-carrier closure theorem. Claim 9.1 constructs its separator independently of the carrier recovery. Claim 9.2 accepts the separator and the compatible-carrier packet. The final source composes these results; it does not use its conclusion to establish either premise.

The two branches do not identify the recovered carrier's supplied local Weil or period structures with restrictions of the transported target family. They remain logically distinct outputs. A simultaneous all-place carrier statement is obtained by applying the pointwise result at every place when the corresponding carrier presentations are supplied.

D.3 Occupancy and inherited support

Zero multiplicities are permitted by the formal endpoint. A nonzero occupied conclusion additionally requires the manuscript's declared nonzero-multiplicity premise and nonzero constituents. The module-support/Fitting-ideal alternative is logically separate: its checked support formalization is inherited and is not re-proved by this final module. No unrestricted companion existence, automorphy, carrier existence, or universal p-adic Hodge comparison follows from this conditional assembly.

D.4 Evidence tied to this exact source

The saved gate records two warnings-as-errors builds from separate copies of the frozen dependency cache, byte-identical corresponding proof objects, and an axiom audit allowing only `propext`, `Classical.choice`, and `Quot.sound`. False and placeholder controls, a missing profile-closure control, and a missing separator/nonisomorphism-premise control were rejected. An injected custom axiom compiled and was detected by the policy audit. The pinned dependency cache and predecessor releases are trusted inputs of that recorded gate.

The October 5 collection audit records a fresh compilation of 30 Paper X project modules, including this endpoint and the arithmetic occurrence closure, using its selected pinned analytic foundation. The present publication migration checks source identity and rebuilds the documents. It performs no new Lean kernel execution.

D.5 A later attempt with a different scope

The retained stronger residual-system attempt reports 19/20 and `FAIL_CLAIM111_NOT_PROMOTED`, with terminal error `NO_PINNED_LEAN`. No endpoint was promoted by that attempt. It is not a compiler rejection or retraction of the earlier checked conditional theorem. Its criterion begins with finite nonempty companion levels, while the checked conditional endpoint assumes actual completed members. These are different source obligations.

References

[1] Michael F. Atiyah and Ian G. Macdonald. *Introduction to Commutative Algebra*. Addison-Wesley, Reading, Massachusetts, 1969.

[2] Joël Bellaïche and Gaëtan Chenevier. *Families of Galois Representations and Selmer Groups*, volume 324 of *Astérisque*. Société Mathématique de France, Paris, 2009.

[3] Nicolas Bourbaki. *Commutative Algebra, Chapters 1 to 7*. Elements of Mathematics. Springer, Berlin, 1989.

[4] Richard Brauer and Cecil Nesbitt. On the modular characters of groups. *Annals of Mathematics*, 42(2):556–590, 1941.

[5] Gaëtan Chenevier. The p-adic analytic space of pseudocharacters of a profinite group and Pseudorepresentations over arbitrary rings. In Fred Diamond, Payman L. Kassaei, and Minhyong Kim, editors, *Automorphic Forms and Galois Representations. Volume 1*, volume 414 of *London Mathematical Society Lecture Note Series*, pages 221–285. Cambridge University Press, Cambridge, 2014.

[6] Charles W. Curtis and Irving Reiner. *Representation Theory of Finite Groups and Associative Algebras*, volume 11 of *Pure and Applied Mathematics*. Interscience, New York, 1962.

[7] Jan Denef. The rationality of the Poincaré series associated to the p-adic points on a variety. *Inventiones Mathematicae*, 77:1–23, 1984.

[8] Reinhard Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Berlin, fifth edition, 2017.

[9] David Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry*, volume 150 of *Graduate Texts in Mathematics*. Springer, New York, 1995.

[10] John L. Kelley. *General Topology*. University Series in Higher Mathematics. Van Nostrand, New York, 1955.

[11] Mark Kisin. Deformations of $G_{\mathbb{Q}_p}$ and $\mathrm{GL}_2(\mathbb{Q}_p)$ representations. In Laurent Berger, Christophe Breuil, and Pierre Colmez, editors, *Représentations p-adiques de Groupes p-adiques II: Représentations de* $\mathrm{GL}_2(\mathbb{Q}_p)$ *et* (φ, Γ)*-modules*, volume 330 of *Astérisque*, pages 511–528. Société Mathématique de France, Paris, 2010.

[12] Michael Larsen and Richard Pink. On ℓ-independence of algebraic monodromy groups in compatible systems of representations. *Inventiones Mathematicae*, 107(3):603–636, 1992.

[13] Hideyuki Matsumura. *Commutative Ring Theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, corrected paperback edition, 1989. First paperback edition (with corrections), 1989; English translation first published 1986.

[14] Barry Mazur. Deforming Galois representations. In Yasutaka Ihara, Kenneth Ribet, and Jean-Pierre Serre, editors, *Galois Groups over* $\mathbb{Q}$, volume 16 of *Mathematical Sciences Research Institute Publications*, pages 385–437. Springer, New York, 1989.

[15] Jürgen Neukirch. *Algebraic Number Theory*, volume 322 of *Grundlehren der Mathematischen Wissenschaften*. Springer, Berlin, 1999.

[16] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of Number Fields*, volume 323 of *Grundlehren der Mathematischen Wissenschaften*. Springer, Berlin, second edition, 2008.

[17] Ravi Ramakrishna. On a variation of Mazur's deformation functor. *Compositio Mathematica*, 87(3):269–286, 1993.

[18] Luis Ribes and Pavel Zalesskii. *Profinite Groups*, volume 40 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*. Springer, Berlin, second edition, 2010.

[19] Michael Schlessinger. Functors of Artin rings. *Transactions of the American Mathematical Society*, 130(2):208–222, 1968.

[20] Jean-Pierre Serre. *Linear Representations of Finite Groups*, volume 42 of *Graduate Texts in Mathematics*. Springer, New York, 1977.

[21] Jean-Pierre Serre. *Local Fields*, volume 67 of *Graduate Texts in Mathematics*. Springer, New York, 1979.

[22] Jean-Pierre Serre. *Abelian ℓ-adic Representations and Elliptic Curves*. A K Peters, Wellesley, Massachusetts, 1998.

[23] Charles A. Weibel. *An Introduction to Homological Algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

E Worked visual atlas

The following thirty pages reproduce the worked atlas from the separate visual guide. They retain their own page labels V1–V30, so the same example has the same address in either reading route. The two opening contract pages of the standalone guide are not repeated here.

The atlas develops exact finite models, not substitutes for universal proofs. Its first part separates residual roots, coherent branches, and occupied carrier packets. Its second part studies residue norms, cancellation, Mellin estimates, and the boundary between a finite calculation and arithmetic occurrence. The final page records the three distinct historical and current contracts.

The build script appends these source-built pages after this leaf and copies their bookmarks and internal links. The mathematical manuscript, its historical checkpoint statements, and the later conditional assembly remain above.

READER'S GUIDE

One root is not the whole problem

Main text: §§1–3, 8–11. Current contract: Appendix D. Earlier scalar/Mellin checkpoints are historical.

Paper X separates three questions that can look deceptively similar. Does a residual object exist? Does it extend coherently through every precision? Does the resulting object occur in the required carrier? A successful answer to one question does not supply the next.

A residual root

$x^2 + 1 = 0$ over $\mathbb{F}_2$



one point mod 2

no point mod 4

A coherent branch

$x^2 + 1 = 0$ over $\mathbb{Z}_5$



reductions agree

An occupied packet

1	0
0	0

$\operatorname{rank} E = 1$

positivity needs a carrier

Three models to keep in view

The scalar equation $x^2 + 1 = 0$ has the root 1 modulo two, but no root modulo four. Over powers of five, the residues $2, 7, 57, 182$ form the beginning of a compatible branch. Neither fact constructs an automorphic carrier. In a supplied finite carrier, an idempotent such as $\operatorname{diag}(1, 0)$ detects an occupied subspace; it does not explain where that carrier came from.

Read the appendix in three passes. Pages V3–V11 explain residual digits, finite counts, and obstruction spaces. Pages V12–V20 explain directed transport, common data, and finite occupancy. Pages V21–V29 give small exact models for the newer arithmetic continuation and its remaining limits.

> **Which ledger applies?** The Block 27 and scalar/Mellin checkpoints record 17/20. The later checked assembly records **20/20 at the amended conditional scope**. Its inputs are stated on the opening pages and in Appendix D. The finite character examples below are not proofs of prescribed Frobenius occurrence.

The figures are exact-model illustrations, not pictures of an entire arithmetic moduli space. New Python checks support these examples. Saved Lean evidence is preserved separately. The October 20 production replay has its own source and compiler records in the Lean guide.

NOTATION BEFORE PICTURES

An atlas of the objects

Main text: Definitions 1.1, 5.1, 6.1, 7.1–7.2, and 8.1.

Keep the coefficient ring, its reduction, and the carrier separate. A residue class is a finite-precision address. A vector in a carrier is a different kind of object, even when both are selected by something called a packet.

Object	What it records	What it does not supply
$\mathcal{O}, \varpi, k$	Complete valuation ring, uniformizer, and residue field k of size q.	A represented deformation functor.
$\bar{x}$	One residual object in the declared moduli problem.	Other residual roots or their transports.
$X_m(\bar{x})$	Admissible objects modulo ϖ^m reducing to $\bar{x}$.	A lift of every individual member.
$(x_m)_m$	A branch satisfying $x_{m+1} \bmod \varpi^m = x_m$.	A global object without the stated reconstruction hypotheses.
w_i	A coordinate constrained to begin in $\varpi^{w_i}\mathcal{O}$.	A classification of all singular deformation rings.
$Z(T)$	The formal series $\sum_{m\geq 1} \#X_m T^m$.	The cardinality of the inverse limit.
$C^\bullet, d$	Cochains and a differential with $d^2 = 0$.	Its arithmetic identification with a Galois problem.
G	The graph of supplied admissible residual transports.	A proof that a proposed arithmetic edge exists.
$\mathscr{C}$	Common algebraic polynomials and declared local data.	A member realizing those data at every prime.
$C\boldsymbol{m} = \boldsymbol{b}$	A finite trace system for carrier multiplicities.	Integrality, positivity, or available capacity automatically.
$e_T V$	The joint eigenspace inside an existing finite carrier V.	Nonzero occupancy for every desired joint label.

Three indices that must not be conflated

The depth m counts retained digits. A coordinate index i selects a parameter or a packet. A coefficient place λ selects a different local field. Increasing m at one λ neither changes i nor constructs the member at another λ.

> **Read this as:** Specify the ambient object before applying a projector, a reduction, a rank, or a counting formula. The same-looking symbol can refer to quite different spaces.

RESIDUAL PARTITION

Higher digits cannot change the root

Main text: Theorems 2.1–2.2 and Corollary 2.3.

Work in $\mathbb{Z}/27\mathbb{Z}$. Every element has a unique form $x = 3q + r$ with $r \in \{0,1,2\}$ and $q \in \{0,\dots,8\}$. The row below is the residual root. The column contains all the remaining information at this precision.

$x = 3q + r \pmod{27}$: columns are higher-digit addresses q

	$q=0$	$q=1$	$q=2$	$q=3$	$q=4$	$q=5$	$q=6$	$q=7$	$q=8$
$r=0$	0	3	6	9	12	15	18	21	24
$r=1$	1	4	7	10	13	16	19	22	25
$r=2$	2	5	8	11	14	17	20	23	26

Adding 3 moves within one row. Adding 1 can change the row.

Read the operation, not just the arrow

An operation $x \mapsto x + 3h$ preserves r, because its correction vanishes modulo three. Repeated corrections of this form can visit every column in a row, but can never enter another row. By contrast, adding one changes the residual class. It is therefore not a positive-depth perturbation in this model.

For example, the addresses $1, 4, 7, 10, \dots, 25$ may be deeply connected by higher-digit changes. They remain disjoint from the addresses reducing to zero. In a genuine deformation problem, the same conclusion applies to operations whose reduction is the identity, not to every operation that happens to be called a deformation.

$$\operatorname{red}(x + 3h) = \operatorname{red}(x), \qquad \mathcal{M}(A) = \bigsqcup_{\bar{x}} \mathcal{M}_{\bar{x}}(A).$$

> **Read this as:** Vertical completeness can fill one entire residual fiber without reaching a single point in a different fiber.

All 27 addresses and their finite corrections are checked by integer reduction. Passing to an infinite perturbation stream additionally requires the convergence condition recorded in the formal no-crossing statement.

INVERSE-LIMIT RECONSTRUCTION

A branch fixes digits permanently

Main text: Theorem 3.2 and the compatible-digit example in §3.

The rational number $1/4$ is a five-adic integer. Its normalized residues are obtained by solving $4x_m \equiv 1 \pmod{5^m}$, not by taking decimal approximations. At the first six depths they are $4, 19, 94, 469, 2344, 11719$.

	5^0	5^1	5^2	5^3	5^4	5^5
$x_1 = 4$	4	—	—	—	—	—
$x_2 = 19$	4	3	—	—	—	—
$x_3 = 94$	4	3	3	—	—	—
$x_4 = 469$	4	3	3	3	—	—
$x_5 = 2344$	4	3	3	3	3	—
$x_6 = 11719$	4	3	3	3	3	3

$$4x_m - 1 = 3 \cdot 5^m$$

digits already fixed never change

Compatibility is stronger than independent solvability

The residues satisfy both

$$x_{m+1} \equiv x_m \pmod{5^m}, \qquad 4x_m - 1 = 3 \cdot 5^m.$$

The latter identity shows that the residual error tends to zero in the five-adic norm. The stabilized digit sequence is

$$\frac{1}{4} = 4 + 3 \cdot 5 + 3 \cdot 5^2 + 3 \cdot 5^3 + \cdots \quad \text{in } \mathbb{Z}_5.$$

This series is not convergent in the ordinary real norm. The topology is part of the statement. To view the branch as a positive-depth coordinate above the residual root 4, use $t = x - 4 \in 5\mathbb{Z}_5$.

An inverse-limit point is the entire compatible family, not its last displayed approximation. The six rows are a finite window; the displayed exact formula supplies a branch at every depth in this particular example. In the represented deformation theorem, compatibility reconstructs maps on all ring elements, with multiplication, continuity, and locality retained.

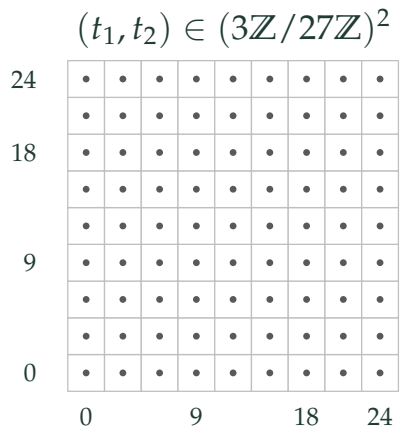
> **Read this as:** A later approximation may append a digit. It may not revise a digit already certified by reduction.

ACTUAL FINITE POINTS

A smooth chart has independent choices

Main text: Proposition 3.4 and Theorem 6.2.

Take $R = \mathbb{Z}_3[[t_1, t_2]]$ and coefficient-preserving local maps to $\mathbb{Z}/27\mathbb{Z}$. Locality forces both parameters into the maximal ideal. Each therefore has nine possible values, $0, 3, 6, \ldots, 24$.



$9 \times 9 = 81$ points

$(t_1, t_2) \mapsto (t_1, t_2) \bmod 9$

Each parent has $3^2 = 9$ children.

The chart hypothesis supplies independent digits.

Why the exponent is $d(m-1)$

At depth m, the image of the maximal ideal contains 3^{m-1} elements. Two independent parameters give $3^{2(m-1)}$ points. The grid displays all 81 points at depth three, not a random sample. Reduction to depth two maps each coordinate to one of $0, 3, 6$, and every parent has nine children.

$$\#X_m = 3^{2(m-1)}, \qquad Z(T) = \sum_{m \geq 1} \#X_m T^m = \frac{T}{1 - 9T}.$$

Counting all 27^2 parameter assignments at depth three would ignore locality. Counting only tangent directions would describe the first layer but not all later layers. The power-series presentation explains why these choices stay independent through every precision.

This is a coordinate model with a proved presentation. A general deformation ring can impose equations among the parameters, changing the counts and causing branches to die. The next pages retain the same depth convention while changing the allowed coordinates or adding a singular equation.

> **Read this as:** The smooth count is not a universal rule for deformation spaces. Its independence comes from the actual formal power-series chart.

WEIGHTED DYADIC CHARTS

Weights delay visibility

Main text: Proposition 5.3, Theorem 6.3, and §12.2.

Over $\mathbb{Z}_2$, require three coordinates to lie in $2\mathbb{Z}_2$, $2^2\mathbb{Z}_2$, and $2^4\mathbb{Z}_2$. At depth m, coordinate i has $\max(m - w_i, 0)$ free binary digits. A zero in the table means only one allowed residue: zero.

	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$
$w_1 = 1$	0	1	2	3	4	5	6
$w_2 = 2$	0	0	1	2	3	4	5
$w_3 = 4$	0	0	0	0	1	2	3
count	1	2	8	32	256	2048	16384

Cell = number of free binary digits, not first separation depth.

The two different thresholds

A coordinate of weight w_i first becomes nontrivial when $m = w_i + 1$. By contrast, the first separation depth of two distinct fixed values a, b is $v_2(a - b) + 1$. Thus 1 and -1 separate at depth two, because their difference has valuation one. Neither threshold is simply the printed coordinate label.

For weights $(1, 2, 4)$,

$$N(m) = 2^{\max(m-1,0)+\max(m-2,0)+\max(m-4,0)}.$$

The first counts are $1, 2, 8, 32, 256, 2048$. Once all three coordinates are active, every additional precision digit multiplies the count by eight. Separating the initial levels from that geometric tail gives

$$Z(T) = \frac{T(1 - 6T - 8T^2 - 32T^3)}{1 - 8T}.$$

Negative coefficients in the numerator do not mean negative numbers of points; expanding the whole rational expression gives positive coefficients.

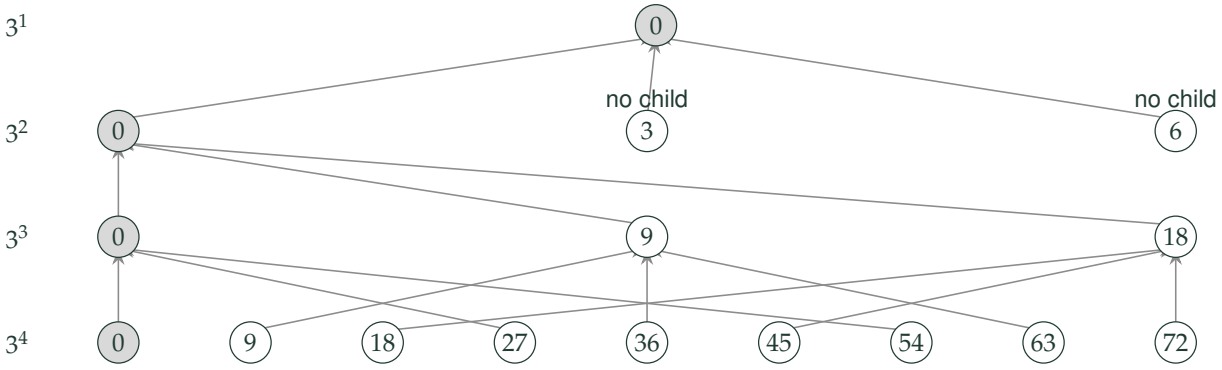
> **Read this as:** Weights change the finite-depth delay. After activation, this chart still has three independent digit streams.

SINGULAR POWER SEEDS

Many approximate roots, one limit

Main text: Theorem 6.4 and Corollary 6.6.

Now impose $x^2 = 0$ modulo 3^m. The condition is exactly $2v_3(x) \geq m$, so the roots are the multiples of $3^{\lceil m/2 \rceil}$ in that quotient. Their number is $3^{m-\lceil m/2 \rceil}$.



More finite roots, but only the zero stream persists through every precision.

Count vertices and count branches separately

There are three roots modulo nine: $0, 3, 6$. But the only roots modulo 27 are $0, 9, 18$, and all three reduce to zero modulo nine. The other two vertices at the previous level have no children. Counting them as permanent branches would be incorrect.

$$\#X_m = 1, 3, 3, 9, 9, 27, \ldots, \qquad Z(T) = \frac{T + 3T^2}{1 - 3T^2}.$$

The count-growth exponent is $\lim_m \log_3(\#X_m)/m = 1/2$. Nevertheless, a compatible branch reconstructs $x \in \mathbb{Z}_3$ with $x^2 = 0$. Since $\mathbb{Z}_3$ has no nonzero nilpotent, the only such branch is the zero branch.

More generally, $x^e = 0$ has count $q^{m-\lceil m/e \rceil}$ and count-growth exponent $1 - 1/e$. These exponents describe finite thickenings. They do not assert that the set of complete roots has a fractional geometric dimension or more than one point.

> **Read this as:** The seed zeta function counts finite-level solutions. It is not a generating function for distinct infinite branches.

A SCALAR COMPANION MODEL

One equation, several local outcomes

Main text: Theorem 8.2, Corollary 8.3, and Example 8.4.

Consider the fixed scalar equation $x^2 + 1 = 0$ in $\mathbb{Z}/p^m\mathbb{Z}$. The following are exact counts for five primes and three depths. The coefficient rings and the scalar nature of the unknown are fixed throughout this page.

	$p = 2$	$p = 3$	$p = 5$	$p = 7$	$p = 13$
$m = 1$	1	0	2	0	2
$m = 2$	0	0	2	0	2
$m = 3$	0	0	2	0	2

Counts of $x \in \mathbb{Z}/p^m\mathbb{Z}$ with $x^2 + 1 = 0$.

Where failure occurs matters

At $p = 3$ and $p = 7$, no residual root exists. At $p = 2$, the residual root exists but dies immediately: every odd square is one modulo four. At $p = 5$ and $p = 13$, the two residual roots are simple, and the usual one-digit equation gives one child above each root at every depth.

At five, one branch begins $2, 7, 57, 182$; the other is its negative modulo each power of five. Arbitrarily many digits in either branch cannot supply the absent scalar root over $\mathbb{Z}_3$. Nor can one root modulo two satisfy the hypothesis that every finite level is nonempty.

The paper's system-completion theorem uses a supplied common algebraic spine, finite nonempty levels at every coefficient place, coherent reductions, and closed local conditions. The scalar example isolates why these are substantive requirements. It is not a disproof of a companion theorem whose coefficients may be enlarged or whose objects are higher-dimensional representations.

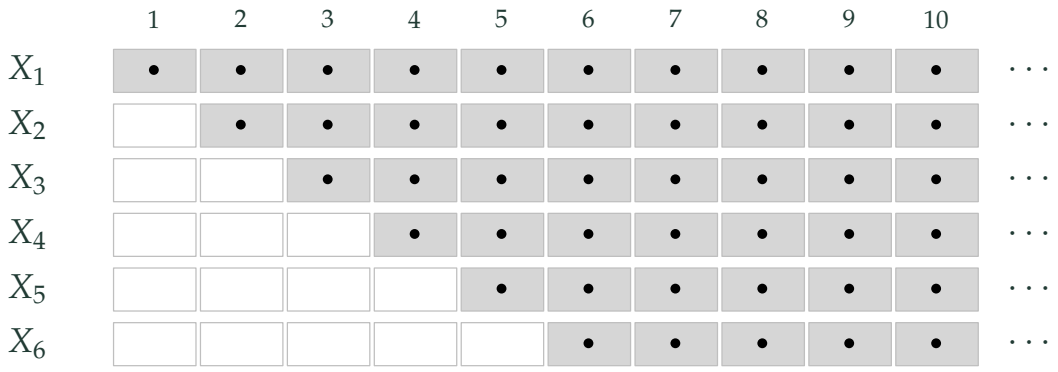
> **Read this as:** A residual root, roots at all depths, and roots at every required coefficient place are three distinct tests.

AN INFINITE-LEVEL COUNTERMODEL

Why finiteness enters compactness

Main text: Theorem 3.3 and its finite-branching hypotheses.

Let $X_m = \{n \in \mathbb{Z} : n \geq m\}$, with the inclusion $X_{m+1} \hookrightarrow X_m$ as transition map. Every level is nonempty. Every vertex has at most one child. Yet there is no compatible infinite branch.



$X_m = \{n \in \mathbb{N} : n \geq m\}$; every reduction is an inclusion.

The missing hypothesis is not surjectivity

Compatibility would force all coordinates of a branch to be one fixed integer n. Membership at every level would then require $n \geq m$ for every m, which is impossible. Each vertical column in the drawing eventually ends. A finite window always contains columns extending beyond it, but no single column survives all windows.

There is no contradiction with finite-branch compactness: these levels are infinite. Attaching a common artificial root would give that root infinitely many children. The finitely branching rooted-tree argument therefore does not apply.

In the paper, finite nonempty levels and actual total reduction maps provide compactness even when some vertices die and some transition maps are not onto. One can retain only vertices with descendants at arbitrarily large depths. Finiteness makes that surviving part nonempty and permits compatible choices.

$$X_m \neq \varnothing \text{ for all } m \quad \not\Longrightarrow \quad \varprojlim_m X_m \neq \varnothing \quad \text{without the compactness hypotheses.}$$

> **Read this as:** Arbitrarily long finite paths are not automatically one infinite path. The finite-level hypothesis prevents escape to new roots.

A TWO-PACKET COCHAIN COMPLEX

See cycles, boundaries, and classes

Main text: Theorem 4.2 and the finite-complex example.

Over $\mathbb{F}_5$, use bases (e_0, e_1), (u_0, u_1, u_2, u_3), and (f_0, f_1) in degrees zero, one, and two. The only nonzero differential values are $d_0 e_0 = u_0$ and $d_1 u_1 = f_0$. Consequently $d_1 d_0 = 0$.

$d_0 : C^0 \to C^1$

	e_0	e_1
u_0	1	0
u_1	0	0
u_2	0	0
u_3	0	0

$d_1 : C^1 \to C^2$

	u_0	u_1	u_2	u_3
f_0	0	1	0	0
f_1	0	0	0	0

$$d_1 d_0 = 0$$

$$H^0 = \langle [e_1] \rangle$$

$$H^1 = \langle [u_2], [u_3] \rangle$$

$$H^2 = \langle [f_1] \rangle$$

Packet A is exact. Packet B has zero differentials and carries all cohomology.

The quotient is the important step

In degree one, the cycles are $\operatorname{span}(u_0, u_2, u_3)$, but u_0 is a boundary. Thus the cohomology is represented by u_2, u_3, not by every vector killed by d_1. In the other degrees the surviving classes are e_1 and f_1.

$$\dim(H^0, H^1, H^2) = (1, 2, 1).$$

Packet A contains e_0, u_0, u_1, and f_0. It is exact and contributes $(0, 0, 0)$. Packet B contains e_1, u_2, u_3, and f_1, with zero differentials, and contributes $(1, 2, 1)$. Adding the packet cohomologies reconstructs the total.

The projectors must be specified in every degree and commute with the differential. Their existence is not a consequence of the printed dimensions. Identifying this finite complex with the actual continuous cohomology of an arithmetic deformation problem would also require a separate comparison.

> **Read this as:** Resolve the differential first, then pass from cycles to classes. Counting packet dimensions alone does not compute cohomology.

THE CHAIN-MAP CONDITION

An arbitrary cut need not descend

Main text: Theorem 4.2 and the scope of the cohomology splitting.

Retain the previous complex. The valid A-packet projectors are $P^0 = \mathrm{diag}(1,0)$, $P^1 = \mathrm{diag}(1,1,0,0)$, and $P^2 = \mathrm{diag}(1,0)$. They satisfy $d_r P^r = P^{r+1} d_r$.

valid P^1_A

1	0	0	0
0	1	0	0
0	0	0	0
0	0	0	0

invalid Q^1

0	0	0	0
0	1	0	0
0	0	0	0
0	0	0	0

$d_0 P^0_A - Q^1 d_0$

1	0
0	0
0	0
0	0

e_0 is retained at the source but its image u_0 is deleted at the target.

All displayed degreewise maps are idempotent. The differential test distinguishes them.

A boundary can cross an incompatible cut

Replace the degree-one projector by $Q^1 = \mathrm{diag}(0,1,0,0)$ while keeping P^0. Then

$$d_0 P^0 e_0 = u_0, \qquad Q^1 d_0 e_0 = 0, \qquad (d_0 P^0 - Q^1 d_0) e_0 = u_0 \neq 0.$$

The selected degree-zero component is sent outside the selected degree-one component. There is no subcomplex with these cuts, so the cohomology packet theorem cannot be applied to them. Both matrices remain perfectly valid idempotents; the failure is compatibility with the differential.

This example separates two kinds of obstruction. The nonzero matrix above obstructs this proposed decomposition. A nonzero class in H^2 may obstruct a lift only after the deformation problem and its obstruction map have been identified. One should not equate every failed projector commutator with an arithmetic obstruction class.

The exact checks verify both commuting diagrams for the valid family and reject the altered family. They also enumerate the finite kernels and images, so the dimension comparison is not inferred from the drawing.

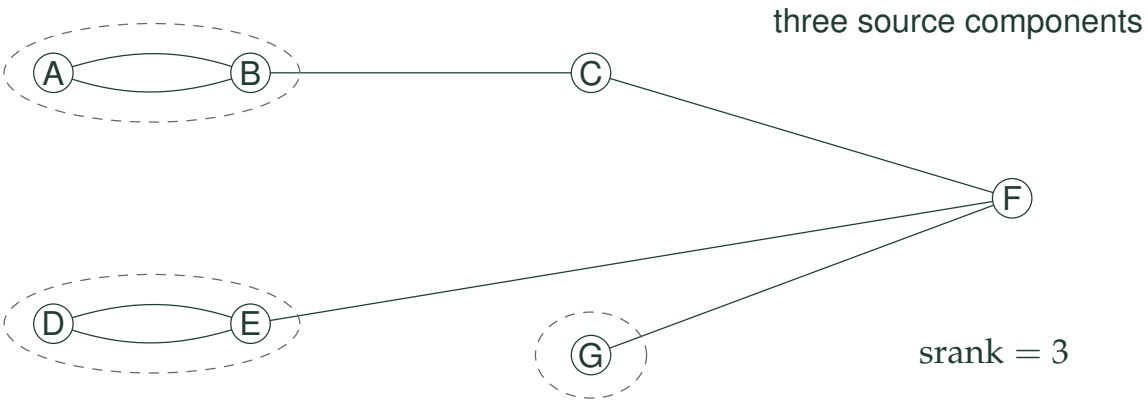
> **Read this as:** A projector on each vector space is not yet a projector on the complex. The differential must respect the chosen packets.

DIRECTED TRANSPORT

How many starting seeds are necessary?

Main text: Theorem 7.3 and the seven-root example.

The arrows below are the entire supplied transport graph. Vertices joined by paths in both directions form strongly connected components. The components $\{A, B\}$, $\{D, E\}$, and $\{G\}$ have no incoming edge from outside themselves.



A minimum set meets each dashed source component.

A lower bound and a construction

Any seed set reaching every vertex must meet all three source components. Otherwise there is no path into the omitted source. Conversely, one representative from each source reaches C and F as well. The directed seed rank is therefore three.

There are precisely four minimum seed sets:

$$\{A, D, G\}, \quad \{A, E, G\}, \quad \{B, D, G\}, \quad \{B, E, G\}.$$

The example checks this by exhaustive enumeration of all 2^7 subsets, independently of the strongly connected component calculation. Adding an admissible edge cannot increase the required seed count. In particular, adding $F \to G$ makes G reachable from the other sources, reducing the count to two.

The graph theorem is exact for the graph supplied. It does not establish that a proposed twist, induction, or arithmetic transfer preserves a determinant, local condition, or occupancy theorem. Those facts justify the arrows before graph optimization begins.

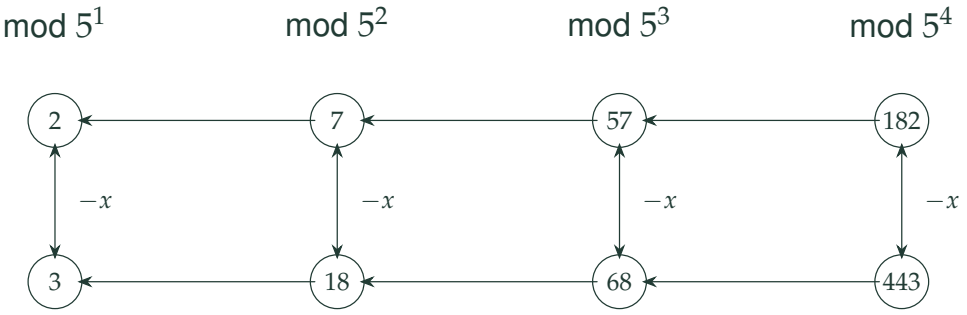
> **Read this as:** Count source components, not all components and not all vertices. But do not add an arithmetic arrow merely because it would improve the count.

ONE REVERSIBLE EXAMPLE

A transport must work at every depth

Main text: Definition 7.1 and the reduction-compatibility requirement.

For the two roots of $x^2 + 1 = 0$ modulo powers of five, negation is an actual reversible transport. It preserves the equation, is its own inverse, and commutes with reduction.



Negation commutes with reduction. Translation by 1 fails already at $2 \mapsto 3 \mapsto 4$ mod 5.

One successful input is not a natural transformation

At every depth m,

$$N_m(x) = -x \pmod{5^m}, \qquad r_{m+1,m} N_{m+1} = N_m r_{m+1,m}.$$

The two compatible branches beginning at 2 and 3 are exchanged. Because $2 \not\equiv 3 \pmod 5$, this transport changes the residual root. It is horizontal rather than a positive-depth correction.

Translation by one is a useful false control. Modulo five it takes the root 2 to the root 3, but takes the root 3 to 4, which is not a root. A single successful arrow between selected points does not define an operation on the whole deformation problem.

Even negation is admissible only for the equation and conditions stated here. An extra condition that fixes the residual root at 2 would not be preserved. Likewise, a representation-theoretic dual or twist needs its own compatibility checks for the chosen local data and carrier.

> **Read this as:** A transport is a family of maps respecting equations and reductions. A convenient pair of endpoints is not enough.

Both branches are computed by exact one-digit lifting. Negation is checked on all roots at the displayed depths; its universal reduction identity follows directly from ring homomorphism laws.

THE ARITHMETIC SPINE

Common coefficients, different residues

Main text: Definition 8.1 and Theorem 8.2.

A common spine records algebraic data before reducing at a coefficient prime. For a small model, take the one polynomial

$$P(T) = T^2 - 5T + 6 = (T-2)(T-3) \in \mathbb{Z}[T].$$

Its reductions at five and seven look different, but they come from the same coefficients.

$$P(T) = T^2 - 5T + 6 = (T-2)(T-3)$$

coefficient images under the declared maps

	T^2	T	1
$\mathbb{Z}$	1	-5	6
$\mathbb{F}_5$	1	0	1
$\mathbb{F}_7$	1	2	6

Different residue rows come from the same algebraic polynomial.

Agreement means descent from supplied data

Modulo five the polynomial is $T^2 + 1$; modulo seven it is $T^2 + 2T + 6$. The roots 2 and 3 are the reductions of the same integer roots in both places. The coefficient lists are not required to be equal as raw residues in different rings.

The paper's common spine is larger: it includes characteristic polynomials at the permitted arithmetic places and the specified determinant, weight, and local labels. Requiring every finite member to match those reductions is a compatibility condition on existing objects. It is not a construction of the objects.

Finite agreement also leaves ambiguity. Adding 35 to an integer coefficient changes neither its residue modulo five nor its residue modulo seven. Thus two finite coefficient observations do not determine a unique global polynomial without a height bound or further exact information. Conversely, one arbitrarily chosen family of local objects need not descend to an algebraic spine.

> **Read this as:** The common spine is input shared across coefficient places. It is not created by comparing unrelated residue labels after the fact.

CHARACTERS BEFORE ARITHMETIC PRIMES

A finite trace matrix can separate types

Main text: Theorem 9.1; retained scalar/Mellin continuation: arithmetic-system separation.

Use the three irreducible characters of S_3: trivial, sign, and standard. On the identity, a transposition, and a three-cycle, their values form the matrix below. Rows are samples; columns are constituent types.

C: actual S_3 character table

	trivial	sign	standard
1	1	1	2
(12)	1	−1	0
(123)	1	1	−1

$$\boldsymbol{m} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \quad = \quad \boldsymbol{b} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

$$\det C = 6$$

$$C^{-1}\boldsymbol{b} = (2, 0, 1)^T$$

Class representatives are finite group elements, not a proof that suitable Frobenius primes occur.

What full rank buys

Since $\det C = 6$, these samples distinguish every rational multiplicity vector. A target with two trivial constituents, no sign constituent, and one standard constituent has

$$\boldsymbol{m} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{b} = C\boldsymbol{m} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}.$$

Solving $C\boldsymbol{m} = \boldsymbol{b}$ recovers the unique rational vector. The standard character is not introduced as an arbitrary list: the checks construct matrices for a transposition and a three-cycle, verify their group relations, and obtain their traces.

This finite example explains the linear algebra after separating samples are available. For the arithmetic theorem, the samples must be genuine unramified Frobenius classes with the required compatibility and allowed-prime conditions. Selecting abstract group elements does not construct such primes.

The scalar/Mellin checkpoint supplies a separator assembly conditional on arithmetic coverage. The later checked source `Claim91ConstructedClosure` discharges that interface for its rational-base contract. Neither result follows from the finite character table displayed here.

> **Read this as:** Invertibility answers reconstruction from these samples. Arithmetic occurrence answers whether the required samples can be taken at actual primes.

INTEGRAL, NONNEGATIVE, WITHIN CAPACITY

A solution is not yet a summand

Main text: Theorem 9.2 and its multiplicity-recovery example.

Keep the same separating matrix C and suppose the available carrier has capacities $\boldsymbol{c} = (2,1,1)$. A recovered vector describes a genuine sub-sum of that carrier only if each entry is an integer and $0 \le m_i \le c_i$.

trace data $\boldsymbol{b}$	$C^{-1}\boldsymbol{b}$	capacity $(2,1,1)$
$(4,2,1)$	$(2,0,1)$	allowed
$(1,0,0)$	$(\frac{1}{6},\frac{1}{6},\frac{1}{3})$	nonintegral
$(2,-2,-1)$	$(-1,1,1)$	negative
$(5,3,2)$	$(3,0,1)$	over capacity

Three independent rejection tests

The trace vector $(1,0,0)$ gives $(1/6,1/6,1/3)$, so it fails integrality. The vector $(2,-2,-1)$ gives $(-1,1,1)$, which is integral but not nonnegative. The vector $(5,3,2)$ gives $(3,0,1)$, which exceeds the available capacities. In each case the linear equation itself has a unique rational solution.

The accepted example $\boldsymbol{b} = (4,2,1)$ recovers $(2,0,1)$. When compatible constituent families are already supplied, the same admissible multiplicities define the corresponding direct sum at every coefficient place. This is the finite algebraic core of summand transport.

It does not supply a compatible family for an arbitrary object, construct all local attachments, or ensure that a proposed carrier contains the desired representation. The historical Claim 9.2 checkpoint left these comparisons partial. The later checked carrier theorem retains explicit constituent-copy presentations and supplied local comparisons rather than manufacturing them.

> **Read this as:** Invert the trace matrix, then check the representation-theoretic constraints. A unique vector in a field need not be a realizable multiplicity vector.

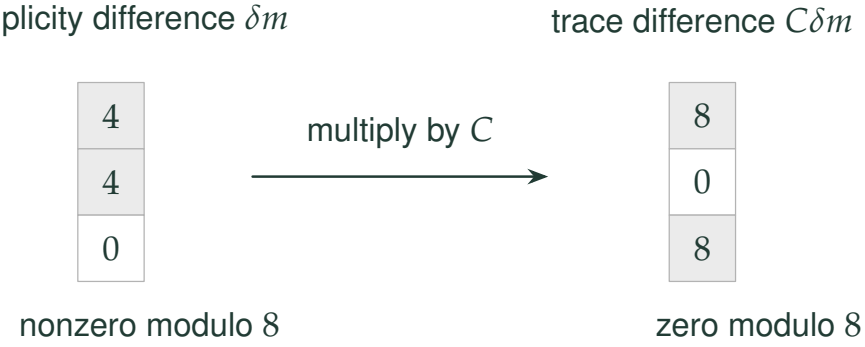
All accepted and rejected vectors are solved over exact rationals. The code rejects the wrong cases by their stated reason, not by a floating-point tolerance.

EXACT PRECISION LOSS

Separation can consume a digit

Main text: The finite trace-recovery discussion in §9.

The character matrix has determinant six, so its inverse contains a factor of two in the denominator. That arithmetic fact is visible as a loss of one binary digit in trace reconstruction.



Modulo 16, the traces separate them. $v_2(\det C) = v_2(6) = 1$.

A concrete ambiguity modulo eight

The multiplicity difference $\delta\boldsymbol{m} = (4, 4, 0)$ is nonzero modulo eight, but

$$C\,\delta\boldsymbol{m} = (8, 0, 8) \equiv 0 \pmod 8.$$

Thus traces known modulo eight do not distinguish those two residue vectors. The same trace difference is nonzero modulo sixteen, so one more trace digit resolves this ambiguity.

More generally, the kernel of C modulo 2^n has exactly two elements:

$$(0, 0, 0), \qquad (2^{n-1}, 2^{n-1}, 0).$$

To see this, the second equation gives $m_1 = m_2$; the third gives $m_3 = 2m_1$; the first then gives $6m_1 = 0$. Three is a unit modulo 2^n, leaving precisely the displayed possibilities.

Conversely, $C\delta\boldsymbol{m} \equiv 0 \pmod{2^{n+1}}$ implies $6\delta\boldsymbol{m} \equiv 0 \pmod{2^{n+1}}$ by the adjugate identity, hence $\delta\boldsymbol{m} \equiv 0 \pmod{2^n}$. Capacity bounds can remove ambiguity earlier in a particular carrier, but do not change this universal linear precision calculation.

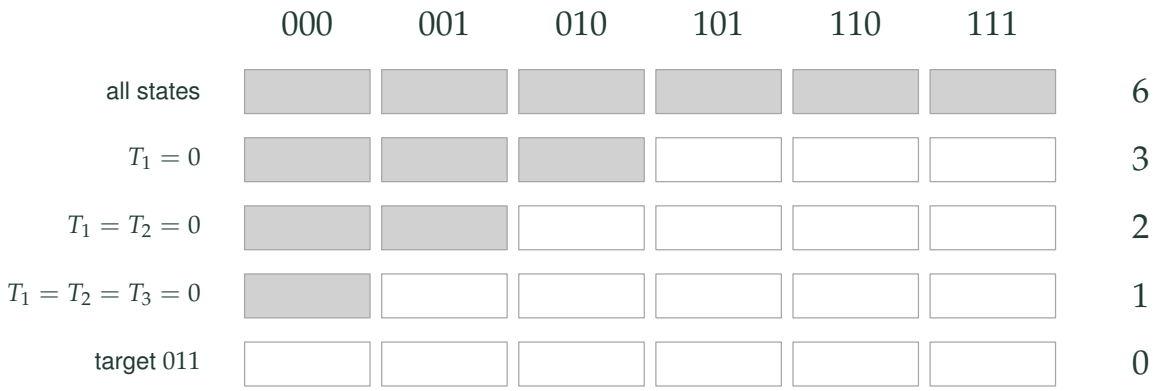
> **Read this as:** Full rank over characteristic zero is not the same as invertibility over every residue ring. Denominator valuations determine the needed extra precision.

EXACT JOINT PROJECTORS

Pairwise occupancy can still fail jointly

Main text: Theorem 10.1 and the finite occupancy models.

Let a six-dimensional rational carrier have the joint eigenvalue labels shown below for three commuting diagonal operators T_1, T_2, T_3. Each coordinate is zero or one. Selecting a zero uses $I - T_i$; selecting a one uses T_i.



Each pair of requirements from 011 is occupied. Their triple intersection is empty.

The full conjunction is a different test

For target $(0,0,0)$, successive conditions reduce the ranks from 6 to 3, then 2, then 1. The joint projector is nonzero, so that label is occupied in this carrier.

For target $(0,1,1)$, every pair of conditions is occupied: $(0,1,0)$ witnesses the first two, $(0,0,1)$ the first and third, and $(1,1,1)$ the second and third. But the triple itself is absent. Multiplying all three selectors gives the zero matrix.

$$E_{\boldsymbol{a}} = \prod_{i=1}^{3} \begin{cases} I - T_i, & a_i = 0, \\ T_i, & a_i = 1. \end{cases} \qquad \operatorname{Tr}(E_{\boldsymbol{a}}) = \operatorname{rank}(E_{\boldsymbol{a}}).$$

The checks compute all eight joint projectors, their pairwise orthogonality, and their sum. The sum is the identity even though two labels are absent: those two projectors are simply zero.

> **Read this as:** A formula for a projector does not force its image to be nonzero. Occupancy must be tested on the complete set of required conditions.

TRACE AND SEMISIMPLICITY

The hypotheses do real work

Main text: Theorem 10.1 and the checked false controls following it.

The finite occupancy theorem uses a split semisimple commuting family in characteristic zero. Two tiny matrices show why neither the coefficient condition nor semisimplicity should disappear from the statement.

I_5 over $\mathbb{F}_5$

1	0	0	0	0
0	1	0	0	0
0	0	1	0	0
0	0	0	1	0
0	0	0	0	1

rank 5, trace 0

$N^2 = 0$

0	1
0	0

only eigenvalue 0

$\dim \ker N = 1$

sole-eigenvalue product $= I_2$

Keep characteristic zero for trace detection, and split semisimplicity for the Lagrange formula.

Trace zero need not mean an empty image

On $\mathbb{F}_5^5$, the identity is an idempotent of rank five, but its field-valued trace is $5 = 0$ in $\mathbb{F}_5$. Rank and localization still detect its nonzero image. A characteristic-zero integer trace cannot be silently transferred to this setting.

An eigenvalue label need not select ordinary eigenvectors

For the nilpotent matrix $N = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, the only eigenvalue is zero. A Lagrange selector that multiplies over all other eigenvalues is an empty product, hence I. Its image has dimension two, while $\ker N$ has dimension one. The selector does not project onto the ordinary eigenspace.

The whole two-dimensional space is the generalized zero-eigenvalue packet. Paper XII's Artin-packet treatment is designed to retain such nilpotent structure. That is a different target from Paper X's split semisimple joint-eigenspace formula.

> **Read this as:** A correct theorem in one coefficient category is not a theorem in all categories. Changing from ordinary eigenspaces to generalized packets also changes the object being detected.

CLOSEDNESS IS NOT DECORATION

A condition must survive the limit

Main text: Theorems 8.2 and 11.1; the inverse-limit closure hypotheses.

In $\mathbb{Z}_3$, every term of the sequence $x_k = 3^k$ is nonzero. Yet $x_k \to 0$ in the three-adic topology. The property “nonzero” is not closed, so a limit of objects satisfying it can lose that property.

	3^0	3^1	3^2	3^3	3^4	3^5	3^6
$a_k = 3^0$	1	0	0	0	0	0	0
$a_k = 3^1$	0	1	0	0	0	0	0
$a_k = 3^2$	0	0	1	0	0	0	0
$a_k = 3^3$	0	0	0	1	0	0	0
$a_k = 3^4$	0	0	0	0	1	0	0
$a_k = 3^5$	0	0	0	0	0	1	0

Every a_k is nonzero. Every fixed finite digit window eventually reads only zeros.

Polynomial conditions behave differently

For a continuous polynomial f, if a convergent sequence satisfies $f(x_k) = 0$ for every k, then continuity gives $f(x) = 0$ at its limit. More generally, a compatible residue branch satisfying $f(x_m) \equiv 0 \pmod{3^m}$ reconstructs an exact root once the complete ring and its reduction maps have been identified.

The sequence 3^k here is *not* presented as a compatible branch of nonzero residues modulo 3^k: those particular residues are zero. It is a convergent sequence of nonzero elements of the complete ring. The distinction prevents an incorrect finite-level interpretation of the picture.

In an arithmetic application, local conditions may be given by quotient equations, integral structures, or other data. The completion theorem requires a proof that the chosen condition is preserved by the limit under construction. The word “admissible” does not itself provide that proof.

> **Read this as:** Compatibility reconstructs an object. Closedness is what carries the declared equations or conditions to that object.

The finite diagram displays the retreat of the first nonzero digit. The convergence assertion follows from $|3^k|_3 = 3^{-k}$, not from extrapolating a finite numerical plot.

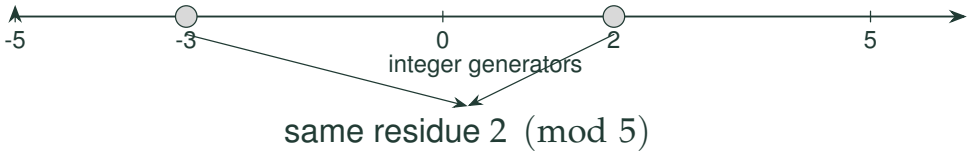
ELEMENTS VERSUS IDEALS

The sign of a norm is arithmetic data

Main text: §9, arithmetic residue-counting continuation; signed-norm comparison.

Even in the number field $\mathbb{Q}$, congruent generators need not have congruent absolute norms. The integers 2 and -3 are congruent modulo five. Their signed field norms are congruent, but their absolute values are not.

$$N_{\mathbb{Q}/\mathbb{Q}}(-3) = -3 \qquad N_{\mathbb{Q}/\mathbb{Q}}(2) = 2$$



$$N((-3)) = 3 \qquad N((2)) = 2$$

The ideal norm is positive. Passing to absolute values must retain the real sign.

The smallest counterexample is already decisive

For $K = \mathbb{Q}$,

$$N_{K/\mathbb{Q}}(2) = 2, \qquad N_{K/\mathbb{Q}}(-3) = -3 \equiv 2 \pmod 5,$$

whereas

$$N((2)) = 2, \qquad N((-3)) = 3 \not\equiv 2 \pmod 5.$$

Here $N((a))$ is the positive index of the ideal (a), equal to $|a|$. The determinant norm of an element is signed. Taking absolute values is not a ring operation and does not commute with reduction in this example.

The retained scalar/Mellin continuation counts the two real norm signs separately before reassembling its generator-to-ideal sums. Its exact comparison also includes the roots-of-unity multiplicity and the norm of the chosen inverse-class representative. Omitting those factors would change the arithmetic sum, not merely its normalization.

The finite-field norm on the following pages is a determinant in a finite algebra. It has no ordering or absolute-value sign. Its role is to control residue characters before the separate real-lattice counting argument.

> **Read this as:** Do not infer an ideal-norm residue from a generator congruence without accounting for the sign in the element-to-ideal comparison.

THE QUOTIENT ALGEBRA $\mathbb{F}_{25}$

A norm is a determinant you can compute

Main text: §9; retained scalar/Mellin continuation: actual quotient determinant norms.

Use $A = \mathbb{F}_5[u]/(u^2 - 2)$. Since two is not a square modulo five, this quotient is a field with 25 elements. Every element is uniquely $a + bu$, with $a, b \in \mathbb{F}_5$.

$$N(a + bu) = a^2 - 2b^2 \pmod 5$$

	$b = 0$	$b = 1$	$b = 2$	$b = 3$	$b = 4$
$a = 0$	0	3	2	2	3
$a = 1$	1	4	3	3	4
$a = 2$	4	2	1	1	2
$a = 3$	4	2	1	1	2
$a = 4$	1	4	3	3	4

$$u^2 = 2 \text{ in } \mathbb{F}_{25}$$

$$M_{a+bu} = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$$

$$\#N^{-1}(0) = 1$$

$$\#N^{-1}(r) = 6 \text{ for } r \neq 0$$

All 25 elements are shown. Every nonzero element is a unit.

Coordinates, multiplication, and norm agree

In the basis $(1, u)$, multiplication by $a + bu$ has matrix

$$M_{a,b} = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}, \qquad N(a + bu) = \det M_{a,b} = a^2 - 2b^2 \pmod 5.$$

The table shows every norm value. Only the zero element has norm zero. Each nonzero norm value has exactly six preimages. This makes the later character cancellation visible without estimating any sums.

The checks enumerate all 25 matrices, verify the determinant formula, and verify norm multiplicativity on every pair of field elements. They also change to a different invertible basis and check that conjugating the multiplication matrix leaves its determinant unchanged.

This model is the reduction of the quadratic integer algebra $\mathbb{Z}[\sqrt{2}]$ modulo five. Its finite arithmetic illustrates the determinant construction used in the continuation; it is not a computation of all ideal classes or a general ideal-counting theorem.

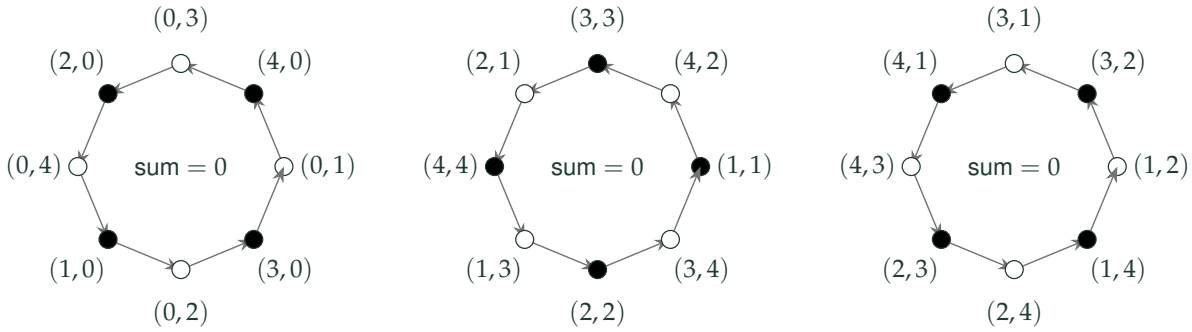
> **Read this as:** A quotient norm is built from the actual multiplication operator. Its basis independence is a determinant identity, not a choice of convenient coordinates.

A NONSCALAR UNIT WITNESS

Permutation forces a zero mean

Main text: §9; retained scalar/Mellin continuation: norm-residue lattice mean cancellation.

Let χ be the quadratic character modulo five, extended by $\chi(0) = 0$. In the field on the preceding page, choose $v = 2u$. Its norm is $N(v) = 2$, so $\chi(N(v)) = -1$.



Multiply by $v = 2u$. Its norm is 2, so $\chi(N(vz)) = -\chi(N(z))$.

Filled node: $+1$. Open node: -1. The omitted zero element has weight 0.

The whole sum is paired with its negative

Multiplication by v permutes the 25 elements. Norm multiplicativity gives

$$\sum_{z \in A} \chi(N(z)) = \sum_{z \in A} \chi(N(vz)) = -\sum_{z \in A} \chi(N(z)).$$

The sum is an integer, so it is zero. The origin contributes zero. The other 24 elements fall into three cycles of length eight under multiplication by v; the character signs alternate around each cycle.

The same reasoning works with a complex root of unity $\chi(N(v)) \neq 1$: subtracting the two equal sums gives $(1 - \chi(N(v)))S = 0$. The essential inputs are an actual permutation, a multiplicative norm, and a nontrivial character value. They are not consequences of drawing an alternating cycle.

In the arithmetic continuation the quotient is the actual additive module J/mJ associated with an ideal representative. This finite-field example isolates the cancellation mechanism while keeping that broader module and lattice comparison separate.

> **Read this as:** The witness is one unit that changes the character value. Once it exists, the permutation cancels the entire finite mean.

THE EXACT SCALAR TEST

A nontrivial character may still be exceptional

Main text: §9; scalar/Mellin checkpoint: scalar-power and nonscalar-witness boundaries.

A scalar $a \in \mathbb{F}_5^\times$ acts on a degree-d algebra with determinant a^d. Therefore the scalar cancellation route tests $\chi(a^d)$, not merely $\chi(a)$.

	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$
trivial	no	no	no	no	no	no
quartic	yes	yes	yes	no	yes	yes
quadratic	yes	no	yes	no	yes	no
conjugate quartic	yes	yes	yes	no	yes	yes

For $\chi_k(2) = i^k$, a witness exists exactly when $kd \not\equiv 0 \pmod 4$.

A failed sufficient test is not an impossibility theorem

Let $\chi_4(2) = i$ generate the character group of $\mathbb{F}_5^\times$, and write $\chi_k = \chi_4^k$. Since 2 generates the unit group,

$$\exists a : \chi_k(a^d) \neq 1 \quad \Longleftrightarrow \quad kd \not\equiv 0 \pmod 4.$$

For the quadratic character $k = 2$ and degree $d = 2$, every scalar test is trivial. Yet page V23 supplies the nonscalar witness $2u$, proving cancellation in that specific algebra. Scalar exceptional does not mean exceptional for every possible quotient unit.

Conversely, the quartic character $k = 1$ works in degree two even though $\gcd(2,4) \neq 1$. Thus the simpler degree–totient coprimality criterion is sufficient, not necessary. With degree three, every nontrivial character passes because exponentiation by three permutes the units.

The retained scalar/Mellin continuation states its public hypothesis in the weaker exact scalar-power form. It does not claim to construct a nonscalar witness for every scalar-exceptional arithmetic character.

> **Read this as:** Test the character after the degree-power map. Nontriviality before that map can disappear after it.

A CRT MODEL FOR REPRESENTATIVE CHOICE

Avoid finitely many primes constructively

Main text: §9; retained scalar/Mellin continuation: finite-prime avoidance and automatic representatives.

A representative needed by an arithmetic argument often has to avoid a prescribed finite collection of primes. A small integer version asks for

$$x \equiv 1 \pmod 2, \qquad x \equiv 2 \pmod 3, \qquad x \equiv 4 \pmod 5.$$

There is exactly one solution modulo 30: $x = 29$.

0	1	2	3	4	5
6	7	8	9	10	11
12	13	14	15	16	17
18	19	20	21	22	23
24	25	26	27	28	29

Choose one residue in each quotient

$$a \equiv 1 \pmod 2$$

$$a \equiv 2 \pmod 3$$

$$a \equiv 4 \pmod 5$$

$$a = 29, \quad \gcd(29, 30) = 1$$

Shaded values avoid 2, 3, and 5 simultaneously.

Avoidance and ideal class are different constraints

The selected residues are nonzero modulo each forbidden prime, so $\gcd(29, 30) = 1$. The eight units modulo 30 are displayed separately. They all avoid the primes, but only one also satisfies the three specific residue prescriptions.

The continuation proves an ideal-theoretic analogue with more structure: in every prescribed inverse ideal class it constructs a nonzero integral representative whose norm is coprime to the nonzero modulus. That removes a representative-choice assumption from the public norm-character bound. It does not remove the remaining character, lattice, or Frobenius-occurrence obligations.

The integer CRT calculation is an illustration, not a proof of the ideal-class representative theorem. In particular, enumerating integers cannot impose a nontrivial ideal class. The full construction uses prime-ideal avoidance and its source comparison to the class group.

> **Read this as:** A finite avoidance argument supplies an admissible representative. It does not by itself make the arithmetic object being counted, or the desired Frobenius class, occur.

All 30 residue classes are tested exactly. The unique CRT result and the unit count are independent finite outputs of the check script.

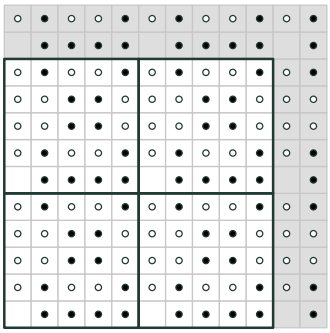
A FULL LATTICE-BOX CALCULATION

Cancellation leaves a boundary problem

Main text: §9; retained scalar/Mellin continuation: additive residue counts and norm-character bounds.

Extend the function $f(a,b) = \chi(a^2 - 2b^2 \bmod 5)$ periodically to $\mathbb{Z}^2$. Every complete 5×5 residue tile has sum zero by the finite-field calculation. Only incomplete tiles can contribute to a box sum.

$[0,12)^2 \cap \mathbb{Z}^2$



four full 5×5 cells

100 bulk points: sum 0

44 boundary points

actual total $S_{12} = 1$

$|S_L| \le L^2 - (L - (L \bmod 5))^2$ for this square-lattice model.

Separate bulk cancellation from boundary size

For $0 \le a, b < 12$, four complete tiles fill a 10×10 square and contribute zero. The remaining 44 points lie in the two boundary strips. Their exact sum is one. Since $|f| \le 1$, a bound of 44 follows without needing that final cancellation.

For a general positive integer L, write $L = 5q + r$, $0 \le r < 5$. The complete tiled square has side $L - r$, so

$$|S_L| \le L^2 - (L - r)^2 = 2Lr - r^2 \le 8L.$$

The area is $X = L^2$, making this a square-root-in-area bound for this square-box model. The proof is the exact tile decomposition; checking many L values is only a regression check of the implementation.

Number-field ideal sums involve different regions, lattice indices, signs, and generator multiplicities. Their $X^{1-1/d}$ exponent cannot be justified simply by relabeling this square picture as an ideal-norm ball.

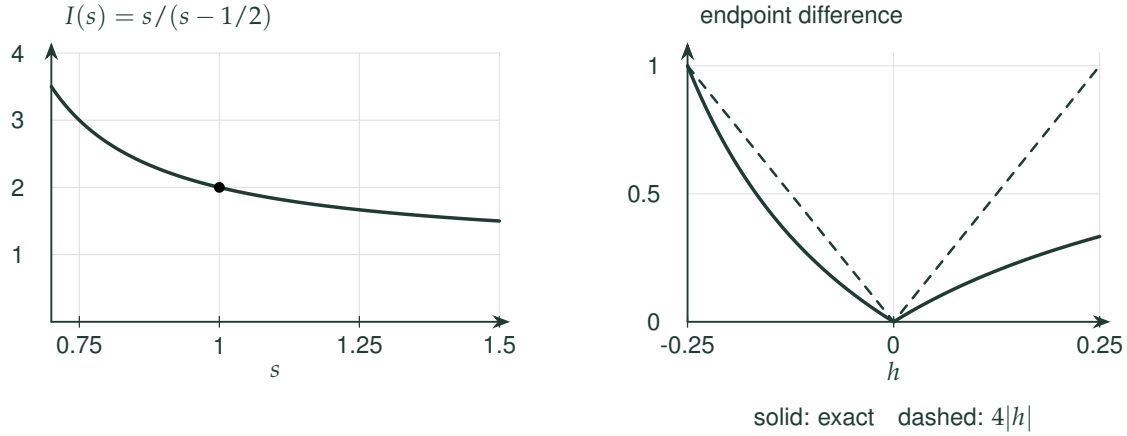
> **Read this as:** A zero finite mean removes the bulk. The remaining estimate depends on the actual geometry of the region and its boundary.

A MELLIN ENDPOINT MODEL

What a power-saving bound buys analytically

Main text: §9; retained scalar/Mellin continuation: summatory measurability and first-order Mellin control.

Use the explicit envelope $A(x) = x^{1/2}$ for $x \geq 1$. It is not an ideal-counting function. It is a model showing why a power saving can make a Mellin expression regular at $s = 1$.



Compute the endpoint rather than just naming it

For real $s > 1/2$,

$$I(s) = s \int_1^\infty A(x) x^{-s-1}\,\mathrm{d}x = \frac{s}{s-1/2}.$$

At $s = 1 + h$ this gives

$$I(1+h) - I(1) = -\frac{h}{h+1/2}, \qquad |I(1+h) - I(1)| \leq 4|h| \quad (|h| \leq 1/4).$$

The derivative at one is -2. The logarithmic moment is finite as well: $\int_1^\infty x^{-3/2} \log x\,\mathrm{d}x = 4$. These are explicit calculations; the drawn curve does not establish differentiability or convergence.

The continuation performs the corresponding analytic steps for its actual summatory functions, with their explicit measurability and growth hypotheses. The arithmetic Euler-product identity, nonzero pole residues, and the required prescribed-prime divergence remain separate. Regularity of one Mellin transform does not construct all of those inputs.

> **Read this as:** An analytic estimate controls a function already tied to arithmetic data. It is not a substitute for proving that tie.

THE REMAINING FROBENIUS STEP

Finite splitting data are not a density theorem

Main text: Theorem 9.1. Latest continuation: GeneralArithmeticCoverage and ArithmeticSystemSeparation.

For the quadratic polynomial X^2+1, reduction modulo an odd prime either gives two roots or no roots. This finite scan displays both behaviors. It is a useful model for the distinction between splitting and nonsplitting classes.

odd primes p

3	5	7	11	13	17	19	23
0	2	0	0	2	2	0	0

roots of x^2+1 in $\mathbb{F}_p$

two factorization types
already appear in this finite list

repeat one class

1	1	2
1	1	2
1	1	2

three rows, rank one

Finite character separation needs enough different evaluations, not just more evaluations.

Existence must survive arbitrary exclusions

The continuation's missing general coverage proposition asks for an actual unramified rational prime in a prescribed conjugacy class, outside any prescribed finite forbidden set, together with the corresponding absolute Frobenius witness. A list of small primes does not establish that quantifier.

The finite character argument on page V15 also needs samples that jointly separate the constituents. Repeating the identity-class row $(1,1,2)$ of the S_3 table gives rank one no matter how often it is repeated. Many samples are not necessarily informative samples.

The later checked rational-base separator composes its constructed arithmetic coverage theorem with finite linear algebra. The earlier scalar/Mellin route retained a coverage premise. The displayed scan certifies neither route: it is a finite example, not an occurrence theorem.

> **Read this as:** A separating group element, a witnessed Frobenius at one prime, and a new prime after every finite exclusion are distinct levels of evidence.

The scan exhaustively checks scalar roots for the shown primes. No density, infinitude, or general prescribed-conjugacy statement is inferred from it.

SCALAR ABSENCE VERSUS REPRESENTATION EXISTENCE

Changing coefficients changes the problem

Main text: Example 8.4 and the declared coefficient categories in §§8–9.

Over $\mathbb{F}_3$, no scalar solves $x^2 + 1 = 0$. But the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{F}_3) \quad \text{satisfies} \quad A^2 + I = 0.$$

And over $\mathbb{F}_9 = \mathbb{F}_3[u]/(u^2 + 1)$, the scalars u and $-u$ solve the same polynomial.

Scalar over $\mathbb{F}_3$	Matrix over $\mathbb{F}_3$	Scalar over $\mathbb{F}_9$
$x = 0, 1, 2$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$u^2 = -1$
$x^2 + 1 = 1, 2, 2$		roots $u, -u$
no solution	$A^2 + I = 0$	coefficient extension

Fix the coefficient field, dimension, and point category before declaring an occupancy failure.

Three outputs with different types

The failed scalar problem over $\mathbb{F}_3$, the successful matrix problem over $\mathbb{F}_3$, and the successful scalar problem over $\mathbb{F}_9$ are not contradictory answers. They have different allowed objects. The matrix is multiplication by u in the $\mathbb{F}_3$-basis $(1, u)$ of the extension field.

This also has a characteristic-zero analogue. The matrix with the same entries gives a rational two-dimensional representation of a cyclic group of order four. Its scalar eigenvalues require a field containing a square root of -1. No one-dimensional rational eigencharacter is created by writing the two-dimensional matrix.

Accordingly, the scalar companion failure on page V8 is a counterexample only to the fixed scalar problem. It is not a counterexample to a representation problem that permits coefficient extension or a different dimension. Those permissions must be declared before comparing fibers across primes.

> **Read this as:** An empty fiber is meaningful only after its coefficient ring and object type are fixed. Do not repair an empty fiber by silently changing either.

FINAL READER'S CHECKLIST

Read the evidence at its actual scope

Main text: Appendix D. Formal guide: statement register, dependency sources, and replay boundary.

The reader carries three distinct contracts. A larger date or a larger numerator does not make their hypotheses interchangeable.

Record	What it establishes
Historical 17/20	Block 27 and the scalar/Mellin continuation. Claims 9.1, 9.2, and 11.1 were partial in those revisions.
Checked 20/20	The amended conditional occupied-system assembly. Completed source members, closed target profiles, compatible carrier presentations, and local comparisons remain explicit inputs.
Later 19/20	A stronger residual-system attempt with no promoted terminal endpoint and an unavailable pinned Lean executable. It is a different contract, not a failed recompilation of the checked source.

Two conclusions, not an unstated identification

The terminal theorem constructs a transported target family and independently recovers a summand of the supplied carrier. To identify them requires an explicit comparison. The carrier result is pointwise at an arbitrary coefficient place. Recovering carriers at every place needs the relevant presentations there. Zero multiplicity is allowed. Nonzero occupancy needs a separate nonzero premise.

What the examples verify

The appendix supplies reproducible finite examples and exact Python checks. No plotted pixel decides a congruence, rank, determinant, root, or character sum. The original Lean sources and recorded audits are preserved. The October 10 editorial release performed no fresh Lean compilation. The separate October 20 production replay is reported in the Lean guide; it introduces no new formal theorem.

> **Read this as:** A completed conditional theorem is a theorem with its hypotheses, not a claim that every arithmetic application supplies them.