OPERATION KOMOREBI

REALIZATION CALCULUS



CORE SERIES IX

Euler Congruences

Completion, factorial strengthening, and sharp dyadic geometry

Traditional paper

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REALIZATION CALCULUS

Abstract

We prove Nishibuchi's generalized Euler congruence conjecture in its full stated parameter range and develop sharper arithmetic consequences. For every prime-power packet width, a charged Hurwitz differential operator transfers integral difference estimates through the reciprocal generating series. In the dyadic case this gives mixed finite-difference bounds at every row, without a starting threshold. For the four addresses of width four, a divided-power argument adds the factorial gain $v_2(k!)$, determines every repeated-step valuation in addresses zero and one, and gives a stronger address-two estimate with exact values at all power-of-two orders. The corresponding first differences yield least periods and bijective isometric continuations onto explicit dyadic residue classes. We also determine exact Taylor radii for three addresses. The manuscript retains the complete earlier arithmetic, lifting, deformation, and analytic-regularity development, with dated historical appendices. A separate appendix states the precise remaining simplicity problem for ordinary Euler tails. Formal coverage is explicit: the external conjecture and several four-address results have inherited Lean verification; the new factorial, mixed-step, all-rank, and radius arguments are written proofs with recorded audits and exact computational checks, and await dedicated Lean formalization.

What this final edition establishes

The original external problem is Nishibuchi's Conjecture 3.9 for generalized Euler numbers. Its full parameter range is resolved. If p is prime, m is positive with $m \mid p-1$ or $m = 2$, $0 \le j < mp$, and $q = \operatorname{lcm}(2, p-1)$, then the original Euler coefficients are p-adically integral and

$$E^{(mp,j)}_{mpn+qp^r} \equiv E^{(mp,j)}_{mpn} \pmod{p^r} \qquad (r \ge 1,\ n \ge r).$$

The paper states the generating function and the conversion between ordinary degree and packet row before using this formula. The normalized coefficient is $C_{N,j}(n)$ and the original coefficient is $E^{(N,j)}_{Nn} = j!C_{N,j}(n)$. The constant coefficient of the original series is $j!$.

The strongest common-step conclusion in width four is particularly compact. For $n \ge 0$ and $k, H \ge 1$,

$$\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}E^{(4,j)}_{4(n+iH)} \equiv 0 \pmod{2^{\,v_2(k!)+kv_2(H)+A_j(k)}},$$

where

$$A_0(k) = A_1(k) = k, \qquad A_2(k) = \max\{k+4,\ 2k+\lfloor k/2\rfloor+1\}, \qquad A_3(k) = k+2.$$

There is no starting-row threshold. The exponent is exact in addresses zero and one for every order and positive step. It is exact in address two at every power-of-two order. The proof chapters also give mixed-step forms, the cancellation criterion in addresses zero and one, and exact first-displacement laws.

Result	Scope	Evidence in this edition
Nishibuchi Conjecture 3.9	Every parameter in the original statement; explicit $n \geq r$	Inherited Lean proof
Width-four all-row common-step bound, first differences, least periods and isometry	Every address, row and positive displacement	Inherited Lean proof
General prime-power inverse transfer and mixed steps	Every prime-power width; signed steps for odd primes	Written proof and audit
Factorial gains and exact higher-order families	Four width-four addresses, with stated sharpness scopes	Written proof and audit
Taylor radii and broader dyadic first differences	Exact radii in addresses $0, 1, 2$; stated all-rank cases	Written proof and audit
Ordinary Euler tail simplicity	Partial families and an explicit remaining discriminant problem	Open in full generality

The complete earlier Paper IX development follows alongside the strengthening. Dated appendices retain the original continuation arguments and recovered historical text. Statements about an earlier open target or an earlier build remain records of that checkpoint; the current theorem and scope statements govern this edition. The Lean companion supplies the exact formal source map and distinguishes source reconciliation from compiler execution. The visual guide supplies exact finite examples and theorem-derived diagrams.

This is a project publication under the name Komorebi P. The recorded parallel proof audits are internal computational-assistant reviews. They are not external peer review. The edition makes no claim of journal acceptance, external certification, or global historical priority.

1 Scope and conventions

The central objects are sequences and polynomial equations equipped with arithmetic coordinates. Residue projectors select addresses in a sequence, finite differences measure compatibility between nearby addresses, and valuation bounds control the passage from finite precision to a p-adic limit. These constructions provide a common language for the Euler packet results and the later lifting results. Each theorem nevertheless retains its own coefficient ring, topology and nonsingularity hypotheses.

This edition integrates the original arithmetic framework with the completed extensions. The stated external Nishibuchi conjecture is proved in its full specified parameter range. The dyadic refinements and polynomial deformation theorems give stronger conclusions under the hypotheses stated with each result. The native conjectures retain their original identifiers, 13.2 and 13.3, when discussed as research questions, although the section numbering of this consolidated manuscript is new.

Throughout, $\mathbb{N} = \{0, 1, 2, \dots\}$. When p is prime, $\mathbb{Z}_p$ denotes the p-adic integers and $\mathbb{Q}_p$ their fraction field. We normalize $v_p(p) = 1$. For a rational or p-adic quantity, a displayed lower valuation bound is understood with the mathematical convention $v_p(0) = +\infty$. The corresponding formal predicates are zero-safe: the quantity is zero or its valuation meets the stated bound. An exact finite valuation assertion requires a nonzero quantity, and this is stated where used. Congruence in $\mathbb{Z}_p$ means ideal divisibility. Thus

$$x \equiv y \pmod{p^r} \quad \Longleftrightarrow \quad x - y \in p^r \mathbb{Z}_p.$$

For vectors, ideal membership and congruence are coordinatewise.

Write $\Delta_u f(n) = f(n+u) - f(n)$ and use Δ_u^k for its k-fold iterate. Euler coefficients indexed by ordinary degree and by packet row are distinguished explicitly. A packet displacement by u rows is a degree displacement by Nu in a packet of width N.

The main text presents theorem statements and their arguments; accompanying source maps identify the Lean declarations and archived files that support them. The verification record keeps the original dates and scope of each build. In particular, the latest 91-check polynomial build covers the polynomial development, while earlier builds support the arithmetic results. The inherited formal development is retained without adding Lean declarations. The strengthening chapters add written proofs and clearly mark their formalization boundary; no historical-priority claim is made. The analytic inverse-function results used in Section 11 come from Mathlib.

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2 Arithmetic packets and coefficient transport

We work with sequences, formal power series, and finite recurrence states. Residue selectors isolate coefficients in a chosen congruence class, while triangular transport expresses the same object in new coordinates. A valuation filtration determines how many coordinates are needed at a given precision. Together, these operations provide a common framework for the Euler congruences and the finite lifting constructions developed later.

The original arithmetic seed notation

$$\mathcal{S} = (A, a_0, \mathcal{O}, \mathcal{R}, \{P_{N,j}\}, w, \tau_p)$$

records a coefficient ring or module, a seed, allowed operators, relations, residue selectors, a depth weight, and a topology. This notation describes the ingredients of the construction; a category of seed presentations and a notion of equivalence still require separate definitions. We therefore state each theorem for a concrete object. Section 12 discusses the additional structure needed for the original native conjectures.

The Lean declarations cited in this section have namespace prefix `KomorebiPaperIX`. Their exact modules and source hashes are recorded in the accompanying statement map. All belong to the preserved baseline formalization; this section brings their statements and proofs together.

2.1 Residue selectors and exact reconstruction

Let R be a commutative ring, $a : \mathbb{N} \to R$, $N \geq 1$, and $0 \leq j < N$. Define

$$(P_{N,j}a)(n) = \begin{cases} a(n), & n \equiv j \pmod{N}, \\ 0, & \text{otherwise.} \end{cases}$$

This definition keeps the original index set. The compressed subsequence $a(Nq + j)$, used below for recurrence packets, is a different representation of the selected coefficients.

Theorem 2.1 (Residue reconstruction)**.** *The selectors satisfy*

$$P_{N,j}^2 = P_{N,j}, \qquad P_{N,i}P_{N,j} = 0 \quad (i \neq j), \qquad \sum_{j=0}^{N-1} P_{N,j} = I.$$

If $N \mid M$ and $M > 0$, then

$$P_{N,j}a = \sum_{\substack{0 \leq r < M \\ r \equiv j \pmod{N}}} P_{M,r}a.$$

The same identities hold coefficientwise for formal power series.

Proof. At each index n, exactly one standard residue label equals n mod N. Consequently a repeated selector changes nothing, two different selectors have zero composite, and the sum retains $a(n)$ exactly once. If $N \mid M$, reducing n mod M modulo N gives n mod N. This proves the refinement identity at every coefficient. Equality of all coefficients proves the formal-series versions. □

Formal correspondence. `ResiduePackets.packet_idempotent`, `ResiduePackets.packet_orthogonal`, `ResiduePackets.packet_reconstruction`, and `ResiduePackets.packet_refinement`; the series variants have the prefix `series_packet` in the same module. The algebraic statements need no convergence hypothesis.

2.2 Formal exponential coordinates and Hurwitz multiplication

Over a characteristic-zero field K, put

$$\mathcal{E}(a) = \sum_{n\geq 0} \frac{a_n}{n!} t^n, \qquad \mathcal{D}(A)_n = n![t^n]A.$$

These inverse maps between sequences and $K[[t]]$ provide *formal* exponential coordinates. No analytic convergence is needed for the coefficient identities below.

Theorem 2.2 (Constructed unit-triangular coordinates)**.** *Let $h : \mathbb{N} \to K$ satisfy $h_0 = 1$, write $H = \mathcal{E}(h)$ and $H^{-1} = \mathcal{E}(k)$, and define*

$$U_H a = \mathcal{D}(H\mathcal{E}(a)).$$

Then U_H is an equivalence, with inverse $U_H^{-1}a = \mathcal{D}(H^{-1}\mathcal{E}(a))$. Its coefficient formulas are

$$(U_H a)_n = \sum_{r=0}^{n} \binom{n}{r} h_r a_{n-r}, \qquad (U_H^{-1} a)_n = \sum_{r=0}^{n} \binom{n}{r} k_r a_{n-r}.$$

The resulting triangular matrix has diagonal entries 1*. The transported selectors*

$$\widehat{P}_{N,j} = U_H P_{N,j} U_H^{-1}$$

are idempotent and pairwise orthogonal, and their sum is the identity.

Proof. The factorials are nonzero in K, so encoding and decoding cancel coefficientwise. The constant coefficient of H is 1; hence its formal power-series inverse exists. Multiplication by H and H^{-1} gives the two inverse laws. The ordinary product coefficient formula, multiplied by $n!$, gives the displayed binomial sums. In row n, the coefficient of a_n is $h_0 = 1$ and coefficients of a_q with $q > n$ vanish. Conjugating the identities of Theorem 2.1 by the constructed equivalence proves the transported projector identities. □

Formal correspondence. `EGF.decode_encode`, `EGF.encode_decode`, `EGF.egf_product`, `EGF.coordinateEquiv`, `EGF.coordinate_forward`, `EGF.coordinate_inverse`, and `EGF.transported_reconstruction`, together with the idempotence and orthogonality declarations in `EGFCoordinates`.

In characteristic p, factorial division is unavailable in general, but the coefficient operation remains meaningful over every commutative ring:

$$(f \star g)_n = \sum_{r=0}^{n} \binom{n}{r} f_r g_{n-r}, \qquad (D^d f)_n = f_{n+d}, \qquad \mathbf{1}_n = \begin{cases} 1, & n = 0, \\ 0, & n > 0. \end{cases}$$

Here $\star$ is Hurwitz multiplication. The symbol D^d denotes a coefficient shift, which agrees with formal differentiation after characteristic-zero encoding.

Theorem 2.3 (Hurwitz product rules)**.** *For $d \geq 0$,*

$$D^d(f \star g) = \sum_{r=0}^{d} \binom{d}{r} (D^r f) \star (D^{d-r} g).$$

If R has prime characteristic p, then

$$D^p(f \star g) = f \star D^p g + D^p f \star g.$$

Over a field, if $g_0 = 1$, a sequence b is constructed recursively with $b \star g = \mathbf{1}$. If also $D^p g = Dg$ in characteristic p, then $D^p b = Db$.

Proof. Pascal's identity proves the first product rule, and induction proves the higher rule. The intermediate binomial coefficients $\binom{p}{r}$ vanish in characteristic p. For the inverse, solve row n using the coefficient of b_n, which is $g_0 = 1$:

$$b_n = \mathbf{1}_n - \sum_{r<n} \binom{n}{r} b_r g_{n-r}.$$

Subtract the first and pth product rules for $b \star g = \mathbf{1}$. The assumed identity for g leaves $(D^p b - Db) \star g = 0$. A second triangular induction shows that convolution with a sequence having constant coefficient 1 has trivial kernel. □

Formal correspondence. `Hurwitz.shift_convolution`, `Hurwitz.prime_shift_convolution`, `Hurwitz.convolution_left_eq_zero`, `HurwitzInverse.inverse_convolution`, and `HurwitzInverse.inverse_prime_difference`. Working directly with coefficient sequences avoids the undefined expression $1/n!$ in $\mathbb{F}_p$ when $p \mid n!$.

2.3 Triangular inversion and a finite precision filtration

Theorem 2.4 (Forward substitution and denominator clearing)**.** *Let T_{nq} be coefficients over a field with $T_{nn} \neq 0$ for every n. For any sequence b, define*

$$a_n = \frac{b_n - \sum_{q<n} T_{nq} a_q}{T_{nn}}.$$

This constructs the unique solution of $\sum_{q=0}^{n} T_{nq} a_q = b_n$ for all n. If T_{nq} and b_n are integers and the solution is taken over $\mathbb{Q}$, then

$$\left(\prod_{r=0}^{n} T_{rr} \right) a_n \in \mathbb{Z}.$$

Proof. The recursion uses only smaller indices, and substitution verifies each row. If a second solution agrees through row $n-1$, subtracting row n and cancelling T_{nn} proves equality at n. For integrality, use induction and promote every earlier clearing product to $\prod_{r<n} T_{rr}$. Multiplication by the additional factor T_{nn} cancels the only new division. □

Formal correspondence. `Triangular.solve_row`, `Triangular.solve_unique`, and `Triangular.solve_denominator_integral`. The clearing product bounds the denominator; the unscaled solution can still be nonintegral.

To isolate the filtration argument, let p be any element of a commutative ring and let $w : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$. Set $p^{[r]} = p^r$ for finite r and $p^{[\infty]} = 0$. The latter notation means the coefficient vanishes.

Theorem 2.5 (Filtered finite reconstruction)**.** *For every $n, L \in \mathbb{N}$ and sequences c, S in that ring,*

$$\begin{aligned} &\sum_{r=0}^{n} \binom{n}{r} c_r p^{[w(r)]} S_{n-r} \\ &\qquad \equiv \sum_{\substack{0 \le r \le n \\ w(r) < L}} \binom{n}{r} c_r p^{[w(r)]} S_{n-r} \pmod{p^L}. \end{aligned}$$

Here congruence means divisibility of the difference by p^L in the coefficient ring. In $\mathbb{Z}_p$, it means membership in $p^L \mathbb{Z}_p$.

Proof. The difference of the two sums consists exactly of terms with $w(r) \geq L$. Every such term is divisible by p^L, including the zero terms with infinite weight. Their finite sum has the same divisibility. □

Formal correspondence. `Filtered.filtered_triangular_reconstruction` and `Filtered.padic_filtered_reconstruction`. For weights $2r$, retaining $r \leq M$ gives the bound p^{2M+2} through `Filtered.even_weight_tail`. A sharper kernel weight can improve this bound, as in the classical dyadic reconstruction later in the paper. Each coefficient row is finite, so no hypothesis $w(r) \to \infty$ is needed here.

2.4 Finite Mahler coordinates and weighted differences

For $f : \mathbb{N} \to R$, define the step-h forward difference and its iterates by

$$\Delta_h f(n) = f(n+h) - f(n), \qquad \Delta_h^0 f = f,$$

and put $c_r(f) = \Delta_1^r f(0)$. Thus the coefficients are determined by the differences of f.

Theorem 2.6 (Finite Newton–Mahler identities)**.** *For every $n, s, h \in \mathbb{N}$,*

$$\begin{aligned} f(n) &= \sum_{r=0}^{n} c_r(f) \binom{n}{r}, \\ c_r(\Delta_1^s f) &= c_{s+r}(f), \\ \Delta_1^s f(n) &= \sum_{j=0}^{n} c_{s+j}(f) \binom{n}{j}, \\ \Delta_h^s f(n) &= \sum_{j=0}^{s} (-1)^{s-j} \binom{s}{j} f(n+jh). \end{aligned}$$

The finite interpolant $I_N f(n) = \sum_{r=0}^{N} c_r(f) \binom{n}{r}$ agrees with f for $n \leq N$. It has the same order-s unit-step difference at n whenever $n + s \leq N$. For arbitrary coefficients $c_0, \ldots, c_N$, the corresponding binomial interpolant has exactly those coefficients and zero coefficients above N.

Proof. Write $E_h f(n) = f(n+h)$ for the shift operator. The unit shift is $I + \Delta_1$. Expanding its nth power and evaluating at zero gives reconstruction. Composing difference operators gives the coefficient shift and applying reconstruction to $\Delta_1^s f$ gives the third identity. The binomial expansion of $(E_h - I)^s$ gives the explicit stencil. Binomial coefficients vanish when their lower index exceeds n, which proves the interpolation claim. Equality of every point of the stencil proves equality of differences. Repeated finite differences of $\binom{n}{r}$ at zero give the exact coefficient extraction. □

Formal correspondence. `Mahler.finite_mahler_reconstruction`, `Mahler.coefficient_difference`, `Mahler.difference_from_coefficients`, `Mahler.difference_explicit`, `Mahler.finite_sample_difference`, and `Mahler.coefficient_interpolant`. Finite samples determine the stencils contained in the sample interval; a universal tail estimate requires additional information.

For $x \in \mathbb{Z}_p$, extend the valuation to zero by setting

$$\operatorname{ord}_p(x) = \begin{cases} v_p(x), & x \neq 0, \\ \infty, & x = 0. \end{cases}$$

Thus $L \leq \operatorname{ord}_p(x)$ is equivalent to $x \in p^L \mathbb{Z}_p$.

Theorem 2.7 (Weighted difference bound)**.** *Let $f : \mathbb{N} \to \mathbb{Z}_p$ and let $w : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ satisfy $w(s+j) \leq \operatorname{ord}_p(c_{s+j}(f))$ for $0 \leq j \leq n$. Then*

$$\operatorname{ord}_p(\Delta_1^s f(n)) \geq \min_{0 \leq j \leq n} \left\{ w(s+j) + v_p\binom{n}{j} \right\}.$$

Proof. Apply the nonarchimedean inequality to the finite coefficient formula in Theorem 2.6. For $0 \leq j \leq n$, the integer $\binom{n}{j}$ is positive. The order of its product with $c_{s+j}(f)$ is the sum of the two orders, including the case where that coefficient is zero. □

Formal correspondence. `PadicBounds.nat_le_order_iff`, `PadicBounds.weighted_difference_bound`, and `PadicBounds.weighted_difference_divisibility`. An expansion on all of $\mathbb{Z}_p$ requires a convergence theorem in addition to these finite algebraic identities and bounds.

2.5 Unit modes and the dyadic distinction

Theorem 2.8 (Differences of unit modes)**.** *Over every commutative ring,*

$$\Delta_h^s(u^n) = u^n(u^h - 1)^s.$$

For any prime p, units $u_i \in \mathbb{Z}_p^\times$, coefficients $a_i \in \mathbb{Z}_p$, and a finite index set I, the sequence $f(n) = \sum_{i \in I} a_i u_i^n$ satisfies

$$\Delta_{p-1}^s f(n) \in p^s \mathbb{Z}_p \qquad (n, s \in \mathbb{N}).$$

Proof. Induction gives the geometric identity. Reduction of a unit to $\mathbb{F}_p^\times$ gives $u_i^{p-1} = 1$ modulo p. Taking s powers supplies p^s in each summand. The difference operator is linear over finite sums. □

Formal correspondence. `UnitModes.geometric_difference`, `UnitModes.unit_fermat_divisibility`, and `UnitModes.finite_unit_modes_divisibility`. The result includes $p = 2$, where the displayed step is 1. Its scope here is finite sums of modes.

Theorem 2.9 (Finite unit coordinates)**.** *Let p be prime, $L \geq 0$, and $a \in (\mathbb{Z}/p^{L+1}\mathbb{Z})^\times$. Define*

$$\omega_L(a) = a^{p^L}, \qquad \langle a \rangle_L = a\omega_L(a)^{-1}.$$

Then

$$a = \omega_L(a)\langle a \rangle_L, \qquad \omega_L(a)^{p-1} = 1, \qquad \langle a \rangle_L \equiv 1 \pmod{p}.$$

The two factors are uniquely determined by these properties and are compatible with reduction to lower precision. For $a \in (\mathbb{Z}/2^{L+2}\mathbb{Z})^\times$, there is instead a unique decomposition

$$a = \epsilon(a)\langle a \rangle_4, \qquad \epsilon(a) \in \{1, -1\}, \qquad \langle a \rangle_4 \equiv 1 \pmod{4}.$$

Proof. The unit-group exponent $p^L(p-1)$ gives the root identity for ω_L. Its residue is that of a, so the remaining factor has residue 1. If two $(p-1)$st roots have equal residue, raising them to p^L makes them equal modulo p^{L+1}, while each root is fixed by this exponent. This proves uniqueness and then compatibility under reduction. In the dyadic construction, an odd residue is exactly 1 or -1 modulo 4. Choose that sign and multiply a by it to obtain the principal factor. The two signs remain distinct modulo 4, which proves uniqueness. □

Formal correspondence. `FiniteUnits.finite_unit_decomposition`, `FiniteUnits.decomposition_unique`, `UnitRefinement.omega_compatible`, `UnitRefinement.principal_compatible`, and `DyadicUnits.dyadic_unit_decomposition` with `DyadicUnits.decomposition_unique`. The finite $p = 2$ version of the first construction has trivial root-of-unity factor. The separate modulo-4 construction retains the sign that reduction modulo 2 cannot detect.

2.6 Recurrence packets and finite arithmetic synchronization

Theorem 2.10 (Companion jumps and packet recurrences)**.** *Let C be a $d \times d$ matrix over a commutative ring and $x_n = C^n x_0$. For $N, j, k \in \mathbb{N}$, put $b_k = x_{Nk+j}$. Then*

$$b_k = C^j (C^N)^k x_0, \qquad b_{k+1} = C^N b_k.$$

If $\chi_{C^N}(X) = X^d + \sum_{r<d} q_r X^r$, then

$$b_{k+d} = -\sum_{r=0}^{d-1} q_r b_{k+r}.$$

Every coordinate and every linear scalar observable satisfies the same recurrence. No invertibility or diagonalization hypothesis is required.

Proof. The jump identity follows from the power law for one matrix. Apply the Cayley–Hamilton identity $\chi_{C^N}(C^N) = 0$ to the state b_k. The leading coefficient is 1, so the final state can be isolated. A linear observable preserves the resulting finite sum. □

Formal correspondence. `Companion.companion_jump`, `Companion.packet_charpoly_recurrence`, and `Companion.packet_observable_recurrence`. The source-sequence versions derive the packet recurrence from the relation $x_{n+1} = Cx_n$. For the Fibonacci matrix $C = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, its jump trace is the Lucas number L_N and its determinant is $(-1)^N$. Thus

$$F_{N(k+2)+j} = L_N F_{N(k+1)+j} - (-1)^N F_{Nk+j}.$$

This specialization is `FibonacciPackets.fibonacci_packet_recurrence`; it is derived from the Fibonacci state.

Theorem 2.11 (Dyadic–odd packet factorization)**.** *Let $a \geq 0$ and let M be a positive odd integer. The Chinese remainder map is a ring isomorphism*

$$\mathbb{Z}/(2^a M)\mathbb{Z} \simeq \mathbb{Z}/2^a\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z}.$$

For $0 \leq r < 2^a M$ and any coefficient sequence f,

$$P_{2^a M, r} f = P_{2^a, r \bmod 2^a} P_{M, r \bmod M} f.$$

The same identity holds for formal-series coefficient selectors.

Proof. The two moduli are coprime. An index has the prescribed combined residue exactly when it has both component residues. Applying this equivalence to the indicator defining each selector proves the formula. □

Formal correspondence. `CRTPackets.addressEquiv`, `CRTPackets.residue_equivalence`, `CRTPackets.packet_factorization`, and `CRTPackets.series_packet_factorization`. This theorem synchronizes finite arithmetic addresses. The passage from a family of p-adic objects to a global integer object is a separate question, discussed in Section 12.3.

2.7 Polyphase notation retained from the recovered manuscript

The recovered Annals text develops a broader polyphase vocabulary. We explain it through elementary identities with explicit algebraic derivations. These identities have not been separately formalized in Lean for this consolidation. For a sequence a and $M \geq 1$, let $a_r^{[M]}(q) = a_{Mq+r}$ for $0 \leq r < M$. If $A(z) = \sum_n a_n z^n$ and $A_r(w) = \sum_q a_{Mq+r} w^q$ are ordinary generating functions, then unique quotient and remainder give

$$A(z) = \sum_{r=0}^{M-1} z^r A_r(z^M), \qquad \left(a_r^{[M]}\right)_s^{[N]} = a_{r+Ms}^{[MN]}.$$

For ordinary Cauchy multiplication, the phase-r product has a term without a carry and a term with one carry:

$$C_r(w) = \sum_{u+v=r} A_u(w)B_v(w) + w \sum_{u+v=r+M} A_u(w)B_v(w), \qquad 0 \leq u, v < M.$$

Adding the two phase addresses gives the formula. It describes ordinary Cauchy multiplication; Hurwitz convolution has its own binomial weights.

For $M \geq 1$ and $s, r, k \in \mathbb{N}$, define the affine Mahler kernel by

$$\Theta_{M,s}(r,k) = [z^r](1+z)^s\left((1+z)^M - 1\right)^k = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \binom{Mj+s}{r}.$$

It vanishes unless $k \leq r \leq Mk + s$. For a finite binomial expansion $f(n) = \sum_{r=0}^{D} c_r \binom{n}{r}$, the kth Newton coefficient of $g(q) = f(Mq + s)$ is therefore

$$c_k(g) = \sum_{r=k}^{\min(D, Mk+s)} c_r \Theta_{M,s}(r,k).$$

Indeed, insert the finite stencil for $\Delta_1^k g(0)$ and interchange its two finite sums. Coefficient extraction of the binomial expansion of $((1+z)^M - 1)^k$ gives the kernel. In the binary zero-address case,

$$\Theta_{2,0}(k+j,k) = 2^{k-j}\binom{k}{j} \qquad (0 \leq j \leq k),$$

since $(2z + z^2)^k = z^k(2+z)^k$. Exact valuations for the general kernel remain to be determined. The recovered infinite analytic extensions and stronger historical identities lie beyond these finite derivations; their status is discussed in Section 12.5.

3 Congruential Euler packets and their arithmetic

Two related constructions run through this section. The congruential Euler family is indexed by a modulus N and an address j. Its exceptional dyadic member has $N = 4$ and four addresses. The classical signed Euler numbers arise from $N = 2$, $j = 0$. Keeping these indices visible distinguishes the four-packet period theorems from the classical edge truncations.

3.1 Original coefficients, normalized packets, and precision

Definition 3.1 (Congruential Euler coefficients)**.** For $N \geq 1$ and $0 \leq j < N$, define rational coefficients by the formal identity

$$\sum_{d \geq 0} E_d^{(N,j)} \frac{t^d}{d!} = \left(\sum_{n \geq 0} \frac{t^{Nn}}{(Nn+j)!} \right)^{-1}. \tag{3.1}$$

The reciprocal exists because its denominator has constant coefficient $1/j! \neq 0$. Set

$$U_{N,j}(n) = E_{Nn}^{(N,j)}, \qquad C_{N,j}(n) = \frac{U_{N,j}(n)}{j!}.$$

Thus $C_{N,j}(0) = 1$ and $U_{N,j}(0) = j!$. All coefficients outside the multiples of N vanish.

The normalized source and its inverse are

$$K_{N,j}(t) = j! \sum_{n\geq 0} \frac{t^{Nn}}{(Nn+j)!}, \qquad \mathcal{P}_{N,j}(t) = \sum_{n\geq 0} C_{N,j}(n) \frac{t^{Nn}}{(Nn)!}. \tag{3.2}$$

Their product is one. Clearing factorials in its coefficient of t^{Nn} gives the triangular system

$$\sum_{q=0}^{n} \binom{Nn+j}{Nq} C_{N,j}(q) = \delta_{n0}. \tag{3.3}$$

In particular, for $n > 0$,

$$C_{N,j}(n) = -\binom{Nn+j}{Nn}^{-1} \sum_{q=0}^{n-1} \binom{Nn+j}{Nq} C_{N,j}(q).$$

The diagonal binomial coefficient is positive, so this recurrence determines the sequence uniquely. The Lean construction begins with this recurrence and derives the reciprocal identity from it. The resulting sequence is then identified with the degree-indexed coefficients of (3.1), so both descriptions refer to the same family.

The relevant declarations are `KomorebiPaperIX.Congruential.triangular_inverse`, `KomorebiPaperIX.Congruential.packet_is_actual_inverse`, and `KomorebiPaperIX.NishibuchiCompletion.generalizedEuler_at_packet`. The cleared identity with the original positive-address exponential section is `KomorebiPaperIX.NishibuchiCompletion.original_generating_identity`.

Proposition 3.2 (A denominator bound for every address)**.** *For $N \geq 1$, $0 \leq j < N$, and $n \in \mathbb{N}$, set*

$$D_{N,j}(n) = \prod_{r=1}^{n} \binom{Nr+j}{Nr}, \qquad D_{N,j}(0) = 1.$$

Then $D_{N,j}(n) C_{N,j}(n) \in \mathbb{Z}$.

Proof. Proceed by induction in the recurrence (3.3). For every $q < n$, the product $D_{N,j}(q)$ divides $D_{N,j}(n-1)$. Multiplying the recurrence by $D_{N,j}(n-1)$ therefore makes every previous term integral. The diagonal factor completes the product $D_{N,j}(n)$, proving the claim. At $n = 0$, both factors are one. This gives a uniform denominator bound without requiring integrality of the normalized coefficients themselves. The Lean statements are `KomorebiPaperIX.Congruential.clearing_product` and `KomorebiPaperIX.Congruential.universal_denominator_clearing`. □

Example 3.3 (The Bernoulli boundary address)**.** At $N = 2$, the address $j = 1$ gives a second classical family. Define the Bernoulli numbers by

$$B(z) = \frac{z}{e^z - 1} = \sum_{r\geq 0} B_r \frac{z^r}{r!}, \qquad B_0 = 1, \quad B_1 = -\frac{1}{2}.$$

All quotients here are regular formal series. The identity

$$2B(z) - B(2z) = \frac{z}{\sinh z} = \sum_{n\geq 0} C_{2,1}(n)\frac{z^{2n}}{(2n)!}$$

follows by putting the two fractions on a common denominator and using (3.1). Comparing even coefficients yields

$$C_{2,1}(n) = (2 - 2^{2n})B_{2n} \qquad (n \in \mathbb{N}).$$

This classical derivation restores the boundary example from the original manuscript. No dedicated Lean bridge for this Bernoulli specialization was identified in the preserved sources.

Definition 3.4 (Differences and valuation conventions)**.** For a sequence f and $h, k, n \in \mathbb{N}$, put

$$\Delta_h f(n) = f(n+h) - f(n), \qquad \Delta_h^0 f(n) = f(n), \qquad \Delta_h^{k+1} f(n) = \Delta_h^k f(n+h) - \Delta_h^k f(n).$$

We use the extended valuation $v_p(0) = +\infty$ in mathematical statements. Thus $v_p(x) \geq L$, including negative integers L, means $x = 0$ or the ordinary valuation of the nonzero rational x is at least L. For integral coefficients and $L \geq 0$, it is equivalent to $x \in p^L\mathbb{Z}_p$. In expressions involving a step multiplier we write $\nu_p(u) = v_p(u)$ for $u > 0$ and set $\nu_p(0) = 0$. This finite convention makes the displayed exponents meaningful at step zero, where all positive-order differences vanish.

The valuation convention at zero is encoded by `KomorebiPaperIX.GlobalEulerBoundary.precision`. For p-adic congruences, divisibility is interpreted in $\mathbb{Z}_p$; statements over other coefficient rings specify the ring explicitly. Field divisibility in $\mathbb{Q}_p$ alone does not encode precision.

3.2 The standard address window and its common reduction

The addresses below the prime have unit diagonal coefficients. In this window the normalized packets are integral, and their reductions agree at every row.

Proposition 3.5 (Lucas reduction and address independence)**.** *Let p be prime, $m > 0$, and $N = mp$. For $0 \leq j < p$ and $n, q \in \mathbb{N}$,*

$$\binom{mpn + j}{mpq} \equiv \binom{mn}{mq} \pmod{p}.$$

Every $C_{mp,j}(n)$ belongs to $\mathbb{Z}_p$. Its reduction is the unique sequence $\overline{C}_m : \mathbb{N} \to \mathbb{F}_p$ satisfying

$$\sum_{q=0}^{n} \binom{mn}{mq} \overline{C}_m(q) = \delta_{n0} \quad \text{in } \mathbb{F}_p.$$

Consequently, for any two addresses $0 \leq j, k < p$,

$$C_{mp,j}(n) \equiv C_{mp,k}(n) \pmod{p} \qquad (n \in \mathbb{N}).$$

Proof. Lucas reduction separates the lowest base-p digits, which are j and zero in the upper and lower arguments. Their binomial contribution is one, leaving the displayed reduced row. On the diagonal the reduced coefficient is one. The recurrence (3.3) therefore constructs an integral packet by successive division by units in $\mathbb{Z}_p$. After reduction, every address satisfies the same triangular system over $\mathbb{F}_p$. Its diagonal entries are one, so its solution is unique. This proves both the common reduction and the congruence between addresses. The row reduction is `KomorebiPaperIX.StandardWindow.lucas_row_reduction`. The integral identification and the two reduction statements are `KomorebiPaperIX.WindowIndependence.standard_window_integral`, `KomorebiPaperIX.WindowIndependence.coeffPadic_coe`, `KomorebiPaperIX.WindowIndependence.residue_independence`, and `KomorebiPaperIX.WindowIndependence.two_window_addresses_congruent`. □

3.3 The full Nishibuchi statement and its starting bound

Theorem 3.6 (Nishibuchi congruence)**.** *Let p be prime, let $m > 0$, and suppose that $m \mid p-1$ or $m = 2$. Put $N = mp$ and $q = \operatorname{lcm}(2, p-1)$. For every $0 \le j < N$, all $U_{N,j}(n)$ belong to $\mathbb{Z}_p$. For $r \ge 1$ and $n \ge r$,*

$$E^{(N,j)}_{Nn+qp^r} \equiv E^{(N,j)}_{Nn} \pmod{p^r}. \tag{3.4}$$

Thus the positive sufficient starting index $n_0 = r$ works uniformly. If $m \mid p-1$, the stronger sufficient condition $r \le mn$ is valid.

This theorem covers the full parameter range and displacement of Conjecture 3.9 in the fixed version of Nishibuchi's paper used for the statement comparison. The corresponding Lean theorem is `KomorebiPaperIX.NishibuchiCompletion.conjecture_3_9`. The stronger divisor-branch threshold is `KomorebiPaperIX.NishibuchiCompletion.divisor_period`. The starting index is sufficient uniformly, although it need not be the earliest one for a given tuple of parameters.

We first prove the difference estimates on the divisor branch, then derive the period from a finite Newton expansion.

Theorem 3.7 (All-address differences on the divisor branch)**.** *Let p be prime, $m > 0$, $m \mid p-1$, $N = mp$, and $0 \le j < N$. Write $h = (p-1)/m$. For every $u, k, n \in \mathbb{N}$,*

$$U_{N,j}(n) \in \mathbb{Z}_p, \tag{3.5}$$

$$\Delta^k_{uh} U_{N,j}(n) \in p^{\min(k,mn)}\mathbb{Z}_p, \tag{3.6}$$

$$v_p\big(\Delta^k_{uh} C_{N,j}(n)\big) \ge \min(k, mn) - v_p(j!). \tag{3.7}$$

The factorial correction is part of the normalized statement, including addresses $j \ge p$.

Proof. Work in the Hurwitz integral ring of formal series

$$\mathcal{H}_p = \left\{ F \in \mathbb{Q}_p[[t]] : d![t^d]F \in \mathbb{Z}_p \text{ for every } d \right\}.$$

For $D = d/dt$, let $B_p = (D^{p^2} - D^p)/p$. Call F flat if $B_p^s F \in \mathcal{H}_p$ for every $s \ge 0$. We will prove that $j!\mathcal{P}_{N,j}$ is flat at every address. The main difficulty is to include the addresses at which the usual unit-pivot argument fails.

Set

$$H_c(t) = \sum_{n\ge 0} \frac{t^{Nn+c}}{(Nn+c)!}, \qquad A_c(t) = \sum_{n\ge 0} \frac{t^{Nn}}{(Nn+c)!}, \qquad R_c(t) = \frac{A_{c-1}(t)}{A_c(t)} \quad (1 \le c < N).$$

The denominator of this last quotient has nonzero constant coefficient. The section identities give

$$R_c H_c = t H_{c-1}, \qquad R_c(0) = c, \qquad j!\mathcal{P}_{N,j} = \mathcal{P}_{N,0} \prod_{c=1}^{j} R_c.$$

Cancellation of consecutive source sections proves the product identity and ties the ratios directly to the family under study.

At the singular addresses $c = ap$, $1 \le a < m$, use the weighted ring

$$\mathcal{W}_p = \left\{ \sum_{d\ge 0} a_d T^d : p^{\lfloor d/p \rfloor} a_d \in \mathbb{Z}_p \right\}.$$

The key estimate is $R_{ap}(\log(1+T)) \in \mathcal{W}_p$. Its arithmetic input comes from the cyclic group ring $\mathbb{Z}[\mathbb{Z}/N\mathbb{Z}]$. For its canonical element x and $I = (p, x^{p-1}-1)$, the relation $(x^{p-1})^p = 1$ implies $I^p \subseteq (p)I$. Induction yields $I^q \subseteq (p^{v_p(q!)})$. Grouping the falling factorial $(x)_d = \prod_{i=0}^{d-1}(x-i)$ into consecutive blocks of length p then gives an integral residue-coordinate factor $p^{v_p(\lfloor d/p \rfloor !)}$. The identity

$$v_p(d!) = \lfloor d/p \rfloor + v_p(\lfloor d/p \rfloor !)$$

accounts for precisely the weight in $\mathcal{W}_p$ after dividing by $d!$. Comparison of the section and group-ring recurrences identifies these coordinates with the logarithmically substituted section.

To pass from the section to the singular ratio, factor out its initial zero: for $c = ap$, the section has the form

$$H_c(\log(1+T)) = \frac{T^c}{c!}\,V(T), \qquad V(0) = 1, \qquad V \in \mathcal{W}_p.$$

The weight of $c!$ compensates for the coefficient shift by c. The weighted ring is closed under inversion of series with constant coefficient one and under $L_T = \log(1+T)(1+T)d/dT$. With $\ell(T) = \log(1+T)/T$, interpreted by coefficient shifting, differentiation of the displayed factorization gives

$$R_c(\log(1+T)) = c(1+T)\ell(T) + (L_T V)V^{-1}.$$

Since $p\ell(T) \in \mathcal{W}_p$ and $p \mid c$, both terms are weighted. This is the singular estimate `KomorebiPaperIX.SingularRatioAmice.paper_singular_amice_estimate`.

To deduce flatness from the weighted estimate, use $\Theta = (1+T)d/dT$. Both Θ and $(\Theta^{p^2} - \Theta^p)/p$ preserve $\mathcal{W}_p$. For the first operator, its coefficient formula is $(d+1)a_{d+1} + da_d$; whenever the floor weight rises, the factor $d+1$ provides the missing p. For the second operator, the proof uses its full Leibniz correction and the scalar congruence $p^2 \mid b^{p^2} - b^p$. Weighted monomials factor as

$$a_d T^d = (p^{\lfloor d/p \rfloor} a_d)(T^p/p)^{\lfloor d/p \rfloor} T^{d \bmod p}.$$

The operator preserves these monomials and hence finite weighted polynomials. Each fixed output coefficient depends on only finitely many input coefficients, which extends the result to the formal series. Substitution $T = e^t - 1$ intertwines D with Θ and maps $\mathcal{W}_p$ into $\mathcal{H}_p$, because

$$d![t^d]F(e^t-1) = [T^0]\Theta^d F.$$

It follows that the singular ratios are flat. This transfer is proved in `KomorebiPaperIX.WeightedDworkTransfer.singular_ratio_flat`.

At a nonsingular address $2 \le c < N$ with $p \nmid c$, differentiation of the section identities gives

$$tR_c' = R_c(1 + R_{c-1} - R_c).$$

Writing $q = R_{c-1} - (c-1)$ and $y = R_c - c$, this becomes

$$(tD + c)y = cq + yq - y^2.$$

On degrees divisible by p, the inverse of $tD + c$ divides the degree-d coefficient by the unit $d + c$. Starting with $y_0 = 0$, form

$$y_{s+1} = (tD + c)^{-1}(cq + y_s q - y_s^2).$$

Flatness and support are preserved. Indeed, put $E_c = tD + c$ and $C_p = pD^{p^2} - D^p$. The commutator is $[B_p, E_c] = C_p$, and iteration gives

$$B_p^r E_c y = E_c B_p^r y + rC_p B_p^{r-1} y.$$

For the inverse equation $E_c y = f$, induction on r makes the correction term Hurwitz-integral. Both operators preserve support in degrees divisible by p. On that support the degree multiplier $d + c$ is a unit, so integrality of $E_c B_p^r y$ implies integrality of $B_p^r y$. Thus the inverse preserves full flatness, as well as support. Since the difference of two right sides factors as $(z - y)(q - z - y)$ and its second factor has zero constant term, agreement with the solution gains one coefficient at each iteration. Consequently $[t^d]y_s = [t^d]y$ for $d \le s$. Every coefficient of $B_p^r y$ can therefore be read from a sufficiently long finite iterate. The argument is entirely formal. This is the content of `KomorebiPaperIX.RiccatiFlatPropagation.riccati_solution_flat`.

The standard window supplies the initial ratio R_1. The weighted estimate handles $p \mid c$, and Riccati propagation handles $p \nmid c$. Flatness is closed under products, so the exact factorization establishes flatness of $j!\mathcal{P}_{N,j}$. Extracting its supported Hurwitz coefficients and applying the proved saturation and step-multiple lemmas yields (3.5)–(3.6).

Here is the coefficient extraction explicitly. If $a_d = d![t^d]F$, then

$$d![t^d]B_p^k F = p^{-k}\Delta^k_{p(p-1)}a_{d+pk}.$$

For $F = j!\mathcal{P}_{N,j}$ and $k \le mn$, take $d = mpn - pk$. Support in multiples of mp turns this into $p^{-k}\Delta_h^k U_{N,j}(n)$, so flatness gives divisibility by p^k. For $k > mn$, expand Δ_h^{k-mn} applied to $\Delta_h^{mn}U_{N,j}(n)$; every shifted row is at least n, so each summand is divisible by p^{mn}. Finally $\Delta_{uh} = ((I + \Delta_h)^u - I)$, whose kth power for $u > 0$ is an integral combination of powers of Δ_h of order at least k. The case $u = 0$ gives zero for $k > 0$, and order zero is the original integral coefficient. This proves the bound for all u and k. Finally $\Delta_{uh}^k U_{N,j} = j!\Delta_{uh}^k C_{N,j}$ gives (3.7), with the zero case retained. The resulting statements are `KomorebiPaperIX.AllResidueEuler.all_residue_flat`, `KomorebiPaperIX.AllResidueEuler.global_unnormalized_bound`, and `KomorebiPaperIX.AllResidueEuler.global_normalized_precision`. □

Example 3.8 (Why the factorial loss is necessary)**.** Take $p = 3$, $m = 2$, $N = 6$, and $j = 3$. The recurrence gives

$$C_{6,3}(1) = -\frac{1}{84}, \qquad C_{6,3}(2) = \frac{703}{5460}, \qquad C_{6,3}(2) - C_{6,3}(1) = \frac{64}{455}.$$

The last difference has 3-adic valuation zero. Here $h = 1$, so at $n = 1$ the order-one bound without factorial loss would require valuation at least one. The corrected normalized bound is sharp, since

$$v_3\big(C_{6,3}(2) - C_{6,3}(1)\big) = 0 = 1 - v_3(3!).$$

Multiplication by 3! gives

$$U_{6,3}(2) - U_{6,3}(1) = \frac{384}{455}, \qquad v_3\big(U_{6,3}(2) - U_{6,3}(1)\big) = 1,$$

consistent with the unnormalized theorem. Thus the factorial correction is already necessary at a single step in the all-address parameter range. These exact values and the refutation of a universal bound without loss are formalized in `KomorebiPaperIX.Normalization.normalized_first`, `KomorebiPaperIX.Normalization.normalized_second`, `KomorebiPaperIX.Normalization.normalized_difference`, `KomorebiPaperIX.Normalization.difference_valuation`, `KomorebiPaperIX.Normalization.factorial_loss_is_sharp`, `KomorebiPaperIX.Normalization.unnormalized_difference`, `KomorebiPaperIX.Normalization.unnormalized_difference_valuation`, and `KomorebiPaperIX.Normalization.universal_no_loss_refuted`.

Corollary 3.9 (A common period in the standard window)**.** *Let p be prime, $m > 0$, $m \mid p - 1$, and $0 \le j < p$. Put $h = (p - 1)/m$. For every $n \ge 1$ and $u \in \mathbb{N}$,*

$$\overline{C}_m(n + uh) = \overline{C}_m(n), \qquad C_{mp,j}(n + uh) \equiv C_{mp,j}(n) \pmod{p}.$$

Thus h is a sufficient period from row one, common to all addresses in the standard window. It need not be the least period.

Proof. Since $j < p$, the factorial $j!$ is a unit in $\mathbb{Z}_p$. Also $mn \geq 1$. The order-one case of Theorem 3.7 therefore gives valuation at least one for the normalized difference at every step uh. This proves the congruence, and Proposition 3.5 identifies its reduction with the equality for $\overline{C}_m$. The preserved Lean development also proves this period directly from the Hurwitz inverse in characteristic p: `KomorebiPaperIX.WindowPeriod.uniform_standard_window_period`, `KomorebiPaperIX.WindowPeriod.multiple_period`, and `KomorebiPaperIX.WindowPeriod.standard_window_period_divisibility`. □

Lemma 3.10 (Prime-power period from finite differences)**.** *Let R be a commutative ring, p a prime, $f : \mathbb{N} \to R$, $h, n \in \mathbb{N}$, and $r \geq 1$. If*

$$p^{\min(k,r)} \mid \Delta_h^k f(n) \quad \text{for every } k \in \mathbb{N},$$

then $p^r \mid f(n + p^{r-1}h) - f(n)$.

Proof. The finite Newton expansion gives

$$f(n + p^{r-1}h) - f(n) = \sum_{k=1}^{p^{r-1}} \binom{p^{r-1}}{k} \Delta_h^k f(n).$$

Terms with $k \geq r$ have the required factor from the difference. For $1 \leq k < r$, the identity

$$k\binom{p^{r-1}}{k} = p^{r-1}\binom{p^{r-1}-1}{k-1}$$

and $v_p(k) < k$ imply $p^r \mid \binom{p^{r-1}}{k} p^k$ in $\mathbb{Z}$. Mapping that integer divisibility to R proves the claim term by term. The argument includes $r = 1$. The declaration is `KomorebiPaperIX.PeriodFromDifferences.prime_power_period`. □

Proof of Theorem 3.6. On the divisor branch, $m \mid q$ and the packet step q/m is an integer multiple of $h = (p-1)/m$. If $r \leq mn$, Theorem 3.7 supplies the hypotheses of Lemma 3.10. Its packet displacement is $p^{r-1}q/m$, which converts to the original degree as follows:

$$mp\left(n + p^{r-1}\frac{q}{m}\right) = mpn + qp^r.$$

The choice $n \geq r$ implies $mn \geq r$ because $m \geq 1$.

For an odd prime, $2 \mid p - 1$, so the option $m = 2$ introduces no further case. The sole remaining branch is $p = 2, m = 2, N = 4$. The all-row theorem proved next gives $2^k \mid \Delta_1^k C_{4,j}(n)$ at every row and address. After multiplying by $j!$, Lemma 3.10 gives packet period 2^{r-1} and degree displacement $4 \cdot 2^{r-1} = 2^{r+1} = qp^r$. It also gives integrality. Hence the complete statement follows. The formal degree conversion is `KomorebiPaperIX.NishibuchiCompletion.original_degree_shift`; the all-parameter degree theorem is `KomorebiPaperIX.NishibuchiCompletion.full_period_degree`. □

3.4 All-row dyadic differences and displacement amplification

For the remainder of the four-packet discussion abbreviate $C_j = C_{4,j}$ and $U_j = U_{4,j}$. Removing the original row cap requires an unshifted coefficient operator.

Theorem 3.11 (All-row dyadic higher differences). *For $0 \le j < 4$ and all $u, k, n \in \mathbb{N}$,*

$$C_j(n) \in \mathbb{Z}_2, \tag{3.8}$$

$$\Delta_u^k C_j(n) \in 2^{k(1+\nu_2(u))}\mathbb{Z}_2, \tag{3.9}$$

$$\Delta_u^k U_j(n) \in 2^{k(1+\nu_2(u))+v_2(j!)}\mathbb{Z}_2. \tag{3.10}$$

There is no restriction $k \le n$ and no minimum with the row n.

Proof. The Hurwitz coefficients of the normalized source $K_{4,j}$ at degrees $4n$ are

$$g_j(n) = \binom{4n+j}{4n}^{-1}.$$

Their explicit expressions are

$$g_0(n) = 1, \qquad g_1(n) = \frac{1}{4n+1},$$
$$g_2(n) = \frac{1}{(4n+1)(2n+1)}, \quad g_3(n) = \frac{3}{(4n+1)(2n+1)(4n+3)}.$$

Every denominator is odd. An affine sequence with even slope satisfies $2^k \mid \Delta_1^k f(n)$ for every k, n. The exact product identity

$$\Delta_1^s(fg)(n) = \sum_{i=0}^{s} \binom{s}{i} \Delta_1^i f(n) \Delta_1^{s-i} g(n+i)$$

proves closure of this property under products. Applying the formula with the factors interchanged to $fg = 1$ isolates $f(n+s)\Delta_1^s g(n)$. Multiplication by $f(n+s)^{-1}$ and induction then prove closure under pointwise unit inverses. Thus all four g_j have the property. The explicit formulas and difference bounds are formalized in `KomorebiPaperIX.DyadicFourKernel.source_coe` and `KomorebiPaperIX.DyadicFourKernel.source_kummer`.

For an integer charge c, put

$$A_c = \frac{D^4 - c}{2}.$$

A series is flat of charge c when applying any power A_c^s to it gives an element of $\mathcal{H}_2$. The required product formula is

$$A_{c+d}(FG) = (A_c F)G + F(A_d G) + 2F'G''' + 3F''G'' + 2F'''G'. \tag{3.11}$$

All middle coefficients are integral. Derivatives preserve $\mathcal{H}_2$ and commute with A_c. Induction on the number of iterations in (3.11) proves that multiplication adds charges. If $FG = 1$, the same formula yields

$$A_{-c}G = -G^2 A_c F - G(2F'G''' + 3F''G'' + 2F'''G').$$

Induction, starting from the independently established integrality of the reciprocal G, proves that inversion negates the charge. Finally $A_{c+2} = A_c - 1$, so increasing a charge by two preserves flatness. These are `KomorebiPaperIX.ChargedDworkClosure.B_mul`, `KomorebiPaperIX.ChargedDworkClosure.flat_inverse`, and `KomorebiPaperIX.ChargedDworkClosure.flat_charge_shift`.

Packing g_j into degrees divisible by four gives $K_{4,j}$. Its source property is precisely charge-one flatness. Inversion gives charge -1, and the charge shift restores charge one for the reciprocal $\mathcal{P}_{4,j}$. If $\operatorname{dec}(F)(d) = d![t^d]F$, the exact coefficient identity is

$$\operatorname{dec}(A_1^s F)(d) = 2^{-s}\Delta_4^s \operatorname{dec}(F)(d).$$

At $d = 4n$, packing turns the right side into $2^{-s}\Delta_1^s C_j(n)$. This extraction requires no negative degree and therefore holds at every n, including zero. It proves the unit-step bound `KomorebiPaperIX.DyadicFourAllRows.base_step_bound`.

To incorporate $v_2(u)$, expand a displacement using finite Newton coefficients. For any prime p, $u > 0$, and $1 \le a \le u$,

$$p^{v_p(u)+1} \mid \binom{u}{a} p^a,$$

as follows from $a\binom{u}{a} = u\binom{u-1}{a-1}$ and $v_p(a) < a$. Repeated displacement, retaining the remaining base-difference order at each induction step, gives a factor $p^{k(1+v_p(u))}$ from k displacements. This induction is formalized in `KomorebiPaperIX.StepAmplification.repeated_precision`. Applying it with $p = 2$ proves (3.9); multiplication by $j!$ proves (3.10). Zero step and zero order follow from their definitions. The final statements are `KomorebiPaperIX.DyadicFourFinalStrengthening.normalized_step_bound` and `KomorebiPaperIX.DyadicFourFinalStrengthening.unnormalized_step_bound`. □

Corollary 3.12 (Step-sensitive divisor-branch bound)**.** *Under the hypotheses of Theorem 3.7, for all $u, k, n \in \mathbb{N}$,*

$$\Delta^k_{u(p-1)/m} U_{mp,j}(n) \in p^{\min(k(1+v_p(u)),mn)}\mathbb{Z}_p.$$

For $C_{mp,j}$ the corresponding valuation lower bound is $\min(k(1+v_p(u)), mn) - v_p(j!)$.

Proof. Apply the same repeated-displacement theorem with the fixed cap mn to (3.6). The argument preserves this cap and improves the contribution from the order of the difference. Division by the fixed factorial gives the rational valuation statement, with the zero case included. These are `KomorebiPaperIX.AllResidueStepAmplification.unnormalized_precision` and `KomorebiPaperIX.AllResidueStepAmplification.normalized_precision`. □

Corollary 3.13 (Dyadic periods from row zero)**.** *For $0 \le j < 4$, $r \ge 1$, and $n \ge 0$,*

$$U_j(n + 2^{r-1}) \equiv U_j(n) \pmod{2^r}.$$

For $j = 2, 3$ and $r \ge 2$, 2^{r-2} is also a sufficient packet period. The respective original degree displacements are 2^{r+1} and 2^r.

Proof. Use Lemma 3.10 and the unit-step case of Theorem 3.11. At $j = 2, 3$ the factorial contributes an additional power of two. The original degree versions are `KomorebiPaperIX.DyadicFourFinalStrengthening.all_rows_period_degree` and `KomorebiPaperIX.DyadicFourFinalStrengthening.all_rows_half_period_degree`. The next theorem sharpens these sufficient periods to a complete classification. □

3.5 Exact displacement valuations and least packet periods

Lemma 3.14 (Finite Newton dominance)**.** *Let $f : \mathbb{N} \to \mathbb{Q}$ and write $a_k = \Delta_1^k f(0)$. Suppose that $c \in \mathbb{N}$, $a_1 \ne 0$, $v_2(a_1) = c$, and*

$$v_2(a_k) \ge c + 1 + \lfloor \log_2 k \rfloor \qquad (k \ge 2).$$

Then for every $n \geq 0$ and $T > 0$,

$$f(n+T) - f(n) \neq 0, \qquad v_2\big(f(n+T) - f(n)\big) = v_2(T) + c.$$

Proof. Finite Newton reconstruction at an arbitrary row gives

$$\Delta_1^k f(n) = \sum_{i=0}^{n} \binom{n}{i} a_{k+i}.$$

The monotonicity of $\lfloor \log_2 k \rfloor$ therefore preserves the higher-coefficient bound after translation. In addition, $\Delta_1 f(n) - a_1 \in 2^{c+1}\mathbb{Z}_2$, so the first difference is nonzero of exact valuation c.

Now expand the displacement:

$$f(n+T) - f(n) = T\Delta_1 f(n) + \sum_{k=2}^{T} \binom{T}{k} \Delta_1^k f(n).$$

For $1 \leq k \leq T$, the binomial denominator identity implies

$$v_2\binom{T}{k} \geq v_2(T) - v_2(k) \geq v_2(T) - \lfloor \log_2 k \rfloor.$$

Every term of the remaining sum consequently has valuation at least $v_2(T) + c + 1$. Its sum cannot cancel the leading term, which has exact valuation $v_2(T) + c$. This proves nonvanishing and the claimed valuation using only finite sums. The general theorem is `KomorebiPaperIX.NewtonDominance.exact_displacement`. □

Theorem 3.15 (Exact four-packet displacement law)**.** *Set*

$$(c_0, c_1, c_2, c_3) = (1, 1, 4, 2), \qquad (d_0, d_1, d_2, d_3) = (1, 1, 5, 3).$$

For $0 \leq j < 4$, $n \geq 0$, and every positive integer T,

$$\begin{aligned} C_j(n+T) - C_j(n) &\neq 0, & v_2\big(C_j(n+T) - C_j(n)\big) &= v_2(T) + c_j, \\ U_j(n+T) - U_j(n) &\neq 0, & v_2\big(U_j(n+T) - U_j(n)\big) &= v_2(T) + d_j. \end{aligned}$$

Proof. The first Newton coefficients, derived from (3.3), are

$$a_{0,1} = -2, \quad a_{1,1} = -\frac{6}{5}, \quad a_{2,1} = -\frac{16}{15}, \quad a_{3,1} = -\frac{36}{35}.$$

Their exact valuations are c_j. Theorem 3.11 gives $v_2(a_{j,k}) \geq k$ for all orders. To obtain the stronger logarithmic envelope required by Lemma 3.14, only a finite initial segment needs improvement. The recurrence proves the following exact valuations:

Address j	Further orders k	$v_2(a_{j,k})$ in that order
0	2	3
1	2	3
2	2, 3, 4, 5, 6	6, 9, 13, 16, 22
3	2, 3, 4	8, 8, 15

For all subsequent k, the elementary inequality $c_j + 1 + \lfloor \log_2 k \rfloor \le k$ completes the envelope. The entries follow by exact rational computation from the defining recurrence, and their valuations are proved in Lean. Together with the inequality for all subsequent orders, they establish the bound at every order. The uniform envelope is `KomorebiPaperIX.DyadicFourPrefix.tail_precision`.

Apply Lemma 3.14 to C_j. Multiplication by the nonzero factorial $j!$ adds $v_2(j!)$ and preserves nonzeroness. Hence $d_j = c_j + v_2(j!)$, giving the four valuation gains for the original coefficients. The declarations are `KomorebiPaperIX.DyadicFourLeastPeriods.normalized_exact_displacement` and `KomorebiPaperIX.DyadicFourLeastPeriods.exact_displacement`. □

Corollary 3.16 (All positive periods and their least element)**.** *For $r \in \mathbb{N}$ and $0 \le j < 4$, a positive integer T is a packet period of U_j modulo 2^r if and only if*

$$2^{\max(0,r-d_j)} \mid T.$$

Here a packet period means $U_j(n+T) \equiv U_j(n) \pmod{2^r}$ for every $n \ge 0$. Consequently the least positive packet periods are

j	0	1	2	3
least period	$2^{\max(0,r-1)}$	$2^{\max(0,r-1)}$	$2^{\max(0,r-5)}$	$2^{\max(0,r-3)}$.

Proof. By the exact displacement law, the congruence at any row is equivalent to $v_2(T) + d_j \ge r$. This is exactly the displayed integer divisibility criterion. Its least positive solution is the indicated power of two, which also covers $r = 0$ and $r \le d_j$. The all-period and least-element statements are `KomorebiPaperIX.DyadicFourLeastPeriods.period_iff_dvd` and `KomorebiPaperIX.DyadicFourLeastPeriods.least_packet_period`. □

For example, $U_2(n) \equiv 2 \pmod{32}$ and $U_3(n) \equiv 6 \pmod{8}$ for every row. At precision 2^{10} the least packet periods are $512, 512, 32, 128$. A packet displacement T has original degree displacement $4T$, as proved by `KomorebiPaperIX.DyadicFourLeastPeriods.original_degree_exact_displacement`. Minimality here concerns packet shifts. Shifts of the full degree-indexed sequence, including its zero coefficients off the packet support, require a separate analysis. The new strengthening chapters determine all repeated-step higher differences in addresses zero and one and a power-of-two-order family in address two. The full exact classification of arbitrary higher-order differences are also a separate question from the first-displacement law.

3.6 Classical odd-length coordinates and their congruence ladder

The classical signed Euler numbers have generating function $\operatorname{sech} t = \sum_{n \ge 0} E_{2n} t^{2n}/(2n)!$. They are the $N = 2$, $j = 0$ member of (3.1). The coordinate theorem reconstructs them from finite alternating sums of odd powers.

Definition 3.17 (Odd-length source and inverse coefficients)**.** For a positive odd integer p, define

$$S_d(p) = \sum_{a=0}^{p-1} (-1)^a (2a+1)^d.$$

Let γ_m be the integral Hurwitz coefficients of sech^2:

$$\operatorname{sech}^2 t = \sum_{m \ge 0} \gamma_m \frac{t^{2m}}{(2m)!}, \qquad \gamma_m = \sum_{b=0}^{2m} \binom{2m}{b} E_b E_{2m-b},$$

where $E_b = 0$ for odd b. Then $\gamma_0 = 1$. Define

$$T^{\mathrm{odd}}_{p,M}(n) = \sum_{m=0}^{\min(M,n)} \binom{2n}{2m} \gamma_m p^{2m} S_{2(n-m)}(p).$$

Here p need only be odd; primality is needed only to interpret the result as a prime-adic congruence ladder.

The source and inverse coefficients are `KomorebiPaperIX.OddPower.source` and `KomorebiPaperIX.EulerCoefficients.inverseEven`; the truncation is `KomorebiPaperIX.OddEuler.truncation`.

The same inverse coefficients can be written in the signed zigzag notation of the original manuscript. Define Z_r by

$$\sec t + \tan t = \sum_{r \geq 0} Z_r \frac{t^r}{r!}.$$

Then

$$\gamma_m = (-1)^m Z_{2m+1} \qquad (m \in \mathbb{N}). \tag{3.12}$$

Indeed, the odd coefficients come from $\tan t$, whose derivative is $\sec^2 t$. Changing the sign of the squared variable in the even series for $\operatorname{sech}^2 t$ gives $\sec^2 t$, so its degree-$2m$ Hurwitz coefficient is $(-1)^m \gamma_m$. Differentiation identifies this coefficient with Z_{2m+1}, proving the formula. The Lean bridge is `KomorebiPaperIX.Zigzag.inverse_is_signed_zigzag`; it uses the formal generating-function definition of the zigzag numbers.

Theorem 3.18 (Classical odd-length reconstruction)**.** *For every positive odd integer p and every $n \in \mathbb{N}$,*

$$E_{2n} = \sum_{m=0}^{n} \binom{2n}{2m} \gamma_m p^{2m} S_{2(n-m)}(p).$$

Equivalently, $E_{2n} = \sum_{q=0}^{n} V_{nq} S_{2q}(p)$ for the lower triangular matrix

$$V_{nq} = \binom{2n}{2q} \gamma_{n-q} p^{2(n-q)} \quad (q \leq n), \qquad V_{nq} = 0 \quad (q > n).$$

It has $V_{nn} = 1$ and $p^{2(n-q)} \mid V_{nq}$ when $q \leq n$.

Proof. The exponential generating series of the source is the finite sum

$$\mathcal{S}_p(t) = \sum_{a=0}^{p-1} (-1)^a e^{(2a+1)t}.$$

Multiplication by $\cosh t$ telescopes. Since p is odd,

$$\mathcal{S}_p(t) = \frac{1 + e^{2pt}}{2} \operatorname{sech} t.$$

Taking its even part gives

$$\sum_{n \geq 0} S_{2n}(p) \frac{t^{2n}}{(2n)!} = \cosh^2(pt) \operatorname{sech} t.$$

Multiplication by the reciprocal $\operatorname{sech}^2(pt)$ yields $\operatorname{sech} t$. The EGF product rule now gives the reconstruction coefficient by coefficient. Replacing m by $n - q$ produces the displayed lower triangular row. Its diagonal and weights follow from $\gamma_0 = 1$ and the explicit factor $p^{2(n-q)}$. The source-series identity is `KomorebiPaperIX.OddPower.odd_source_even_egf`; the reconstruction and row statement are `KomorebiPaperIX.OddEuler.odd_euler_reconstruction` and `KomorebiPaperIX.OddEuler.row_action`. □

Corollary 3.19 (Odd-prime truncation ladder)**.** *For every positive odd integer p and all $M, n \in \mathbb{N}$,*

$$p^{2M+2} \mid E_{2n} - T^{\mathrm{odd}}_{p,M}(n) \quad \text{in } \mathbb{Z}.$$

If $M \geq n$, the truncation equals E_{2n} exactly. In particular, for an odd prime p this is a congruence modulo p^{2M+2} in $\mathbb{Z}_p$.

Proof. Every omitted summand has $m \geq M+1$, hence contains the integer factor p^{2m} divisible by p^{2M+2}. All its other factors are integral. The finite sum of these omitted terms proves the assertion, with the empty-tail case giving exact equality. These are `KomorebiPaperIX.OddEuler.odd_euler_supercongruence` and `KomorebiPaperIX.OddEuler.full_depth_exact`. □

The reconstruction and its truncation bound already hold over the integers. For the dyadic construction we use a different source. Its original bound of 2^{2M+2} follows from integral reciprocal coefficients, and the next subsection sharpens that bound to the optimal uniform exponent $4M+3$.

3.7 The optimal classical dyadic edge tower

Return now to the classical signed Euler numbers

$$\operatorname{sech} t = \frac{1}{\cosh t} = \sum_{n \geq 0} E_{2n} \frac{t^{2n}}{(2n)!}.$$

Thus $E_{2n} = E^{(2,0)}_{2n}$ in the convention (3.1). In particular $E_0 = 1$, $E_2 = -1$, and $E_4 = 5$. The following construction concerns these $N = 2$ coefficients, not $C_{4,0}$.

Definition 3.20 (Dyadic truncation)**.** Define β_m by the formal unit-series identity

$$\sum_{m \geq 0} \beta_m z^{2m} = \left(\frac{z}{\sinh z}\right)^2, \qquad \beta_0 = 1,$$

where the quotient means the inverse of the unit series $(\sinh z/z)^2$. Define

$$A(n) = \frac{3^{2n+2} - 1}{4(2n+1)(2n+2)}, \qquad S_m(n) = \beta_m 2^{2m} \frac{(2n)!}{(2n-2m)!} A(n-m) \quad (0 \leq m \leq n),$$

and

$$T_M(n) = \sum_{m=0}^{\min(M,n)} S_m(n) \qquad (M, n \in \mathbb{N}). \tag{3.13}$$

These definitions correspond to `KomorebiPaperIX.Dyadic.beta`, `KomorebiPaperIX.Dyadic.sourceEven`, and `KomorebiPaperIX.DyadicReconstruction.truncation`. The formal summand uses a falling factorial, and `KomorebiPaperIX.DyadicReconstruction.falling_factorial_fraction` proves its equality with the quotient displayed above.

Lemma 3.21 (Exact reconstruction and source integrality)**.** *For every $n \in \mathbb{N}$,*

$$E_{2n} = \sum_{m=0}^{n} S_m(n).$$

Every β_m is 2-adically integral and every $A(n)$ is a nonzero 2-adic unit. If $M \geq n$, then $T_M(n) = E_{2n}$.

Proof. The ordinary kernel coefficients of $(\sinh z/z)^2$, in z^2, are

$$\frac{2^{2m+1}}{(2m+2)!}.$$

Legendre's strict factorial bound $v_2((2m+2)!) < 2m+2$ makes them integral. Their constant coefficient is one. The coefficient recurrence for its reciprocal therefore proves integrality of every β_m. For the source factor, lifting the exponent at the even power $2n+2$ gives

$$v_2(3^{2n+2}-1) = 2 + v_2(2n+2).$$

Its denominator has the same valuation, since $2n+1$ is odd. Thus $A(n)$ is a nonzero unit.

Its exponential generating function is

$$\mathcal{A}(t) = \sum_{n\geq 0} A(n)\frac{t^{2n}}{(2n)!} = \frac{\cosh(3t)-\cosh t}{4t^2}.$$

The quotients by powers of t are well defined because the required initial coefficients vanish. The hyperbolic identity

$$\mathcal{A}(t)\cosh t = \left(\frac{\sinh(2t)}{2t}\right)^2$$

then gives the exact formal-series product

$$\left(\sum_{m\geq 0}\beta_m(2t)^{2m}\right)\mathcal{A}(t) = \operatorname{sech} t.$$

Taking the factorial-scaled coefficient of t^{2n} produces the stated finite sum. Full depth leaves no omitted terms. The relevant declarations are `KomorebiPaperIX.Dyadic.beta_integral`, `KomorebiPaperIX.Dyadic.source_valuation`, `KomorebiPaperIX.DyadicSeries.dyadic_reconstruction_series`, and `KomorebiPaperIX.DyadicReconstruction.dyadic_reconstruction`. □

Lemma 3.22 (Exact weighted inverse coefficient)**.** *Let $\kappa_m = \beta_m 2^{2m}(2m)!$. For every $m \geq 1$,*

$$\kappa_m = -2^{4m}(2m-1)B_{2m} \neq 0, \qquad v_2(\kappa_m) = 4m-1,$$

where B_{2m} is the usual even Bernoulli number.

Proof. Let $Q(z) = z\coth z$, the regular formal series with constant coefficient one. The elementary differential identity

$$\left(\frac{z}{\sinh z}\right)^2 = Q(z) - zQ'(z)$$

and the Bernoulli expansion

$$Q(z) = 1 + \sum_{m\geq 1}\frac{2^{2m}B_{2m}}{(2m)!}z^{2m}$$

give $\beta_m = -2^{2m}(2m-1)B_{2m}/(2m)!$. Multiplication by $2^{2m}(2m)!$ proves the identity for κ_m. The Bernoulli denominator theorem gives $v_2(B_{2m}) = -1$ for $m > 0$: its denominator contains each prime q with $(q-1) \mid 2m$ exactly once, including 2[3]. Since $2m-1$ is odd, the valuation is exactly $4m-1$. This also proves nonvanishing. In Lean, the hyperbolic identity is proved after clearing removable zeros. The library Bernoulli theorem then determines the valuation of κ_m. The declarations are `KomorebiPaperIX.DyadicSharpTower.kappa_bernoulli` and `KomorebiPaperIX.DyadicSharpTower.kappa_valuation`. □

Theorem 3.23 (Optimal uniform dyadic tower)**.** *For all $M, n \in \mathbb{N}$,*

$$E_{2n} - T_M(n) \in 2^{4M+3}\mathbb{Z}_2. \tag{3.14}$$

At the first omitted row $n = M + 1$, the error is nonzero and

$$v_2\big(E_{2M+2} - T_M(M+1)\big) = 4M + 3.$$

Thus the exponent $4M + 3$ cannot be increased uniformly in the Euler row.

Proof. Rewrite the summand exactly as

$$S_m(n) = \kappa_m \binom{2n}{2m} A(n-m).$$

For $m > M$, its first factor is divisible by 2^{4m-1}, the binomial coefficient is integral, and $A(n-m)$ is a unit. Hence every omitted summand is divisible by 2^{4M+3}. Summing the finite tail proves (3.14). If $M \geq n$, the tail is empty and the statement includes the resulting zero error.

At $n = M + 1$ there is exactly one omitted summand. Here $\binom{2n}{2n} = 1$ and $A(0) = 1$, so the error is κ_{M+1}. Lemma 3.22 gives its exact valuation $4M + 3$ and nonvanishing. The statements are `KomorebiPaperIX.DyadicSharpTower.dyadic_supercongruence_sharp`, `KomorebiPaperIX.DyadicSharpTower.first_omitted_error`, and `KomorebiPaperIX.DyadicSharpTower.first_omitted_error_valuation`. □

Proposition 3.24 (Row-dependent summand weights)**.** *For $1 \leq m \leq n$, the summand is nonzero and*

$$v_2(S_m(n)) = 4m - 1 + v_2\binom{2n}{2m}.$$

More generally, if $L \in \mathbb{N}$ satisfies

$$L \leq 4m - 1 + v_2\binom{2n}{2m} \qquad \text{for every } M < m \leq n,$$

then $E_{2n} - T_M(n) \in 2^L\mathbb{Z}_2$.

Proof. Use the same factorization of $S_m(n)$, the nonzero binomial coefficient for $m \leq n$, the exact valuation of κ_m, and the unit property of $A(n-m)$. The tail conclusion follows by addition in $2^L\mathbb{Z}_2$. These are `KomorebiPaperIX.DyadicSharpTower.summand_valuation` and `KomorebiPaperIX.DyadicSharpTower.dyadic_supercongruence_row_weight`. □

The minimum summand weight bounds the tail valuation from below, but cancellation may raise that valuation. Appendix A gives a later exact factorization and finite-polynomial reduction for each fixed depth. A universal explicit root and simplicity classification for that factor remains open. Uniform optimality follows from the equality already established at $n = M + 1$.

3.8 Isometric continuation of the four dyadic packets

The exact displacement law also determines the images of the packets after continuous extension to the dyadic integers.

Theorem 3.25 (Isometric packet bijections)**.** *For $0 \le j < 4$, define on natural numbers*

$$f_j(n) = \frac{U_j(n) - j!}{2^{d_j}}, \qquad (d_0, d_1, d_2, d_3) = (1, 1, 5, 3).$$

There is a unique continuous map $\Phi_j : \mathbb{Z}_2 \to \mathbb{Z}_2$ whose value at every $n \in \mathbb{N}$ equals the embedded rational $f_j(n)$. It is an isometric bijection and $\Phi_j(0) = 0$:

$$|\Phi_j(x) - \Phi_j(y)|_2 = |x - y|_2 \qquad (x, y \in \mathbb{Z}_2).$$

In particular, every element of $\mathbb{Z}_2$ has exactly one preimage.

Proof. For $n > 0$, the displacement law at row zero gives $v_2(U_j(n) - j!) = v_2(n) + d_j$, so $f_j(n) \in \mathbb{Z}_2$. The value at zero is zero. Applying the same law to the smaller of two unequal natural numbers shows

$$|f_j(n) - f_j(m)|_2 = |n - m|_2.$$

The natural numbers are dense in $\mathbb{Z}_2$, and the target is complete. The map therefore extends continuously. The distance identity extends on the dense product of natural-number points, making the extension an isometry. Density also proves uniqueness among all continuous maps with these prescribed values.

To prove surjectivity, use compactness. For an isometry g of a compact metric space into itself, the sequence of iterates $g^n(x)$ has a convergent subsequence. Two sufficiently close iterates, with exponents $a < b$, yield $d(x, g^{b-a}(x))$ equally small after removing a isometric iterations. Since $b - a > 0$, the latter point is in the image of g. Thus the image is dense. It is also compact and hence closed, so it is the whole space. Apply this to Φ_j on $\mathbb{Z}_2$.

The general ingredients are `KomorebiPaperIX.DensePadicIsometry.existsUnique_extensionEquiv` and `KomorebiPaperIX.CompactSelfIsometry.surjective`. The packet specialization is `KomorebiPaperIX.DyadicFourIsometry.existsUnique_isometry`; uniqueness among continuous maps is `KomorebiPaperIX.DyadicFourIsometry.unique_continuous`. □

Corollary 3.26 (Exact image of the original coefficients)**.** *The continuous extension of the original sequence is*

$$\widetilde{U}_j(x) = j! + 2^{d_j}\Phi_j(x).$$

It is injective, its image is exactly $j! + 2^{d_j}\mathbb{Z}_2$, and

$$|\widetilde{U}_j(x) - \widetilde{U}_j(y)|_2 = 2^{-d_j}|x - y|_2.$$

Every point of that affine image has exactly one preimage in $\mathbb{Z}_2$.

Proof. The formula agrees with U_j on every natural number. Continuity and the distance equality follow from the corresponding properties of Φ_j. Its surjectivity gives precisely the displayed affine image, and multiplication by the nonzero scalar 2^{d_j} preserves injectivity. The declarations are `KomorebiPaperIX.DyadicFourIsometry.originalExtension_nat`, `KomorebiPaperIX.DyadicFourIsometry.originalExtension_dist`, and `KomorebiPaperIX.DyadicFourIsometry.originalExtension_range`. □

The four rational packet sequences thus extend to bijections onto their exact dyadic images. The proof uses density, completeness, and the displacement law. The strengthening chapters supply the further factorial and Taylor-coefficient argument for analyticity, including exact radii in addresses zero, one and two. The analytic polynomial solution maps constructed later rest on separate hypotheses and a separate proof.

4 Threshold-free reciprocal transfer at prime-power section orders

The completion of Nishibuchi's conjecture fixes the requested eventual congruence. The stronger question is whether the sufficient starting row can be removed, and how much precision the difference order and the step actually provide. This section answers the first question at every prime-power section order. Subsequent sections give a factorial improvement and exact results at section order four.

Throughout these sections, the normalized source and its reciprocal are

$$G_{N,j}(t) = j! \sum_{n\ge 0} \frac{t^{Nn}}{(Nn+j)!}, \qquad F_{N,j}(t) = G_{N,j}(t)^{-1} = \sum_{n\ge 0} C_{N,j}(n) \frac{t^{Nn}}{(Nn)!}, \tag{4.1}$$

where $N \ge 1$ and $0 \le j < N$. In particular, $C_{N,j}(0) = 1$. The coefficients in the original convention satisfy

$$E^{(N,j)}_{Nn} = j! C_{N,j}(n), \qquad E^{(N,j)}_{d} = 0 \quad (N \nmid d). \tag{4.2}$$

Thus every valuation of a nonzero difference gains $v_p(j!)$ on returning to the original coefficients. This factor must be retained when stating optimal periods or the final congruence.

Write $Sf(n) = f(n+1)$ and $\Delta_H = S^H - 1$. A mixed difference $\Delta_{H_1} \cdots \Delta_{H_k}$ acts on the row variable n, rather than on the full Euler subscript. We use $v_p(0) = +\infty$. Orders and starting rows are nonnegative integers; steps are positive integers unless stated otherwise. An empty mixed difference is the original value, and a zero ordinary step makes every nonempty product containing it zero.

4.1 The Hurwitz lattice and translated differential flatness

Definition 4.1 (Hurwitz integral series)**.** For a prime p, let

$$\mathcal{H}_p = \left\{ \sum_{d\ge 0} f_d \frac{t^d}{d!} : f_d \in \mathbb{Z}_p \right\} \subset \mathbb{Q}_p[[t]].$$

For $N = p^a$ and $\lambda \in \mathbb{Z}_p$, put

$$D = \frac{d}{dt}, \qquad L_\lambda = \frac{D^N - \lambda}{p}.$$

A series A is λ-flat if $L_\lambda^k A \in \mathcal{H}_p$ for every $k \ge 0$.

Products preserve $\mathcal{H}_p$ because the coefficient at factorial degree d is

$$(AB)_d = \sum_{i=0}^{d} \binom{d}{i} a_i b_{d-i}.$$

The derivative D acts by the coefficient shift $(DA)_d = a_{d+1}$. Consequently every nonnegative power of D preserves the lattice. If $A \in \mathcal{H}_p$ has unit constant term, its reciprocal belongs to the same lattice: with $B = A^{-1}$,

$$b_0 = a_0^{-1}, \qquad b_d = -a_0^{-1} \sum_{i=1}^{d} \binom{d}{i} a_i b_{d-i}. \tag{4.3}$$

If A is supported at degrees divisible by N, this recurrence also proves that B has the same support.

Lemma 4.2 (Translated product formula)**.** *For $N = p^a$, $a \geq 1$, and $\lambda, \mu \in \mathbb{Z}_p$,*

$$L_{\lambda+\mu}(AB) = (L_\lambda A)B + A(L_\mu B) + \sum_{i=1}^{N-1} c_i (D^i A)(D^{N-i}B), \qquad c_i = \frac{1}{p}\binom{N}{i} \in \mathbb{Z}. \tag{4.4}$$

Flatness types add under multiplication.

Proof. The identity $(1+X)^{p^a} = 1 + X^{p^a}$ over $\mathbb{F}_p$ shows that every interior binomial coefficient $\binom{N}{i}$ is divisible by p. Subtract $(\lambda+\mu)AB$ from the ordinary Nth derivative product formula and divide by p to obtain (4.4).

For completeness, the iterated formula is an integral multinomial formula, rather than a product rule with unspecified remainders. On the tensor of the two factors use the mutually commuting operators

$$L_\lambda \otimes 1, \quad 1 \otimes L_\mu, \quad c_i D^i \otimes D^{N-i} \quad (1 \leq i < N).$$

Applying multiplication after the kth power of their sum gives exactly $L^k_{\lambda+\mu}(AB)$. Expanding it yields terms indexed by $a, b, e_1, \ldots, e_{N-1} \geq 0$, with $a + b + \sum_i e_i = k$:

$$\frac{k!}{a!b!\prod_i e_i!} \prod_i c_i^{e_i} (D^\alpha L^a_\lambda A)(D^\beta L^b_\mu B), \quad \alpha = \sum_i i e_i, \quad \beta = \sum_i (N-i)e_i. \tag{4.5}$$

Every scalar is integral. If A and B have the stated flatness types, both factors in every term belong to $\mathcal{H}_p$, as does their product. □

Lemma 4.3 (Reciprocal transfer)**.** *If A is λ-flat and has unit constant term, then A^{-1} is $(-\lambda)$-flat.*

Proof. Set $B = A^{-1}$. Its order-zero integrality was established in (4.3). Assume by induction that $L^b_{-\lambda}B \in \mathcal{H}_p$ for $b < k$. Since $AB = 1$,

$$L^k_0(AB) = \left(\frac{D^N}{p}\right)^k 1 = 0 \qquad (k > 0).$$

Use (4.5) with $\mu = -\lambda$. Exactly one term has inverse order $b = k$: it has $a = 0$ and all $e_i = 0$, and is $AL^k_{-\lambda}B$. In every other term $b < k$, so its second factor is integral by induction and derivative-shift integrality. Its first factor is integral by the flatness of A, and its scalar is integral. Hence $AL^k_{-\lambda}B \in \mathcal{H}_p$. Multiplying by the already integral B proves $L^k_{-\lambda}B \in \mathcal{H}_p$. This induction takes place in the entire Hurwitz lattice. It places no restriction on the coefficient degree or the starting row. □

4.2 The actual Euler source

The packed factorial coefficient of $G_{N,j}$ is

$$g_{N,j}(n) = \frac{j!(Nn)!}{(Nn+j)!} = \prod_{i=1}^{j} \left(1 + \frac{N}{i}n\right)^{-1}. \tag{4.6}$$

The empty product is one. If $N = p^a$ and $1 \leq i < N$, then $v_p(i) < a$, so $N/i \in p\mathbb{Z}_p$. Every affine denominator in (4.6) is therefore a unit on $\mathbb{Z}_p$.

For $s \in p\mathbb{Z}_p$, an exact finite-difference identity is

$$\Delta_1^k \frac{1}{1+sn} = \frac{(-s)^k k!}{\prod_{h=0}^{k}(1+s(n+h))}. \tag{4.7}$$

It follows by induction, subtracting adjacent rational functions; each successive subtraction adds the factor $-(k+1)s$ in the numerator. Its denominator is a unit, so it belongs to $p^k\mathbb{Z}_p$. The discrete Leibniz formula

$$\Delta_1^k(fg)(n) = \sum_{r=0}^{k} \binom{k}{r} (\Delta_1^r f)(n)(\Delta_1^{k-r} g)(n+r) \tag{4.8}$$

then proves $\Delta_1^k g_{N,j}(n) \in p^k\mathbb{Z}_p$ for every k, n. The coefficient at degree Nn of $L_1^k G_{N,j}$ is precisely $p^{-k}\Delta_1^k g_{N,j}(n)$, and the other residue classes vanish. Thus the actual source is 1-flat.

Theorem 4.4 (All-row prime-power transfer)**.** *Let p be prime, $a \geq 1$, $N = p^a$, and $0 \leq j < N$. The coefficients $C_{N,j}(n)$ belong to $\mathbb{Z}_p$ for all $n \geq 0$. Set $A_{N,j}(n) = (-1)^n C_{N,j}(n)$ and $T_H = S^H - (-1)^H$. For $k \geq 1$ and positive $H_1, \ldots, H_k$,*

$$v_p(\Delta_{H_1} \cdots \Delta_{H_k} A_{N,j}(n)) \geq k + \sum_{\nu=1}^{k} v_p(H_\nu), \tag{4.9}$$

$$v_p(T_{H_1} \cdots T_{H_k} E_{Nn}^{(N,j)}) \geq v_p(j!) + k + \sum_{\nu=1}^{k} v_p(H_\nu). \tag{4.10}$$

At $p = 2$, the ordinary differences of $C_{N,j}$ and of the original coefficients satisfy the same bounds. These dyadic statements also hold for $N = 1$.

Proof. Reciprocal transfer gives $L_{-1}^k F_{N,j} \in \mathcal{H}_p$. Extraction of the packed coefficient gives $(S+1)^k C_{N,j}(n) \in p^k\mathbb{Z}_p$. Moreover,

$$\Delta_1^k A_{N,j}(n) = (-1)^{n+k}(S+1)^k C_{N,j}(n),$$

so $A_{N,j}$ has the ordinary p^k difference property. At $p = 2$, $L_1 = L_{-1} - 1$; its integral binomial expansion on $F_{N,j}$ gives $\Delta_1^k C_{N,j}(n) \in 2^k\mathbb{Z}_2$ directly. For $N = 1$ the interior sum in the product formula is empty, so the same argument applies without change; equivalently, $F_{1,0}(t) = e^{-t}$.

To retain the valuation of a step, let B be any sequence satisfying $\Delta_1^m B \in p^m\mathbb{Z}_p$ at every order and row. Use the finite operator identity

$$\Delta_H = \sum_{i=1}^{H} \binom{H}{i} \Delta_1^i.$$

The relation $i\binom{H}{i} = H\binom{H-1}{i-1}$ implies

$$v_p\left(p^i \binom{H}{i}\right) \geq i + v_p(H) - v_p(i) \geq 1 + v_p(H).$$

Multiplying these identities for the k steps, every term of total order M has its scalar multiplied by p^M, and is divisible by $p^{k+\sum_\nu v_p(H_\nu)}$. This proves the mixed-step bound. Finally,

$$\Delta_H A_{N,j}(n) = (-1)^{n+H} T_H C_{N,j}(n).$$

Iteration contributes only a global sign. Multiplying by $j!$ proves (4.10). □

Corollary 4.5 (Threshold-free dyadic congruence)**.** *For $N = 2^a$, $a \geq 0$, all permitted j, and $n \geq 0$,*

$$v_2(\Delta_{H_1} \cdots \Delta_{H_k} E_{Nn}^{(N,j)}) \geq v_2(j!) + k + \sum_{\nu=1}^{k} v_2(H_\nu). \tag{4.11}$$

In particular, Δ_H^k gains $k(1+v_2(H))$ bits before normalization. For $r > v_2(j!)$, the row displacement $H = 2^{\max(0,r-v_2(j!)-1)}$ is a sufficient period modulo 2^r from row zero. If $r \le v_2(j!)$, all original coefficients vanish modulo 2^r.

Remark 4.6 (Necessary boundaries). At an odd prime the ordinary step-one difference cannot replace the sign-twisted operator in the general theorem. For $j = 0$, $C_{N,0}(0) = 1$ and $C_{N,0}(1) = -1$, so their ordinary difference is -2, a unit at every odd prime. The proof also does not extend unchanged to non-prime-power N, whose interior binomial coefficients need not be divisible by the chosen prime. Neither observation is a counterexample to Nishibuchi's original eventual congruence.

5 Factorial precision at section order four

Write $C_j = C_{4,j}$ and $F_j = F_{4,j}$. The all-row theorem removes the threshold, but at $N = 4$ it leaves a further factorial gain unused. The exact algebra below proves that gain for all four addresses and identifies several infinite families where it is sharp.

Theorem 5.1 (Four-address factorial bound)**.** *For $j \in \{0,1,2,3\}$, $n \ge 0$, $k \ge 1$, and positive mixed steps,*

$$v_2(\Delta_{H_1} \cdots \Delta_{H_k} C_j(n)) \ge k + v_2(k!) + \sum_{\nu=1}^{k} v_2(H_\nu) + \beta_j, \quad (\beta_0,\beta_1,\beta_2,\beta_3) = (0,0,3,1). \tag{5.1}$$

Address two also satisfies

$$v_2(\Delta_{H_1} \cdots \Delta_{H_k} C_2(n)) \ge 2k + v_2(k!) + \left\lfloor \frac{k}{2} \right\rfloor + \sum_{\nu=1}^{k} v_2(H_\nu). \tag{5.2}$$

For the original coefficients, the offsets in (5.1) *are* $(0,0,4,2)$*. No positive-order offset is asserted for the empty difference.*

The proof of the first two addresses uses one classical arithmetic input: Frobenius's exact congruence for the ordinary Euler numbers in the sech convention, as stated and derived by Gessel [2], Section 9. The address-two and address-three bounds below are finite Bernoulli calculations and do not require that input. A divided-power argument is necessary: ordinary integral moments alone do not justify division by $k!$.

5.1 Euler moments and an integral divided-power algebra

Let $\mathcal{L}$ be the ordinary Euler evaluation functional on a variable $\mathcal{E}$, defined by

$$\mathcal{L}(\mathcal{E}^d) = \mathrm{Euler}_d, \qquad \sum_{d\ge 0} \mathrm{Euler}_d \frac{t^d}{d!} = \operatorname{sech} t.$$

The functional is linear; it is not a multiplicative evaluation. Independent variables mean that moments of polynomials in separate variables factor. Odd moments vanish. Frobenius's congruence is

$$v_2\left(\sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} \mathrm{Euler}_{2n+2r}\right) = m + v_2(m!) \qquad (n,m \ge 0). \tag{5.3}$$

For $m = 0$, the assertion is that every even Euler number is odd. With $P = (\mathcal{E}^2 - 1)/2$ and $\gamma_m(P) = P^m/m!$, the case $n = 0$ says

$$\mathcal{L}(\gamma_m(P)) \in \mathbb{Z}_2^\times \qquad (m \ge 0). \tag{5.4}$$

Introduce an independent Euler variable $\mathcal{F}$ and $Q = (\mathcal{F}^2 - 1)/2$.

Put $K = \mathbb{Q}_2(i)$, $R = \mathbb{Z}_2[i]$, $a = (1+i)/2$, and $b = (1-i)/2$. The exact identity

$$F_0(t) = \frac{2}{\cosh t + \cos t} = \operatorname{sech}(at)\operatorname{sech}(bt) \tag{5.5}$$

gives, for $U = a\mathcal{E} + b\mathcal{F}$, $\mathcal{L}(U^{4n}) = C_0(n)$. Although a and b have negative valuation, their denominators disappear from the following exact polynomial identities:

$$\begin{aligned} U^2 &= \mathcal{E}\mathcal{F} + i(P - Q), \\ H_0 := \frac{U^4 - 1}{2} &= P + Q - \frac{P^2}{2} - \frac{Q^2}{2} + 3PQ + i\mathcal{E}\mathcal{F}(P - Q). \end{aligned} \tag{5.6}$$

Let $\Gamma_R(P, Q)$ be the R-subalgebra generated by all $\gamma_m(P)$ and $\gamma_r(Q)$. Adjoin $\mathcal{E}, \mathcal{F}$ with

$$\mathcal{E}^2 = 1 + 2P, \qquad \mathcal{F}^2 = 1 + 2Q,$$

and call the resulting algebra $\mathcal{A}_0$. Its elements are finite expressions; no completion is being presumed. It is generated as a $\Gamma_R(P, Q)$-module by $1, \mathcal{E}, \mathcal{F}, \mathcal{E}\mathcal{F}$. The functional maps this module to R. For an even basis term this is (5.4) and independence. For a basis term with an odd Euler factor the moment is zero, since P and Q are even polynomials in their respective variables.

Every displayed component of H_0 has all its divided powers in $\mathcal{A}_0$. Besides the generators, the needed identities are

$$\begin{aligned} \gamma_m(P^2/2) &= \frac{(2m)!}{2^m m!}\gamma_{2m}(P), \\ \gamma_m(3PQ) &= 3^m m!\gamma_m(P)\gamma_m(Q), \\ \gamma_m(P - Q) &= \sum_{r=0}^{m} (-1)^r \gamma_{m-r}(P)\gamma_r(Q). \end{aligned} \tag{5.7}$$

The first scalar counts partitions into pairs and is an integer. For the last term of H_0, multiply the third identity by $(i\mathcal{E}\mathcal{F})^m$ and reduce the Euler powers using the integral quadratic relations. Finally,

$$\gamma_k(X + Y) = \sum_{r=0}^{k} \gamma_r(X)\gamma_{k-r}(Y)$$

proves $\gamma_k(H_0) \in \mathcal{A}_0$. Applying the functional and putting $W = U^4 - 1 = 2H_0$ gives

$$\mathcal{L}(W^m) \in 2^m m! R \qquad (m \geq 0). \tag{5.8}$$

These moments are rational. The identity $R \cap \mathbb{Q}_2 = \mathbb{Z}_2$, obtained from the linear independence of $1, i$ over $\mathbb{Q}_2$, descends every such divisibility statement to $\mathbb{Z}_2$. No field property of $R/2R$ is needed.

For every starting row,

$$\Delta_1^k C_0(n) = \mathcal{L}(U^{4n} W^k) = \sum_{s=0}^{n} \binom{n}{s} \mathcal{L}(W^{k+s}). \tag{5.9}$$

Each summand lies in $2^k k! \mathbb{Z}_2$, proving the factorial bound for address zero from row zero onward.

5.2 A finite step-expansion lemma

The conversion from unit steps to arbitrary mixed steps is used several times, so it is useful to make it explicit. Write $\Delta = \Delta_1$. For $H > 0$,

$$\Delta_H = \sum_{m=1}^{H} \binom{H}{m} \Delta^m. \tag{5.10}$$

A mixed term with local orders $m_1, \dots, m_k \geq 1$ and $M = \sum_\nu m_\nu$ therefore has valuation at least

$$M + v_2(M!) + \beta + \sum_\nu \bigl(v_2(H_\nu) - v_2(m_\nu)\bigr)$$

if the unit-order estimate has offset β for every positive order. The inequalities

$$m - 1 \geq v_2(m), \qquad M \geq k, \qquad v_2(M!) \geq v_2(k!)$$

make this at least $k + v_2(k!) + \beta + \sum_\nu v_2(H_\nu)$. Every expansion is finite. This proves the desired mixed-step transfer without an exchange of limits.

5.3 An elementary Bernoulli denominator bound

Let B_m denote the Bernoulli numbers with $B_1 = -1/2$. The identity obtained by substituting $u = e^t - 1$ into $\log(1+u)/u$ is

$$B_m = \sum_{r=0}^{m} (-1)^r \frac{r!}{r+1} \left\{ {m \atop r} \right\}, \tag{5.11}$$

where the braces are Stirling numbers of the second kind. Each $2r!/(r+1)$ is dyadically integral. Indeed, if $s = v_2(r+1) > 0$, then $r \geq 2^s - 1$ and

$$v_2(r!) \geq v_2((2^s-1)!) = 2^s - 1 - s \geq s - 1.$$

The last inequality is $2^s \geq 2s$; the case $s = 0$ is immediate. Consequently

$$2B_m \in \mathbb{Z}_2 \qquad (m \geq 0). \tag{5.12}$$

Thus the transfer factors below require no appeal to the von Staudt–Clausen theorem.

5.4 Address one and the scaled-binomial transfer

The address-one source is $G_{4,1}(t) = (\sinh t + \sin t)/(2t)$. The identities $a + b = 1$, $a - b = i$, and $ab = 1/2$ give

$$D_1(t) := G_{4,1}(t) F_0(t) = \frac{1}{2} \left(\frac{\tanh(at)}{at} + \frac{\tanh(bt)}{bt} \right). \tag{5.13}$$

For example, its numerator follows from

$$\sinh t + \sin t = 2b \sinh(at) \cosh(bt) + 2a \cosh(at) \sinh(bt),$$

while $\cosh t + \cos t = 2\cosh(at)\cosh(bt)$. Only degrees divisible by four survive. Its actual Hurwitz packets are

$$D_1(t) = \sum_{i \geq 0} d_i \frac{t^{4i}}{(4i)!}, \qquad d_i = \left(-\frac{1}{4} \right)^i \frac{2^{4i+2}(2^{4i+2} - 1) B_{4i+2}}{(4i+2)(4i+1)}. \tag{5.14}$$

The constant is one. Since $v_2(4i+2) = 1$, the elementary Bernoulli bound shows $v_2(d_i) \geq 2i$. Hurwitz inversion therefore gives

$$D_1(t)^{-1} = \sum_{i \geq 0} h_i \frac{t^{4i}}{(4i)!}, \qquad h_0 = 1, \quad v_2(h_i) \geq 2i.$$

This follows by induction from the inverse recurrence: for packet index i, every product $d_r h_{i-r}$ has valuation at least $2i$, and its binomial multiplier is integral. Because $F_1 = F_0 D_1^{-1}$,

$$C_1(n) - C_0(n) = \sum_{i \geq 1} h_i \binom{4n}{4i} C_0(n-i). \tag{5.15}$$

At a natural n this is a finite sum, with $i \leq n$.

Here is the polynomial estimate that supplies one additional bit. For $i \geq 1$, put $P_i(x) = 4^i \binom{4x}{4i}$. Its constant is zero, and its degree-d coefficient is

$$[x^d] P_i(x) = \frac{4^i 4^d s(4i, d)}{(4i)!}, \tag{5.16}$$

where $s(m, d)$ is a signed Stirling number of the first kind. In each product contributing to $s(4i, d)$, at most $d-1$ of the $2i-1$ even integers in $1, \ldots, 4i-1$ can be omitted. Thus

$$v_2(s(4i, d)) \geq \max(0, 2i - d).$$

Writing $s_2(i)$ for the number of ones in the binary expansion of i, and using $v_2((4i)!) = 4i - s_2(i)$, gives

$$v_2([x^d] P_i) \geq 2d - 2i + s_2(i) + \max(0, 2i - d) \geq d + s_2(i) \geq d + 1 \quad (d \geq 1). \tag{5.17}$$

For $k \geq 1$, $\Delta_1^k x^d$ vanishes when $d < k$ and is an integer polynomial with values divisible by $k!$. To see the latter, expand $x^d = \sum_r \left\{ {d \atop r} \right\} r! \binom{x}{r}$ and apply $\Delta_1^k \binom{x}{r} = \binom{x}{r-k}$. It follows that

$$\Delta_1^k P_i(x) \in 2^{k+1} k! \mathbb{Z}_2 \qquad (k \geq 0,\ x \in \mathbb{Z}_2). \tag{5.18}$$

For $k = 0$ the bound follows directly from the coefficient estimate.

To apply discrete products in (5.15) without a moving truncation, first extend C_0 to $\mathbb{Z}_2$ by its uniformly convergent Newton series. The all-row bound $v_2(\Delta_1^m C_0(0)) \geq m$ proves this convergence. Its factorial bounds extend to every dyadic starting point by continuity and approximation with natural rows. The sum in (5.15) is then uniformly convergent on $\mathbb{Z}_2$: both the binomial polynomial and $C_0(x-i)$ are integral-valued and $v_2(h_i) \geq 2i$.

For the pointwise product $P_i(x) C_0(x-i)$, the discrete Leibniz formula has scalar divisibility

$$\binom{k}{r} (2^{r+1} r!)(2^{k-r}(k-r)!) = 2^{k+1} k!.$$

Multiplication by $h_i / 4^i \in \mathbb{Z}_2$ and uniform summation now give

$$v_2(\Delta_1^k (C_1 - C_0)(n)) \geq k + v_2(k!) + 1 \qquad (k, n \geq 0). \tag{5.19}$$

The same extra bit holds for positive mixed steps by (5.10). Combining this with the address-zero bound proves Theorem 5.1 for address one.

5.5 Central factorials and integral Bernoulli divided powers

Lemma 5.2 (Bernoulli divided-power moments)**.** *Let Z be the ordinary Bernoulli variable, $\mathcal{L}(Z^r) = B_r$, with exponential transform $t/(e^t-1)$. For $P = Z(Z+1)$ and $\mu_m = \mathcal{L}(P^m/m!)$,*

$$\mu_m \in \mathbb{Z}_2, \qquad v_2(\mu_m) \geq s_2(m) - 1 \quad (m \geq 1). \tag{5.20}$$

Moreover, μ_m is odd exactly when $m = 0$ or m is a positive power of two.

Proof. Define the central-factorial basis polynomials

$$P_r(X) = \prod_{a=0}^{r-1} (X - a(a+1)).$$

At $X = Z(Z+1)$, they have the exact factorization

$$P_r(Z(Z+1)) = \prod_{a=0}^{r-1} (Z-a)(Z+a+1) = (2r)! \binom{Z+r}{2r}. \tag{5.21}$$

The Bernoulli transform gives $\mathcal{L}((1+T)^Z) = \log(1+T)/T$, whence $\mathcal{L}(\binom{Z}{j}) = (-1)^j/(j+1)$. Finite Vandermonde expansion therefore gives

$$\begin{aligned} \mathcal{L}\left(\binom{Z+r}{2r}\right) &= (-1)^r \sum_{q=0}^{r} (-1)^q \binom{r}{q} \frac{1}{r+q+1} \\ &= (-1)^r \frac{(r!)^2}{(2r+1)!}. \end{aligned} \tag{5.22}$$

The last equality is the rational identity obtained by expanding $\int_0^1 t^r (1-t)^r \, dt$. This use of an ordinary finite integral makes no assertion about a positive or bounded p-adic measure. In particular,

$$\mathcal{L}(P_r(P)) = (-1)^r \frac{(r!)^2}{2r+1}.$$

Expand $X^m = \sum_{r=0}^{m} T(m,r) P_r(X)$. Since $XP_r(X) = P_{r+1}(X) + r(r+1)P_r(X)$, these integer coefficients satisfy

$$T(m+1,r) = T(m,r-1) + r(r+1)T(m,r), \quad T(0,0) = 1. \tag{5.23}$$

Every $r(r+1)$ is even. Induction, including the boundary indices, proves $2^{m-r} \mid T(m,r)$ for $0 \leq r \leq m$. For $m > 0$ the $r = 0$ term vanishes. Each remaining summand of $\mathcal{L}(P^m/m!)$ has valuation at least

$$m - r + 2v_2(r!) - v_2(m!) = r + s_2(m) - 2s_2(r) \geq s_2(m) - 1 \geq 0. \tag{5.24}$$

Here $r \geq 2s_2(r) - 1$ follows from $r \geq 2^{s_2(r)} - 1$ and $2^u \geq 2u$ for $u \geq 1$. This proves integrality and the stronger bound.

If m is not a power of two, $s_2(m) \geq 2$ makes μ_m even. If m is a positive power of two, the only central indices which can contribute an odd residue are $r = 1$ and $r = 3$, since $r - 2s_2(r) = -1$ only at these two positive integers. The $r = 1$ term is odd: $T(m,1) = 2^{m-1}$ and $v_2(m!) = m - 1$. For $m \geq 4$ a power of two, the $r = 3$ term is even. More explicitly,

$$\frac{T(m,3)}{2^{m-3}} = h_{m-3}(1,3,6), \qquad \sum_{d \geq 0} h_d(1,3,6) t^d = \frac{1}{(1-t)(1-3t)(1-6t)}.$$

Reduction modulo two gives $(1-t)^{-2}$, whose degree $m-3$ coefficient has parity $m - 2 = 0$. At $m = 1, 2$ the $r = 3$ term is absent, and the $r = 1$ term is again odd. Finally $\mu_0 = 1$. □

5.6 Address two and its stronger filtration

The variable $\mathcal{B} = 2Z + 1$ has transform

$$e^t \frac{2t}{e^{2t} - 1} = \frac{t}{\sinh t}.$$

Its odd moments vanish, and $P = (\mathcal{B}^2 - 1)/4 = Z(Z + 1)$ has the divided-power moments of Lemma 5.2. Introduce an independent copy $\mathcal{D}$ and $Q = (\mathcal{D}^2 - 1)/4$. The exact factorization

$$F_2(t) = \frac{t^2}{\cosh t - \cos t} = \frac{at}{\sinh(at)} \frac{bt}{\sinh(bt)} \tag{5.25}$$

shows that $U = a\mathcal{B} + b\mathcal{D}$ satisfies $\mathcal{L}(U^{4n}) = C_2(n)$. Direct expansion gives

$$H_2 := \frac{U^4 - 1}{4} = P + Q - P^2 - Q^2 + 6PQ + i\mathcal{B}\mathcal{D}(P - Q). \tag{5.26}$$

Let $\mathcal{A}_2$ be the divided-power algebra over R generated by P, Q, adjoining $\mathcal{B}, \mathcal{D}$ with $\mathcal{B}^2 = 1 + 4P$, $\mathcal{D}^2 = 1 + 4Q$. The functional is integral on its module basis $1, \mathcal{B}, \mathcal{D}, \mathcal{B}\mathcal{D}$ by the Bernoulli lemma, independence, and vanishing of odd moments. Every component in (5.26) has integral divided powers, by the same finite identities as before; P^2 is even easier than $P^2/2$. Consequently

$$\mathcal{L}((U^4 - 1)^m) \in 4^m m! \mathbb{Z}_2, \qquad v_2(\Delta_1^k C_2(n)) \geq 2k + v_2(k!). \tag{5.27}$$

The starting-row conclusion follows from the finite base expansion (5.9), now with $W = 4H_2$.

The low Newton coefficients of the actual reciprocal are

$$C_2(0) = 1, \quad C_2(1) = -\frac{1}{15}, \quad C_2(2) = \frac{13}{45}, \qquad a_1 = -\frac{16}{15}, \quad a_2 = \frac{64}{45}, \tag{5.28}$$

where $a_m = \Delta_1^m C_2(0)$. Thus $v_2(a_1) = 4$ and $v_2(a_2) = 6$. For $m \geq 3$, (5.27) bounds a_m by $2m + v_2(m!)$. The finite shifted Newton formula

$$\Delta_1^k C_2(n) = \sum_{m=k}^{n+k} a_m \binom{n}{m-k} \tag{5.29}$$

therefore gives $v_2(\Delta_1^k C_2(n)) \geq k + v_2(k!) + 3$ for every $k \geq 1$. For $k \geq 3$ this is $2k \geq k + 3$; for $k = 1, 2$ it uses the exact initial coefficients and the higher tail bound. The finite step expansion proves the offset three in (5.1).

There is a still stronger filtration. Put

$$X = P + Q, \qquad \tau = i\mathcal{B}\mathcal{D}, \qquad \epsilon = 1 + \tau.$$

Then the exact identities are

$$H_2 = \epsilon X + 2G, \quad G = -\gamma_2(P) - \gamma_2(Q) + 3PQ - \tau Q, \quad \epsilon^2 = 2V, \quad V = \tau - 2X - 8PQ. \tag{5.30}$$

Every divided power of G is integral component by component. The finite addition identity gives

$$\gamma_k(H_2) = \sum_{r=0}^{k} 2^r \gamma_r(G) \epsilon^{k-r} \gamma_{k-r}(X) \in 2^{\lfloor k/2 \rfloor} \mathcal{A}_2, \tag{5.31}$$

because the rth term belongs to $2^{r+\lfloor (k-r)/2 \rfloor}\mathcal{A}_2$. Applying the functional, rational descent, and the base-index expansion proves the unit-step version of (5.2). For a mixed term of total order M, its lower bound exceeds the target at order k by at least

$$2(M-k) + \left\lfloor \frac{M}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor + v_2(M!) - v_2(k!) - \sum_{\nu} v_2(m_\nu) \geq 0.$$

Indeed $\sum_\nu v_2(m_\nu) \leq M-k$. This proves the stronger mixed bound. The earlier offset-three bound is retained, since it is stronger at the lowest orders.

5.7 Address three and cancellation of the apparent pole

The address-three source is $G_{4,3}(t) = 3(\sinh t - \sin t)/t^3$. Addition identities give

$$\sinh t - \sin t = 2a\sinh(at)\cosh(bt) + 2b\cosh(at)\sinh(bt), \qquad \cosh t - \cos t = 2\sinh(at)\sinh(bt).$$

Consequently

$$D_3(t) := G_{4,3}(t)F_2(t) = \frac{3}{2}\left(\frac{\coth(at)}{at} + \frac{\coth(bt)}{bt}\right). \tag{5.32}$$

This expression is a power series: its two polar terms cancel because $a^{-2} + b^{-2} = 0$. Its constant is one, only degrees divisible by four survive, and the Bernoulli expansion gives

$$D_3(t) = \sum_{i\geq 0} d_i \frac{t^{4i}}{(4i)!}, \qquad d_i = 3\left(-\frac{1}{4}\right)^i \frac{2^{4i+2}B_{4i+2}}{(4i+2)(4i+1)}. \tag{5.33}$$

Again $v_2(d_i) \geq 2i$, and the inverse coefficients h_i have the same bound. Since $F_3 = F_2 D_3^{-1}$,

$$C_3(n) - C_2(n) = \sum_{i\geq 1} h_i \binom{4n}{4i} C_2(n-i). \tag{5.34}$$

The scaled-binomial argument (5.18) applies unchanged. The basic bound for C_2 includes order zero; its continuous extension is available from the threshold-free theorem. Thus the shifted sums converge uniformly on $\mathbb{Z}_2$, and the pointwise product rule proves

$$v_2(\Delta_1^k(C_3 - C_2)(n)) \geq k + v_2(k!) + 1 \qquad (k, n \geq 0). \tag{5.35}$$

At every positive order C_2 has three extra bits, so this gives the one-bit offset for C_3. Finite step expansion finishes the proof of Theorem 5.1.

6 Sharp higher-difference families

The lower bounds do not mean that every mixed stencil has exactly the stated valuation. Addresses zero and one have a complete classification of when the leading factorial precision is attained. Address two has exact mixed precision at every positive power-of-two order.

Theorem 6.1 (Exact factorial precision at addresses zero and one)**.** *Let $j \in \{0,1\}$, $k \geq 1$, $n \geq 0$, and $H_1, \ldots, H_k > 0$. Put*

$$L = k + v_2(k!) + \sum_{\nu=1}^{k} v_2(H_\nu), \qquad e = \#\{\nu : 2 \mid H_\nu\}.$$

The mixed difference has valuation exactly L if k is odd or e is even. If k is even and e is odd, it has valuation at least $L+1$. In particular, for every homogeneous positive step,

$$v_2(\Delta_H^k C_j(n)) = k + v_2(k!) + kv_2(H). \tag{6.1}$$

Proof. Return to the Euler variables of address zero and put $X = P + Q$, $\tau = i\mathcal{E}\mathcal{F}$, $\epsilon = 1 + \tau$. Direct rearrangement of (5.6) gives

$$H_0 = \gamma_2(X) + \epsilon X + 2G, \qquad G = -\gamma_2(P) - \gamma_2(Q) + PQ - \tau Q, \qquad \epsilon^2 = 2(\tau - P - Q - 2PQ). \quad (6.2)$$

The divided powers of G are integral by the same component formulas used above. Thus modulo $2\mathcal{A}_0$, $\gamma_k(H_0)$ equals the divided power of $\gamma_2(X) + \epsilon X$. The identity

$$\gamma_r(\gamma_2(X)) = \frac{(2r)!}{2^r r!}\gamma_{2r}(X)$$

has an odd integer scalar. Every term involving ϵ^j for $j \geq 2$ vanishes modulo two. The surviving zero and first powers give

$$\gamma_k(H_0) \equiv \gamma_{2k}(X) + \epsilon\gamma_{2k-1}(X) \pmod{2\mathcal{A}_0}. \quad (6.3)$$

The multiplication coefficient in the second term is $2k - 1$, also odd. By the odd moments (5.4) and independence,

$$\mathcal{L}(\gamma_m(P + Q)) \equiv m + 1 \pmod 2.$$

The τ-weighted moment vanishes because it contains an odd power of each Euler variable. Therefore the first term of (6.3) evaluates to one modulo two and the second to zero. Hence $\mathcal{L}(\gamma_k(H_0))$ is odd at every order, including order zero. This proves

$$v_2(\Delta_1^k C_0(0)) = k + v_2(k!).$$

In the base-row expansion (5.9), every $s > 0$ term has at least one extra bit after division by $2^k k!$. Thus the same exact valuation holds at every starting row. The extra-bit transfer (5.19) proves it for address one as well.

It remains to track possible ties in the mixed-step expansion. For local orders $m_\nu \geq 1$ and total M, the excess over L is at least

$$\left(M - k - \sum_\nu v_2(m_\nu)\right) + \left(v_2(M!) - v_2(k!)\right). \quad (6.4)$$

The first bracket is nonnegative, and equality requires every m_ν to be one or two. Two or more occurrences of two make the second bracket positive, since the intervening factorial interval then contains an even number. Exactly one occurrence of two can tie only when k is even; its binomial coefficient can have the minimally allowed valuation only if its associated step is even.

The all-one multi-index contributes one after normalization by 2^L. When k is even, each even step supplies exactly one further tied term, again with normalized residue one. For example, $\binom{H}{2} = H(H-1)/2$ attains $v_2(H) - 1$ exactly when H is even; all remaining factors and the exact normalized order-$k+1$ difference are odd. Thus the total normalized parity is one for odd k, and $1 + e$ for even k. This proves the classification. In the homogeneous case, e is zero or k, so it is even whenever k is even, proving (6.1). □

Remark 6.2 (A mixed cancellation). The exceptional class can gain more than one bit. The exact values

$$\Delta_1\Delta_2 C_0(0) = -33728, \qquad \Delta_1\Delta_2 C_1(0) = -\frac{150208}{585}$$

both have valuation six, whereas their baseline L is four. The theorem classifies attainment of the baseline; it does not give a complete deeper valuation in the cancellation class.

Theorem 6.3 (Exact address-two power orders)**.** *For every starting row $n \geq 0$ and all positive mixed steps,*

$$v_2(\Delta_{H_1} \cdots \Delta_{H_k} C_2(n)) = V_2(k) + \sum_{\nu=1}^{k} v_2(H_\nu) \tag{6.5}$$

whenever k is a positive power of two, where

$$V_2(1) = 4, \qquad V_2(2) = 6, \qquad V_2(k) = 2k + v_2(k!) + \frac{k}{2} \quad (k \geq 4).$$

For the original coefficients, the right side gains one bit.

Proof. For $k = 2^r \geq 4$, put $q = k/2$, which is even. Normalize the exact expansion (5.31) by 2^q before reducing modulo two. Its uth auxiliary term has excess $u + \lfloor (k-u)/2 \rfloor - q = \lfloor u/2 \rfloor$. Only auxiliary indices zero and one can therefore survive:

$$\frac{\gamma_k(H_2)}{2^q} \equiv V^q \gamma_k(X) + \epsilon V^{q-1} G \gamma_{k-1}(X) \pmod{2\mathcal{A}_2}. \tag{6.6}$$

Here $V \equiv \tau \pmod 2$ and $\tau^2 \equiv 1 \pmod 2$. Since q is even and $\epsilon\tau \equiv \epsilon \pmod 2$, this reduces to $\gamma_k(X) + \epsilon G \gamma_{k-1}(X)$. Moreover, $G \equiv \gamma_2(X) + \tau Q \pmod 2$. After evaluation, every term with a single τ vanishes, so the second contribution reduces to

$$\mathcal{L}\big((\gamma_2(X) + Q)\gamma_{k-1}(X)\big).$$

The $\gamma_2(X)$ product has scalar $\binom{k+1}{2}$, which is even because $4 \mid k$. The other contribution expands as

$$\mathcal{L}(Q\gamma_{k-1}(P+Q)) = \sum_{a=0}^{k-1} (k-a)\mu_a \mu_{k-a}. \tag{6.7}$$

An odd summand would require $k-a$ to be odd and μ_{k-a} to be odd. By Lemma 5.2, this requires $k-a = 1$. Then $a = k-1$ is not a power of two and μ_a is even. Hence the second contribution is zero modulo two.

For the first contribution, independence gives $\mathcal{L}(\gamma_k(P+Q)) = \sum_{a=0}^{k} \mu_a \mu_{k-a}$. The parity generating series of these convolution moments is

$$(1 + t + t^2 + t^4 + t^8 + \cdots)^2 = 1 + t^2 + t^4 + t^8 + \cdots \quad \text{in } \mathbb{F}_2[[t]].$$

Its coefficient is one at every $k = 2^r \geq 4$. Thus the normalized moment in (6.6) is odd and

$$v_2(\Delta_1^k C_2(0)) = 2k + v_2(k!) + k/2.$$

Inserting $U^{4n} = (1 + 4H_2)^n$ introduces only corrections with strictly higher valuation, by the filtered bound at orders $k + s$ for $s > 0$. The equality consequently holds at every starting row.

For mixed steps the all-one local multi-index gives the claimed leading term. A higher term of total order $M > k$ has excess at least

$$2(M-k) + \left\lfloor \frac{M}{2} \right\rfloor - \frac{k}{2} + v_2(M!) - v_2(k!) - \sum_{\nu} v_2(m_\nu) \geq M - k \geq 1.$$

Hence the leading term cannot cancel.

At order one, a_1 in (5.28) has valuation four and the Newton tail starts at valuation six; its shifted value remains exact. In the step expansion a local order two can lose at most one bit through its binomial coefficient, so every correction is still above $4 + v_2(H_1)$. At order two, a_2 has valuation six and the next coefficient has lower bound eight from the stronger filtration. A total-order-three correction loses at most one bit, so it stays above the leading valuation six, while later orders gain still more. The same argument proves exactness for both mixed steps. □

Remark 6.4 (What the power orders do not decide). The exact orders in Theorem 6.3 form an infinite family. They do not supply a complete valuation formula at non-power-of-two orders in address two, nor an exact higher-difference formula at address three. Numerical patterns at other orders remain bounded observations unless proved separately.

7 Exact first differences, least periods, and dyadic images

The factorial estimates are stronger than needed for the optimal first-difference law. A general Newton criterion gives the exact law from the threshold-free bound and finitely many rational coefficients. This route also yields all addresses at ranks two and eight, and an infinite family at every larger dyadic rank.

Lemma 7.1 (A first-difference criterion). *Let $C : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_2$ have Newton coefficients $a_k = \Delta_1^k C(0)$. Suppose for an integer $b \geq 0$ that*

$$v_2(a_1) = b, \qquad v_2(a_k) \geq b + 1 + \lfloor \log_2 k \rfloor \quad (k \geq 2). \tag{7.1}$$

Then for every $n \geq 0$ and $H > 0$,

$$v_2(C(n+H) - C(n)) = b + v_2(H). \tag{7.2}$$

The Newton interpolation extends continuously to $\mathbb{Z}_2$ and satisfies the same identity for any two distinct dyadic arguments.

Proof. At a natural argument the exact Newton expansion is finite. Vandermonde gives

$$\binom{n+H}{k} - \binom{n}{k} = \sum_{i=1}^{k} \binom{H}{i}\binom{n}{k-i}.$$

For $1 \leq i \leq H$, $v_2(\binom{H}{i}) \geq v_2(H) - v_2(i) \geq v_2(H) - \lfloor \log_2 k \rfloor$; terms with $i > H$ vanish. Thus each Newton summand with $k \geq 2$ has valuation at least $b + v_2(H) + 1$. The $k = 1$ summand is Ha_1, of exact valuation $b + v_2(H)$, proving (7.2).

The coefficient hypothesis implies $a_k \to 0$ dyadically, so the Newton series converges uniformly on $\mathbb{Z}_2$, where $\binom{x}{k}$ is integral-valued. For distinct $x, y \in \mathbb{Z}_2$, approximate the pair by natural integer pairs with their difference valuation eventually equal to $v_2(x-y)$. The corresponding function differences have the fixed exact valuation $b + v_2(x-y)$. A limit of elements with that fixed valuation has the same valuation: the normalized residues are units, and $\mathbb{Z}_2^\times$ is closed. □

Theorem 7.2 (Exact four-address first differences). *For every $n \geq 0$, $H > 0$, and $j \in \{0,1,2,3\}$,*

$$v_2(C_j(n+H) - C_j(n)) = v_2(H) + b_j, \qquad (b_0, b_1, b_2, b_3) = (1,1,4,2), \tag{7.3}$$

$$v_2(E^{(4,j)}_{4(n+H)} - E^{(4,j)}_{4n}) = v_2(H) + B_j, \qquad (B_0, B_1, B_2, B_3) = (1,1,5,3). \tag{7.4}$$

Consequently, the least positive row period modulo 2^r is exactly

$$P_j(r) = 2^{\max(0,r-B_j)} \qquad (r \geq 1). \tag{7.5}$$

The corresponding displacement in the original Euler subscript is $4P_j(r)$.

Proof. The actual reciprocal recurrence is

$$C_{N,j}(n) = -\sum_{q=1}^{n} \binom{Nn}{Nq} g_{N,j}(q) C_{N,j}(n-q), \qquad C_{N,j}(0) = 1. \tag{7.6}$$

For $N = 4$, the four kernels are

$$1, \qquad \frac{1}{4q+1}, \qquad \frac{1}{(4q+1)(2q+1)}, \qquad \frac{3}{(4q+1)(2q+1)(4q+3)}.$$

The threshold-free theorem gives $v_2(a_k) \geq k$ in every address. The following exact values cover all orders where that bound does not already imply (7.1):

j	k	a_k	$v_2(a_k)$
0	1	-2	1
0	2	72	3
1	1	$-6/5$	1
1	2	$184/45$	3
2	1	$-16/15$	4
2	2	$64/45$	6
2	3	$-14848/1365$	9
2	4	$71426048/69615$	13
2	5	$-25509691392/85085$	16
2	6	$149113198870528/765765$	22
3	1	$-36/35$	2
3	2	$256/231$	8
3	3	$-65792/35035$	8
3	4	$161644544/4849845$	15

They are finite rational evaluations of (7.6), followed by $a_k = \sum_{h=0}^{k} (-1)^{k-h} \binom{k}{h} C_j(h)$. For addresses zero and one, $k \geq 3$ is covered by the generic bound; for address two, $k \geq 7$ is covered; for address three, $k \geq 5$ is covered. In each range the elementary inequality is $k - \lfloor \log_2 k \rfloor \geq b_j + 1$. Lemma 7.1 proves the normalized identity. Multiplication by $j!$ proves the original identity.

A row displacement H preserves the original coefficient modulo 2^r at even one row exactly when $v_2(H) + B_j \geq r$. The least positive such integer is $2^{\max(0,r-B_j)}$, and the same displacement works at every row. This proves both optimality and the absence of a starting threshold. □

Corollary 7.3 (Exact image and residue enumeration)**.** *Let $\widetilde{C}_j$ denote the dyadic Newton interpolation. The map*

$$T_j(x) = \frac{\widetilde{C}_j(x) - 1}{2^{b_j}}$$

is a bijective isometry $\mathbb{Z}_2 \to \mathbb{Z}_2$, and

$$\widetilde{C}_j(\mathbb{Z}_2) = 1 + 2^{b_j}\mathbb{Z}_2, \qquad j!\widetilde{C}_j(\mathbb{Z}_2) = j! + 2^{B_j}\mathbb{Z}_2. \tag{7.7}$$

For $r > B_j$, one least period modulo 2^r visits every residue in $j! + 2^{B_j}\mathbb{Z}/2^r\mathbb{Z}$ exactly once. For $r \le B_j$, there is one residue and the least row period is one.

Proof. The exact difference identity gives $v_2(T_j(x) - T_j(y)) = v_2(x - y)$ and $T_j(0) = 0$. For every $s \ge 1$, the map induces an injection of $\mathbb{Z}_2/2^s\mathbb{Z}_2$ into itself. It is a bijection because both sets have 2^s elements. These quotient bijections are compatible with reduction. Inverting them gives a compatible preimage system for every dyadic integer, proving surjectivity. Multiplication by $j!$ changes the image ideal by exactly $v_2(j!)$. The finite quotient bijection is also precisely the residue enumeration in a least row period. □

7.1 An infinite exact family at all dyadic ranks

Theorem 7.4 (Lower-half exact laws)**.** *Let $N = 2^a \ge 4$ and $0 \le j < N/2$. Then for all $n \ge 0$ and $H > 0$,*

$$v_2(C_{N,j}(n + H) - C_{N,j}(n)) = 1 + v_2(H). \tag{7.8}$$

Its normalized interpolation is a bijection onto $1 + 2\mathbb{Z}_2$. For the original coefficients, the least row period modulo 2^r is $2^{\max(0,r-1-v_2(j!))}$.

Proof. Every slope N/i in the source has valuation at least two, since $i \le j < N/2$. Hence $g(1) \equiv 1 \pmod 4$ and $g(2) \equiv 1 \pmod 8$. The first two Newton coefficients from the reciprocal recurrence are

$$a_1 = -g(1) - 1, \qquad a_2 = \binom{2N}{N} g(1)^2 - g(2) + 2g(1) + 1. \tag{7.9}$$

The first has valuation exactly one. Also $\binom{2N}{N} \equiv 6 \pmod 8$ for every $N = 2^a \ge 2$. To see this, write

$$B_a = \binom{2^{a+1}}{2^a} = 2 \prod_{h=1}^{2^a-1} \left(1 + \frac{2^a}{h}\right).$$

The even-h factors give B_{a-1}; the odd-h factors are one modulo four for $a \ge 2$. Since B_{a-1} has valuation one, $B_a \equiv B_{a-1} \pmod 8$, and $B_1 = 6$. Therefore $a_2 \equiv 6 - 1 + 2 + 1 = 0 \pmod 8$. The global Mahler bound covers all $k \ge 3$, completing the criterion with $b = 1$. The period and image deductions are the same as above. The proof also covers $N = 2, j = 0$, whose kernel is identically one. □

7.2 Complete exact laws at ranks two and eight

Proposition 7.5 (Small-rank complete isometry table)**.** *Every address at $N \in \{2, 4, 8\}$ satisfies*

$$v_2(C_{N,j}(n + H) - C_{N,j}(n)) = b_{N,j} + v_2(H).$$

The following entries are ordered by $j = 0, \ldots, N - 1$:

N	*normalized shifts $b_{N,j}$*	*original shifts $B_{N,j} = b_{N,j} + v_2(j!)$*
2	1, 2	1, 2
4	1, 1, 4, 2	1, 1, 5, 3
8	1, 1, 1, 1, 4, 3, 2, 2	1, 1, 2, 2, 7, 6, 6, 6

For each entry, the least original row period is $2^{\max(0,r-B_{N,j})}$, its Euler subscript displacement is N times that period, and the exact-image and residue-enumeration conclusions hold with the corresponding shifts.

Proof. Rank four was proved above. At ranks two and eight the finite criterion is verified by the actual rational recurrence (7.6). The following valuation lists record every initial Newton coefficient needed in addition to the generic tail $v_2(a_k) \geq k$:

N	j	$b_{N,j}$	$v_2(a_1), v_2(a_2), \ldots$ through the finite cutoff
2	0	1	1, 3
2	1	2	2, 5, 9, 11
8	0	1	1, 3
8	1	1	1, 3
8	2	1	1, 3
8	3	1	1, 3
8	4	4	4, 6, 9, 13, 16, 20
8	5	3	3, 6, 10, 13, 17
8	6	2	2, 6, 8, 13
8	7	2	2, 11, 8, 15

The first entries can also be read directly from $a_1 = -1 - \binom{N+j}{j}^{-1}$. At rank eight they are respectively

$$-2,\ -\frac{10}{9},\ -\frac{46}{45},\ -\frac{166}{165},\ -\frac{496}{495},\ -\frac{1288}{1287},\ -\frac{3004}{3003},\ -\frac{6436}{6435}.$$

All later entries in the table are finite rational arithmetic, with full numerator and denominator records retained in the accompanying exact certificate `all_rank_extensions_verified.json`. Every entry with $k \geq 2$ meets $b_{N,j} + 1 + \lfloor \log_2 k \rfloor$. Beyond the displayed cutoff, the inequality $k \geq b_{N,j} + 1 + \lfloor \log_2 k \rfloor$ holds at every order. Thus the finite table and the proved infinite tail establish the full hypothesis of Lemma 7.1. This is a finite coefficient proof followed by a universal tail argument, rather than extrapolation from sampled rows or steps. □

Remark 7.6 (Failure of a universal upper-half exact law). The all-rank lower bound remains valid in every address. The exact first-difference law does not automatically extend to all upper-half addresses. Exact rational recurrence calculations give

(N, j)	$v_2(C(1) - C(0))$	$v_2(C(2) - C(0))$	naive $b + v_2(2)$
$(16, 9)$	5	7	6
$(32, 18)$	6	6	7
$(32, 19)$	5	7	6

For example, at $(N, j) = (16, 9)$ the first difference is $-2042976/2042975$, whereas the two-row difference is $-2215603929728/2215923048625$. These exact counterexamples refute the additional universal isometry proposal. They preserve the threshold-free lower bound and the original conjecture.

8 Analytic interpolation and exact Taylor radii

The threshold-free Mahler theorem gives continuous interpolation. Analytic interpolation needs a stronger estimate, and the factorial bounds provide it directly. The ordinary Taylor radius is determined by the coefficient growth, including infinite exact subsequences.

Theorem 8.1 (Taylor coefficient bounds and radii)**.** *For $j = 0, 1, 2, 3$, the dyadic interpolation has an ordinary power series*

$$\widetilde{C}_j(x) = 1 + \sum_{\ell \geq 1} d_{j,\ell} x^\ell, \qquad v_2(d_{j,\ell}) \geq \ell + \beta_j.$$

Its Taylor series converges for $|x|_2 < 2$. At addresses zero and one,

$$v_2(d_{j,\ell}) = \ell \quad (\ell \geq 1), \qquad R_0 = R_1 = 2. \tag{8.1}$$

At address two,

$$v_2(d_{2,\ell}) \geq \max\left(2\ell + \left\lfloor \frac{\ell}{2} \right\rfloor, \ell + 3\right), \qquad R_2 = 2^{5/2}. \tag{8.2}$$

The radius at address three is at least two; its exact radius is not asserted here. The stated exact radii at addresses zero, one, and two hold at every center in $\mathbb{Z}_2$.

Proof. Put $a_m = \Delta_1^m C_j(0)$ and expand the falling factorial in the Newton series. The ordinary coefficients are

$$d_{j,\ell} = \sum_{m \geq \ell} \frac{a_m}{m!} s(m, \ell). \tag{8.3}$$

Since $s(m, \ell)$ is integral and $v_2(a_m/m!) \geq m + \beta_j$, the coefficient sum converges and has valuation at least $\ell + \beta_j$. The sum exchange is justified on $\mathbb{Z}_2$ by these uniform geometric coefficient bounds. The ordinary power series agrees there with the Newton interpolation and converges whenever $|x|_2 < 2$.

For $j = 0, 1$, the term $m = \ell$ in (8.3) has exact valuation ℓ, since $s(\ell, \ell) = 1$ and the Newton valuation is exact. Every later term has valuation at least $\ell + 1$. Thus $v_2(d_{j,\ell}) = \ell$, and the coefficient radius is exactly two. At a center $x_0 \in \mathbb{Z}_2$, the coefficient of y^ℓ in the translated series is

$$\sum_{m \geq \ell} d_{j,m} \binom{m}{\ell} x_0^{m-\ell}. \tag{8.4}$$

Its initial term again has exact valuation ℓ, and all later terms have at least one more bit. The exact radius is therefore two at every integral center.

For address two, the stronger factorial filtration gives the lower bound in (8.2). At each $\ell = 2^r \geq 4$, Theorem 6.3 makes the $m = \ell$ term in (8.3) have valuation $2\ell + \ell/2 = 5\ell/2$. Every $m > \ell$ term has strictly greater lower bound $2m + \lfloor m/2 \rfloor$. Hence

$$v_2(d_{2,\ell}) = 5\ell/2 \qquad (\ell = 2^r \geq 4).$$

The universal lower bound and this unbounded exact subsequence prove that the coefficient radius is exactly $2^{5/2}$. Translation by an integral center preserves the same unique leading coefficients at these degrees in (8.4), proving the exact radius at every such center. □

Remark 8.2 (Boundary interpretation). For addresses zero and one, at an argument of absolute value two every positive-degree Taylor term has valuation zero, so that ordinary Taylor series diverges. For address two, over a complete extension admitting arguments of absolute value $2^{5/2}$, the power-of-two-degree terms have valuation zero and fail to tend to zero. These statements concern the Taylor expansions just constructed; they do not prohibit a different analytic continuation. On $\mathbb{Q}_2$-valued arguments, open-disc notation must be read with its discrete value group. The coefficient radius is the precise invariant in the theorem.

9 The final congruence in the original Euler convention

The preceding bounds can be stated as one uniform congruence without suppressing the address normalization. This is the strengthened formula for the four-address family, valid from the first row.

Corollary 9.1 (Final common-step congruence)**.** *Let $j \in \{0,1,2,3\}$, $n \geq 0$, and $k, H \geq 1$. Define*

$$A_0(k) = A_1(k) = k, \qquad A_2(k) = \max\left(k+4,\ 2k + \left\lfloor \frac{k}{2} \right\rfloor + 1\right), \qquad A_3(k) = k+2.$$

Then

$$\boxed{\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} E^{(4,j)}_{4(n+iH)} \equiv 0 \pmod{2^{\,v_2(k!)+kv_2(H)+A_j(k)}}.} \tag{9.1}$$

For $j = 0, 1$, this displayed exponent is the exact valuation for every n, k, H in the stated range. For $j = 2$, the exact positive power-of-two-order valuations are given by Theorem 6.3, with one added bit.

Proof. The left side is $j!\Delta_H^k C_j(n)$. At addresses zero and one, $v_2(j!) = 0$ and the exact homogeneous formula gives the exponent. At address two, $v_2(2!) = 1$; the maximum of (5.1) and (5.2) yields $\max(k+4, 2k + \lfloor k/2 \rfloor + 1)$ after removing the common $v_2(k!) + kv_2(H)$. At address three, the one-bit normalized offset and $v_2(3!) = 1$ give $k+2$. □

For mixed steps the same corollary holds with $kv_2(H)$ replaced by $\sum_\nu v_2(H_\nu)$ and the binomial stencil replaced by $j!\Delta_{H_1} \cdots \Delta_{H_k} C_j(n)$. At addresses zero and one, exactness is then governed by the parity classification in Theorem 6.1. At first order these bounds are exact in every address, with original valuation offsets $(1, 1, 5, 3)$; the least periods are those in (7.5).

9.1 Proof coverage and remaining questions

The results in these sections are mathematical theorems with detailed proofs and preserved audit evidence. Their formal status must be read statement by statement. The recovered Paper IX release already contains Lean results for the rank-four threshold-free homogeneous bound, the sharp first-difference laws, the least periods, and the dyadic isometry and image conclusions. Those inherited results should not be described as missing merely because the earlier short completion supplement did not include their later modules.

The all-rank dyadic and general prime-power transfer proof presented here, the factorial improvements, the mixed leading-parity classification, the Bernoulli power-order sharpness, the all-rank lower-half exact laws, the complete rank-eight extension, and the exact Taylor-radius arguments have not received new Lean formalization in this strengthening. A mathematical audit and an exact computation are neither a Lean compilation nor external peer review. The separate Lean guide records the recovered formal declarations and any fresh release checks without extending their coverage to these new claims.

The completed external statement is Nishibuchi's Conjecture 3.9 in its prescribed parameter family and coefficient normalization. These stronger congruences enlarge its dyadic branch and remove its starting threshold there. They do not settle every Euler problem. In particular, the following questions remain separate:

- a complete exact higher-difference law at all non-power-of-two orders in address two;
- complete higher-difference valuations and the exact ordinary Taylor radius in address three;

- a classification of the upper-half dyadic addresses having an exact first-difference metric law at arbitrary larger ranks;
- the distinct universal simplicity or nonvanishing question for the later Euler-tail residuals, and the original native Paper IX conjectures beyond the completed external statement.

The exact counterexamples above rule out the naive universal upper-half isometry, and mixed cancellations rule out unconditional baseline sharpness for arbitrary stencils. No global historical-priority or novelty assertion is made for these strengthened statements.

10 Finite obstructions, compatible lifts, and deformation

The arithmetic results above concern particular families. We now study the underlying correction problem, from finite polynomial lifts to continuous parameter charts and quantitative deformation. Throughout, we distinguish three objects: a residual obstruction, the set of correcting digits, and a root in a completion. A vanishing obstruction at one precision gives a lift to the next. Continuing indefinitely requires additional hypotheses.

Unless a different coefficient ring is specified, a polynomial system is a finite list of integer polynomials. All divisibility conditions on vectors are coordinatewise. For a positive integer q, write $S_q = \mathbb{Z}/q\mathbb{Z}$ and use an overline for reduction modulo q. For a prime p, write $\mathbb{F}_p = S_p$, normalize $|p|_p = p^{-1}$, and put

$$B_L(x) = x + p^L\mathbb{Z}_p^d.$$

An integer center is included in this notation by its natural image in $\mathbb{Z}_p^d$. All dimensions below are finite and may be zero. A statement over S_q allows composite q and $q = 1$ when explicitly stated. A cardinality formula in terms of vector-space rank uses prime p.

10.1 The polynomial correction equation

For $F = (F_1, \dots, F_e) \in \mathbb{Z}[X_1, \dots, X_d]^e$, define its evaluated Jacobian by the formal partial derivatives:

$$DF(x)_{ij} = (\partial_j F_i)(x), \qquad J_x = DF(x) \bmod q : S_q^d \longrightarrow S_q^e.$$

The obstruction therefore uses the Jacobian of the given polynomials. The following algebraic Taylor identity relates this linear map to polynomial evaluation.

Lemma 10.1 (Polynomial Taylor remainder)**.** *Let R be a commutative ring, $P \in R[X_1, \dots, X_d]$, $a \in R^d$, and $c \in R$. There is a polynomial $Q \in R[U_1, \dots, U_d]$, with $Q(0) = 0$, such that for every $u \in R^d$,*

$$P(a + cu) = P(a) + c\sum_{j=1}^{d}(\partial_j P)(a)u_j + c^2 Q(u). \tag{10.1}$$

In particular, when $R = \mathbb{Z}$ and a, u, c are integral, subtracting the first two terms on the right from $P(a + cu)$ gives a multiple of c^2.

Proof. Apply polynomial induction using constants, addition, and multiplication by a variable. Constants have zero remainder, and remainders add. For the multiplication step, suppose the assertion is known for P and write $D_P(U) = \sum_j(\partial_j P)(a)U_j$. Multiplication of its expansion by $a_i + cU_i$ gives constant term $P(a)a_i$, linear term $c(a_i D_P(U) + P(a)U_i)$, and remainder

$$c^2\big(Q(U)(a_i + cU_i) + D_P(U)U_i\big).$$

The product rule identifies the linear term with the derivative of PX_i. The displayed remainder vanishes at $U = 0$. This constructs the polynomial remainder before evaluation at u, even over rings where division by c would be invalid. □

The general polynomial identity is formalized by `Komorebi.PolynomialTaylorRing.exists_eval_scaled_remainder`. Its integer divisibility form is `Komorebi.PolynomialTaylor.taylor_dvd`.

Theorem 10.2 (Finite polynomial obstruction)**.** *Let $q \geq 1$, $L \geq 1$, $x \in \mathbb{Z}^d$, and suppose $F(x) \in q^L\mathbb{Z}^e$. Define the exact integer residual r_x by $F(x) = q^L r_x$, and put*

$$\mathcal{O}_x = S_q^e / \operatorname{im} J_x, \qquad o_x = [\overline{r_x}].$$

Then $\mathcal{O}_x$ is finite and

$$o_x = 0 \iff \exists y \in \mathbb{Z}^d : \quad y \equiv x \pmod{q^L}, \qquad F(y) \equiv 0 \pmod{q^{L+1}}. \tag{10.2}$$

No surjectivity or nonsingularity assumption is required.

Proof. Every point in the lower residue class has the form $y = x + q^L t$ with $t \in \mathbb{Z}^d$. By Lemma 10.1 and $2L \geq L + 1$,

$$F(x + q^L t) \equiv q^L \big(r_x + DF(x)t\big) \pmod{q^{L+1}}.$$

The integer q^L is nonzero. Cancelling it in the corresponding divisibility condition proves

$$F(x + q^L t) \equiv 0 \pmod{q^{L+1}} \iff \overline{r_x} + J_x \bar{t} = 0. \tag{10.3}$$

Integer vectors reduce onto S_q^d. The equation on the right is solvable exactly when $\overline{r_x}$ belongs to $\operatorname{im} J_x$, since that image is closed under negation. This is precisely $o_x = 0$. Finiteness follows because $\mathcal{O}_x$ is a quotient of the finite module S_q^e. □

The formal theorem `Komorebi.PolynomialObstruction.finite_polynomial_obstruction` presents the image using the integer-domain map $\mathbb{Z}^d \xrightarrow{DF(x)} \mathbb{Z}^e \to S_q^e$. Reduction onto S_q^d identifies that image with $\operatorname{im} J_x$. The class-preserving comparison is explicit in `Komorebi.PolynomialLiftCounts.obstructionEquiv`; the change of presentation does not change the distinguished residual class.

Proposition 10.3 (Affine depth zero)**.** *Let $F(x) = Ax - b$ for an integer linear map $A : \mathbb{Z}^d \to \mathbb{Z}^e$. The criterion of Theorem 10.2 holds for every $L \geq 0$, with $J = A \bmod q$.*

Proof. The affine identity $F(x + q^L t) = F(x) + q^L At$ has no quadratic remainder. The proof of (10.3) therefore needs no inequality involving $2L$. The residual and the finite quotient are obtained as above. □

The affine statement is formalized by `Komorebi.NativeAffine.finite_affine_obstruction`. Positive depth is essential in the nonlinear theorem: for $F(X) = X^2 - 1$, $q = 2$, $L = 0$, and $x = 0$, the Jacobian is zero and the residual class is nonzero, yet $y = 1$ is a root modulo two in the same residue modulo one.

Proposition 10.4 (Independence of the integer representative)**.** *Under the hypotheses of Theorem 10.2, let $x' = x + q^L s$ and write $F(x') = q^L r_{x'}$. Then*

$$J_{x'} = J_x, \qquad \overline{r_{x'}} = \overline{r_x} + J_x \bar{s}.$$

Consequently the two residual classes coincide in the common quotient $S_q^e / \operatorname{im} J_x$.

Proof. Polynomial evaluation respects coordinatewise congruence. Apply this first to every formal partial derivative: $x' \equiv x \pmod{q}$ gives $J_{x'} = J_x$. Next apply (10.1) to F at scale q^L. Subtract the two exact residual identities and cancel q^L to obtain the displayed relation modulo q. Its additional term belongs to the common Jacobian image, so it vanishes in the quotient. □

This comparison is formalized by `Komorebi.PolynomialRepresentative.representative_independence`. This proves independence of the integer representative within a fixed polynomial presentation. It makes no identification of the raw cokernels of different equation lists. For example, adjoining a zero equation leaves the roots unchanged while adding an S_q summand to the raw cokernel.

10.2 One correction shared by two systems

The next obstruction concerns compatibility between individually attainable corrections. Its algebraic form works over an arbitrary commutative ring. Let U, V, W be R-modules and let $J : U \to V, K : U \to W$ be linear. Define

$$C(J, K) = U / (\ker J + \ker K). \tag{10.4}$$

An attainable value Ju has label $[u] \in C(J, K)$; this is independent of the choice of its preimage because $\ker J$ is killed in the quotient. Similarly, Kv has label $[v]$. Denote these maps by $\lambda_J : \operatorname{im} J \to C(J, K)$ and $\lambda_K : \operatorname{im} K \to C(J, K)$.

Theorem 10.5 (Common correction and fiber product)**.** *For every $u, v \in U$,*

$$[u] = [v] \text{ in } C(J, K) \quad \Longleftrightarrow \quad \exists w \in U : \ Jw = Ju, \quad Kw = Kv. \tag{10.5}$$

Both label maps are surjective, and the image of the joint map is exactly the fiber product:

$$\operatorname{im}\big(U \to \operatorname{im} J \times \operatorname{im} K, \ u \mapsto (Ju, Ku)\big) = \operatorname{im} J \times_{C(J,K)} \operatorname{im} K. \tag{10.6}$$

If either V or W is finite, then $C(J, K)$ is finite.

Proof. If $[u] = [v]$, write $u - v = s + t$ with $s \in \ker J$ and $t \in \ker K$. Then $w = u - s = v + t$ satisfies both required equalities. Conversely, a common w gives

$$u - v = (u - w) + (w - v) \in \ker J + \ker K.$$

Every class $[u]$ is the label of Ju and of Ku, proving surjectivity. For attainable $a = Ju$ and $b = Kv$, the first equivalence says exactly that equal labels are equivalent to a simultaneous preimage. This proves (10.6). Finally, a finite target has finite image, and the associated surjective label map makes $C(J, K)$ finite. □

These statements are `Komorebi.SharedCorrection.quotient_mk_eq_iff_common` and `Komorebi.SharedCorrection.joint_range_eq_fiber`, together with the finite-target corollaries in the same module. The quotient also has the expected universal property: every linear map from U annihilating both kernels factors uniquely through $C(J, K)$. Equivalently, compatible linear observations on $\operatorname{im} J$ and $\operatorname{im} K$ factor uniquely through their label maps. This is formalized by `Komorebi.SharedCorrection.common_paired_universal`.

Corollary 10.6 (Simultaneous polynomial lifting)**.** *Let $F \in \mathbb{Z}[X_1, \ldots, X_d]^e$ and $G \in \mathbb{Z}[X_1, \ldots, X_d]^f$ have a common lower root x modulo q^L, where $q \geq 1$ and $L \geq 1$. Use the integer-domain reduced Jacobians $J : \mathbb{Z}^d \to S_q^e$ and $K : \mathbb{Z}^d \to S_q^f$. Suppose $u, v \in \mathbb{Z}^d$ are separately valid corrections for F and G, respectively. Then*

$$[u] = [v] \text{ in } \mathbb{Z}^d / (\ker J + \ker K)$$

if and only if a single $y \equiv x \pmod{q^L}$ solves both systems modulo q^{L+1}. Each separate label is independent of the chosen valid correction. The common quotient is finite.

Proof. For each system, (10.3) translates a valid next root into a linear correction equation. Once u corrects F, another digit w corrects F precisely when $Jw = Ju$. The analogous condition for G is $Kw = Kv$. Theorem 10.5 now gives the criterion. Two valid corrections of one block differ by a vector in its Jacobian kernel, so they have equal labels. Finiteness follows from either finite residue target. An unconditional formulation quantifies over separately valid u, v with equal labels, thereby including their existence in the criterion rather than assuming it. □

See `Komorebi.PolynomialShared.joint_lift_iff_class_eq` and `Komorebi.PolynomialShared.finite_polynomial_pair_compatibility`.

For affine integer systems $F(X) = AX - b$ and $G(X) = BX - c$, where $A : \mathbb{Z}^d \to \mathbb{Z}^e$ and $B : \mathbb{Z}^d \to \mathbb{Z}^f$ are linear, Corollary 10.6 holds for $q \geq 1$ and every $L \geq 0$, including depth zero. Here x is a common root modulo q^L, and J and K are the reductions of A and B with domain $\mathbb{Z}^d$. Given separately valid corrections u, v, a simultaneous next lift exists if and only if their classes in $\mathbb{Z}^d / (\ker J + \ker K)$ agree. The quotient is finite, and each label is independent of the chosen valid correction. Indeed, Proposition 10.3 supplies the correction equation without a positive-depth restriction, and the proof of Theorem 10.5 then applies unchanged. The formal statement is `Komorebi.AffinePair.joint_lift_iff_class_eq`. Quantifying over separately valid corrections with equal labels also includes their existence in the criterion, as in `Komorebi.AffinePair.finite_affine_pair_compatibility`. Separate liftability alone is insufficient. At $x = 0$ modulo two, $F(X) = X$ and $G(X) = X - 2$ both lift to modulus four, but their correcting digits are respectively 0 and 1 modulo two. No single residue solves both equations modulo four.

10.3 Correction torsors and exact residue counts

Integer representatives and residue classes at the next level must also be distinguished. For $q \geq 1$ and $L \geq 0$, the map

$$S_q^d \longrightarrow \{y \in (\mathbb{Z}/q^{L+1}\mathbb{Z})^d : y \equiv x \pmod{q^L}\}, \qquad t \longmapsto x + q^L \widetilde{t}$$

is a bijection, where $\widetilde{t}$ is any integer representative. Independence of the representative and injectivity both follow by cancelling q^L in integer divisibility; surjectivity follows by expressing the lower congruence as a multiple of q^L. This is the equivalence `Komorebi.ResidueLiftDigits.digitEquivResidue`.

At positive polynomial depth, restrict it to the correction fiber

$$\mathcal{T}_x = \{t \in S_q^d : \overline{r_x} + J_x t = 0\}$$

and the lift set

$$\mathcal{L}_{x,L} = \{y \in (\mathbb{Z}/q^{L+1}\mathbb{Z})^d : y \equiv x \pmod{q^L},\ F(y) = 0\}.$$

The polynomial predicate in the second display is evaluation in the residue ring. Evaluation on canonical integer representatives gives the same predicate, because integer polynomial evaluation respects congruence.

Theorem 10.7 (Torsor and lift cardinalities)**.** *Under the hypotheses of Theorem 10.2, there is an equivalence $\mathcal{T}_x \simeq \mathcal{L}_{x,L}$. Every nonempty fiber is a torsor for* $\ker J_x$ *under addition of correction digits. If $q = p$ is prime and $\rho_x = \operatorname{rank}_{\mathbb{F}_p} J_x$, then*

$$\#\mathcal{L}_{x,L} = \begin{cases} p^{d-\rho_x}, & o_x = 0, \\ 0, & o_x \neq 0, \end{cases} \qquad \#\mathcal{O}_x = p^{e-\rho_x}. \tag{10.7}$$

Proof. Equation (10.3) identifies exactly which points of the residue fiber satisfy the equations, proving the restricted equivalence. If $u, t \in \mathcal{T}_x$, subtraction gives $J_x(t - u) = 0$. Conversely, adding any element of $\ker J_x$ preserves the correction equation. The unique kernel vector carrying u to t is $t - u$, so the action is free and transitive. Choosing one valid digit identifies the fiber with the kernel, without making that choice part of its translation action.

For prime p, rank-nullity gives $\dim \ker J_x = d - \rho_x$ and $\dim(\mathbb{F}_p^e / \operatorname{im} J_x) = e - \rho_x$. The cardinality of an n-dimensional $\mathbb{F}_p$-space is p^n. Combine these identities with the obstruction criterion and the residue equivalence. □

The cardinality formula is formalized by `Komorebi.PolynomialLiftCounts.finite_polynomial_lift_count`. The explicit torsor and chosen-point equivalence are developed in `Komorebi.FiniteFiberCounts.translate_exists_unique` and `Komorebi.PolynomialLiftCounts.liftEquivKernel`. For composite q, the torsor and kernel cardinality remain valid, while the vector-space rank formula is unavailable. Multiplication by two on $\mathbb{Z}/4\mathbb{Z}$ already has a kernel of size two.

10.4 Smooth compatible towers and exact roots

Surjectivity removes every residual obstruction and persists throughout a positive-depth residue ball. This supplies the additional hypothesis needed to iterate the finite theorem.

Theorem 10.8 (Compatible polynomial tower)**.** *Let $q \geq 1$, $L \geq 1$, $F \in \mathbb{Z}[X_1, \ldots, X_d]^e$, and $x \in \mathbb{Z}^d$ satisfy $F(x) \in q^L\mathbb{Z}^e$. Suppose $J_x : S_q^d \to S_q^e$ is surjective. There is a single sequence $z_n \in \mathbb{Z}^d$ such that*

$$z_0 = x, \qquad F(z_n) \in q^{L+n}\mathbb{Z}^e,$$
$$z_{n+1} - z_n \in q^{L+n}\mathbb{Z}^d, \qquad J_{z_n} = J_x.$$

For $a \leq b$, it also satisfies $z_b - z_a \in q^{L+a}\mathbb{Z}^d$.

Proof. At level $N = L + n$, divide the residual by q^N. Surjectivity supplies $t \in S_q^d$ with $J_{z_n} t = -\overline{F(z_n)/q^N}$. Choose an integer representative and apply (10.3) to obtain z_{n+1}. The new point is congruent to z_n modulo q. Evaluation of each formal partial derivative therefore gives $J_{z_{n+1}} = J_{z_n}$.

Construct the sequence recursively, retaining the root condition and the fixed Jacobian at each step. Surjectivity at every later stage then follows from the single initial assumption. The cross-level congruence follows by induction, adding consecutive differences whose divisibilities all imply divisibility by q^{L+a}. □

The full tower statement is `Komorebi.PolynomialIntegerTower.exists_compatible_tower_strong`. The proof constructs a compatible path using classical choice. It does not claim an optimized procedure for selecting its digits.

Theorem 10.9 (Completion and smooth branching)**.** *In Theorem 10.8, let $q = p$ be prime. The tower can be chosen with a limit $a \in \mathbb{Z}_p^d$ such that*

$$F(a) = 0, \qquad z_n \longrightarrow a, \qquad a - z_n \in p^{L+n}\mathbb{Z}_p^d \quad (n \geq 0).$$

At each stage, the number of distinct next-level residue roots over z_n is exactly p^{d-e}, and the finite obstruction module has one element. If $d = e$, there is exactly one root of F in $B_L(x)$.

Proof. Cross-level compatibility bounds the supremum distance between later terms by $p^{-(L+n)}$. Completeness gives a limit, and closed norm bounds retain the congruence to every z_n. View the coefficients of the original integer polynomials in $\mathbb{Z}_p$. Their evaluations satisfy both $F_i(z_n) \to F_i(a)$, by continuity, and $F_i(z_n) \to 0$, by their growing divisibility. Uniqueness of limits gives exact vanishing.

Surjectivity gives rank $J_x = e \leq d$. Theorem 10.7 then gives p^{d-e} next residue roots and obstruction cardinality one at every stage.

For square uniqueness, let $a, b \in B_L(x)$ be arbitrary exact roots. Their canonical integer truncations are genuine roots in the respective finite residue rings, since evaluation commutes with reduction. They

agree at depth L. If they agree at depth $N \geq L$, both next residues belong to the same lift set, which has cardinality $p^{d-d} = 1$. Induction gives agreement at every depth. The residue maps separate p-adic integers, so $a = b$. Existence was already supplied by the tower. □

The endpoint construction is `Komorebi.PolynomialPadicLift.exists_padic_root_with_tower`, the branching statement is `Komorebi.PolynomialSmoothCounts.exists_regular_tower`, and uniqueness for all roots in the ball is `Komorebi.PolynomialPadicUnique.exists_unique_padic_root`.

Remark 10.10. The one-step singular criterion and the smooth infinite theorem have different hypotheses. The polynomial $X^2 - 5$ has odd roots modulo two and four but has no root modulo eight. Thus a previous zero obstruction does not force the next one to vanish. Also, rectangular surjectivity alone does not imply uniqueness: $F(X, Y) = X$ has the distinct dyadic roots $(0, 0)$ and $(0, 2)$ in the same residue ball modulo two. These examples show why the hypotheses of the tower theorem and its uniqueness clause are needed.

10.5 Coefficient stability and continuous root charts

Proposition 10.11 (Smooth coefficient stability)**.** *Let p be prime, $M \geq 1$, $F, H \in \mathbb{Z}[X_1, \ldots, X_d]^e$, and $c \in p^M\mathbb{Z}$. Suppose $DF(x) \bmod p$ is surjective for an integer $x \in \mathbb{Z}^d$. If $a \in \mathbb{Z}_p^d$ satisfies $F(a) = 0$ and $a \equiv x \pmod{p}$, then there is a root b of $G = F + cH$ with $b - a \in p^M\mathbb{Z}_p^d$.*

If $d = e$ and $M \geq L \geq 1$, every pair of roots $a, b \in B_L(x)$ of F and G, respectively, satisfies $b - a \in p^M\mathbb{Z}_p^d$. The perturbed root is unique in that ball whenever it exists.

Proof. Since $p \mid c$, differentiation gives $DG(x) \equiv DF(x) \pmod{p}$. Truncate a to an integer vector y at depth M. Then $F(y) \equiv 0 \pmod{p^M}$, hence also $G(y) \equiv 0 \pmod{p^M}$. Congruence of y and x modulo p preserves the reduced Jacobian. The smooth root theorem gives $b \equiv y \equiv a \pmod{p^M}$.

In the square case this constructed root lies in $B_L(x)$ because $M \geq L$. Square uniqueness for G identifies it with every competing root in that ball. The construction therefore gives the precision assertion for an arbitrary root there. □

See `Komorebi.PolynomialRootStability.exists_nearby_perturbed_root` and `Komorebi.PolynomialRootStability.perturbed_roots_congruent`. The scalar c may be negative or zero. The weighted theorems below allow arbitrary integral p-adic coefficients and give an explicit first-order displacement.

Theorem 10.12 (Continuous charts with a selected Jacobian minor)**.** *Let p be prime, $L \geq 1$, $F \in \mathbb{Z}[X_1, \ldots, X_{e+k}]^e$, and $x \in \mathbb{Z}^{e+k}$ satisfy $F(x) \in p^L\mathbb{Z}^e$. Suppose the full reduced Jacobian is surjective. There is a permutation σ of the coordinates such that, writing $\sigma x = (y_0, t_0)$ with e solved coordinates and k parameter coordinates, the parameter ball $T = t_0 + p^L\mathbb{Z}_p^k$ is homeomorphic to the entire root locus*

$$\mathcal{R} = \{z \in B_L(x) : F(z) = 0\}.$$

The inverse homeomorphism is the actual free-coordinate projection $z \mapsto \mathrm{free}(\sigma z)$. If $h : T \to \mathcal{R}$ is the forward map, then for every $N \geq 0$,

$$h(t) \equiv h(u) \pmod{p^N} \iff t \equiv u \pmod{p^N}. \tag{10.8}$$

Equivalently, $\sigma h(t) = (\phi(t), t)$ for a continuous $\phi : T \to y_0 + p^L\mathbb{Z}_p^e$ whose graph is the entire reordered root locus.

Proof. First consider a square integer polynomial map P whose reduced Jacobian at x is surjective. Its Jacobian is invertible over $\mathbb{F}_p$. A digit comparison using Taylor expansion shows that equal outputs modulo p^{N+1} force equal inputs modulo p^{N+1} whenever the inputs already agree modulo p^N and belong to $B_1(x)$. Induction and integer truncation give, for arbitrary $a, b \in B_1(x)$,

$$P(a) \equiv P(b) \pmod{p^N} \quad \Longleftrightarrow \quad a \equiv b \pmod{p^N}.$$

More generally, for a rectangular map with surjective reduced Jacobian,

$$P(B_L(x)) = P(x) + p^L \mathbb{Z}_p^e. \tag{10.9}$$

One inclusion follows from polynomial congruence. For integer targets in the right-hand ball, subtract the target from the constant terms and use the smooth root theorem. For arbitrary p-adic targets, approximate by integer truncations. The input ball is compact, so its continuous image is closed. It contains all these approximants and hence their limit. This argument does not assume that independently chosen preimages form a compatible sequence.

For square P, the resulting bijection between the two compact residue balls is continuous, and its target is Hausdorff. It is therefore a homeomorphism, with the same congruence preservation for its inverse.

Now suppose a chosen set of e columns of $DF(x) \bmod p$ is independent, and put those coordinates first. Apply the square result to the augmented polynomial map

$$\Psi(Y, T) = (F(Y, T), T).$$

Its reduced Jacobian has upper blocks $(D_Y F, D_T F)$ and lower blocks $(0, I_k)$, so it is invertible. For $t \in T$, the target $(0, t)$ belongs to $\Psi(x) + p^L \mathbb{Z}_p^{e+k}$ because $F(x) \equiv 0 \pmod{p^L}$. The inverse of Ψ at that target gives the unique root with parameters t. The inverse congruence property gives one direction of (10.8); projection onto the parameter coordinates gives the other.

Finally, surjectivity of the original rectangular Jacobian guarantees a basis among its columns. Extend the corresponding injection of coordinate indices to a permutation σ. Renaming the variables by σ^{-1} transports polynomial evaluation and every formal partial derivative. The fixed-coordinate construction therefore applies to the original system after undoing the permutation. □

The main theorem is `Komorebi.PolynomialGeneralChart.exists_parameter_homeomorph`. The square local inverse and surjectivity for arbitrary targets are `Komorebi.PolynomialLocalInverse.localHomeomorph` and `Komorebi.PolynomialLocalSurjective.exists_padic_preimage`. The center x is only an approximate root: the chart need not satisfy $\phi(t_0) = y_0$ unless $F(x) = 0$ exactly. The assertion selects suitable coordinates; it does not permit an arbitrary prescribed solved block.

10.6 Prescribed parameters and their obstruction

Suppose now that the split into solved variables and parameter variables is fixed in advance. Let

$$F \in \mathbb{Z}[Y_1, \ldots, Y_m, T_1, \ldots, T_k]^e, \qquad x = (y_0, t_0),$$

and assume $F(x) = q^L r_x$ with $q \geq 1$ and $L \geq 1$. Write the reduced Jacobian as

$$DF(x) \bmod q = (J_x \; K_x), \qquad J_x : S_q^m \to S_q^e, \quad K_x : S_q^k \to S_q^e.$$

For a prescribed parameter digit $v \in S_q^k$, let $\mathcal{L}_{x,L,v}$ be the set of next residue roots over x whose parameter coordinates are $t_0 + q^L \widetilde{v}$ modulo q^{L+1}.

Theorem 10.13 (Prescribed-parameter obstruction and count)**.** *Define*

$$o_x(v) = [\overline{r_x} + K_x v] \in S_q^e / \operatorname{im} J_x.$$

Then $\mathcal{L}_{x,L,v}$ is nonempty if and only if $o_x(v) = 0$. When nonempty, it is a torsor for $\ker J_x$*. If $q = p$ is prime and $\rho_x = \operatorname{rank}_{\mathbb{F}_p} J_x$, then*

$$\#\mathcal{L}_{x,L,v} = \begin{cases} p^{m-\rho_x}, & o_x(v) = 0, \\ 0, & o_x(v) \neq 0, \end{cases} \qquad \#(\mathbb{F}_p^e / \operatorname{im} J_x) = p^{e-\rho_x}. \tag{10.10}$$

The criterion holds uniformly for every prescribed v without a surjectivity hypothesis.

Proof. A candidate for the next lift has the form $x + q^L(\widetilde{u}, \widetilde{v})$, with solved digit $u \in S_q^m$. Its Taylor correction equation is

$$\overline{r_x} + J_x u + K_x v = 0.$$

Solvability is exactly vanishing of the displayed cokernel class. Restrict the residue-digit equivalence to the prescribed parameter coordinates. This identifies the lift fiber with the solved-digit fiber. Translation by $\ker J_x$ is free and transitive on any nonempty such fiber. The prime-field count follows from rank-nullity as before. □

The formal statements start from the lower root and construct its residual: `Komorebi.PolynomialPrescribedObstruction.finite_prescribed_obstruction` and `Komorebi.PolynomialPrescribedObstruction.finite_prescribed_lift_count`. For $F(Y, T) = T$ at the dyadic origin, the full Jacobian $(0\ 1)$ is surjective but $J_x = 0$. Prescribing $v = 1$ forces $T = 2 \pmod 4$, so no root exists; prescribing $v = 0$ gives two choices of Y modulo four over the initial zero residue. This example explains why the criterion depends on the solved block, rather than merely on the full Jacobian.

Theorem 10.14 (Every compatible integer parameter path)**.** *In the preceding notation, assume J_x is surjective. For every supplied integer path $t_n \in \mathbb{Z}^k$ satisfying*

$$t_0 = \operatorname{free}(x), \qquad t_{n+1} - t_n \in q^{L+n}\mathbb{Z}^k,$$

there is a compatible polynomial tower $z_n \in \mathbb{Z}^{m+k}$ with

$$z_0 = x, \quad \operatorname{free}(z_n) = t_n, \quad F(z_n) \in q^{L+n}\mathbb{Z}^e, \quad z_{n+1} - z_n \in q^{L+n}\mathbb{Z}^{m+k}.$$

The free-coordinate equality is exact in integers. Both reduced Jacobian blocks remain fixed, and $z_b - z_a \in q^{L+a}\mathbb{Z}^{m+k}$ whenever $a \leq b$.

Proof. At level $N = L + n$, write the supplied parameter increment exactly as $t_{n+1} - t_n = q^N w$ with $w \in \mathbb{Z}^k$. Surjectivity of the solved block gives a digit u satisfying $J_{z_n} u = -\overline{F(z_n)/q^N} - K_{z_n}\overline{w}$. Choose an integer representative $\widetilde{u}$ and set $z_{n+1} = z_n + q^N(\widetilde{u}, w)$. Its parameter coordinates equal the supplied target exactly. Taylor expansion proves the next root condition, while congruence modulo q preserves the full reduced Jacobian. Construct the sequence recursively, preserving the exact parameter coordinates and the fixed solved Jacobian. The earlier cross-level argument proves all the stated integer congruences. □

This is `Komorebi.PolynomialPrescribedIntegerTower.exists_prescribed_tower_strong`. In particular, no assumption about solvability at each later level has been added to the initial surjectivity condition.

Theorem 10.15 (Arbitrary p-adic parameter endpoints)**.** *Let $q = p$ be prime in Theorem 10.14. For every $\tau \in t_0 + p^L\mathbb{Z}_p^k$, there are an integer tower z_n and $b \in \mathbb{Z}_p^{m+k}$ such that*

$$F(b) = 0, \qquad \mathrm{free}(b) = \tau, \qquad z_n \longrightarrow b, \qquad b - z_n \in p^{L+n}\mathbb{Z}_p^{m+k}.$$

The initial value is $z_0 = x$. At every stage and for every proposed parameter digit $v \in \mathbb{F}_p^k$, the corresponding next residue fiber has exactly p^{m-e} elements. If $m = e$, then for each τ there is exactly one such root in $B_L(x)$, independently of all choices of integer approximants and correcting digits.

Proof. Set the first integer parameter approximation equal to the prescribed t_0. For $n \geq 1$, use canonical representatives of τ mod p^{L+n}. The initial congruence and truncation precision give

$$\tau - t_n \in p^{L+n}\mathbb{Z}_p^k.$$

Subtracting consecutive relations gives compatibility in $\mathbb{Z}_p$. Faithfulness of integer divisibility under $\mathbb{Z} \hookrightarrow \mathbb{Z}_p$ gives compatibility in $\mathbb{Z}$, and the same norm bounds give $t_n \to \tau$. Theorem 10.14 and completion now produce b with exact polynomial vanishing and all retained precisions. Each free coordinate has both the corresponding coordinate of b and that of τ as its limit, proving exact parameter equality.

The solved Jacobian remains surjective, so every shifted obstruction $[\bar{r} + K_{z_n}v]$ vanishes. Formula (10.10) therefore gives p^{m-e} choices for every parameter digit at every stage. When $m = e$, the fixed-coordinate chart in the proof of Theorem 10.12 contains every root in the initial ball with the specified parameters. Its uniqueness proves the endpoint assertion. This is uniqueness of the actual endpoint, not of the integer representatives used to approach it. □

The endpoint theorems are `Komorebi.PolynomialPrescribedPadicTower.exists_prescribed_padic_root_with_tower` and `Komorebi.PolynomialPrescribedPadicTower.exists_unique_prescribed_padic_root`. The separate supplied-path theorem `Komorebi.PolynomialPrescribedPadicTower.exists_root_with_prescribed_path` accepts any compatible integer parameter path with limit τ. For $m > e$, endpoint uniqueness is false in general: the system $F(Y_1, Y_2, T) = Y_1$ has two distinct roots $(0, 0, 0)$ and $(0, 2, 0)$ in the same dyadic ball with identical parameter value.

10.7 Weighted lifting for a nonunit Jacobian

The remaining theorems in this section allow arbitrary coefficients and centers in $R = \mathbb{Z}_p$. They concern square solved Jacobians invertible over $\mathbb{Q}_p$, even when their reductions modulo p are singular. The exponent controlling an integral inverse is retained explicitly.

Theorem 10.16 (Weighted nonlinear lifting)**.** *Let p be prime, $d, s, r \in \mathbb{N}$, $F \in R[X_1, \ldots, X_d]^d$, and $a \in R^d$. Set $J = DF(a)$. Suppose*

$$J \in \mathrm{GL}_d(\mathbb{Q}_p), \qquad p^s J^{-1} \in M_d(R), \qquad F(a) \in p^r R^d, \qquad r > 2s. \tag{10.11}$$

Then there is exactly one root b of F in $a + p^{s+1}R^d$, and it satisfies

$$b - a \in p^{r-s}R^d. \tag{10.12}$$

Uniqueness holds throughout the full ball of depth $s + 1$.

Proof. Let $A = p^s J^{-1} \in M_d(R)$. Inclusion into $\mathbb{Q}_p$ and its injectivity give the two integral identities

$$AJ = JA = p^s I_d.$$

These identities are also an equivalent integral certificate for the first two hypotheses of (10.11). In particular, A is injective: $Aw = 0$ implies $p^s w = JAw = 0$, hence $w = 0$.

Precondition the polynomial system by A and set $c = p^{s+1}$. Its derivative at a is $AJ = p^s I_d$. Lemma 10.1 gives a polynomial vector Q with $Q(0) = 0$ and

$$(AF)(a + cu) = (AF)(a) + p^{2s+1}u + p^{2s+2}Q(u).$$

Since A is integral, $(AF)(a) \in p^r R^d$. The strict integer inequality $r > 2s$ permits $k = r - (2s + 1) \geq 0$ and a vector $C \in p^k R^d$ such that $(AF)(a) = p^{2s+1}C$. The normalized root equation is therefore

$$u = -C - pQ(u). \tag{10.13}$$

Integral polynomial evaluation is 1-Lipschitz for the supremum p-adic metric: a common divisibility of input differences is preserved under addition and multiplication by integral elements. Consequently the right side of (10.13) has contraction constant $|p|_p < 1$ on the complete space R^d. It has a unique fixed point u.

To obtain the sharper displacement, begin with $u \in R^d$. If $u \in p^n R^d$, then $Q(0) = 0$ and polynomial congruence give $Q(u) \in p^n R^d$. For $n < k$, both C and $pQ(u)$ are divisible by p^{n+1}. Equation (10.13) therefore improves the divisibility of u by one digit. Induction gives $u \in p^k R^d$. Thus $b = a + p^{s+1}u$ satisfies (10.12), because $s + 1 + k = r - s$. Its normalized preconditioned value vanishes, and injectivity of A gives $F(b) = 0$.

Every competing root $z \in a + p^{s+1}R^d$ can be written $z = a + p^{s+1}w$. Its root equation implies the same fixed-point equation after cancellation of the nonzero scalar p^{2s+1}. Contraction uniqueness gives $w = u$, proving uniqueness on the entire stated ball. □

The existence and uniqueness statements are `Komorebi.PolynomialWeightedLift.exists_weighted_root` and `Komorebi.PolynomialWeightedLift.exists_unique_weighted_root`. The inverse hypothesis in Lean specifies an integral matrix whose image is exactly $p^s J^{-1}$; it does not substitute a potentially larger determinant valuation for s.

Example 10.17 (Strict threshold and displacement boundary)**.** For $p = 2$, $F(X) = X^2 + 3$, and $a = 1$, one has $F(a) = 4$ and $DF(a) = 2$. Thus $s = 1$ and $r = 2$ satisfy every displayed hypothesis except the strict inequality. No root exists in $\mathbb{Z}_2$, since none exists modulo eight. Hence $r \geq 2s$ cannot replace $r > 2s$ uniformly. Also, $F(X) = p^s X + p^r$ has root $-p^{r-s}$, whose displacement is not divisible by p^{r-s+1}. The guaranteed displacement cannot uniformly gain a digit.

10.8 First-order coefficient and parameter motion

Theorem 10.18 (Weighted coefficient deformation)**.** *Let $F, H \in R[X_1, \ldots, X_d]^d$, $F(a) = 0$, and $J = DF(a)$. Suppose $J \in \mathrm{GL}_d(\mathbb{Q}_p)$, $p^s J^{-1} \in M_d(R)$, and $M > 2s$. For $G = F + p^M H$, there is exactly one root $b \in a + p^{s+1}R^d$. It satisfies $b - a \in p^{M-s}R^d$ and, in $\mathbb{Q}_p^d$,*

$$b - a = -p^M J^{-1} H(a) + \varepsilon, \qquad \varepsilon \in p^{2M-3s}R^d. \tag{10.14}$$

Only the original Jacobian $DF(a)$ is subject to an inverse hypothesis.

Proof. Retain the original certificate $A = p^s J^{-1}$ and put $c = p^{s+1}$. Taylor expansion and a polynomial difference identity yield polynomial vectors Q_F, Q_H, both zero at the origin, such that

$$(AF)(a + cu) = p^{2s+1}u + p^{2s+2}Q_F(u), \qquad (AH)(a + cu) = AH(a) + cQ_H(u).$$

Consequently

$$(AG)(a + cu) = p^{2s+1}\big(C + u + pQ(u)\big),$$

where

$$C = p^{M-(2s+1)}AH(a), \qquad Q = Q_F + p^{M-s-1}Q_H, \qquad Q(0) = 0.$$

All exponents are nonnegative. The same contraction and divisibility argument as in Theorem 10.16 gives the root, the displacement p^{M-s}, and full-ball uniqueness. This constructs the root without introducing an inverse hypothesis for $DG(a)$.

For the error estimate, let $\delta = b - a$ and $D = M - s$. Taylor's theorem for the preconditioned polynomial gives

$$E_F = (AF)(b) - (AF)(a) - p^s\delta \in p^{2D}R^d.$$

Polynomial congruence gives $E_H = (AH)(b) - (AH)(a) \in p^D R^d$. The equations $F(a) = 0$ and $G(b) = 0$ imply

$$p^s\delta + p^M AH(a) = -E_F - p^M E_H \in p^{2D}R^d,$$

because $M + D \geq 2D$. Cancel p^s in the integral domain R to obtain

$$\delta + p^{M-s}AH(a) \in p^{2M-3s}R^d.$$

This integral vector is ε. After inclusion into $\mathbb{Q}_p$, $p^{M-s}A = p^M J^{-1}$ gives (10.14). □

The complete theorem is `Komorebi.PolynomialRootMotion.exists_unique_root_motion`. The error depth exceeds the displacement depth by $M - 2s \geq 1$. It is a guaranteed divisibility, not an assertion of exact valuation for every system.

Theorem 10.19 (Joint coefficient and parameter deformation)**.**
Let $F, H \in R[Y_1, \ldots, Y_e, T_1, \ldots, T_k]^e$ and $a = (y_0, t_0) \in R^{e+k}$ satisfy $F(a) = 0$. Define the actual blocks

$$J = D_Y F(a) \in M_e(R), \qquad K = D_T F(a) \in M_{e\times k}(R).$$

Suppose $J \in \mathrm{GL}_e(\mathbb{Q}_p)$, $p^s J^{-1} \in M_e(R)$, and $M > 2s$. For every $v \in R^k$, set

$$\tau = t_0 + p^M v, \qquad G = F + p^M H.$$

There is exactly one $y \in y_0 + p^{s+1}R^e$ with $G(y, \tau) = 0$. It satisfies

$$\begin{aligned} &y - y_0 \in p^{M-s}R^e, \\ &y - y_0 = -p^M J^{-1}\big(H(a) + Kv\big) + \varepsilon, \qquad \varepsilon \in p^{2M-3s}R^e. \end{aligned} \tag{10.15}$$

The equality involving J^{-1} is in $\mathbb{Q}_p^e$, with an integral error vector. The parameter equality is exact.

Proof. Use the augmented polynomial systems

$$\mathcal{F}(Y, T) = (F(Y, T), T - t_0), \qquad \mathcal{H}(Y, T) = (H(Y, T), -v).$$

Then $\mathcal{F}(a) = 0$, and its derivative is

$$B = D\mathcal{F}(a) = \begin{pmatrix} J & K \\ 0 & I_k \end{pmatrix}.$$

For $A = p^s J^{-1} \in M_e(R)$ and $c = p^s$, define

$$C = \begin{pmatrix} A & -AK \\ 0 & cI_k \end{pmatrix}.$$

The identities $AJ = JA = cI_e$ give $CB = BC = cI_{e+k}$. In the second product, the upper right block is $-JAK + cK = 0$; in the first, it is $AK - AK = 0$. Thus augmentation preserves the original inverse bound s.

Theorem 10.18 gives a unique root b of $\mathcal{F} + p^M\mathcal{H}$ in $a + p^{s+1}R^{e+k}$, with displacement depth $M - s$. Its last equations say exactly $b_T = t_0 + p^M v = \tau$. Its first equations are $G(b) = 0$. The integral first-order expression uses

$$C\begin{pmatrix} H(a) \\ -v \end{pmatrix} = \begin{pmatrix} A(H(a) + Kv) \\ -p^s v \end{pmatrix}.$$

Projection onto the solved coordinates gives

$$b_Y - y_0 + p^{M-s}A(H(a) + Kv) \in p^{2M-3s}R^e.$$

The same inclusion identity as before converts this to (10.15).

Finally, if $z \in y_0 + p^{s+1}R^e$ is any competing solved root with parameters τ, then (z, τ) belongs to the full augmented ball. Indeed, $M > 2s$ implies $M \geq s + 1$. Full-point uniqueness therefore gives $(z, \tau) = b$. This comparison proves uniqueness on exactly the solved ball stated in the theorem. □

The exact solved-coordinate statement is `Komorebi.PolynomialJointDeformation.exists_joint_deformation`, with its unique-existence interface `Komorebi.PolynomialJointDeformation.exists_unique_joint_deformation`. The triangular inverse identities hold over arbitrary commutative rings, so they apply in particular over the p-adic integers.

Example 10.20 (First-order sign and error precision)**.** For $p = 2$, $s = 0$, $M = 2$, $F(Y, T) = Y + T$, $H = 1$, $a = 0$, and $v = 1$, the prescribed parameter is $\tau = 4$ and the solved root is $y = -8$. Here $J = K = 1$, and the forcing term is $H(a) + Kv = 2$. Omitting the parameter term would predict -4, and reversing its sign would predict 0. The corresponding errors -4 and -8 are both outside $2^{2M}\mathbb{Z}_2 = 16\mathbb{Z}_2$. Thus either change would violate the stated error bound.

The error exponent cannot uniformly be increased. For $F(Y, T) = Y + Y^2 + T$, $H = 0$, $a = 0$, $p = 2$, $s = 0$, $M = 1$, and $v = 1$, there is an even root y of $y + y^2 + 2 = 0$. Its equation forces $y \equiv 2 \pmod 4$ and $y + 2 = -y^2$, so

$$4 \mid y + 2, \qquad 8 \nmid y + 2.$$

Thus the error depth $2M - 3s = 2$ is attained even for parameter-only motion. The same quadratic with a constant coefficient perturbation gives the coefficient-only boundary. These examples refute a uniform extra error digit; they do not assert optimality for every individual choice of p, s, M.

The combined results establish finite singular obstructions, smooth compatible towers, full continuous root charts after a suitable coordinate choice, and quantitative deformation for a rationally invertible solved Jacobian. Rationally rank-deficient persistence remains a separate problem. The analytic solution-map theorem in the next section supplies the regularity conclusion for these same derivative blocks.

11 Analytic regularity of the local solution map

The deformation estimates control how far an exact root moves under a small perturbation. We now show that its local dependence on the parameters is analytic, and compute the derivative from the polynomial Jacobian. The analytic argument takes place over a complete normed field. For integral p-adic data, we first include the coefficients and base point in $\mathbb{Q}_p$; $\mathbb{Z}_p$ itself is not a vector space over $\mathbb{Q}_p$.

Let $\mathbb{K}$ be a complete nontrivially normed field. Let $e, k \in \mathbb{N}$, and write the variables as $(Y, T) \in \mathbb{K}^e \times \mathbb{K}^k$. For

$$F \in \mathbb{K}[Y_1, \dots, Y_e, T_1, \dots, T_k]^e, \qquad a = (y_0, t_0),$$

define the matrices of evaluated formal partial derivatives

$$J = D_Y F(a) \in M_e(\mathbb{K}), \qquad K = D_T F(a) \in M_{e\times k}(\mathbb{K}). \tag{11.1}$$

Rows index equations and columns index variables. We first establish that these formal derivatives are the derivatives of the evaluation map.

Lemma 11.1 (Derivative of polynomial evaluation)**.** *For $P \in \mathbb{K}[X_1, \dots, X_n]$, polynomial evaluation is analytic at every point. At $a \in \mathbb{K}^n$ its strict Fréchet derivative is*

$$v \longmapsto \sum_{j=1}^{n} (\partial_j P)(a) v_j.$$

For a finite system, its strict derivative is the continuous linear map represented by the matrix of evaluated formal partial derivatives.

Proof. For the derivative statement, use polynomial induction on constants, addition, and multiplication by a variable. Constants have zero derivative. The sum rule agrees with the additivity of formal partial differentiation. For PX_j, the product rule gives

$$d(PX_j)_a(v) = a_j\, dP_a(v) + P(a) v_j,$$

which agrees with the evaluated formal product rule for every coordinate. The finite product rule for maps to coordinate spaces then assembles the system derivative. Each matrix defines a continuous linear map because the coordinate spaces are finite-dimensional over the complete field. Polynomial analyticity follows from the corresponding polynomial induction in Mathlib and its finite-coordinate assembly. □

The derivative bridge and analytic statements are `Komorebi.PolynomialAnalyticEvaluation.hasStrictFDerivAt_evalSystem`, `Komorebi.PolynomialAnalyticEvaluation.analyticAt_evalSystem`, and `Komorebi.PolynomialAnalyticEvaluation.analyticOnNhd_evalSystem`.

Lemma 11.2 (The augmented derivative)**.** *If $J \in \mathrm{GL}_e(\mathbb{K})$, the polynomial map*

$$\Phi(Y, T) = (F(Y, T), T)$$

has invertible strict derivative at a, with

$$D\Phi(a) = \begin{pmatrix} J & K \\ 0 & I_k \end{pmatrix}, \qquad D\Phi(a)^{-1} = \begin{pmatrix} J^{-1} & -J^{-1}K \\ 0 & I_k \end{pmatrix}. \tag{11.2}$$

The local inverse supplied by this derivative is analytic near $\Phi(a)$.

Proof. The lower coordinates of Φ are the parameter variables. Differentiating them gives the blocks 0 and I_k. Lemma 11.1 identifies the remaining blocks with (11.1). Multiplication of the two matrices in (11.2), in both orders, gives the identity. The resulting linear equivalence is continuous, as is its inverse. Apply the strict inverse-function theorem to Φ. The analytic inverse theorem applies to the same open partial homeomorphism, because Φ is analytic and its derivative is that continuous linear equivalence. It supplies a convergent local power-series representation of this inverse. □

The augmented Jacobian is identified in `Komorebi.PolynomialJointSystem.jacobian_augmented`. The continuous inverse block and its restriction are `Komorebi.PolynomialAnalyticBlocks.upperEquiv` and `Komorebi.PolynomialAnalyticBlocks.inverse_solved_zero`. Analyticity of the same constructed inverse is `Komorebi.PolynomialAnalyticInverse.analyticAt_inverse`.

Theorem 11.3 (Analytic local root graph)**.** *Suppose $F(a) = 0$ and $J \in \mathrm{GL}_e(\mathbb{K})$. There are open neighborhoods $t_0 \in U \subseteq \mathbb{K}^k$ and $y_0 \in V \subseteq \mathbb{K}^e$, and a map $\psi : \mathbb{K}^k \to \mathbb{K}^e$, such that*

$$\psi(t_0) = y_0, \qquad \psi(U) \subseteq V, \qquad \psi \text{ is analytic at every point of } U, \tag{11.3}$$

$$F(y,t) = 0 \iff y = \psi(t) \qquad (t \in U,\ y \in V). \tag{11.4}$$

At the base parameter, the strict Fréchet derivative is

$$\boxed{D\psi(t_0) = -J^{-1}K.} \tag{11.5}$$

In particular, ψ is differentiable on U.

Proof. Let g be the analytic local inverse of Φ supplied by Lemma 11.2. Since $F(a) = 0$, $\Phi(a) = (0, t_0)$ and $g(0, t_0) = a$. Define

$$\psi(t) = \pi_Y g(0, t),$$

where π_Y projects to the solved coordinates. This gives the base value and analyticity at t_0. The chain rule and (11.2) give

$$D\psi(t_0)v = \pi_Y \begin{pmatrix} J^{-1} & -J^{-1}K \\ 0 & I_k \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = -J^{-1}Kv.$$

To obtain the root equivalence on a product neighborhood, we choose the neighborhoods as follows. The source of the open partial homeomorphism defined by Φ is open around a; its inverse is g. Choose open sets V and U_0 containing y_0 and t_0 such that $V \times U_0$ lies in that source. Restrict U_0 further by the preimage of an open analytic inverse neighborhood under $t \mapsto (0, t)$, the open locus where this same ψ is analytic, and the interior of $\psi^{-1}(V)$. All of these sets contain t_0. For the last set, use the continuity of ψ at t_0 and $\psi(t_0) = y_0 \in V$. Their intersection is an open neighborhood U with $\psi(U) \subseteq V$.

If $t \in U$, $y \in V$ and $F(y, t) = 0$, then $\Phi(y, t) = (0, t)$. The left inverse identity forces $g(0, t) = (y, t)$, hence $y = \psi(t)$. Conversely, the right inverse identity $\Phi(g(0, t)) = (0, t)$ says both that its solved equations vanish and that its parameter coordinates are exactly t. Thus $g(0, t) = (\psi(t), t)$ and $F(\psi(t), t) = 0$. Analyticity at every point of U implies differentiability there. □

This is `Komorebi.PolynomialAnalyticImplicit.exists_analytic_implicit`. Differentiability is also included explicitly in the statement contract `Komorebi.PolynomialRegularityContracts.actual_padic_analytic_root_graph`.

Theorem 11.4 (Regularity of a supplied root selection)**.** *Under the hypotheses of Theorem 11.3, let $h : \mathbb{K}^k \to \mathbb{K}^e$ be continuous at t_0, with $h(t_0) = y_0$. Suppose $F(h(t), t) = 0$ for every t in some neighborhood of t_0. Then h and ψ agree in a neighborhood of t_0. Consequently h is analytic there as a germ and has strict derivative $Dh(t_0) = -J^{-1}K$.*

Proof. Continuity at t_0 places $h(t)$ inside V for all sufficiently close parameters. Intersect this neighborhood with U and the neighborhood on which the root equation holds. Equation (11.4) gives $h(t) = \psi(t)$ throughout the intersection. Analyticity and strict differentiability at a point are unchanged by agreement on a neighborhood. □

The formal result is `Komorebi.PolynomialAnalyticSelection.continuous_selection_regularity`. Its formulation derives $F(a) = 0$ from the root equation near the base parameter and the specified base value.

Corollary 11.5 (Regularity throughout an open parameter set)**.** *Let $U \subseteq \mathbb{K}^k$ be open, and let h be continuous on U. Suppose that for every $t \in U$,*

$$F(h(t), t) = 0, \qquad J_t = D_Y F(h(t), t) \in \mathrm{GL}_e(\mathbb{K}).$$

Then h is analytic at every point of U and its strict derivative is

$$Dh(t) = -J_t^{-1} D_T F(h(t), t) \qquad (t \in U).$$

Proof. At each $t \in U$, openness turns continuity on U into continuity at t and makes the root equation hold on a neighborhood of t. Apply Theorem 11.4 at $(h(t), t)$. □

This is `Komorebi.PolynomialAnalyticSelection.continuous_selection_analyticOnNhd`. The pointwise invertibility assumption is part of this corollary.

Corollary 11.6 (Integral coefficients and centers)**.** *Let p be any prime. Suppose $F \in \mathbb{Z}_p[Y, T]^e$, $a \in \mathbb{Z}_p^{e+k}$, and $F(a) = 0$. Let $\iota : \mathbb{Z}_p \hookrightarrow \mathbb{Q}_p$ be the inclusion. If $\iota(D_Y F(a)) \in \mathrm{GL}_e(\mathbb{Q}_p)$, the polynomial system obtained by including the original coefficients in $\mathbb{Q}_p$ has the analytic local root graph of Theorem 11.3 at $\iota(a)$. The strict derivative at the base is*

$$-\iota(D_Y F(a))^{-1} \iota(D_T F(a)).$$

The open neighborhoods and the output map are in the ambient spaces over $\mathbb{Q}_p$.

Proof. Coefficient inclusion commutes with evaluation and formal partial differentiation. Consequently both Jacobian blocks after inclusion are the entrywise inclusions of the original blocks, and the original exact root remains an exact root. Apply Theorem 11.3 to the system with included coefficients. □

This is `Komorebi.PolynomialRegularityIntegral.exists_analytic_integral_implicit`. It requires no invertibility modulo p. In particular, it includes solved Jacobians singular modulo p but invertible over the fraction field.

Example 11.7 (The derivative sign and a nonlinear control)**.** For $F(Y, T) = Y + T$, the exact branch is $\psi(T) = -T$ and $J = K = 1$. The derivative is -1. Uniqueness of the derivative rules out $+1$ over $\mathbb{Q}_p$, including $p = 2$. The nonlinear polynomial $F(Y, T) = Y + Y^2 + T$ has $F(0, 0) = 0$ and $J = K = 1$ there. Applying the formal theorem to this polynomial constructs an analytic local root graph with derivative -1. Thus the example checks the existence conclusion as well as the derivative formula. The declarations and the deliberately rejected positive-sign statement are in the regularity control and statement-contract files.

The results cover arbitrary finite e and k, including zero. The local inverse machinery explicitly handles a zero-dimensional domain. The analytic theorem does not assign a uniform numerical radius to its neighborhoods. The earlier weighted lifting and joint deformation theorems retain their quantitative radii and error bounds, while the present statements establish regularity of the local solution germ. A direct Lean identification with each historical subtype-valued chart is an additional interface task. General persistence at a fraction-field singular solved block is a separate problem.

12 Open questions and further directions

The arithmetic results above resolve Nishibuchi's conjecture throughout its stated parameter range. The local analyticity and differentiability theorem is also complete under its stated invertibility hypothesis. What remains is a set of extensions and sharper classifications, together with the broader questions raised by the original native seed-calculus program.

12.1 Singular persistence beyond an invertible solved block

The weighted lifting theorem allows the solved Jacobian to be noninvertible modulo p provided that it is invertible over $\mathbb{Q}_p$ with a controlled inverse. This includes a substantial class of lifts often called singular modulo p. The next problem is to understand Jacobians that remain rank-deficient over $\mathbb{Q}_p$, where the inverse construction is no longer available.

For a fixed polynomial system $F \in \mathbb{Z}[X_1, \ldots, X_d]^e$, a precision $L \geq 1$, and a root $\bar{x}$ modulo p^L, let

$$\mathcal{L}_{F,L}(\bar{x}) = \{x' \in (\mathbb{Z}/p^{L+1}\mathbb{Z})^d : F(x') = 0,\ x' \equiv \bar{x} \pmod{p^L}\}.$$

The finite obstruction theorem decides whether this set is empty from the residual digit and reduced Jacobian. When it is nonempty, the theorem also determines its cardinality and affine kernel structure. The conclusion concerns one level: vanishing of the obstruction permits the next step, but does not guarantee that subsequent steps are possible.

For example, $X^2 - 5$ has the root 1 modulo 2, and both 1 and 3 lift it to roots modulo 4. Every odd square is 1 modulo 8, so neither branch continues to a root modulo 8. Thus even a successful first correction can lead to branches that stop at the following level.

We seek structural hypotheses and higher obstruction data that control an entire singular branch, including its behavior under coefficient and parameter deformation. One route is to propagate suitable conditions from level to level. Another is to prove existence at all levels in a form that yields compatible branches. The finite obstruction calculus supplies the tests for individual levels, but a persistence theorem needs a further argument linking them. Any general formulation must allow multiple branches, later obstructions, and changes in the rank of the Jacobian.

12.2 Native 13.2: presentation invariance

The original conjecture asks for a canonical finite obstruction packet whose vanishing is necessary and sufficient for a one-level lift in a finitely presented arithmetic seed system. For a fixed finite polynomial presentation, the paper now supplies the precise obstruction

$$o_{F,\bar{x}} = [r] \in \operatorname{coker}\left(J_{F,\bar{x}} : \mathbb{F}_p^d \longrightarrow \mathbb{F}_p^e\right),$$

where r is the residual digit. We have proved independence from the chosen integer representative. Invariance under a change of equations or presentation requires more.

The raw cokernel already reveals the difficulty. The presentations $F = (X)$ and $G = (X, 0)$ describe the same zero locus over every coefficient ring. At zero their Jacobians are, respectively,

$$\mathbb{F}_p \xrightarrow{1} \mathbb{F}_p, \qquad \mathbb{F}_p \xrightarrow{(1,0)} \mathbb{F}_p^2.$$

Their raw cokernels are 0 and $\mathbb{F}_p$. Both distinguished obstruction classes vanish, but the ambient obstruction groups are not isomorphic. The example rules out the raw cokernel as a presentation-independent object while leaving open the possibility of an intrinsic obstruction construction.

A precise form of Native 13.2 therefore needs four ingredients:

1. a category of seed objects, including how the operators, relations, projectors, weights, and topology enter an object;
2. specified changes of presentation and a proof that they preserve the actual lift problem;
3. an obstruction object that respects dependencies among the relations, together with comparison maps or a universal property;
4. a precise meaning of "finite" and "canonical", and a proof that vanishing is equivalent to the intended one-step continuation.

For commutative algebra presentations, the relation module offers a natural way to account for dependencies among listed equations. To extend this approach to the full seed vocabulary, we would need an encoding that preserves the lift problem, together with a proof of that preservation. This is one of the links still needed between the concrete obstruction theorem and Native 13.2.

12.3 Native 13.3: finite compatibility, completion, and descent

The original proposed form is

$$X \simeq X_2^{\mathrm{wt}} \times_C {\prod_{p>2}}' X_p.$$

It leaves X, the distinguished subsets defining the restricted product, the maps to C, the ambient category, and the universal meaning of "smallest" unspecified. In particular, one must decide whether X is a global object or a completed object.

The linear compatibility theorem provides one concrete part of this program. If two local observations $J : U \to V$ and $K : U \to W$ are linear maps from the *same* correction module, define

$$C_{J,K} = U/(\ker J + \ker K).$$

The attainable observations $\operatorname{im} J$ and $\operatorname{im} K$ have canonical labels in this quotient. Equality of labels is equivalent to simultaneous realization by a single correction. The joint image is exactly the fiber product over $C_{J,K}$, with the proved linear universal property. The declarations are `Komorebi.SharedCorrection.labels_eq_iff_common`, `Komorebi.SharedCorrection.joint_range_eq_fiber`, and `Komorebi.SharedCorrection.common_universal`.

To apply this result arithmetically, one must construct the common correction module and the observation maps, then show that the resulting compatibility condition agrees with the intended arithmetic one.

Why the global/completed distinction is necessary. Consider constant sequences with integer initial value. Their global parameter space is $\mathbb{Z}$, while their local parameter spaces are $\mathbb{Z}_p$. Under the canonical

local maps, the image of the global objects is the diagonal

$$D = \{(n)_p : n \in \mathbb{Z}\} \subseteq \prod_p \mathbb{Z}_p.$$

This diagonal is dense: a basic open set specifies finitely many congruences, and the Chinese remainder theorem provides an integer satisfying them. It is proper: the tuple with 2-adic coordinate zero and all odd-adic coordinates one is not the image of an integer. Indeed an integer zero in $\mathbb{Z}_2$ must be zero, contradicting its 3-adic coordinate.

Consequently this canonical global diagonal cannot be an equality locus

$$\{(a,b) : f(a) = g(b)\} \subseteq \mathbb{Z}_2 \times \prod_{p>2} \mathbb{Z}_p$$

for continuous maps to a Hausdorff space C. Such an equality locus is closed, as the inverse image of the closed diagonal in $C \times C$, whereas D is proper and dense. This topological argument, retained from the native progress note, concerns the stated topological interpretation. It has not been formalized in Lean for this manuscript.

Finite CRT reconstruction leads to the familiar completed object $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$, but does not identify which local tuples come from a single global integer. A complete form of Native 13.3 must therefore specify finite-level compatibility, transition maps and completion, and any additional global descent condition. If recurrence, spectral, or Frobenius information is to supply that condition, it must be expressed through defined maps or constraints on the local data, with a precise account of how they enter C.

The accompanying `native/Native_Conjectures_Progress.md` records further results in restricted settings: an intrinsic finite Ext^1 obstruction for finitely presented commutative integer algebras, finite CRT reconstruction for polynomial solution sets, and a characterization of rectangular finite relations by equality of common labels. The obstruction result is a stated specialization of a classical external theorem; the CRT and rectangularity results have prose proofs in the note. These are preserved mathematical progress, with the restrictions and source records given there, rather than Lean completions of the general Native 13.2 and 13.3 statements.

12.4 Arithmetic refinements

Write $U_{N,j}(n) = j!C_{N,j}(n)$ and $U_j = U_{4,j}$, and use $v_p(0) = \infty$ when discussing valuation bounds. The following classification questions remain open in this program.

1. **The remaining higher-difference classification.** The new strengthening chapters determine every repeated-step valuation in addresses $j = 0, 1$, give the exact criterion for equality in the common mixed-step lower bound there, and determine address-two valuations at all power-of-two orders. What remains is a complete exact formula in address two at the other orders, a full classification in address three, and the amount of additional cancellation in mixed-step cases where the leading term vanishes. Numerical patterns through a finite range do not settle these questions.
2. **Upper-half addresses at larger dyadic rank.** The exact first-difference laws now include $N = 2, 4, 8$ and every lower-half address for $N = 2^a \geq 4$. Larger upper-half addresses need a finer classification. Explicit counterexamples show that extrapolating one low-order Newton coefficient to a universal metric law is false. A least period is measured in packet rows; its Euler-degree displacement is NP.
3. **Exact affine Mahler-kernel valuations.** For the kernel of Section 2.7, determine exact values of

$$v_p\big(\Theta_{M,s}(r,k)\big), \qquad \Theta_{M,s}(r,k) = [z^r](1+z)^s((1+z)^M - 1)^k,$$

beyond the explicit special families. The minimum valuation of the summands gives a lower bound. Equality requires a separate argument excluding cancellation.

4. **Classical tail root classification and simplicity.** The later tail program gives an exact factorization, finite preparation for each fixed depth, and several infinite regular-wall and simplicity families, as summarized in Appendix A. A universal explicit root classification and universal simplicity of the remaining analytic factor are still open. In the exceptional address-one quadratic reduction, the residual task is a nonvanishing discriminant condition. Empty tails remain exactly zero. This problem is distinct from the solved Nishibuchi congruence conjecture.

These refinements build on the proved estimates and exact cases, including the sharp uniform classical dyadic bound and the resolution of Nishibuchi's conjecture.

12.5 Historical material and interface work

The original Paper IX, with corrected proof-status labels, survives as a 28-page PDF. Its recovered formalization supports the framework, Euler coordinates, Mahler differences, unit coordinates, recurrence packets, and CRT results retained in this manuscript. The exact original TeX bytes, original figure-generation files, and complete historical experiment package were not recovered with that PDF. The reconstructed Annals text is reproduced in Appendix D, with its separate provenance and local qualifications. Their older figure and test counts remain historical records; those computations have not been reproduced for this consolidation.

A reconstructed textual copy of the Annals manuscript was also recovered. It includes finite-field mode and subgroup inversion, Fourier invariance, Carmichael and matrix transfer, Amice pullbacks, depleted analytic Euler branches, and extensive polyphase formulas. The sharp classical dyadic kernel and truncation claims from that text have since been connected to checked Lean proofs. The finite polyphase formulas used for explanation here have short explicit algebraic derivations. The remaining claims in that larger historical text have not all been formalized. The source archive retains the text with its provenance, and the statement map distinguishes these historical claims from the certified claims in the present manuscript.

Remaining recovery work includes the precise older odd-prime edge identities, their stated ranges, and the binary digit-sum form of the checked weights. For each item, the work is to recover the formula with its stated range, check the proof, and identify any corresponding Lean declaration. Where the historical source is still missing, the record remains incomplete.

One further task is to connect the older chart constructions to the new regularity theorem. That theorem applies to polynomials with arbitrary coefficients in the ambient field $\mathbb{Q}_p$ and identifies the analytic germ of every continuous ambient-field root selection under its hypotheses. Several older rectangular charts were defined using integer coefficients, integer centers, or subtype-valued maps. Comparing each such historical interface explicitly with the new ambient formulation remains to be done. These comparisons would connect the existing constructions to the proved analytic implicit-function theorem. General singular persistence remains the separate problem described in Section 12.1.

13 Formal evidence and reproduction

The verification record supports the manuscript in several ways. The Lean source states the quantified theorem and its hypotheses. A compiler report records acceptance of that source by a specified Lean and Mathlib environment. An axiom audit identifies the axioms used by the resulting named declarations.

Statement contracts and mathematical controls check selected interfaces and boundary cases. Finally, source correspondence reviews compare the formal objects with the intended mathematical statement. Each check has a specific role: numerical examples test particular cases, whereas a universal claim requires a proof, and correspondence with an external conjecture requires a comparison of the statements.

This edition revises the exposition and checks the evidence index against the preserved sources. The Lean sources are unchanged. Their compiler acceptance is supported by the earlier successful reports, whose source and log hashes are checked by the accompanying index. The most recent compilation covers local analytic regularity and its polynomial dependencies; earlier reports cover the arithmetic and native obstruction developments.

The proofreading pass also restores several results omitted from the first consolidation and records the comparison with the original paper. Its coverage matrix and semantic review appear in `consolidation/editorial/`. The Bernoulli boundary is accompanied by a classical derivation, without a claim of a dedicated Lean certificate. The reviews distinguish this addition from restored results already present in the checked sources.

13.1 Verification frontiers

Table 1 gives the recorded scope of each major verification frontier. An *inventoried module* is a source file in that frontier's formalization list. Older lists sometimes include files of examples or positive controls in this group. A *check* is an entry in the report's run list, including expected rejection of invalid claims. An *axiom closure* is the recorded axiom dependency set of an explicitly named declaration; definitions and abbreviations contribute to this count as well as theorems. None of these columns is a count of distinct mathematical results across the paper.

Preserved frontier	Modules	Checks	Axiom closures
Original completion baseline	79	$79 + 37 + 3$	1000
Dyadic strengthening and isometry	22	38	246
Standalone finite affine obstruction	2	4	17
Shared correction quotient	6	13	60
Polynomial one-step lifting	9	18	91
Exact polynomial lift counts	9	17	106
Smooth infinite towers	14	22	127
Coefficient deformation	17	26	159
Continuous parameter charts	25	40	244
Weighted lifting	30	50	304
First-order root motion	35	58	326
Prescribed parameter obstruction	39	65	375
Prescribed parameter towers	45	74	408
Joint coefficient and parameter motion	49	82	460
Local analytic regularity	55	91	515

Table 1: Recorded verification scopes. Rows overlap extensively through inherited sources and must not be added. The baseline separates 79 module checks, 37 control checks, and three additional recorded runs; its nine dependency revision comparisons are recorded separately.

The baseline inventory contains 836 theorem commands, 162 definitions and two abbreviations, matching its 1000 audited names. The latest polynomial inventory contains 322 theorem commands, 76 definitions and seven abbreviations in its 55 formalization modules. Its eight positive-control support modules and

13 statement-contract modules bring the full audit to 415 theorem commands, 92 definitions and eight abbreviations, hence 515 closures. The 91 checks also include 12 negative controls. Six new regularity modules contribute 30 theorem commands and eight definitions; all 49 inherited formalization modules are unchanged in that release.

The arithmetic strengthening inventory exposes a small historical metadata error. Its summary fields say 179 theorems and 12 definitions, while its module entries list 215 theorem commands, ten lemma commands, 20 definitions and one abbreviation. These 246 named entries exactly match the preserved axiom audit. The consolidation uses the entries and audit, and records the stale summary fields explicitly. The original inventory remains unchanged as historical evidence. This is a correction to the reporting metadata, not a newly discovered proof or a modification of a checked theorem.

13.2 Claim-to-source correspondence

The source references attached to the mathematical sections identify the principal declarations for each result. Their complete names, exact source hashes, roles and verification frontiers are collected in `consolidation/audit/DECLARATIONS.csv`. The accompanying `consolidation/audit/EVIDENCE_INDEX.json` records the corresponding inventories, reports and retained logs. In particular, the following interfaces distinguish the principal completed results from generic auxiliary lemmas:

- The external conjecture is assembled by `NishibuchiCompletion.conjecture_3_9` in the original completion baseline. Its coefficient identification, integrality, parameter split and explicit threshold are part of that proof chain.
- Exact $N = 4$ packet periods are supplied by `DyadicFourLeastPeriods.least_packet_period`; the isometric extension is supplied by `DyadicFourIsometry.existsUnique_isometry`. The normalization of the original coefficients is an explicit part of these interfaces.
- The classical sharp edge tower is supplied by `DyadicSharpTower.dyadic_supercongruence_sharp`, together with its first omitted error valuation. Its optimality concerns the stated uniform truncation bound.
- Prescribed parameter lifting uses the actual residual and actual partial Jacobians in `PolynomialPrescribedObstruction.prescribed_lift_iff_obstruction_zero`. The quotient criterion and exact lift counts are not assumptions of the later tower results.
- Joint deformation is supplied by `PolynomialJointDeformation.exists_unique_joint_deformation`. Local regularity is supplied by `PolynomialAnalyticImplicit.exists_analytic_implicit`, with the actual polynomial derivative and the derivative formula $-J^{-1}K$.
- `PolynomialAnalyticSelection.continuous_selection_regularity` transfers the analytic germ and strict derivative to any continuous local root selection satisfying the stated root and base-point conditions.

Here arithmetic declaration names have namespace prefix `KomorebiPaperIX.`; polynomial and native declarations have prefix `Komorebi.`. The section maps and CSV retain the full names.

The declaration index covers every explicitly named theorem, lemma, definition and abbreviation in the 14 release-frontier inventories, together with the earlier standalone affine report. Byte-identical copies of a source file are collapsed, while all their locations remain recorded. If the same qualified name occurs in different source versions, the versions remain separate and are marked. Anonymous examples, compiler-generated declarations, failed development attempts, and the contents of earlier nested historical snapshots are not counted as named results in this index. Negative-control source files

are checked for integrity but are not recorded as proved mathematical assertions. Consequently, the number of CSV rows must not be quoted as a global unique theorem count.

13.3 External statement and library dependencies

The original conjecture correspondence was reviewed against Nishibuchi's arXiv version `2605.10677v2`, dated 1 June 2026. The retained source record identifies the official version, the full title, the PDF checksum and the extracted-text checksum. The detailed review compares Definition 2.1, Notation 3.2 and Conjecture 3.9 with the actual reciprocal EGF coefficients, the factor $j!$, the residue range, the degree displacement, and the threshold in the formal capstone. Nishibuchi's conjecture is the conclusion to be proved; it is not an axiom imported into Lean.

The full external paper is not redistributed in this source release. Consolidation preserves the earlier source correspondence audit and its version record. The earlier consolidation recorded a blocked request to the official arXiv endpoints. For the preserved prior edition, the official unversioned PDF was inspected; its internal version is v2, dated 1 June 2026. Definition 2.1 and Remark 2.3 (printed page 3), Notation 3.2 (page 5), and Conjecture 3.9 (page 8) confirm the parameter, index, degree-displacement and normalization match. Some direct versioned-endpoint requests were blocked in that earlier check. The later strengthening source review reached the primary PDF and HTML through the official record and confirmed the same v2 statement. These are dated source-access records, not mathematical hypotheses. This source check is distinct from the inherited compiler acceptance. No broad literature or priority claim is made.

The mathematical environment is Lean `4.34.0-rc2`, with Mathlib revision `aeb72f1e9981b6d35b426b16dca7759be3e7ba06`. The allowed axiom dependencies are `propext`, `Classical.choice` and `Quot.sound`. The reports do not describe a rebuild of Mathlib from source. The local regularity audit separately records the applicable Mathlib analytic inverse and strict derivative interfaces and their hypotheses. In particular, the field-valued analysis is performed over a complete nontrivially normed field; its integral specialization embeds the data into $\mathbb{Q}_p$. It does not use $\mathbb{Z}_p$ as a vector space over $\mathbb{Q}_p$.

13.4 Reproduction and limits of the consolidation check

The archive contains the mathematical sources, verifier scripts, controls, contracts, prior logs and exact inventories. The pinned Lean and Mathlib kit is supplied separately. From the consolidated archive root, the latest polynomial gate can be reproduced with a new output directory:

```
python3 replay_paper9_regularity.py \
  --kit /absolute/path/to/pinned-kit \
  --output /absolute/path/to/new-polynomial-run
```

The preserved arithmetic, native and shared-correction chain has a separate entry point:

```
python3 replay_shared_correction.py --all \
  --kit /absolute/path/to/pinned-kit \
  --output /absolute/path/to/new-arithmetic-run
```

These commands reproduce the recorded checks; they were not rerun during manuscript consolidation or proofreading. The second command rebuilds the preserved baseline before the arithmetic continuation. Each output path must be new. The reports retain the original full commands, runtime fingerprints and

output hashes. The executable-path compatibility adapter used in the recorded environment restores runtime path discovery; it does not alter Lean's mathematical kernel or the source statements.

For the narrower evidence reconciliation carried out for this edition, use

```
python3 consolidation/tools/evidence_index.py \
  --source . --output /absolute/path/to/new-evidence-index
```

This check compared 644 source-file occurrences with their recorded hashes and 713 retained log occurrences with their recorded hashes. Four additional logs from the older standalone affine report were fingerprinted, but that legacy report does not contain prior log hashes; those four are not promoted to hash comparisons. Every named declaration in the selected inventories matches the appropriate recorded standard-axiom dependency set. These are integrity and provenance checks, not another Lean compilation.

Expected negative-control failures are retained as evidence that specific invalid strengthenings are rejected. Examples include an unjustified extra precision digit, omitted parameter terms, a wrong derivative sign, and uniqueness without the appropriate hypotheses. Development failures and interrupted runs remain historical records; only the successful designated release reports support the verification table. Only accepted declarations in the designated successful reports support claims of formal verification.

Finally, the proof of the original stated conjecture does not resolve every extension mentioned in this manuscript. The open questions concerning rationally rank-deficient singular points, presentation invariance, global or completed descent, and further exact arithmetic have no certificate supplied by these counts. Section 12 states the remaining implications explicitly.

13.5 Inherited checks for the prior final edition

The preserved prior final-edition review reconstructed the arguments in its 54 written proof blocks under their stated hypotheses. The two expanded paragraphs in the all-address proof make the inverse-flatness commutator and the coefficient-extraction saturation explicit. The recovered historical text received five local repairs: integral observation vectors for matrix congruences, positive precision for matrix amplification, integral Mahler coefficients for the sparsification consequence, the corrected modulo-seven example, and the even-subsequence range for source changes. Its remaining historical and external-source evidence limits are retained in Appendix D; preservation of its 82 proof bodies is not a blanket formal certificate.

An independent Python gate using arbitrary-precision integers and `fractions.Fraction` passed 116,492 deterministic assertions in nine finite families: packet construction, foundational identities, divisor-branch congruences, four-packet sharpness, classical truncations, polynomial digit fibers, shared affine quotients, finite towers and charts, and weighted deformation. Nineteen invalid strengthenings were rejected by concrete controls. The exact ranges, coefficient domains, scripts and output hashes are in the accompanying `verification/computation` manifests. These finite tests do not establish a universal theorem.

The unchanged early-row script separately reproduced 312 finite period cases and its odd-prime row-zero boundary control. Other research-script replays, unchanged baseline regression gates and recovered-text controls are recorded separately in the same audit directory; their totals are not added to the independent gate. Lean and Lake were unavailable in this execution environment. No fresh compiler run, infinite p-adic computation, analytic-continuation computation or hardware benchmark is asserted.

13.6 Checks for the strengthened Euler edition

The new proof routes are recorded in the threshold-free strengthening checkpoint, whose manifest pins 52 files. Its six principal proof audits are internal computational-assistant reviews and identify the exact proof revisions they checked. The manuscript delta review compares the new chapters and current scope statements with those proofs. It does not replace the earlier Lean kernel or constitute external peer review.

The dedicated arithmetic experiment computed the four reciprocal rows through index 512 modulo 2^{4096}, with independent exact-rational inversion through index 16. It checked 525,312 first differences, 1,141,424 repeated-step factorial bounds, 139,000 mixed-step bounds, and 2,048 Newton coefficients. The supplemental experiments cover additional mixed-step and sharpness cases and retain their separate tested ranges. Their outputs and source hashes are preserved under `preserved/threshold_free/computation`. All these are bounded implementation checks; the written arguments establish the universal claims.

The visual guide has its own exact computation contract and figure manifest. Its modular recurrence is compared with independent rational inversion, and each theorem-derived plot is marked as such. The fresh static Lean review reconciles the unchanged formal sources and inherited compiler logs, with counts stated in the Lean guide. No Lean or Lake executable was available for this release, and no new compiler acceptance is asserted for the new proof routes.

The release includes the editable three-document sources, exact figure generator, the strengthened proof checkpoint, unchanged earlier source packages, and manifests for the final outputs. PDF compilation, figure arithmetic, manifest validation, source reconciliation, and written proof auditing are separate checks with separate scopes.

A Exact classical Euler tails and the remaining simplicity problem

This appendix records the related classical Euler tail program at its stationary-separation checkpoint of 5 October 2026. Its purpose is to state the strongest proved reduction and its exact remaining obligation. The universal simplicity problem below is a separate question from Nishibuchi's Conjecture 3.9, which the congruence results of this paper resolve. It is also separate from the presentation and gluing questions called Native 13.2 and Native 13.3 in the original Paper IX.

The tail results have mathematical proofs, supplemented in the indicated families by complete coefficient certificates and proved bounds for every omitted coefficient. Bounded sampled values support implementation checks; they do not supply the universal quantifiers. The latest continuation did not run Lean. Its mathematical claims must therefore be read with the formal-coverage distinctions in the Lean guide.

A.1 The ordinary Euler packet identity

To distinguish ordinary signed Euler numbers from the generalized numbers of the main text, write

$$\operatorname{sech} t = \sum_{n\ge 0} \mathcal{E}_n \frac{t^n}{n!}, \qquad \frac{t}{e^t-1} = \sum_{n\ge 0} B_n \frac{t^n}{n!}, \qquad B_1 = -\frac{1}{2}.$$

Throughout this appendix, $v_2(2) = 1$ and $v_2(0) = \infty$. Let $s_2(n)$ be the number of ones in the binary expansion of the nonnegative integer n. Define

$$A(N) = \frac{3^{2N+2}-1}{4(2N+1)(2N+2)}, \qquad \left(\frac{z}{\sinh z}\right)^2 = \sum_{m\ge 0} \beta_m z^{2m}, \qquad \kappa_m = 2^{2m}(2m)!\beta_m.$$

The initial values are $\beta_0 = \kappa_0 = 1$. For $m \ge 1$, differentiation of $z\coth z$ gives

$$\beta_m = -\frac{2^{2m}(2m-1)B_{2m}}{(2m)!}, \qquad \kappa_m = -2^{4m}(2m-1)B_{2m}. \tag{A.1}$$

The classical dyadic Bernoulli denominator law $v_2(B_{2m}) = -1$ and Legendre's formula yield

$$v_2(\beta_m) = s_2(m) - 1, \qquad v_2(\kappa_m) = 4m - 1. \tag{A.2}$$

The source identity

$$\sum_{N\ge 0} A(N)\frac{t^{2N}}{(2N)!} = \frac{\sinh(2t)\sinh t}{2t^2}$$

implies, after multiplication by $4t^2/\sinh^2(2t)$,

$$\mathcal{E}_{2N} = \sum_{m=0}^{N} \kappa_m \binom{2N}{2m} A(N-m). \tag{A.3}$$

For a retained depth $M \ge 0$, put

$$\mathcal{T}_M(N) = \mathcal{E}_{2N} - \sum_{m=0}^{\min(M,N)} \kappa_m \binom{2N}{2m} A(N-m).$$

An empty tail, $N \le M$, is exactly zero.

Proposition A.1 (Sharp uniform classical truncation bound)**.** *For every $M, N \geq 0$,*

$$v_2\big(\mathcal{T}_M(N)\big) \geq 4M + 3.$$

For each fixed M, equality holds at $N = M + 1$.

Proof. The source $A(n)$ is a dyadic unit for every nonnegative integer n. Each nonempty summand has $m \geq M + 1$, so (A.2) and integrality of the binomial coefficient give the asserted bound. At $N = M + 1$, only the first omitted summand remains; its binomial coefficient and $A(0)$ both equal one. □

Thus later row-dependent cancellation can improve a particular tail, but cannot increase the uniform exponent required to hold at every row.

A.2 A finite carry minimum and an exact all-row factorization

Let $\mathbb{Z}_2\langle x\rangle$ denote the restricted power series: their integral coefficients tend to zero dyadically. Extend A by

$$\mathscr{A}(x) = \frac{9^{x+1} - 1}{8(x+1)(2x+1)}, \qquad \mathscr{A}(-1) = -\frac{\log 9}{8}.$$

The apparent singularity at $x = -1$ is removable. Since $v_2(\log 9) = 3$, its convergent expansion is

$$\mathscr{A}(x) = \frac{\log 9/8}{1 + 2x} \sum_{h \geq 0} \frac{(\log 9)^h}{(h+1)!}(x+1)^h. \tag{A.4}$$

It follows coefficientwise that $\mathscr{A} \in \mathbb{Z}_2\langle x\rangle^{\times}$, $\mathscr{A} \equiv 1 \pmod 2$, and $\mathscr{A}(N) = A(N)$.

Set $r = M + 1 \geq 1$ and define

$$P_q(x) = \prod_{i=0}^{q-1} (x - i)(2x - (2i+1)), \qquad P_0 = 1,$$

$$H_r(x) = \sum_{q \geq 0} 8^q \frac{\beta_{r+q}}{\beta_r} P_q(x) \mathscr{A}(x - q).$$

The coefficient valuations tend to infinity. At any nonnegative integer x, terms with $q > x$ vanish. Cancelling the factorials in (A.3) therefore gives the exact identity

$$\mathcal{T}_{r-1}(N) = \kappa_r \binom{2N}{2r} H_r(N - r) \qquad (N \geq r). \tag{A.5}$$

This is a finite rational identity at every original Euler row.

The scalar packet weights have exact valuations

$$\lambda_r(q) = 3q + s_2(r + q) - s_2(r).$$

Put

$$\mu_r = \min_{q \geq 0} \lambda_r(q), \qquad J_r = \{q : \lambda_r(q) = \mu_r\}, \qquad G_r = 2^{-\mu_r} H_r.$$

Since $\lambda_r(0) = 0$ and $\lambda_r(q) \geq 3q + 1 - s_2(r)$, every minimizing address satisfies

$$0 \leq q \leq \left\lfloor \frac{s_2(r) - 1}{3} \right\rfloor. \tag{A.6}$$

In particular $\mu_r \le 0$, and the minimum is found by a finite carry calculation. Reduction of the whole series gives

$$\overline{G}_r(X) = \sum_{q \in J_r} X^{\underline{q}} \quad \text{in } \mathbb{F}_2[X]. \tag{A.7}$$

Its highest degree is $\max J_r$, with leading coefficient one. Thus the normalization is the actual Gauss normalization.

Theorem A.2 (Exact classical tail valuation)**.** *For every $r \ge 1$ and $N \ge r$,*

$$v_2\big(\mathcal{T}_{r-1}(N)\big) = 4r - 1 + v_2\binom{N}{r} + \mu_r + v_2 G_r(N-r). \tag{A.8}$$

There are unique factors

$$G_r(X) = W_r(X)U_r(X),$$

where $W_r \in \mathbb{Z}_2[X]$ is monic of degree $d_r = \max J_r$ and $U_r \in \mathbb{Z}_2\langle X\rangle^\times$ satisfies $U_r \equiv 1 \pmod 2$. Consequently $v_2 G_r(x) = v_2 W_r(x)$ for every $x \in \mathbb{Z}_2$, and

$$\deg W_r \le \left\lfloor \frac{s_2(r)-1}{3} \right\rfloor.$$

Proof. Equation (A.5), the exact weight in (A.2), and the digit-sum identity $v_2(\binom{2N}{2r}) = v_2(\binom{N}{r})$ prove (A.8). The binomial coefficients themselves need not be equal.

For preparation, start from the monic polynomial $g = \overline{G}_r$ and its zero-one lift W_1, with $U_1 = 1$. Suppose $G_r - W_k U_k$ is coefficientwise divisible by 2^k. Its divided reduction is a finite polynomial because G_r is restricted. Divide it by g in $\mathbb{F}_2[X]$:

$$\overline{(G_r - W_k U_k)/2^k} = gq + s, \qquad \deg s < d_r.$$

With zero-one coefficient lifts, set $W_{k+1} = W_k + 2^k s$ and $U_{k+1} = U_k + 2^k q$. The error is now divisible by 2^{k+1}, the polynomial remains monic of degree d_r, and the unit reduces to one. The sequences converge in the Gauss norm. Uniqueness of quotient and remainder at successive binary digits proves uniqueness of the factors. For $d_r = 0$, take $W_r = 1$. The degree bound follows from (A.6). □

This theorem reduces every depth to a finite polynomial without discarding the infinite analytic tail. It does not explicitly classify every W_r, decide every repeated root from a fixed precision, or prove that all its dyadic roots are simple.

A.3 Regular walls: exact root emergence

For a wall label $j \ge 1$ and $0 \le q < j$, define

$$e_j(q) = 3q + s_2(j-1) - s_2(j-q-1), \qquad E_j = \min_{q<j} e_j(q),$$

$$\Theta(j) = 3j + 1 + s_2(j-1), \qquad L_j = \Theta(j) - E_j.$$

Here E_j is a carry minimum, not an Euler number. A wall label is *regular* when $e_j(q) > 0$ for $1 \le q < j$. For $r = 2^L - j$ with $L > L_j$, put $\delta = L - L_j$. The carry identities give

$$\mu_r = E_j - \delta, \qquad J_r = \{j\}, \qquad G_r(x) = P_j(x)U(x-j) + 2^\delta V(x),$$

where $U \equiv 1 \pmod 2$. For a regular label, also $E_j = 0$ and $V \equiv 1 \pmod 2$.

For $0 \le i < j$, let

$$d_i = v_2(i!) + v_2((j-1-i)!), \qquad k_i = 1 + \lfloor \log_2 \max(i, j-1-i) \rfloor ,$$
$$B_i = i + 2^{k_i}\mathbb{Z}_2, \qquad t_i = d_i + k_i.$$

At $j = 1$ use $d_0 = k_0 = t_0 = 0$. The balls B_i partition $\mathbb{Z}_2$. Indeed a residue represented twice among $0, \dots, j-1$ splits into its two children; every other represented residue remains one ball. On each such ball,

$$P_j(i + 2^{k_i} y) = 2^{t_i} y Q_i(y), \qquad Q_i \in \mathbb{Z}_2\langle y \rangle^{\times}, \qquad Q_i \equiv 1 \pmod 2.$$

Theorem A.3 (Regular-wall root law)**.** *For every regular label j and every integer $\delta \ge 1$,*

$$v_2 G_r(x) = \begin{cases} \delta, & x \in B_i,\ \delta < t_i, \\ d_i + v_2(x - \alpha_i), & x \in B_i,\ \delta \ge t_i. \end{cases}$$

In the second case there is exactly one root α_i in B_i, it is simple, and $v_2(\alpha_i - i) = \delta - d_i$. Thus the number of dyadic roots is exactly

$$\#\{i : 0 \le i < j,\ t_i \le \delta\}.$$

Proof. If $\delta < t_i$, the unit perturbation has smaller valuation than the skeleton term throughout B_i, so the valuation is exactly δ. Otherwise, after the ball substitution and division by 2^{t_i}, the coefficientwise reduction is $y + c$ for a constant $c \in \mathbb{F}_2$. A restricted integral series with this reduction is a dyadic isometry: division of its difference by $y - z$ gives an odd linear coefficient plus even higher contributions. Binary lifting therefore gives its unique root and exact distance law. Evaluating at $x = i$ gives the stated displacement, since $G_r(i) = 2^{\delta} V(i)$ has exact valuation δ. The partition proves exhaustion of the roots. □

The largest emergence threshold is

$$\max_i t_i = v_2((j-1)!) + \lceil \log_2 j \rceil .$$

For $j = 2^a$, all thresholds equal $j - 1$. There are no roots before that distance and exactly j simple roots at and after it, within the positive-distance range. The equality and pre-wall cases are also resolved: regular labels $j \ge 2$ have no roots there; $j = 1$ has one simple odd root at equality.

For example, $j = 5$ has $(t_0, \dots, t_4) = (6, 3, 4, 3, 6)$. The exact root counts at distances $1, 2, 3, 4, 5, \ge 6$ are $0, 0, 2, 3, 3, 5$. The older sufficient stages $3, 5, 7$ are not the sharp emergence stages, which are $3, 4, 6$. The first irregular label, $j = 17$, has $V \equiv 1 + X \pmod 2$; the regular-wall theorem does not apply. Complete coefficient certificates close distances 1 through 19, and the general separation theorem gives all seventeen simple roots for $\delta \ge 20$. Those two kinds of evidence together resolve that family.

A.4 What the later simplicity reductions prove

For a general depth r, call $j_r = \max J_r$ its *true terminal*. This definition depends on the actual carry minimum; rewriting the same integer as another wall expression does not change j_r. The universal simplicity target is

$$\text{for every integer } r \ge 1 \text{ and } x \in \mathbb{Z}_2, \qquad G_r(x) = 0 \implies G_r'(x) \ne 0. \tag{A.9}$$

The following are completed restricted results of the later continuations. Their family statements include a proof that the indicated label is the true terminal throughout the displayed ranges.

Depth r	Odd multiplier range	Outer scales
$B2^L - 129$	$0 < B < 2^{89}$	$L \geq 392$
$B2^L - 161$	$0 < B < 2^{932}$	$L \geq 487$

All dyadic roots are simple in both infinite integer families. For arbitrary larger admissible multipliers, a multiple root with true terminal 129 can occur only at $L = 515$ and in two necessary multiplier classes modulo 2^{90}. With true terminal 161, it can occur only at $L = 640$ and in one necessary multiplier class modulo 2^{935}. These classes are necessary conditions. The records do not assert existence of a singular ordinary Euler index in them.

For terminal 161, the neighboring phases are resolved uniformly: $L = 639$ has exactly one simple root and $L = 641$ exactly three simple roots, for every positive odd multiplier with that true terminal. The middle phase is reduced, before the deeper certificate is applied, to

$$B \equiv 17659 \pmod{2^{18}}, \qquad x \equiv 10202 \pmod{2^{15}}.$$

The 935-bit multiplier condition composes this whole-phase reduction with an inherited 917-bit local parameter certificate. It is neither a new 935-bit reconstruction nor an all-precision interpolation of the undepleted Euler family.

The structural argument also extends beyond lower address one. For a certified true-terminal wall, let a be its last lower minimizing address, $k = j_r - a$ the regular complement, and $t = v_2(k)$. Under the regular lower-address hypotheses of the source theorem, all roots above emerged complement balls are simple. Every root is simple at every wall distance in the band

$$\Theta(a) \leq t < \Theta(a) + v_2(a!).$$

Equality walls are simple even outside that band. This includes certified terminal-258 walls with $a = 2$, $k = 256$, and $L \geq 777$. For $a = 3$, $k = 4096$, only $L = 16385$ remains a possible critical scale. Outside the band, the theorem gives necessary local valuation conditions; it does not exclude the remaining unit cancellation. Irregular lower addresses need their own prefix analysis.

A.5 The actual stationary discriminant

The current focused obstruction is the admitted ordinary-index family

$$r = (17659 + 2^{18}T)2^{640} - 161, \qquad T \in \mathbb{Z}_{\geq 0}, \qquad j_r = 161. \tag{A.10}$$

On its surviving chart define the *actual*, undepleted function

$$K_r(Z) = 2^{-178} G_r(10202 + 2^{15}Z).$$

The complete coefficient theorem gives

$$K_r(Z) = a_r + 2b_r Z + u_r Z^2 + 256 Z^3 R_r(Z), \quad a_r, b_r \in \mathbb{Z}_2, \quad u_r \in \mathbb{Z}_2^\times, \quad R_r \in \mathbb{Z}_2\langle Z \rangle. \tag{A.11}$$

Its half derivative is a dyadic isometry. It therefore has a unique critical point $c_r \in \mathbb{Z}_2$. Set $\Phi_r^{\mathrm{act}} = K_r(c_r)$. The earlier independent-undepletion notation for this same scalar is $\Phi_r(2^{2r-25})$. Excluding a multiple root on the surviving chart is exactly the assertion $\Phi_r^{\mathrm{act}} \neq 0$.

Proposition A.4 (Exact quadratic compression)**.** *There is a unique preparation*

$$K_r(Z) = \mathcal{U}_r(Z)\mathcal{W}_r(Z), \qquad \mathcal{W}_r(Z) = Z^2 + 2\beta_r Z + \alpha_r, \qquad \mathcal{U}_r \equiv u_r \pmod{256}.$$

With $\delta_r = \beta_r^2 - \alpha_r$ *and* $\tau_r = \mathcal{U}_r'(c_r)/\mathcal{U}_r(c_r)$,

$$\delta_r = -\frac{\Phi_r^{\mathrm{act}}}{\mathcal{U}_r(c_r)}\left(1 - \frac{\tau_r^2 \Phi_r^{\mathrm{act}}}{4\mathcal{U}_r(c_r)}\right), \qquad v_2(\delta_r) = v_2(\Phi_r^{\mathrm{act}}). \tag{A.12}$$

The equality includes the zero case.

Proof. The same successive monic polynomial division used in Theorem A.2, started at precision eight, preserves the quadratic and unit reductions of (A.11). Write $W = Z^2 + 2\beta_r Z + \alpha_r$ and $w = W(c_r)$. Stationarity gives $W'(c_r) = -\tau_r w$, while $w = \Phi_r^{\mathrm{act}}/\mathcal{U}_r(c_r)$. The quadratic identity $\beta_r^2 - \alpha_r = W'(c_r)^2/4 - W(c_r)$ proves the formula. Since $\tau_r \in 256\mathbb{Z}_2$, its bracket is a unit. □

This compression retains the unit-cofactor derivative. It does not assert that c_r, α_r, or β_r is rational or algebraic over $\mathbb{Q}$. The remaining arithmetic obligation is therefore

$$\delta_r \neq 0 \quad \text{for every admitted ordinary index in (A.10).} \tag{A.13}$$

Even this assertion would still leave the terminal-129 and higher-address obligations in the universal target (A.9).

A.6 Approximation reaches every depth; separation remains open

The latest continuation retains the exact Bernoulli undepletion factor

$$e_{r,q} = \frac{1 - 2^{2r-1}}{1 - 2^{2r-1}4^q}, \qquad q \geq 0.$$

It is a dyadic unit, including $e_{r,0} = 1$. Multiplication by this factor preserves the all-q error of the shifted Jacobi moment model. Consequently the actual model critical values satisfy, for every $m \geq 25$,

$$v_2\big(\Phi_{r,m}^{\mathrm{act}} - \Phi_r^{\mathrm{act}}\big) \geq L_m = 8m - 4s_2(m) - 178, \qquad L_m \longrightarrow \infty. \tag{A.14}$$

The estimate includes every packet and controls motion of the critical point. There is no ceiling at the undepletion exponent $2r - 25$. For each fixed admitted r it implies the exact equivalence

$$\Phi_r^{\mathrm{act}} \neq 0 \quad \Longleftrightarrow \quad \text{there exists } m \geq 25 \text{ such that } v_2(\Phi_{r,m}^{\mathrm{act}}) < L_m. \tag{A.15}$$

Indeed a passing strict test has the same valuation as the limit; conversely any finite limit valuation is eventually smaller than L_m. The existence of a passing depth for every ordinary index is unproved. Nonzero models at every depth are compatible with an exactly zero limit.

The degree-two preparation also reduces the old representative-height cost to a linear cost, about $2P$ bits at precision P. This remains larger than the P bits needed for a strict nonzero residue certificate. The fixed rational control $(Z - 1/3)^2$ has zero critical value while its least-residue integer discriminant representatives are nonzero at every even $P \geq 6$. Thus a nonzero integer representative alone cannot prove a nonzero dyadic discriminant.

One sufficient new arithmetic input would be an odd-denominator projective quadratic

$$AZ^2 + 2BZ + C \equiv A\mathcal{W}_r(Z) \pmod{2^P}, \qquad A \text{ odd}, \qquad 0 < |B^2 - AC| < 2^P,$$

at a proved precision for every admitted ordinary r. Then $(B^2 - AC)/A^2 \equiv \delta_r \pmod{2^P}$ cannot vanish modulo 2^P. No depth-independent theorem supplying this inequality has been established. The tested recurrence and exterior-contact identities retain balancing terms and do not force a contradiction. Finite exponential models approximate the actual target at every depth, but their order grows with precision, their clearing polynomial has zeros on the actual chart, and their proved construction-specific height cost does not give the required discriminant bound.

A.7 Status and source map

Claim	Status at this edition
Nishibuchi Conjecture 3.9 in its stated parameter range	Resolved in the main paper
Sharp uniform ordinary Euler tail exponent $4M + 3$	Proved, with earlier formal coverage
Exact tail factorization and finite polynomial at every depth	Proved; later prose development
Regular-wall emergence and indicated integer simplicity families	Proved within their stated hypotheses
Actual quadratic compression and unbounded stationary approximation	Proved; no fresh Lean build
Nonzero actual stationary discriminant for every admitted index	Open
Universal simplicity of every G_r	Open
Native 13.2 presentation invariance and Native 13.3 global/completed gluing	Separate open general programs

The tail factorization, regular-wall proofs, finite preparation, and coefficient cutoffs are recorded in the preserved `sources/euler_v4.md` of the regular-address checkpoint. The two integer families and multiplier conditions are proved in its `next/terminal129/PROOF.md` and `next/critical_phases/PROOF.md`; `local_parameter/RECENTER_PROOF.md` supplies the inherited local precision certificate. The higher-address extension is in `next/higher_address/regular_address_transversality.md`. The stationary checkpoint records the exact discriminant identity in `stationary_separation/ordinary/QUADRATIC_COMPRESSION_AND_HEIGHT.md` and actual convergence in `stationary_separation/root/ACTUAL_LIMIT_AND_SEPARATION.md`. Its synthesis and proof reviews retain the failed implications and the precise remaining arithmetic input.

The source archive's `evidence/root_scope_inventory.md` gives the full checkpoint paths and content hashes used for this appendix. The 2026-10-05 complete-work archive preserves the original layouts and historical releases. Its stored hashes and verification receipts are evidence of integrity and recorded execution; they are not substitutes for a proof of the open assertions above.

B Detailed continuation derivations

The following mathematical release notes preserve the detailed arguments behind the continuation. They are source-grounded exposition of the completed blocks, and retain their original hypotheses. Source-declaration and build inventories are supplied separately in the archive. References to earlier blocks and then-pending work are dated checkpoint statements; current scope is given in the main text.

B.1 Paper IX: finite polynomial lifting and shared corrections

Author: Komorebi P. — 8 September 2026.

This block supplies the actual nonlinear Taylor argument previously missing from the Lean formalization. The statements concern a fixed finite list of integer polynomials. They hold for every positive integer modulus, including composite moduli and modulus one. No Jacobian-surjectivity assumption is needed.

B.1.1 Statement and notation

Let

$$F = (F_1, \dots, F_e) \in \mathbb{Z}[X_1, \dots, X_d]^e, \quad q \geq 1, \quad L \geq 1, \quad x \in \mathbb{Z}^d, \quad F(x) \equiv 0 \pmod{q^L}.$$

Write $R_q = \mathbb{Z}/q\mathbb{Z}$, let ρ_q denote coordinatewise reduction, and put

$$r_x = F(x)/q^L \in \mathbb{Z}^e, \qquad A_x = DF(x) : \mathbb{Z}^d \longrightarrow \mathbb{Z}^e, \qquad J_x = \rho_q \circ A_x.$$

Here A_x is computed from the actual formal partial derivatives of the input polynomials. The Lean map J_x has domain $\mathbb{Z}^d$ and codomain R_q^e, viewed as integer modules. It factors through reduction to R_q^d; the cokernel below is presented directly using its integer-domain image.

Define the finite obstruction module and its distinguished class by

$$\mathcal{O}_x = R_q^e / \operatorname{im} J_x, \qquad o_x = [\rho_q(r_x)] \in \mathcal{O}_x.$$

The checked criterion is

$$\boxed{o_x = 0 \iff \exists y \in \mathbb{Z}^d : \quad y \equiv x \pmod{q^L}, \qquad F(y) \equiv 0 \pmod{q^{L+1}}.}$$

The residual exists by the lower congruence and is unique because $q > 0$. The quotient is finite since its ambient module R_q^e is finite. The theorem permits zero variables or zero equations as well.

B.1.2 The checked Taylor bridge

For every $P \in \mathbb{Z}[X_1, \dots, X_d]$, integer a, and $x, t \in \mathbb{Z}^d$, Lean proves

$$\boxed{a^2 \mid P(x + at) - P(x) - a \sum_{i=1}^{d} (\partial_i P)(x) t_i.}$$

The proof uses polynomial induction: constants, addition, and multiplication by a variable. The product rule for the actual partial derivatives controls the multiplication step. It does not assume a Taylor identity or a liftability condition.

Taking $a = q^L$, the inequality $2L \geq L + 1$ gives

$$F(x + q^L t) \equiv q^L (r_x + A_x t) \pmod{q^{L+1}}.$$

Since $q^L \neq 0$ in the integers, cancellation in divisibility yields

$$F(x + q^L t) \equiv 0 \pmod{q^{L+1}} \iff \rho_q(r_x) + J_x t = 0.$$

Every integer representative of the lower residue has the form $x + q^L t$, and this correction equation has a solution exactly when $o_x = 0$.

The positive-depth hypothesis is necessary for the general nonlinear theorem. For $F(X) = X^2 - 1$, $q = 2$, $L = 0$, $x = 0$, the Jacobian is zero and the residual class is nonzero, yet $y = 1$ is a solution modulo two in the same residue modulo one. Affine systems retain the previously checked $L \geq 0$ theorem.

B.1.3 Independence of the integer representative

Suppose $x' = x + q^L s$ represents the same lower residue, with exact residual $r_{x'}$. Polynomial evaluation respects coordinatewise congruences, including when applied to each partial derivative. Hence

$$J_{x'} = J_x, \qquad \rho_q(r_{x'}) = \rho_q(r_x) + J_x s.$$

The two cokernels therefore have the same ambient module and the same defining submodule, and

$$[\rho_q(r_{x'})] = [\rho_q(r_x)].$$

Lean states this using one explicit common reduced map, avoiding an implicit comparison between differently indexed quotient types. This establishes independence of the integer representative within the fixed polynomial presentation.

Changing the polynomial presentation is a separate question. Appending a zero equation can change the raw cokernel by adding a copy of R_q, even though it does not change the solutions.

B.1.4 Two polynomial blocks and one shared lift

Let $G \in \mathbb{Z}[X_1, \ldots, X_d]^f$ be a second fixed system with the same variables and the same lower root x. Let $K_x = \rho_q \circ DG(x)$, and define

$$\mathcal{C}_x = \mathbb{Z}^d / (\ker J_x + \ker K_x).$$

This quotient is finite: it is a quotient of the finite image of J_x, or equivalently of the finite image of K_x.

Suppose $u, v \in \mathbb{Z}^d$ are separately valid corrections:

$$F(x + q^L u) \equiv 0 \pmod{q^{L+1}}, \qquad G(x + q^L v) \equiv 0 \pmod{q^{L+1}}.$$

Then

$$\boxed{\begin{aligned} [u] = [v] \text{ in } \mathcal{C}_x \iff \exists y \equiv x \pmod{q^L} : \\ F(y) \equiv 0 \pmod{q^{L+1}}, \quad G(y) \equiv 0 \pmod{q^{L+1}}. \end{aligned}}$$

The labels are independent of the choices of valid u and v. An unconditional version asks for separately valid corrections with equal labels; it does not assume that each block is liftable.

Taylor transport turns a common correction t into the simultaneous equations $J_x t = J_x u$ and $K_x t = K_x v$. The previously checked common-quotient theorem gives precisely the displayed equivalence. Separate solvability alone does not imply joint solvability; the controls include $F(X) = X$, $G(X) = X - 2$, starting at $x = 0$ modulo two and lifting to modulus four.

B.2 Paper IX: correction torsors and exact polynomial lift counts

Author: Komorebi P. — 8 September 2026.

This block formalizes the correction-digit torsor and prime-field counts from the previous checklist. The counts apply to distinct next-level residue classes satisfying the actual polynomial equations. They do not count integer representatives, of which there can be infinitely many.

B.2.1 The theorem

Let p be prime, $L \geq 1$, and

$$F \in \mathbb{Z}[X_1, \ldots, X_d]^e, \qquad x \in \mathbb{Z}^d, \qquad F(x) \equiv 0 \pmod{p^L}.$$

Write

$$r_x = F(x)/p^L, \qquad b_x = r_x \bmod p, \qquad A_x = DF(x) \bmod p : \mathbb{F}_p^d \longrightarrow \mathbb{F}_p^e, \qquad \rho_x = \operatorname{rank}_{\mathbb{F}_p} A_x.$$

The actual lift set is

$$\mathcal{L}_{x,L} = \left\{ y \in (\mathbb{Z}/p^{L+1}\mathbb{Z})^d : y \equiv x \pmod{p^L}, \quad F(y) = 0 \right\},$$

where polynomial evaluation in the displayed set is modulo p^{L+1}. Define

$$\mathcal{O}_x = \mathbb{F}_p^e / \operatorname{im} A_x, \qquad o_x = [b_x].$$

Lean proves

$$\boxed{\#\mathcal{L}_{x,L} = \begin{cases} p^{d-\rho_x}, & o_x = 0, \\ 0, & o_x \neq 0, \end{cases} \qquad \#\mathcal{O}_x = p^{e-\rho_x}.}$$

These formulas permit singular Jacobians and zero variables or zero equations. Nonsingularity is not assumed. In particular, rank alone gives the size of a nonempty lift fiber; the residual obstruction determines whether it is empty.

B.2.2 Why these are actual residue-class counts

For every positive integer q, including composite moduli and modulus one, and every $L \geq 0$, the checked map

$$\Phi_{x,L} : (\mathbb{Z}/q\mathbb{Z})^d \longrightarrow \{ y \bmod q^{L+1} : y \equiv x \pmod{q^L} \}, \qquad t \longmapsto x + q^L \widetilde{t}$$

is a bijection. Here $\widetilde{t}$ is the canonical integer representative of the digit vector. The entire residue fiber therefore has q^d elements. Injectivity is proved by cancelling q^L in the corresponding integer divisibility statement; surjectivity comes from the lower-residue congruence.

The formal type `ActualLifts` is a subtype of these genuine `ZMod (q^(L+1))` residue vectors. Its polynomial predicate is evaluated on canonical integer representatives and tested for divisibility by q^{L+1}. The checked polynomial congruence theorem makes that predicate independent of the integer representative. Thus using canonical representatives does not discard or duplicate residue classes.

At positive depth, the preceding actual multivariable Taylor theorem gives

$$F(x + p^L \widetilde{t}) \equiv p^L (r_x + DF(x)\widetilde{t}) \pmod{p^{L+1}}.$$

The new bridge proves that the evaluated integer Jacobian reduces to the actual $\mathbb{F}_p$-linear map A_x. Restricting $\Phi_{x,L}$ therefore gives an explicit equivalence

$$\boxed{\{t \in \mathbb{F}_p^d : b_x + A_x t = 0\} \simeq \mathcal{L}_{x,L}.}$$

The proof uses the actual formal partial derivatives of the original polynomial list.

B.2.3 The torsor

Let $\mathcal{T}_x = \{t : b_x + A_x t = 0\}$. Its natural kernel action is

$$k \cdot t = k + t, \qquad k \in \ker A_x.$$

For every pair $u, t \in \mathcal{T}_x$, exactly one kernel vector sends u to t, namely $t - u$. Thus every nonempty correction fiber is a torsor for $\ker A_x$. Choosing one valid digit u gives

$$\mathcal{T}_x \simeq \ker A_x, \qquad t \longmapsto t - u.$$

Lean constructs an explicit `AddTorsor` structure for the digit fiber, proves the unique translation property, and supplies a chosen-point equivalence from the actual residue lift set to the kernel. A basepoint is required for this equivalence, but the translation action itself does not depend on choosing one.

The torsor and residue bijection remain valid over every positive modulus at positive polynomial depth. The displayed powers involving vector-space rank are restricted to prime moduli.

B.2.4 Connecting the old obstruction to the new count

The earlier obstruction used the integer-domain map

$$J_x = \rho_p \circ DF(x) : \mathbb{Z}^d \to \mathbb{F}_p^e.$$

The new count uses $A_x : \mathbb{F}_p^d \to \mathbb{F}_p^e$. Reduction of integer vectors onto finite vectors is surjective, so the two maps have the same image after restricting scalars. Lean constructs a class-preserving integer-linear equivalence

$$\mathbb{F}_p^e / \operatorname{im} J_x \simeq \mathbb{F}_p^e / \operatorname{im} A_x.$$

Consequently the cardinality theorem counts the original obstruction group, preserving its distinguished residual class. It is not merely a formula for a separately introduced comparison object.

Rank-nullity gives $\dim \ker A_x = d - \rho_x$, and the quotient dimension is $e - \rho_x$. Cardinalities of finite vector spaces complete both formulas.

B.3 Paper IX: compatible polynomial towers and p-adic roots

Author: Komorebi P. — 8 September 2026.

This block completes the nonsingular branch of the passage from one-step polynomial lifting to an infinite compatible tower and an actual p-adic root. Surjectivity is required only of the reduced Jacobian at the initial root. The proof derives its persistence, constructs the tower, proves convergence and verifies the original polynomial equations at the limit.

B.3.1 Main theorem

Let p be prime, $L \geq 1$, and

$$F \in \mathbb{Z}[X_1, \ldots, X_d]^e, \qquad x \in \mathbb{Z}^d, \qquad F(x) \equiv 0 \pmod{p^L}.$$

Assume the actual reduced Jacobian

$$A_x = DF(x) \bmod p : \mathbb{F}_p^d \longrightarrow \mathbb{F}_p^e$$

is surjective. Then Lean proves that there is a point

$$\boxed{a \in \mathbb{Z}_p^d, \qquad F(a) = 0, \qquad a \equiv x \pmod{p^L}.}$$

Here the coefficients of the original integer polynomials are cast into $\mathbb{Z}_p$, and their multivariable evaluation is exactly zero. This is an existence theorem for a fixed polynomial presentation with the stated Jacobian hypothesis.

The stronger checked statement constructs $z_n \in \mathbb{Z}^d$, $n \geq 0$, with

$$\begin{aligned} z_0 &= x, \\ F(z_n) &\equiv 0 \pmod{p^{L+n}}, \\ z_{n+1} &\equiv z_n \pmod{p^{L+n}}, \\ DF(z_n) \bmod p &= A_x, \\ z_n &\longrightarrow a \quad \text{in } \mathbb{Z}_p^d, \\ a - z_n &\in p^{L+n}\mathbb{Z}_p^d. \end{aligned}$$

The last inclusion retains every prescribed precision. It is a divisibility lower bound, not an assertion that the error has exactly that valuation. An approximation can already equal the limiting root.

B.3.2 Why the single initial hypothesis suffices

The one-step Taylor theorem already supplies the exact correction equation

$$b_n + A_{z_n} t = 0, \qquad b_n = F(z_n)/p^{L+n} \bmod p.$$

Surjectivity supplies a correcting digit for every residual b_n. The chosen correction gives an actual new integer point z_{n+1}, using the proved nonlinear Taylor remainder.

Every coordinate change is divisible by p^{L+n}, hence by p. Each formal partial derivative is itself an integer polynomial, so its value is unchanged modulo p. Therefore

$$A_{z_{n+1}} = A_{z_n} = A_x.$$

The proof assumes no separate surjectivity or solvability statement at later levels. A dependent recursion chooses each next point from the proved one-step existence theorem. It produces one compatible sequence, rather than unrelated witnesses at different precisions.

The integer tower construction works more generally for every positive modulus q, including composite q and $q = 1$, with a surjective map over $\mathbb{Z}/q\mathbb{Z}$. The p-adic completion and the rank-based branching formulas in this block use prime p.

B.3.3 Convergence and exact polynomial vanishing

Compatibility implies, for $m \geq n$,

$$p^{L+n} \mid z_m - z_n.$$

After casting to $\mathbb{Z}_p$, this bounds each coordinate distance by $p^{-(L+n)}$, so the sequence is Cauchy. Completeness supplies its limit. Closed norm bounds show that the limit retains the congruence at every stage.

The generic convergence theorem is proved independently of the polynomial application. It also proves

$$p^{L+n} \mid v_n \text{ for all } n \quad \Longrightarrow \quad v_n \longrightarrow 0 \text{ in } \mathbb{Z}_p.$$

Apply this to each integer sequence $F_i(z_n)$. Continuity of multivariable polynomial evaluation also gives $F_i(z_n) \to F_i(a)$. Uniqueness of limits then yields $F_i(a) = 0$.

The cast-evaluation identity is explicitly checked, so the continuous function is the evaluation of the original polynomials. Convergence and polynomial vanishing are conclusions, not assumptions of the main existence theorem.

B.3.4 Exact number of choices at each level

Surjectivity implies

$$\operatorname{rank}_{\mathbb{F}_p} A_x = e, \qquad e \leq d.$$

Combining this with the preceding exact residue-lift count gives

$$\boxed{\#\mathcal{L}_{z_n, L+n} = p^{d-e} \quad \text{for every } n.}$$

These are distinct solutions modulo p^{L+n+1} lying above the selected point modulo p^{L+n}. The original finite obstruction group at each smooth point has cardinality one.

The theorem constructs one path through these finite choices. When $d > e$, it does not claim that all paths have the same limit. The generic uniqueness theorem concerns the limit of a fixed sequence only.

B.3.5 Boundary controls

The positive control theorems and rejected invalid proofs cover three distinctions.

1. **An initial lift does not guarantee an infinite tower without the Jacobian condition.** $X^2 - 5$ has the odd lower root modulo two and a lift modulo four, but no root modulo eight. Lean proves that no integer tower can continue through all precisions in this example.
2. **Surjective rectangular Jacobians do not give uniqueness of all roots.** The system $F(X, Y) = X$ has a surjective Jacobian and distinct dyadic roots $(0, 0)$ and $(0, 2)$, both in the same residue modulo two.
3. **The retained initial precision cannot be increased without justification.** For $F(X) = X - 2$, zero is a lower root modulo two with surjective derivative, but its exact root two does not retain zero modulo four.

The controls are theorem checks of these explicit examples. They supplement the universal proof audit and do not replace it.

For several polynomial blocks, the hypothesis applies to their combined system. Surjectivity of each block separately does not imply surjectivity of the combined Jacobian or shared liftability. The preceding shared-correction obstruction remains relevant in that situation.

B.4 Polynomial roots under coefficient perturbation

Komorebi P. — 8 September 2026

This block formalizes square root uniqueness and nearby-root stability for finite systems of integer polynomials. The polynomials and their partial derivatives are evaluated directly. No root-uniqueness or perturbation-stability premise is assumed.

B.4.1 Notation

Let p be prime, $L \geq 1$, and

$$F, H \in \mathbb{Z}[X_1, \ldots, X_d]^e, \qquad x \in \mathbb{Z}^d.$$

Write

$$A_x = DF(x) \bmod p : \mathbb{F}_p^d \longrightarrow \mathbb{F}_p^e, \qquad B_L(x) = x + p^L \mathbb{Z}_p^d.$$

For an integer c, the perturbed system is the actual polynomial list

$$G = F + cH.$$

There is no restriction on the sign of c, and $c = 0$ is allowed. Divisibility conclusions give lower bounds on precision and allow zero differences.

B.4.2 1. Unique roots for square systems

If $d = e$, A_x is surjective, and $F(x) \equiv 0 \pmod{p^L}$, then

$$\boxed{\exists!\, a \in B_L(x) : F(a) = 0.}$$

The stronger comparison lemma proves that any two exact roots of F in $B_L(x)$ coincide, without needing a separate lower-root hypothesis.

Proof implemented. Map every arbitrary p-adic root into the actual residue rings $\mathbb{Z}/p^n\mathbb{Z}$, and choose its canonical integer representative. Evaluation commutes with reduction, so these representatives are genuine finite polynomial roots. The preceding count theorem gives exactly $p^{d-d} = 1$ next-level residue lift. Inductively any two roots agree at every precision. The p-adic residue maps separate points, so the roots are equal. Existence is the preceding checked tower theorem.

This extends uniqueness from the limit of a selected sequence to **all exact roots in the prescribed ball**.

B.4.3 2. Coefficient perturbations preserve smooth liftability

If $p^L \mid c$, the actual identities give

$$G(x) = F(x) + cH(x), \qquad DG(x) \equiv DF(x) \pmod{p}.$$

Thus if $F(x) \equiv 0 \pmod{p^L}$ and A_x is surjective, then

$$\exists b \in B_L(x) : G(b) = 0.$$

This theorem allows rectangular systems. The evaluation, derivative and Jacobian identities are proved from the polynomial definition $G_i = F_i + C(c)H_i$, rather than supplied as hypotheses. Algebraic congruence helpers also permit composite moduli; exact p-adic root existence uses prime p.

B.4.4 3. Nearby roots for rectangular systems

Let $M \geq 1$ and $p^M \mid c$. Suppose A_x is surjective and $a \in \mathbb{Z}_p^d$ is an exact root of F with $a \equiv x \pmod{p}$. Then

$$\boxed{\exists b \in \mathbb{Z}_p^d : \quad G(b) = 0, \qquad b \equiv a \pmod{p^M}.}$$

No uniqueness is claimed when $d > e$.

Proof implemented. Truncate a to an integer vector y modulo p^M. It is an actual lower root of F, and its reduced Jacobian agrees with A_x. Apply the coefficient-perturbed existence theorem at precision M, then combine the two congruences through y.

B.4.5 4. Full precision for the unique square root

Suppose $d = e$, $M \geq L \geq 1$, $p^M \mid c$, and A_x is surjective. If

$$a, b \in B_L(x), \qquad F(a) = 0, \qquad G(b) = 0,$$

then

$$\boxed{b - a \in p^M \mathbb{Z}_p^d.}$$

In particular, for $G = F + p^M H$, no p-adic digits below depth M are changed in the selected root.

The formalization also proves a combined existence theorem. Starting only with $F(x) \equiv 0 \pmod{p^L}$, it constructs an original root $a \in B_L(x)$ and proves that there is exactly one $b \in B_L(x)$ satisfying $G(b) = 0$ and $b \equiv a \pmod{p^M}$. The comparison theorem shows that the precision bound holds for every root of G in that ball, so uniqueness is not obtained merely by adding a restrictive displacement condition.

Proof implemented. Section 3 produces a nearby root b'. Since $M \geq L$, it belongs to $B_L(x)$. The perturbed Jacobian remains surjective, and square uniqueness identifies b' with the arbitrary root b.

B.5 Paper IX: continuous p-adic parameter charts

Komorebi P. — 8 September 2026.

Fresh verification: PASS — 40 checks, 244 explicit declaration audits.

B.5.1 Main statement

Let p be prime, $L \geq 1$, $e, q \geq 0$, and

$$F \in \mathbb{Z}[X_1, \ldots, X_{e+q}]^e, \qquad x \in \mathbb{Z}^{e+q}, \qquad F(x) \equiv 0 \pmod{p^L}.$$

Assume the actual reduced Jacobian

$$A_x = DF(x) \bmod p : \mathbb{F}_p^{e+q} \longrightarrow \mathbb{F}_p^e$$

is surjective. All vector congruences are coordinatewise.

There is a coordinate permutation σ, with $(\sigma z)_i = z_{\sigma(i)}$. Write $\sigma z = (y, t)$, using e solved coordinates and q free coordinates. Define

$$(y_0, t_0) = \sigma x, \qquad B = x + p^L \mathbb{Z}_p^{e+q}, \qquad T = t_0 + p^L \mathbb{Z}_p^q, \qquad \mathcal{R} = \{z \in B : F(z) = 0\}.$$

There is a homeomorphism

$$h : T \xrightarrow{\sim} \mathcal{R}, \qquad h^{-1}(z) = \text{free}(\sigma z).$$

Equivalently, a continuous function $\phi : T \to y_0 + p^L \mathbb{Z}_p^e$ gives the entire root locus through $\sigma h(t) = (\phi(t), t)$. Every parameter has exactly one corresponding root in B. For every $N \geq 0$,

$$\boxed{h(t) \equiv h(u) \pmod{p^N} \iff t \equiv u \pmod{p^N}.}$$

The coordinate choice is a conclusion. A suitable minor and a preexisting chart are not hypotheses. The theorem covers the original root locus after undoing the coordinate reordering. Since x is an approximate root, it does not assert $\phi(t_0) = y_0$ unless $F(x) = 0$ exactly. The cases $p = 2$, $L = 1$, $e = 0$, and $q = 0$ are included.

B.5.2 Supporting theorems

For a rectangular integer polynomial system with a surjective reduced Jacobian at x, and $L \geq 1$,

$$F(x + p^L \mathbb{Z}_p^d) = F(x) + p^L \mathbb{Z}_p^e.$$

This covers arbitrary p-adic output targets. No lower-root hypothesis is needed.

When $d = e$, the actual restricted polynomial map is a homeomorphism. Moreover, for arbitrary $a, b \in x + p\mathbb{Z}_p^d$ and every $N \geq 0$,

$$F(a) \equiv F(b) \pmod{p^N} \iff a \equiv b \pmod{p^N}.$$

Its inverse therefore preserves every congruence depth.

B.5.3 Proof structure

1. The actual integer Taylor formula gives one additional input digit from one additional output digit when the reduced square Jacobian is surjective. Its finite domain and codomain have the same size, so it is injective. Induction gives every integer precision.
2. Canonical integer truncations of arbitrary p-adic vectors preserve evaluation modulo each power of p. Applying the integer result to truncations proves the p-adic congruence equivalence.
3. Subtracting an integer target from the constant terms leaves the actual Jacobian unchanged. The previous root-lifting theorem supplies a preimage. The image of the compact input ball is closed and contains the convergent truncations of every admissible p-adic target, hence contains the target. No compatibility of arbitrarily chosen preimages is assumed.
4. In the square case, the resulting continuous bijection from a compact ball to a Hausdorff ball is a homeomorphism. Its forward map is actual polynomial evaluation, and its inverse solves the output equation.

5. For a specified surjective solved-variable block, apply this inverse to the actual augmented polynomial map
$$\Psi(y,t) = (F(y,t),t).$$
Its derivative has an identity free-coordinate block. The inverse at $(0,t)$ gives the root. The lower-root congruence places this target in the output ball. Injectivity proves uniqueness and full graph coverage. Projection identifies the inverse chart and reflects congruences.
6. The columns of a surjective finite-dimensional linear map contain a basis. Select those columns and extend their injection to a coordinate permutation. Polynomial renaming by σ^{-1} transports actual evaluations and partial derivatives. The fixed-coordinate chart then gives the main theorem on the entire original root locus.

B.5.4 Scope

Coefficients are integers; roots and parameters are arbitrary p-adic points. This supplies continuous polynomial parameter charts. Analyticity, differentiability, arbitrary p-adic coefficient interfaces, weighted lifting for nonunit Jacobians, and first-order root-motion estimates are separate.

Full Jacobian surjectivity need not allow an arbitrary prescribed parameter choice. At the dyadic origin, $F(X,T) = X^2 - T$ has a surjective full Jacobian but zero X-derivative. Prescribing $T = 2$ gives no root even modulo 4. Suitable coordinates instead solve for T.

B.5.5 Verification and provenance

The eight new mathematical modules contain 75 declarations: 60 theorems and 15 definitions. The active polynomial dependency set contains 25 mathematical modules. Counts including inherited support and theorem contracts are recorded separately in the final report.

The exact universal statements are Lean theorems, not conclusions from sampled numerical tests. The fresh gate rebuilds project sources with warnings treated as errors, checks expanded theorem contracts, audits explicit declarations, and checks rejected invalid extensions. Only propext, Classical.choice, and Quot.sound are permitted as axiom dependencies.

Pinned runtime: Lean 4.34.0-rc2, Mathlib revision `aeb72f1e9981b6d35b426b16dca7759be3e7ba06`.

See `inventory.json`, `verification/polynomial_charts_verification.json`, and `INDEPENDENT_PROOF_AUDIT.md` for actual evidence. RECOVERY_RECORD.md explains workspace maintenance, the intact previous release, and reconstruction of the unpublished chart sources. Lost logs are not represented as recovered.

B.6 Paper IX: weighted nonlinear lifting

Komorebi P. — 8 September 2026.

B.6.1 The theorem

Let p be prime, $R = \mathbb{Z}_p$, $K = \mathbb{Q}_p$, and $d, s, r \in \mathbb{N}$. Let $F = (F_1, \ldots, F_d) \in R[X_1, \ldots, X_d]^d$, $a \in R^d$, and let
$$J = DF(a) = \left(\partial_j F_i(a)\right)_{i,j}.$$

Assume

$$J \in \mathrm{GL}_d(K), \qquad p^s J^{-1} \in M_d(R), \qquad F(a) \in p^r R^d, \qquad r > 2s.$$

Then

$$\boxed{\exists!\, b \in a + p^{s+1}R^d :\ F(b) = 0,} \qquad \boxed{b - a \in p^{r-s}R^d.}$$

All vector divisibility assertions are coordinatewise. The uniqueness predicate is the entire ball of depth $s+1$: competitors need not be assumed to have depth $r-s$. The theorem permits arbitrary integral p-adic coefficients and centers, every prime including two, and dimension zero. It retains the supplied inverse-denominator exponent s; there is no replacement by a determinant valuation. Integrality of $p^s J^{-1}$ is expressed in Lean by existence of an integral matrix with exactly that image under the coefficient inclusion $R \hookrightarrow K$.

B.6.2 Proof

Write $A = p^s J^{-1} \in M_d(R)$. Then

$$AJ = JA = p^s I.$$

The formal matrix bridge proves the equivalence with the stated fraction-field hypotheses, including nonsingularity. In particular, multiplication by A is injective, since multiplication by J sends $Av = 0$ to $p^s v = 0$.

Put $G = AF$, $t = p^{s+1}$, and $k = r - (2s+1)$. Taylor expansion of the actual preconditioned polynomials gives one polynomial system Q such that

$$G(a + tu) = G(a) + p^{2s+1}u + p^{2s+2}Q(u), \qquad Q(0) = 0.$$

This expansion is proved first over every commutative ring. Its remainder is a polynomial valid for every u, not a pointwise quotient depending on u. Since $G(a) \in p^r R^d$, it can be written $p^{2s+1}C$ with $C \in p^k R^d$. Thus

$$G(a + tu) = p^{2s+1}\big(C + u + pQ(u)\big).$$

Integral polynomial evaluation is 1-Lipschitz in the supremum p-adic metric. Consequently $T(u) = -C - pQ(u)$ has contraction constant $|p|_p < 1$ on the complete space R^d, and has exactly one fixed point u.

For completeness, the sharp precision is additional to this fixed-point existence. Inductively, if $u \in p^n R^d$, then polynomial congruence and $Q(0) = 0$ imply $Q(u) \in p^n R^d$. For $n < k$, the equation $u = -C - pQ(u)$ therefore gives $u \in p^{n+1}R^d$. Starting at $n = 0$, this proves $u \in p^k R^d$. Hence $b = a + tu$ satisfies

$$b - a \in p^{s+1+k}R^d = p^{r-s}R^d.$$

The normalization and injectivity of A give $F(b) = 0$.

Every other $z \in a + p^{s+1}R^d$ has the form $a + tw$. If $F(z) = 0$, the same normalization, with cancellation of the nonzero scalar p^{2s+1}, gives $w = T(w)$. Fixed-point uniqueness implies $w = u$, hence $z = b$.

B.6.3 Boundary and sharpness

The strict inequality cannot in general be weakened to $r \geq 2s$: for $p = 2$, $F(X) = X^2 + 3$, and $a = 1$, we have $F(a) = 4$, $DF(a) = 2$, and may take $s = 1, r = 2$. Yet no root exists in $\mathbb{Z}_2$, since there is none modulo eight. The auxiliary Lean controls prove this obstruction and the exact residual depth.

The displacement exponent cannot uniformly be increased by one. For $F(X) = p^s X + p^r$ at $a = 0$, with $r > 2s$, its root is $b = -p^{r-s}$, which is not divisible by p^{r-s+1}. The controls prove the linear root and the absence of that extra digit for all primes.

B.7 Paper IX: weighted first-order root motion

Komorebi P. — 8 September 2026.

B.7.1 The theorem

Let p be prime, $R = \mathbb{Z}_p$, $K = \mathbb{Q}_p$, and $d, s, M \in \mathbb{N}$. Let $F, H \in R[X_1, \dots, X_d]^d$, let $a \in R^d$ satisfy $F(a) = 0$, and set

$$J = DF(a), \qquad G = F + p^M H.$$

Assume

$$J \in \mathrm{GL}_d(K), \qquad p^s J^{-1} \in M_d(R), \qquad M > 2s.$$

Then there is exactly one root b of G in $a + p^{s+1}R^d$. It satisfies

$$b - a \in p^{M-s}R^d,$$

and there is $\varepsilon \in p^{2M-3s}R^d$ such that, in K^d,

$$\boxed{b - a = -p^M J^{-1} H(a) + \varepsilon.}$$

The inverse in this formula belongs to the original Jacobian $DF(a)$. The theorem constructs the perturbed root from these hypotheses and proves uniqueness throughout the full ball of depth $s + 1$. It does not require an additional Jacobian hypothesis on G. Coefficients and centers are arbitrary integral p-adic elements. Every prime and finite square dimension are included, including $p = 2$ and dimension zero.

The error depth exceeds the displacement depth by

$$(2M - 3s) - (M - s) = M - 2s \geq 1.$$

Thus the first-order term determines strictly more digits than the displacement bound alone. These are guaranteed divisibility bounds; equality of valuations is not asserted for every system.

B.7.2 Root construction with the original inverse

Write $A = p^s J^{-1} \in M_d(R)$, so $AJ = JA = p^s I$. Let $t = p^{s+1}$ and $k = M - (2s + 1) \geq 0$. The actual polynomial Taylor and difference identities give polynomial systems Q_F, Q_H, both vanishing at zero, such that

$$(AF)(a + tu) = p^{2s+1}u + p^{2s+2}Q_F(u),$$
$$(AH)(a + tu) = (AH)(a) + tQ_H(u).$$

Consequently

$$(AG)(a + tu) = p^{2s+1}\big(C + u + pQ(u)\big),$$

where

$$C = p^k AH(a), \qquad Q = Q_F + p^{M-s-1}Q_H, \qquad Q(0) = 0.$$

All exponents are nonnegative under $M > 2s$. Actual integral polynomial evaluation is 1-Lipschitz, so $u \mapsto -C - pQ(u)$ is a strict contraction on the complete space R^d. Its unique fixed point inherits the depth k of C, because $Q(0) = 0$. Thus $b = a + tu$ has displacement depth $s + 1 + k = M - s$.

The certificate $JA = p^s I$ makes multiplication by A injective, so the normalized zero is an actual root of G. Every root in $a + tR^d$ has the form $a + tw$ and satisfies the same fixed-point equation. This proves uniqueness on the entire stated ball using the original matrix certificate.

B.7.3 The first-order error

Put $\delta = b - a$ and $L = M - s$. The displacement form of Taylor's theorem, proved over any commutative ring, gives

$$E_F = (AF)(b) - (AF)(a) - p^s\delta \in p^{2L}R^d.$$

Polynomial congruence gives

$$E_H = (AH)(b) - (AH)(a) \in p^L R^d.$$

Since $F(a) = 0$ and $G(b) = 0$,

$$p^s\delta + p^M AH(a) = -E_F - p^M E_H.$$

We have $M + L \geq 2L$, so the right side lies in $p^{2L}R^d$. Cancelling the nonzero scalar p^s proves the integral estimate

$$\delta + p^{M-s}AH(a) \in p^{2M-3s}R^d.$$

Define this integral vector to be ε. Under $R \hookrightarrow K$, the exact identity $p^{M-s}A = p^M J^{-1}$ gives the displayed first-order formula. The Lean bridge checks matrix evaluation and all coefficient inclusions explicitly.

B.7.4 Sharpness and the strict threshold

For $p = 2$, take $F(X) = X + X^2$, $H(X) = 1$, $a = 0$, $s = 0$, $M = 1$. There is a unique root $b \in 2\mathbb{Z}_2$ of $X + X^2 + 2$. Its equation gives $b + 2 = -b^2$. It also forces $b \equiv 2 \pmod 4$. Therefore

$$v_2(b + 2) = 2 = 2M - 3s.$$

The accompanying Lean controls show that the error is divisible by four and is not divisible by eight. This refutes a uniform replacement of $2M - 3s$ by $2M - 3s + 1$, even for a one-variable unit Jacobian.

The prior strict-threshold example also applies to perturbations: take $F(X) = X^2 - 1$, $H(X) = 1$, $a = 1$, $p = 2$, $s = 1$, $M = 2$. Then $F(a) = 0$, $DF(a) = 2$, and $F + 2^2H = X^2 + 3$ has no root modulo eight, hence none in $\mathbb{Z}_2$. The new explanatory identification uses the previously checked no-root polynomial. The strict inequality $M > 2s$ cannot uniformly be replaced by $M \geq 2s$.

B.8 Paper IX: prescribed-parameter polynomial lifting

Komorebi P. — 8 September 2026.

B.8.1 Setup and exact lift set

Let $q > 0$, $L \geq 1$, and $m, k, e \geq 0$. Write $S = \mathbb{Z}/q\mathbb{Z}$. Let

$$F \in \mathbb{Z}[Y_1, \ldots, Y_m, T_1, \ldots, T_k]^e, \qquad x = (y_0, t_0) \in \mathbb{Z}^{m+k}, \qquad F(x) \equiv 0 \pmod{q^L}.$$

The actual lower root supplies the unique integral residual

$$r = q^{-L}F(x) \in \mathbb{Z}^e, \qquad \bar{r} = r \bmod q \in S^e.$$

Let the actual reduced Jacobian blocks be

$$J = (D_Y F)(x) \bmod q : S^m \to S^e, \qquad K = (D_T F)(x) \bmod q : S^k \to S^e.$$

For a prescribed parameter digit $v \in S^k$, choose its canonical integral representative $\widehat{v}$. Define $\mathcal{L}_v$ to be the set of actual residue vectors $z \in (\mathbb{Z}/q^{L+1}\mathbb{Z})^{m+k}$ satisfying

$$z \equiv x \pmod{q^L}, \qquad F(z) \equiv 0 \pmod{q^{L+1}}, \qquad z_T \equiv t_0 + q^L\widehat{v} \pmod{q^{L+1}}.$$

The formal predicate uses canonical integer representatives for evaluation and the lower congruence. The previously checked polynomial congruence theorem and digit-to-residue equivalence establish the actual residue-class interpretation. Changing an integral representative of a parameter digit by a multiple of q does not change its prescribed residue modulo q^{L+1}.

B.8.2 The obstruction theorem

Define the finite solved-block cokernel and its parameter-dependent class by

$$\mathcal{C}_x = S^e / \operatorname{im} J, \qquad o_x(v) = [\bar{r} + Kv] \in \mathcal{C}_x.$$

Then

$$\boxed{\mathcal{L}_v \neq \varnothing \iff o_x(v) = 0.}$$

More precisely, the checked digit-to-lift equivalence is

$$\boxed{\mathcal{L}_v \cong \{u \in S^m : Ju = -\bar{r} - Kv\}.}$$

It sends a solved digit u to the actual next residue represented by $x + q^L(\widehat{u}, \widehat{v})$. The obstruction depends on the solved block J, not on the range of the full map $(J\ K)$.

Every nonempty $\mathcal{L}_v$ has a checked free and transitive action of $\ker J$. The action is defined without choosing a base lift. Choosing one valid solved digit then gives an explicit bijection $\mathcal{L}_v \cong \ker J$. In particular, for every positive modulus,

$$o_x(v) = 0 \quad \Longrightarrow \quad \#\mathcal{L}_v = \#\ker J.$$

These results permit composite q, the degenerate modulus $q = 1$, and zero dimensions. Neither J nor the full reduced Jacobian is assumed surjective.

B.8.3 Exact prime-field counts

When $q = p$ is prime, put $\rho = \operatorname{rank}_{\mathbb{F}_p} J$. Then

$$\boxed{\#\mathcal{L}_v = \begin{cases} p^{m-\rho}, & o_x(v) = 0, \\ 0, & o_x(v) \neq 0, \end{cases}} \qquad \boxed{\#\mathcal{C}_x = p^{e-\rho}.}$$

The count is of distinct actual residue classes at level $L + 1$. The solved dimension m, rather than the total number $m + k$ of variables, appears in the exponent because the parameter digit is fixed. These rank formulas are asserted only over prime fields.

The main wrapper assumes only the original lower polynomial congruence. It constructs one residual r and supplies the criterion and count uniformly for every v, together with the exact obstruction cardinality.

B.8.4 Proof and dependency chain

The actual derivative calculation gives

$$(DF(x) \bmod q)(u, v) = Ju + Kv.$$

Taylor expansion at depth $L \geq 1$ makes the quadratic remainder divisible by q^{L+1}. Therefore the digit (u, v) is a polynomial lifting correction exactly when

$$\bar{r} + Ju + Kv = 0.$$

The quotient class $[\bar{r} + Kv]$ vanishes exactly when this equation admits u. This uses additive inverses in the image of J, so the sign agrees with the actual Taylor equation.

The earlier unrestricted correction-digit equivalence is then restricted to the predicate that the parameter digit equals v. Taking free coordinates commutes with forming digit residues, and digit residues are injective in the digit. Consequently the restricted equivalence enforces exactly the actual parameter residue condition on $\mathcal{L}_v$.

Translation by $\ker J$ acts freely and transitively on the solved-digit fiber. Transporting this explicit action through the residue equivalence yields the torsor on $\mathcal{L}_v$. Rank-nullity over $\mathbb{F}_p$ then gives the two displayed cardinalities. No numerical enumeration is used to establish the general theorem.

B.8.5 Why the solved block matters

Take $F(Y, T) = T$, $q = 2$, $L = 1$, and $x = (0, 0)$. The full Jacobian $(0\ 1)$ is surjective, but $J = 0$. Prescribing $v = 1$ forces $T \equiv 2 \pmod 4$, which cannot satisfy $F = 0 \pmod 4$. Prescribing $v = 0$ gives exactly two lifts, corresponding to $Y \equiv 0, 2 \pmod 4$.

The Lean controls check this actual polynomial example. Thus full-Jacobian surjectivity alone does not make every prescribed coordinate choice admissible. The automatic coordinate selection in the earlier chart theorem is compatible with this obstruction.

B.9 Paper IX: compatible towers with prescribed parameters

Author: Komorebi P. — 8 September 2026.

This block constructs actual compatible polynomial towers from initial solved-block surjectivity, for every compatible integer parameter path. For a prime base, it constructs such a path for every p-adic parameter in the initial ball and proves an exact polynomial endpoint with all retained precisions. The six new mathematical modules compile; the release verification report records the authoritative fresh-build and axiom-audit results.

B.9.1 Statement and notation

Let $q > 0$, $L \geq 1$, and

$$F \in \mathbb{Z}[Y_1, \ldots, Y_m, T_1, \ldots, T_k]^e, \qquad x = (y_0, t_0) \in \mathbb{Z}^{m+k}, \qquad F(x) \in q^L \mathbb{Z}^e.$$

Write $S_q = \mathbb{Z}/q\mathbb{Z}$, and let the actual reduced Jacobian blocks be

$$J_x = D_Y F(x) \bmod q : S_q^m \longrightarrow S_q^e, \qquad K_x = D_T F(x) \bmod q : S_q^k \longrightarrow S_q^e.$$

Assume J_x is surjective. A vector divisibility condition below is coordinatewise.

Integer prescribed-path theorem. For every supplied path $t : \mathbb{N} \to \mathbb{Z}^k$ satisfying

$$t(0) = t_0, \qquad t(n+1) - t(n) \in q^{L+n}\mathbb{Z}^k,$$

there is a single tower $z : \mathbb{N} \to \mathbb{Z}^{m+k}$ with

$$z(0) = x, \qquad \text{free}(z(n)) = t(n),$$

$$F(z(n)) \in q^{L+n}\mathbb{Z}^e, \qquad z(n+1) - z(n) \in q^{L+n}\mathbb{Z}^{m+k}.$$

The parameter equality is exact in integers. For $a \le b$,

$$z(b) - z(a) \in q^{L+a}\mathbb{Z}^{m+k}, \qquad DF(z(n)) \bmod q = DF(x) \bmod q.$$

Consequently $J_{z(n)} = J_x$ at every level. No later surjectivity or per-step solvability is assumed. This theorem includes positive composite bases and $q = 1$.

Arbitrary p-adic target theorem. Now let $q = p$ be prime. For every

$$\tau \in t_0 + p^L\mathbb{Z}_p^k,$$

there are a compatible integer tower z as above and $b \in \mathbb{Z}_p^{m+k}$ such that

$$z(n) \longrightarrow b, \qquad F(b) = 0, \qquad \text{free}(b) = \tau, \qquad b - z(n) \in p^{L+n}\mathbb{Z}_p^{m+k} \quad (n \ge 0).$$

A compatible integer parameter path converging to τ is constructed in the proof, with its initial representative fixed exactly. A separate theorem accepts every supplied compatible integer path with limit τ and retains its exact coordinates in the constructed tower.

Exact branching. At any stage n, put $N = L + n$. For every parameter digit $v \in \mathbb{F}_p^k$, let

$$\mathcal{L}_{n,v} = \left\{ \bar{w} \in (\mathbb{Z}/p^{N+1}\mathbb{Z})^{m+k} : \begin{array}{l} \bar{w} \equiv z(n) \pmod{p^N}, \\ F(\bar{w}) = 0, \\ \text{free}(\bar{w}) = t(n) + p^N v \pmod{p^{N+1}} \end{array} \right\}.$$

Here $p^N v$ means any integer representative of v, multiplied by p^N; the residue class does not depend on the representative. Then

$$\#\mathcal{L}_{n,v} = p^{m-e} \qquad \text{for every } n \text{ and every } v.$$

Surjectivity implies $e \le m$. This is a count of distinct actual next-level residue classes. It does not count integer representatives or entire infinite towers.

Square solved-block uniqueness. If $m = e$, every prescribed τ has exactly one root in the original ball:

$$\exists!\, b \in x + p^L\mathbb{Z}_p^{e+k} : \quad F(b) = 0, \qquad \text{free}(b) = \tau.$$

Thus all valid integer parameter approximations and all lifting choices with the same target have the same endpoint. This does not assert uniqueness of the integer representatives in the tower. All prime statements include $p = 2$.

B.9.2 Proof

At depth $N \geq 1$, suppose $F(a) = q^N r$ and a desired next integer parameter target satisfies $t' - \mathrm{free}(a) = q^N w$, with $w \in \mathbb{Z}^k$. Surjectivity supplies a solved digit $u \in S_q^m$ satisfying

$$J_a u = -\bar{r} - K_a \bar{w}.$$

Choose an integer representative $\widetilde{u}$, and set

$$a' = a + q^N(\widetilde{u}, w).$$

The parameter quotient is the supplied integer w. Therefore $\mathrm{free}(a') = t'$ exactly. The checked polynomial Taylor identity gives

$$F(a') \equiv F(a) + q^N DF(a)(\widetilde{u}, w) \equiv 0 \pmod{q^{N+1}}.$$

Its quadratic remainder is divisible by q^{2N}, and $2N \geq N+1$. Also $a' \equiv a \pmod{q}$, so evaluation of every polynomial partial derivative gives the same reduced Jacobian. Induction on the level, carrying the root, exact parameters, and fixed solved Jacobian as its state, constructs the entire integer tower. The existence proof uses ordinary classical choice; it does not supply an optimized computational lifting algorithm.

For a p-adic target τ, use $t(0) = t_0$ and, for $n \geq 1$, the canonical integer representatives of $\tau \bmod p^{L+n}$. The initial congruence hypothesis and the truncation precision imply

$$\tau - t(n) \in p^{L+n}\mathbb{Z}_p^k.$$

Subtracting successive precision relations gives compatibility in $\mathbb{Z}_p$; the checked faithful integer-divisibility theorem gives compatibility in $\mathbb{Z}$. These precision relations also give convergence to τ.

The compatible vector-tower completion theorem constructs b, proves convergence, and retains every stage precision. Continuous polynomial evaluation and the increasing root precision give $F(b) = 0$. Each parameter coordinate has both the corresponding coordinate of b and τ as its limit, so uniqueness of limits gives exact parameter equality.

For every proposed next parameter digit, the previously checked obstruction is $[\bar{r} + K_a v] \in S_p^e / \operatorname{im} J_a$. The fixed solved Jacobian is surjective, so this class vanishes for every digit at every stage. The actual lift fiber is a torsor for $\ker J_a$. Rank-nullity gives its cardinality p^{m-e}.

When $m = e$, the new solved-Jacobian definition is definitionally the existing square block used by the checked continuous parameter chart. Every root in the initial ball with the given parameters is that chart point. Applying this to any two roots proves equality, including endpoints obtained by different constructions.

B.9.3 Scope controls

The inherited checked example $F(Y, T) = T$, $p = 2$, $x = 0$, $L = 1$ has a surjective full Jacobian but zero solved block. Prescribing parameter digit 1 forces $T = 2 \pmod 4$, so no next root exists. Full-Jacobian surjectivity cannot replace the hypothesis on the prescribed solved block.

The new checked example $F(Y_1, Y_2, T) = Y_1$ has a surjective solved block $(1, 0)$. Nevertheless $(0, 0, 0)$ and $(0, 2, 0)$ are distinct exact roots in the same modulo-two ball with identical parameters. Uniqueness therefore requires the square solved case. The corresponding negative control deliberately attempts the stronger false conclusion and must be rejected by Lean.

Depth $L \geq 1$ is essential for the Taylor step used here. The general singular case can have a nonzero next obstruction after an earlier zero obstruction. Neither the finite one-step criterion nor this smooth theorem establishes unrestricted singular persistence.

B.10 Paper IX: simultaneous coefficient and parameter deformation

Author: Komorebi P. — 8 September 2026.

This block combines coefficient perturbation with prescribed parameter motion. The original solved Jacobian controls existence, local uniqueness, displacement and the first-order error. The four new mathematical modules compile. The fresh verification report and independent audit record the authoritative release status and exact source revisions.

B.10.1 The theorem

Let p be any prime, $R = \mathbb{Z}_p$, and $Q = \mathbb{Q}_p$. Let

$$F, H \in R[Y_1, \ldots, Y_e, T_1, \ldots, T_k]^e, \qquad a = (y_0, t_0) \in R^{e+k}, \qquad F(a) = 0.$$

Here $e, k \geq 0$ are arbitrary finite dimensions. There are as many solved coordinates as equations. Define the actual derivative blocks at the original root:

$$J = D_Y F(a) \in M_e(R), \qquad K = D_T F(a) \in M_{e\times k}(R).$$

Suppose, for natural numbers s, M, that

$$J \in \mathrm{GL}_e(Q), \qquad p^s J^{-1} \in M_e(R), \qquad M > 2s.$$

For every $v \in R^k$, set

$$\tau = t_0 + p^M v, \qquad G = F + p^M H.$$

Then there is exactly one $y \in y_0 + p^{s+1}R^e$ with

$$G(y, \tau) = 0.$$

It satisfies

$$y - y_0 \in p^{M-s}R^e$$

and there is an integral vector $\varepsilon \in p^{2M-3s}R^e$ such that, in Q^e,

$$\boxed{y - y_0 = -p^M J^{-1}\big(H(a) + Kv\big) + \varepsilon.}$$

The same original exponent s appears throughout. No inverse or rank assumption on the perturbed Jacobian is imposed. The statement includes arbitrary integral p-adic coefficients and centers, the dyadic prime, and zero dimensions.

The full-point theorem constructs $b = (y, \tau)$ and also proves

$$b - a \in p^{M-s}R^{e+k} \cap p^{s+1}R^{e+k}.$$

Parameter equality $b_T = \tau$ is exact. Uniqueness concerns every root in the stated solved-coordinate ball with these parameters, not only roots obtained by the construction. No global uniqueness is claimed.

B.10.2 Actual augmented system

Define polynomial systems with $e + k$ equations and variables by

$$\mathcal{F}(Y,T) = \big(F(Y,T), T - t_0\big), \qquad \mathcal{H}(Y,T) = \big(H(Y,T), -v\big).$$

Both are actual polynomials over R. Direct evaluation proves

$$\mathcal{F}(a) = 0, \qquad (\mathcal{F} + p^M\mathcal{H})(b) = 0 \iff F(b) + p^M H(b) = 0 \text{ and } b_T = t_0 + p^M v.$$

Actual polynomial partial differentiation gives

$$D\mathcal{F}(a) = B = \begin{pmatrix} J & K \\ 0 & I_k \end{pmatrix}.$$

These identities are proved from evaluation and partial derivatives, rather than supplied as additional hypotheses.

Choose the integral matrix $A = p^s J^{-1}$. The existing inverse-certificate equivalence gives

$$AJ = JA = p^s I_e.$$

Put $c = p^s$ and define an integral block matrix

$$C = \begin{pmatrix} A & -AK \\ 0 & cI_k \end{pmatrix}.$$

Then

$$CB = \begin{pmatrix} AJ & AK - AK \\ 0 & cI_k \end{pmatrix} = cI_{e+k},$$

$$BC = \begin{pmatrix} JA & -JAK + cK \\ 0 & cI_k \end{pmatrix} = cI_{e+k}.$$

The second equality uses $JA = cI_e$. The generic block proof works over any commutative ring. The p-adic specialization also explicitly proves that B is invertible over Q and that $p^s B^{-1}$ is integral. There is no denominator loss from appending the parameter equations.

B.10.3 Construction, error and uniqueness

Apply the previously checked weighted coefficient-perturbation construction to $\mathcal{F}, \mathcal{H}, a, C$. Its hypotheses follow from the actual augmented identities and the two-sided certificate. It constructs b, proves the full displacement bound p^{M-s}, and proves uniqueness throughout $a + p^{s+1}R^{e+k}$. The augmented root equivalence gives the required original equations and exact parameter values.

The checked integral first-order estimate gives

$$b - a + p^{M-s}C\mathcal{H}(a) \in p^{2M-3s}R^{e+k}.$$

The decisive block action is

$$C\begin{pmatrix} H(a) \\ -v \end{pmatrix} = \begin{pmatrix} A\big(H(a) + Kv\big) \\ -p^s v \end{pmatrix}.$$

Projecting the solved coordinates therefore gives

$$y - y_0 + p^{M-s}A\big(H(a) + Kv\big) \in p^{2M-3s}R^e.$$

The existing rational-conversion theorem turns $p^{M-s}A$ into $p^M J^{-1}$ and retains an integral error vector. This proves the displayed first-order formula with the original derivative blocks.

For any competing solved-coordinate root $z \in y_0 + p^{s+1}R^e$, the joined point (z, τ) lies in the augmented uniqueness ball: $M > 2s$ implies $M \geq s+1$, so $\tau - t_0 = p^M v \in p^{s+1}R^k$. Full-point uniqueness gives $(z, \tau) = b$, hence $z = y$. This is the additional comparison needed to obtain uniqueness in exactly the ball stated in the solved-coordinate theorem.

B.10.4 Checked controls and precision boundary

For $p = 2$, $s = 0$, $F(Y, T) = Y + T$, $H = 1$, $a = 0$, and $v = 1$, both original blocks satisfy $J = K = 1$. At $M = 1$, the exact parameter is $\tau = 2$, and the unique solved value is $y = -4$. The combined correction is $-2(1+1) = -4$, with zero error.

The sign controls use $M = 2$: the actual augmented root is $(y, \tau) = (-8, 4)$, and the required error ideal is $16\mathbb{Z}_2$. Omitting Kv predicts -4, with error $-4 \notin 16\mathbb{Z}_2$. Reversing its sign predicts 0, with error $-8 \notin 16\mathbb{Z}_2$. Both failures are checked. At binary depth $M = 1$, the reversed-sign difference would lie in the allowed error ideal, so that shallower example alone would not distinguish the sign. This control-design limitation was identified and corrected before the release gate.

For parameter-only motion, use

$$F(Y, T) = Y + Y^2 + T, \quad H = 0, \quad a = 0, \quad v = 1, \quad p = 2,\ s = 0,\ M = 1.$$

A checked transfer from the prior scalar quadratic root constructs an even y with

$$y + y^2 + 2 = 0, \qquad 4 \mid y + 2, \qquad 8 \nmid y + 2.$$

Thus the error exponent $2M - 3s = 2$ is attained even when only parameters move. No uniform extra error digit can be added to the general theorem. This example establishes that limitation; it does not assert optimality at every choice of p, s, M.

The inherited controls continue to record the necessity of the strict weighted threshold and other earlier scope boundaries. General rationally rank-deficient systems are outside the inverse hypothesis.

B.11 Paper IX: analytic regularity of the local parameter solution

Author: Komorebi P.

This block establishes both analyticity and differentiability of the actual local polynomial solution map at a root with an invertible solved Jacobian. The statements apply to every prime, including $p = 2$. They also hold over any complete nontrivially normed field.

Let $\mathbb{K}$ be such a field, let $e, k \in \mathbb{N}$, and let

$$F \in \mathbb{K}[Y_1, \ldots, Y_e, T_1, \ldots, T_k]^e, \qquad a = (y_0, t_0) \in \mathbb{K}^e \times \mathbb{K}^k.$$

Evaluation of F means coordinatewise evaluation of its actual polynomials. Define the actual derivative blocks

$$J = D_Y F(a) \in M_e(\mathbb{K}), \qquad K = D_T F(a) \in M_{e\times k}(\mathbb{K}),$$

where each matrix entry is the corresponding formal partial derivative evaluated at a. The hypotheses are exactly

$$F(a) = 0, \qquad J \in \mathrm{GL}_e(\mathbb{K}).$$

Neither analyticity of a proposed solution nor a derivative formula is assumed.

Local analytic graph theorem. There exist a map

$$\psi : \mathbb{K}^k \longrightarrow \mathbb{K}^e$$

and open sets $U \subseteq \mathbb{K}^k$, $V \subseteq \mathbb{K}^e$ such that

$$t_0 \in U, \qquad y_0 \in V, \qquad \psi(t_0) = y_0, \qquad \psi(U) \subseteq V.$$

The map is analytic at every point of U, and its graph describes all roots in the product neighborhood:

$$\forall t \in U \; \forall y \in V, \qquad F(y,t) = 0 \quad \Longleftrightarrow \quad y = \psi(t).$$

At the base parameter it has the strict Fréchet derivative

$$\boxed{D\psi(t_0) = -J^{-1}K.}$$

Consequently it is differentiable on U. The map is defined on the ambient parameter space, but its root, analyticity and uniqueness conclusions are restricted to the specified neighborhood. Uniqueness is among solved coordinates in V.

The main Lean theorem is `Komorebi.PolynomialAnalyticImplicit.exists_analytic_implicit`. The matrix in its `HasStrictFDerivAt` conclusion is the continuous linear map represented by the displayed matrix. `differentiableOn_of_analytic_graph` records the differentiability consequence.

Proof chain. The scalar polynomial derivative bridge is proved by induction on actual multivariate polynomials: constants, sums, and multiplication by a variable. For a scalar polynomial P, its strict derivative is

$$v \longmapsto \sum_j \frac{\partial P}{\partial x_j}(a) v_j.$$

Assembling the coordinates gives the strict derivative of the evaluated system as its actual Jacobian matrix. Polynomial analyticity is supplied by the pinned Mathlib theorem for multivariate polynomial evaluation, followed by finite coordinate assembly.

Consider the actual augmented polynomial map

$$\Phi(y,t) = (F(y,t), t).$$

Its derivative and inverse derivative at a are

$$D\Phi(a) = \begin{pmatrix} J & K \\ 0 & I_k \end{pmatrix}, \qquad D\Phi(a)^{-1} = \begin{pmatrix} J^{-1} & -J^{-1}K \\ 0 & I_k \end{pmatrix}.$$

The proof constructs the continuous linear equivalence from these explicit matrices and verifies both inverse identities. The strict inverse theorem gives a local inverse g of Φ, and the analytic inverse result shows that this same inverse is analytic near $\Phi(a) = (0, t_0)$. Set

$$\psi(t) = \pi_Y g(0,t).$$

The two inverse identities and a sufficiently small product neighborhood give the exact local graph statement. Composing the derivative of g with parameter insertion and solved-coordinate projection yields $-J^{-1}K$. The chosen parameter neighborhood is also restricted so that the inverse identities, analyticity and inclusion $\psi(U) \subseteq V$ hold there simultaneously.

Transfer to a supplied continuous solution. Suppose $g : \mathbb{K}^k \to \mathbb{K}^e$ is continuous at t_0, $g(t_0) = y_0$, and

$$F(g(t), t) = 0$$

for all t in some neighborhood of t_0. Under the same invertibility hypothesis at a, continuity places $g(t)$ in the uniqueness neighborhood V for all sufficiently close parameters. Hence g and ψ agree on a neighborhood of t_0. The theorem `continuous_selection_regularity` transfers both analyticity and the strict derivative formula to g.

There is also an explicit open-set version. Let U be open, let g be continuous on U, and suppose for every $t \in U$ that

$$F(g(t), t) = 0, \qquad J_t := D_Y F(g(t), t) \in \mathrm{GL}_e(\mathbb{K}).$$

Writing $K_t = D_T F(g(t), t)$, the theorem `continuous_selection_analyticOnNhd` proves

$$g \text{ is analytic at every } t \in U, \qquad Dg(t) = -J_t^{-1} K_t \quad (t \in U),$$

with a strict Fréchet derivative at each point. The per-point invertibility hypothesis is part of this theorem. This is a transfer statement for continuous local root selections satisfying its hypotheses; it does not assert equality of arbitrary global branch choices.

Integral p-adic specialization. For any prime p, let

$$F \in \mathbb{Z}_p[Y, T]^e, \qquad a \in \mathbb{Z}_p^{e+k}, \qquad F(a) = 0.$$

Let $\iota : \mathbb{Z}_p \hookrightarrow \mathbb{Q}_p$ be the coefficient inclusion, applied entrywise to points and matrices. Assume only that

$$\iota(D_Y F(a)) \in \mathrm{GL}_e(\mathbb{Q}_p).$$

The theorem `PolynomialRegularityIntegral.exists_analytic_integral_implicit` constructs the preceding analytic graph in the ambient spaces $\mathbb{Q}_p^k$ and $\mathbb{Q}_p^e$. Its root equation evaluates the original integral polynomial coefficients using ι, and its base derivative is

$$-\iota(D_Y F(a))^{-1}\, \iota(D_T F(a)).$$

The source proves that polynomial evaluation, formal partial differentiation and both Jacobian blocks commute with this inclusion. No invertibility modulo p or integral inverse is required. This permits a solved Jacobian that is singular modulo p while remaining invertible over $\mathbb{Q}_p$. The new specialization is an ambient-field theorem; it does not treat $\mathbb{Z}_p$ as a vector space over $\mathbb{Q}_p$ or assert a new explicit integral radius.

Source module	Mathematical role
`PolynomialAnalyticEvaluation.lean`	Actual polynomial analyticity and strict derivative equals the evaluated formal Jacobian.
`PolynomialAnalyticInverse.lean`	Analyticity of the same local inverse supplied by an invertible strict derivative.

Source module	Mathematical role
`PolynomialAnalyticBlocks.lean`	Explicit augmented derivative equivalence and its solved-coordinate restriction.
`PolynomialAnalyticImplicit.lean`	Open local root graph, analyticity, base strict derivative and differentiability.
`PolynomialAnalyticSelection.lean`	Transfer to continuous root selections, including the per-point open-set theorem.
`PolynomialRegularityIntegral.lean`	Integral coefficients and centers included in the ambient p-adic field, with actual Jacobian compatibility.

This block closes the local analytic regularity obligation stated in `TARGET_CONTRACT.json`. It supplies no new uniform numerical radius. The previous weighted lifting and joint deformation results retain their own explicit radii and error estimates. It does not establish a global analytic branch or a result at a point whose solved Jacobian is singular over the fraction field. The proof uses established analytic inverse machinery and makes no novelty claim for the general implicit-function theorem.

The remaining Paper IX work includes general singular persistence with higher obstructions, native presentation invariance (13.2), global or completed compatibility and descent (13.3), the outstanding sharper arithmetic statements, and manuscript consolidation. None of those is closed by this local regularity theorem. Release-level checks and audit counts are recorded in the separate release verification and audit files.

C Restricted native progress and complete prose proofs

This appendix reproduces the mathematical parts of the 8 September 2026 native progress note. The polynomial obstruction, finite reconstruction, rectangularity and topological arguments are prose proofs. The intrinsic commutative-algebra obstruction is the stated classical Stacks specialization, with its original source record and finiteness argument. None of these prose arguments is promoted to a new Lean run or to a completion of unrestricted Native 13.2 or 13.3.

C.1 2. A complete polynomial obstruction theorem

Scope. Fix a prime p, an integer $L \geq 1$, finite natural numbers d, r, and polynomials $F_1, \ldots, F_r \in \mathbb{Z}[X_1, \ldots, X_d]$. Let $\bar{x} \in (\mathbb{Z}/p^L\mathbb{Z})^d$ satisfy every equation $F_i(\bar{x}) = 0$. Choose any integer representative $x \in \mathbb{Z}^d$. Set

$$e_i = F_i(x)/p^L \pmod{p}, \qquad J_{ij} = \frac{\partial F_i}{\partial X_j}(x) \pmod{p}.$$

The divisions are exact integer divisions. Regard J as a linear map $\mathbb{F}_p^d \to \mathbb{F}_p^r$. Define

$$O_{F,\bar{x}} = \mathbb{F}_p^r / \operatorname{im} J, \qquad o_{F,\bar{x}} = [e].$$

Theorem. For this fixed polynomial presentation, the group and distinguished class are independent of the chosen integer representative. The class vanishes if and only if $\bar{x}$ lifts to a solution modulo p^{L+1}. The obstruction group has $p^{r-\operatorname{rank} J}$ elements. When lifts exist, their set is a torsor for $\ker J$ and contains exactly $p^{d-\operatorname{rank} J}$ elements.

Proof. For every integer correction vector t, expansion of each monomial gives

$$F(x + p^L t) \equiv F(x) + p^L J(x) t \pmod{p^{L+1}}. \tag{1}$$

Indeed, each term involving at least two correction factors is divisible by p^{2L}, and $2L \geq L + 1$. Constant and linear correction terms give the two displayed summands. Adding the monomial expansions proves (1) for every polynomial, with no denominator or analytic convergence issue.

Every element reducing to $\bar{x}$ is uniquely expressible modulo p^{L+1} as $x + p^L t$ with $t \in \mathbb{F}_p^d$. By (1), it solves the equations exactly when

$$e + Jt = 0 \quad \text{in } \mathbb{F}_p^r. \tag{2}$$

Equation (2) is soluble exactly when $e \in \operatorname{im} J$: the sign makes no difference because the image is a subspace. This is equivalent to $[e] = 0$.

Another representative has the form $x' = x + p^L s$. Since $L \geq 1$, its Jacobian is the same modulo p. Dividing (1) by p^L shows that its residual digit is $e' = e + Js$, so it gives exactly the same quotient class. If t_0 solves (2), all its solutions are $t_0 + \ker J$, and no others. The finite cardinalities follow from rank-nullity. This also covers no variables, no equations, zero Jacobian and singular solutions. □

The proof is a universal mathematical argument. The accompanying Python checks are independently bounded evidence about an implementation, not a substitute for that argument. The standalone Lean

specialization does not formalize multivariable polynomial expansion (1), unless its own checked source explicitly says so.

C.1.1 Necessary boundaries and exact examples

- **A failed singular lift:** $p = 3$, $L = 1$, $F(X) = X^2 - 3$, $x = 0$. Here $e = -1 \neq 0$ in $\mathbb{F}_3$ and $J = 0$. There are no lifts of zero modulo 3 to a root modulo 9.
- **Representative independence is a quotient assertion:** for $F(X) = X^2 - 1$, $p = 3$, $L = 1$, representatives 1 and 4 have residual digits 0 and 2. Their common Jacobian is 2, with full image, and both give zero obstruction. Their corrected representatives 1 and 10 are the same root modulo 9.
- **A successful step need not give indefinite continuation:** for $F(X) = X^2 - 5$, $p = 2$, the root 1 modulo 2 has zero obstruction and lifts to $1, 3$ modulo 4. Both have nonzero obstruction at the next step. There is no root modulo 8 and no compatible infinite lift.
- **The raw Jacobian cokernel need not be presentation independent:** the presentations $F = (X)$ and $G = (X, 0)$ define the same solution system. Their raw obstruction groups at zero are respectively zero and $\mathbb{F}_p$. Both actual distinguished obstructions are zero. Thus one cannot claim a canonical isomorphism of these two raw groups merely because the equations define the same seed.

C.2 3. An intrinsic finite obstruction under an algebraic restriction

Let B be a finitely presented commutative $\mathbb{Z}$-algebra, p prime and $L \geq 1$. For a ring map $f : B \to \mathbb{Z}/p^L\mathbb{Z}$, use its reduction to make $\mathbb{F}_p$ a B-module. There is a canonical class

$$o(f) \in \mathrm{Ext}^1_B(\mathrm{NL}_{B/\mathbb{Z}}, \mathbb{F}_p)$$

whose vanishing is equivalent to a lift $B \to \mathbb{Z}/p^{L+1}\mathbb{Z}$. This follows from the ring-map lifting theorem in Stacks Project, Lemma 91.2.1, tag 08S5. Specialize the source extension to $0 \to 0 \to B \to B$ and the target extension to the kernel $p^L/p^{L+1} \cong \mathbb{F}_p$. The base maps are identities of $\mathbb{Z}$. The kernel is square zero because $L \geq 1$.

The comparison in Stacks Project, Lemma 10.134.2, tag 00S1 identifies the complex from a polynomial presentation with the canonical naive cotangent complex up to homotopy. Thus this construction is intrinsic, up to the stated canonical comparison.

The group is finite. Choose a finite presentation $P = \mathbb{Z}[X_1, \ldots, X_d] \twoheadrightarrow B$ with finitely generated kernel I. The presentation description in Stacks Project, Remark 91.2.4, tag 0GPU identifies the group as

$$\mathrm{coker}\left(\mathrm{Hom}_B(\Omega_{P/\mathbb{Z}} \otimes_P B, \mathbb{F}_p) \longrightarrow \mathrm{Hom}_B(I/I^2, \mathbb{F}_p)\right).$$

If I has r chosen generators, evaluation on their classes injects the latter Hom module into $\mathbb{F}_p^r$. Hence its quotient is finite. The relation module I/I^2 retains dependencies between equations, including redundant zero equations, which the raw target $\mathbb{F}_p^r$ does not.

Status: externally proved after the exact stated specialization, with the finiteness argument above. This theorem is not independently formalized in the accompanying Lean file. Extending it to all operator/recurrence/module systems admitted by manuscript Definition 1.1 requires a specified encoding and a proof that its notion of lift is preserved. That is the missing implication for applying it to native Conjecture 13.2 without restriction.

Source record. The Stacks Project authors, live primary text inspected 2026-09-08, tags 08S5, 0GPU and 00S1. Full theorem/proof text was inspected, not only search snippets. No immutable source revision was pinned or downloaded. The use is a classical specialization, not a novelty search or a claim of a new obstruction theory. No broader paper or source cache was changed.

C.3 4. A topological obstruction to the proposed spine formula

Consider the recurrence $a_{n+1} = a_n$ with integer initial value. Its global parameter space is $X = \mathbb{Z}$. Its local parameter spaces are $X_p = \mathbb{Z}_p$, and every local realization has the same prime-independent recurrence. Give the local factors their usual p-adic topologies and the product its product topology. For these integral local factors, the restricted product relative to all $\mathbb{Z}_p$ is the ordinary product. Write

$$A = \mathbb{Z}_2, \qquad B = \prod_{p>2} \mathbb{Z}_p, \qquad D = \{(n, (n)_{p>2}) : n \in \mathbb{Z}\} \subset A \times B.$$

Theorem. There are no Hausdorff space C and continuous maps $f : A \to C$, $g : B \to C$ for which the equality locus

$$\{(a, b) : f(a) = g(b)\}$$

is exactly D. Thus the canonical local realizations of global integer constant sequences cannot satisfy the manuscript's fiber-product formula in this Hausdorff topological interpretation.

Proof. The equality locus is closed: it is the inverse image of the diagonal of $C \times C$ under the continuous map $f \times g$, and a Hausdorff space has closed diagonal.

The set D is dense in $A \times B$. A nonempty basic open set prescribes congruences at only finitely many primes and finite precision at each. The moduli are pairwise coprime. The finite Chinese remainder theorem supplies an integer meeting all those congruences, so every such open set meets D.

However, D is not the whole product. The element with 2-adic coordinate zero and every odd-adic coordinate one is not diagonal. If an integer had zero 2-adic coordinate, it would be divisible by every power of two, forcing it to be zero. Its 3-adic coordinate would then be zero, not one. Consequently D is a proper dense subset and is not closed. It cannot be the equality locus above. □

This is not asserted to refute the original, underspecified sentence under every possible reading. It refutes the explicit and natural interpretation just stated. The assumptions of continuity, Hausdorffness and identification through the canonical local maps are essential to this claim.

C.3.1 A precise reconstruction that does hold

For a finite set S of primes, positive precisions L_p, and a fixed integer polynomial system F, put

$$N = \prod_{p \in S} p^{L_p}, \qquad \mathrm{Sol}_F(R) = \{x \in R^d : F(x) = 0\}.$$

Then coefficientwise CRT gives a canonical bijection

$$\mathrm{Sol}_F(\mathbb{Z}/N\mathbb{Z}) \cong \prod_{p \in S} \mathrm{Sol}_F(\mathbb{Z}/p^{L_p}\mathbb{Z}).$$

The reason is exact: the CRT ring isomorphism preserves polynomial evaluation, and zero in a finite product is equivalent to zero in every component. For F empty, passing to all compatible finite congruences identifies the profinite completion $\widehat{\mathbb{Z}}$ with $\prod_p \mathbb{Z}_p$. The same finite reconstruction is compatible with coefficient projections and integer recurrence matrices because these operations commute with reduction and CRT.

Here the polynomial equations or recurrence matrices are fixed common data. The theorem reconstructs finite or profinite objects. It does not identify a "smallest" spine, impose additional spectral descent, or recover global $\mathbb{Z}$ from arbitrary tuples of local points. A completed version of 13.3 must say exactly which object is completed and specify its universal property.

C.3.2 A finite compatibility criterion with a precise universal property

Fix finite sets A, B of local packets and an explicitly given relation $R \subseteq A \times B$. Say that R is rectangular when

$$R(a,b),\ R(a',b),\ R(a',b') \quad \Longrightarrow \quad R(a,b')$$

for all $a, a' \in A$ and $b, b' \in B$.

Theorem. The relation R can be represented exactly as

$$R(a,b) \quad \Longleftrightarrow \quad f(a) = g(b)$$

for maps from A and B to a common set if and only if R is rectangular. In that case a canonical finite choice of common set is the connected-component set C_R of the bipartite graph with vertex set $A \sqcup B$ and edges given by R.

The component maps $f_R : A \to C_R$ and $g_R : B \to C_R$ have the following universal property. For every set Y and maps $h : A \to Y$, $k : B \to Y$ satisfying $h(a) = k(b)$ whenever $R(a,b)$, there is a unique map $u : C_R \to Y$ with $u \circ f_R = h$ and $u \circ g_R = k$.

Proof. Any equality representation is rectangular, since the three assumed equalities imply $f(a) = g(b) = f(a') = g(b')$.

Conversely, suppose R is rectangular. An edge always gives equal component labels. If $a \in A$ and $b \in B$ have equal component labels, there is a finite path between them. Its length is odd. A path of length one is already an R-edge. A longer path begins with three edges $a - b_1 - a_1 - b_2$. Rectangularity gives the edge $a - b_2$. Replacing these three edges by that edge shortens the path by two. Induction therefore yields $R(a,b)$. This proves the exact equality representation. Finiteness follows because the component set is a quotient of the finite set $A \sqcup B$.

For the universal property, combine h and k into a function on $A \sqcup B$. It is constant across every edge by hypothesis, hence along every finite path. It therefore descends to a well-defined function on components. The two required factorization identities hold by construction. They determine the value on every component, proving uniqueness. □

For example, the three pairs $(a_0, b_0), (a_1, b_0), (a_1, b_1)$ without (a_0, b_1) fail rectangularity. Every equality-label representation agreeing on those three pairs also admits the fourth pair, so it cannot represent that relation exactly.

This is a precise finite gluing obligation: future arithmetic applications must define A, B and R and prove rectangularity. The universal property holds for the component quotient even when rectangularity fails, but then the equality locus strictly enlarges R. The theorem establishes neither a minimum number of labels nor a smallest spine in the manuscript's unspecified category. It carries no automatic topology, spectral information or Frobenius compatibility. **Evidence: prose proof in this note, not a Lean formalization.**

D Recovered historical local polyphase manuscript

This is the complete substantive body of the reconstructed Annals textual copy, preserved to retain its derivations and original parameter ranges. The recovered text is not certified as the exact original TeX bytes. Its historical theorem headings distinguish these statements from the current numbered core. Preservation is not a blanket proof audit or Lean certificate: in particular the broader Amice analytic pullback and depleted branches, finite-field modal minimality and unrestricted infinite polyphase claims retain their historical evidence scope. The current formal statement map records only the individually identified interfaces in the main text.

The earlier reconstruction lacked all 14 original figure files. This reading edition supplies new reconstructions at their retained labels and captions. These are explanatory schematics and exact finite examples, not recovered original artwork. The reproducible program `tools/reconstructed_figures.py` checks the displayed finite data. The modulo-seven inverse plot uses the corrected positive tail $6, 5, 2, 6, 5, 2, \ldots$. A reconstructed illustration does not upgrade the historical proof or formalization status. The original historical source package, including all 14 vector figures and the bibliography, has now been recovered and is preserved unchanged in the accompanying release. The current reading edition retains its explicitly identified reconstructions and corrected finite data. The 22 historical references now have recovered bibliographic entries and checked identities. Superseded claims about the divisor-branch finite-difference problem have local notices directing the reader to the current theorem. The remaining higher kernel-valuation and unrestricted local-to-global questions remain open in their stated scopes.

D.1 Introduction

Euler numbers carry several kinds of local information at once. Their indices have residue addresses. Their coefficient transforms have triangular depth. Their local expansions have Mahler degree. Their congruences depend on the valuation of the transfer kernel and on the valuation of the source value in the row being reconstructed. These pieces should not be merged into one undifferentiated congruence calculation.

This article begins with the Hurwitz product

$$(a \star b)_n = \sum_{r=0}^{n} \binom{n}{r} a_r b_{n-r}.$$

Over a field of characteristic zero it is multiplication of exponential generating functions. Over a finite field it realizes the additive group algebra through the exponential modes $\epsilon_u = (u^n)_{n\geq 0}$. This finite algebra gives an exact proof of the standard window period. Hurwitz series rings are treated in [15, 7]. The finite difference and umbral background may be found in [21, 10, 13].

Fix an odd prime p, a divisor $m \mid p-1$, and put

$$h = \frac{p-1}{m}.$$

The reduced standard window inverse is defined by

$$\sum_{q=0}^{n} \binom{mn}{mq} e_q^{(m,p)} = \delta_{n0} \quad \text{in } \mathbb{F}_p.$$

We prove that

$$e_{n+h}^{(m,p)} = e_n^{(m,p)} \qquad (n \geq 1).$$

The result first follows from subgroup symmetry in the finite Hurwitz algebra. A second proof shows that the standard window operator preserves the finite space of sequences that are h periodic after their constant term. The two arguments expose different parts of the same finite character structure.

The central change in this edition is the treatment of the odd and dyadic Euler formulas. They are no longer presented as unrelated reconstruction devices. For each prime p we define an even source series H_p and an inverse kernel K_p satisfying

$$K_p(t)H_p(t) = \operatorname{sech} t.$$

At an odd prime,

$$H_p(t) = \cosh^2(pt)\operatorname{sech} t, \qquad K_p(t) = \operatorname{sech}^2(pt).$$

At the dyadic prime,

$$H_2(t) = \frac{\sinh(2t)\sinh t}{2t^2}, \qquad K_2(t) = \frac{4t^2}{\sinh^2(2t)}.$$

The coefficient convolution is identical in form, but the kernels have different integral weights. This is the correct sense in which the local theories unify.

The dyadic kernel is substantially sharper than the separate coefficient bound alone suggests. If

$$\frac{z^2}{\sinh^2 z} = \sum_{r\geq 0} \beta_r z^{2r}$$

and

$$\kappa_{2,r} = \beta_r 2^{2r}(2r)!,$$

then

$$v_2(\kappa_{2,r}) = 4r - 1 \qquad (r \geq 1).$$

Consequently, truncation after depth M is uniformly valid modulo

$$2^{4M+3},$$

and this exponent is optimal. The rowwise estimate records the additional binary carries in the binomial coefficient. Classical congruence theory for Euler families is extensive [8, 26, 27, 24, 9]. The formulas here are stated as exact coordinate identities. Historical priority requires a separate theorem by theorem comparison.

The second expansion concerns local interpolation. The exponent of the finite unit group moves every unit mode into a principal unit neighborhood. Binomial expansion then gives a Mahler series whose depth r coefficient gains at least Lr digits when the chosen jump is the identity modulo p^L. The same argument applies to polynomial exponential modes and to invertible matrix state systems. Mahler interpolation and the Amice transform are classical [19, 6, 20]. The local unit structure and valuation tools used here are standard [16, 22, 25, 12].

The odd Euler source contains one nonunit base. Removing that base gives an analytic unit source and an interpolation of a depleted Euler sequence on every even character branch. The dyadic source is already analytic on $\mathbb{Z}_p$ after a removable singularity is filled. It yields a direct analytic interpolation of E_{2n} on $\mathbb{Z}_2$. These branches are then refined by the substitution

$$x = Mq + s.$$

coordinate layer	what is retained	coordinate
$n = Nk + j$	residue address	j
$u = \omega(u)\langle u\rangle$	finite character	$\omega(u)$
$f(x) = \sum c_r \binom{x}{r}$	Mahler depth	r
$KH = \operatorname{sech}$	weighted reconstruction	$v_p(\kappa_r)$

Figure 1: New reconstruction. The arithmetic coordinate chain. Residue address, local character, Mahler depth, transfer weight, and reconstruction are recorded separately.

The induced Mahler coefficient transform is finite and explicit. Its kernels compose exactly. At prime power radix they become sparse modulo the residue prime. Every monotone valuation ladder survives the refinement.

The remaining sections treat arithmetic lift obstructions, recurrence packets, and Chinese remainder reconstruction. The lifting obstruction is the residual defect modulo the image of the declared correction map. A recurrence packet is governed by a jump matrix C^N. If one integral recurrence is fixed, all of its local realizations have one common source. These statements do not prove that arbitrary unrelated local objects have a global origin.

Four boundaries remain explicit. A period modulo p does not by itself lift to all powers of p. The full finite difference estimate for every congruential reciprocal packet still requires control of the triangular inverse. Exact higher valuations of the affine polyphase kernel are not yet known in general. Chinese remainder reconstruction and finite adelic synchronization do not prove a global source for arbitrary local systems.

D.2 Hurwitz series and finite field subgroup inversion

Current status. This dated statement is superseded for the stated divisor-branch family by Theorem 3.7, with the $j!$ normalization and factorial valuation loss explicitly retained. It makes no claim for unrestricted families.

D.2.1 The Hurwitz ring

Historical definition D.1 (Hurwitz series ring). Let R be a commutative ring with identity. The Hurwitz series ring $\mathcal{H}(R)$ is the set $R^{\mathbb{N}}$ with componentwise addition and product

$$(a \star b)_n = \sum_{r=0}^{n} \binom{n}{r} a_r b_{n-r}.$$

Its identity is

$$\delta = (1, 0, 0, \ldots).$$

When the factorials are invertible, the sequence a may be written as the exponential generating series

$$A(t) = \sum_{n \geq 0} a_n \frac{t^n}{n!},$$

and the Hurwitz product becomes ordinary multiplication. The sequence definition remains valid in every characteristic.

Historical definition D.2 (Exponential mode). For $u \in R$, define

$$\epsilon_u = (u^n)_{n \geq 0},$$

where $0^0 = 1$.

Historical proposition D.3 (Addition law for modes). For every $u, v \in R$,

$$\boxed{\epsilon_u \star \epsilon_v = \epsilon_{u+v}.}$$

Proof. The nth coefficient is

$$\sum_{r=0}^{n} \binom{n}{r} u^r v^{n-r} = (u+v)^n$$

by the binomial theorem. □

D.2.2 The finite additive group algebra

Let p be prime. Write $\mathbb{F}_p^+$ for the additive group of the finite field.

Historical theorem D.4 (Finite field mode algebra). The map

$$\Theta : \mathbb{F}_p[\mathbb{F}_p^+] \longrightarrow \mathcal{H}(\mathbb{F}_p), \qquad [u] \longmapsto \epsilon_u,$$

is an injective algebra homomorphism. Its image is the p dimensional subalgebra spanned by the modes ϵ_u.

Proof. The product law follows from historical proposition D.3. To prove injectivity, suppose

$$\sum_{u \in \mathbb{F}_p} c_u \epsilon_u = 0.$$

Looking at the coordinates $n = 0, 1, \ldots, p-1$ gives a Vandermonde system with matrix

$$(u^n)_{0 \leq n \leq p-1,\ u \in \mathbb{F}_p}.$$

Its determinant is the product of all differences between distinct field elements, so it is nonzero. Hence every c_u vanishes. □

Historical proposition D.5 (Augmentation criterion). Let

$$x = \sum_{u \in \mathbb{F}_p} c_u [u] \in \mathbb{F}_p[\mathbb{F}_p^+].$$

Then x is a unit if and only if

$$\operatorname{aug}(x) = \sum_u c_u$$

is nonzero. If J is the augmentation ideal, then

$$J^p = 0.$$

Proof. Choose a generator g of the cyclic additive group. Then

$$\mathbb{F}_p[\mathbb{F}_p^+] \cong \mathbb{F}_p[y]/(y^p - 1) = \mathbb{F}_p[y]/(y-1)^p.$$

The augmentation ideal is generated by $y - 1$, so its pth power is zero. Every element has the form $a + n$ with $a \in \mathbb{F}_p$ and $n \in J$. It is invertible exactly when $a \neq 0$, which is the augmentation condition. □

Historical corollary D.6 (Finite inverse formula). If x has augmentation one and $n = x - 1$, then

$$\boxed{x^{-1} = \sum_{r=0}^{p-1} (-n)^r.}$$

Proof. By historical proposition D.5, $n^p = 0$. The finite geometric sum gives the inverse. □

D.2.3 Subgroup selectors

Fix a divisor m of $p - 1$. Let

$$H_m = \{\zeta \in \mathbb{F}_p^\times : \zeta^m = 1\}.$$

This group has order m.

Historical definition D.7 (Multiple selector). Define $a^{(m)} \in \mathcal{H}(\mathbb{F}_p)$ by

$$a_n^{(m)} = \begin{cases} 1, & m \mid n, \\ 0, & m \nmid n. \end{cases}$$

Historical lemma D.8 (Subgroup average). The multiple selector has the mode expansion

$$\boxed{a^{(m)} = \frac{1}{m} \sum_{\zeta \in H_m} \epsilon_\zeta.}$$

Proof. The nth coefficient on the right is

$$\frac{1}{m} \sum_{\zeta \in H_m} \zeta^n.$$

The character sum is one when $m \mid n$ and zero otherwise. □

Since the subgroup average has augmentation one, it has a unique Hurwitz inverse. Write

$$b^{(m,p)} = (a^{(m)})^{-1}.$$

Historical theorem D.9 (Support and positive degree period). The inverse $b^{(m,p)}$ has the following properties.

1. $b_n^{(m,p)} = 0$ unless $m \mid n$.
2. For every positive n,

$$\boxed{b_{n+p-1}^{(m,p)} = b_n^{(m,p)}.}$$

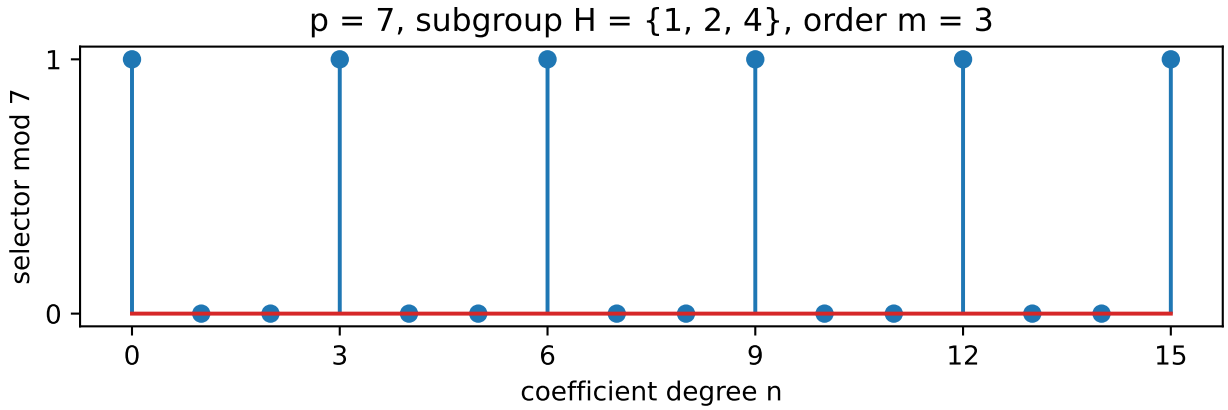


Figure 2: New reconstruction. A subgroup average in the additive group algebra. The highlighted modes select coefficient degrees divisible by the subgroup order.

Proof. Multiplication by an element $\eta \in H_m$ gives an automorphism of the additive group and therefore an automorphism

$$\epsilon_u \longmapsto \epsilon_{\eta u}$$

of the mode algebra. The selector $a^{(m)}$ is fixed by every such automorphism, so its unique inverse is fixed as well.

Write

$$b^{(m,p)} = \sum_{u \in \mathbb{F}_p} c_u \epsilon_u.$$

Invariance gives $c_{\eta u} = c_u$. After the change of variable $u \mapsto \eta u$,

$$b_n^{(m,p)} = \eta^n b_n^{(m,p)}$$

for every $\eta \in H_m$. If $m \nmid n$, some η satisfies $\eta^n \neq 1$, so the coefficient vanishes.

For positive n, the zero mode contributes nothing. Every nonzero mode satisfies $u^{p-1} = 1$. Hence

$$b_{n+p-1}^{(m,p)} = \sum_{u \neq 0} c_u u^{n+p-1} = \sum_{u \neq 0} c_u u^n = b_n^{(m,p)}.$$

□

Historical definition D.10 (Compressed inverse). Set

$$e_n^{(m,p)} = b_{mn}^{(m,p)}.$$

Historical proposition D.11 (Compressed inverse recurrence). The compressed coefficients satisfy

$$\boxed{\sum_{q=0}^{n} \binom{mn}{mq} e_q^{(m,p)} = \delta_{n0}.}$$

In particular, $e_0^{(m,p)} = 1$, and the positive coefficients are determined recursively because the diagonal is one.

Proof. Evaluate the Hurwitz identity

$$a^{(m)} \star b^{(m,p)} = \delta$$

at degree mn. Both factors vanish outside degrees divisible by m. Reindexing the surviving terms gives the displayed recurrence. □

Historical corollary D.12 (Uniform compressed period). Let

$$h = \frac{p-1}{m}.$$

Then

$$\boxed{e^{(m,p)}_{n+h} = e^{(m,p)}_n \qquad (n \geq 1).}$$

Proof. Since $m(n+h) = mn + p - 1$, the conclusion follows from historical theorem D.9. □

Historical corollary D.13 (Rational ordinary generating function). In $\mathbb{F}_p[[z]]$,

$$\boxed{\sum_{n\geq 0} e^{(m,p)}_n z^n = 1 + \frac{\sum_{r=1}^{h} e^{(m,p)}_r z^r}{1 - z^h}.}$$

Proof. Separate the constant coefficient and group the positive indices into residue classes modulo h. □

Historical theorem D.14 (Exponential form and minimal period). Let

$$G_m = (\mathbb{F}_p^{\times})^m.$$

This cyclic group has order h. There are uniquely determined coefficients $d_v \in \mathbb{F}_p$, indexed by $v \in G_m$, such that

$$\boxed{e^{(m,p)}_n = \sum_{v \in G_m} d_v v^n \qquad (n \geq 1).}$$

The minimal positive period is the least common multiple of the orders of those v for which $d_v \neq 0$. In particular, it divides h.

Proof. The coefficients c_u in the proof of historical theorem D.9 are constant on the cosets of H_m. For $u \neq 0$, the value u^m depends only on the coset, and every element of G_m arises. Grouping the modes by cosets gives the displayed expansion.

Distinct elements of G_m define linearly independent functions on one full period, by the Vandermonde determinant. Therefore a positive integer r is a period exactly when $v^r = 1$ for every active mode. The least such r is the stated least common multiple. □

Historical example D.15 (A compressed inverse modulo seven). **Editorial numerical correction.** The positive-index standard-window tail printed below is erroneous. The defining recurrence gives $e_1 = -1 \equiv 6$, $e_2 = 5$ and $e_3 = -61 \equiv 2 \pmod 7$, so the correct positive tail is $6, 5, 2, 6, 5, 2, \ldots$. The period-three conclusion remains correct. The original line is retained only as a corrected historical record.

Take $p = 7$ and $m = 2$, so $h = 3$. The recurrence gives

$$e^{(2,7)}_n = 1, 6, 4, 1, 6, 4, \ldots \qquad (n \geq 1).$$

The positive tail has exact period three.

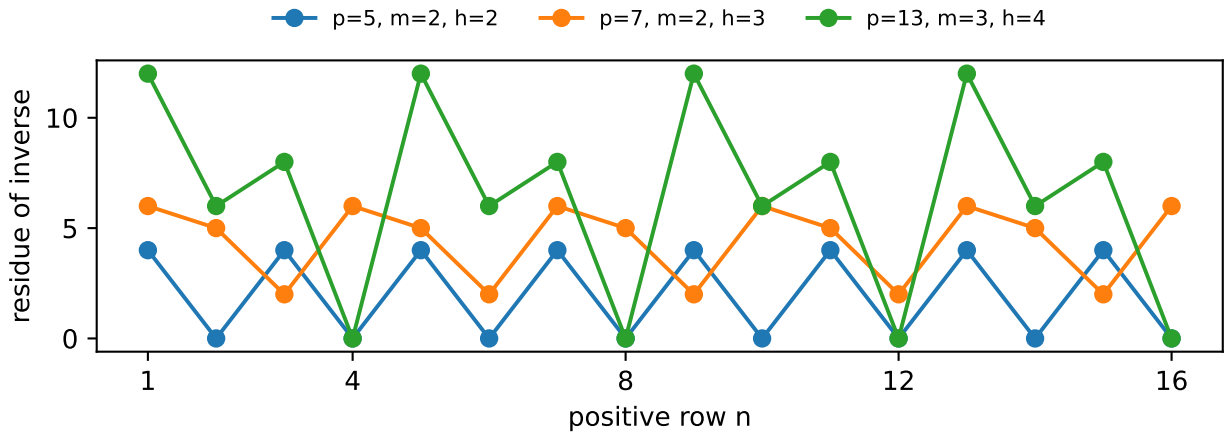


Figure 3: New reconstruction. Several compressed inverses. Every row is periodic after the constant term, as proved in historical corollary D.12.

D.3 Unit triangular coefficient transport

The examples later in the paper use different kernels, but their coefficient algebra has one common form. This section records that form once. Let R be a commutative algebra over $\mathbb{Q}$, and take all coefficients in R.

Let

$$H(t) = \sum_{r \geq 0} h_r \frac{t^r}{r!}, \qquad h_0 = 1.$$

There is a unique inverse

$$K(t) = H(t)^{-1} = \sum_{r \geq 0} k_r \frac{t^r}{r!}.$$

Historical theorem D.16 (Exact coefficient transport). Let

$$A(t) = \sum_{n \geq 0} A_n \frac{t^n}{n!}, \qquad S(t) = H(t)A(t).$$

Then

$$\boxed{S_n = \sum_{r=0}^{n} \binom{n}{r} h_r A_{n-r}.}$$

The inverse coordinate formula is

$$\boxed{A_n = \sum_{r=0}^{n} \binom{n}{r} k_r S_{n-r}.}$$

Both transforms are lower triangular with diagonal one.

Proof. The first identity is the coefficient rule for the product of exponential generating functions. Multiply by $K(t)$ and use $K(t)H(t) = 1$ for the inverse identity. The diagonal coefficient in each transform is the constant coefficient of its kernel, which is one. □

Historical corollary D.17 (Composition of coefficient transports). If U_H and U_G are the triangular transforms associated with unit kernels H and G, then

$$\boxed{U_H U_G = U_{HG}.}$$

In particular, the inverse of U_H is $U_{H^{-1}}$.

Proof. Multiplication by G, followed by multiplication by H, is multiplication by HG. □

Historical theorem D.18 (Filtered triangular reconstruction). Let p be prime. Suppose

$$A_n = \sum_{r=0}^{n} \binom{n}{r} c_r p^{w(r)} S_{n-r},$$

where $c_r \in \mathbb{Z}_p$, $S_n \in \mathbb{Z}_p$, and

$$w(r) \in \mathbb{N}.$$

For every precision $L \geq 1$,

$$\boxed{A_n \equiv \sum_{\substack{0 \leq r \leq n \\ w(r) < L}} \binom{n}{r} c_r p^{w(r)} S_{n-r} \pmod{p^L}.}$$

When the omitted set is nonempty, the tail has valuation at least

$$\boxed{\min_{\substack{0 \leq r \leq n \\ w(r) \geq L}} \left(w(r) + v_p\binom{n}{r} + v_p(S_{n-r}) \right).}$$

Proof. Every omitted summand belongs to $p^L \mathbb{Z}_p$. The sharper statement follows by taking the minimum of the exact lower bounds for the summands and applying the nonarchimedean triangle inequality. □

Historical remark D.19. An arbitrary invertible triangular matrix can encode an arbitrary sequence. The transforms retained here have an explicit generating kernel and a visible filtration. Those two features make exact inversion and uniform truncation possible.

D.4 Congruential reciprocal packets

D.4.1 Residue sections of the exponential series

For integers $N \geq 1$ and $0 \leq j < N$, define

$$\mathcal{E}_{N,j}(t) = \sum_{n \geq 0} \frac{t^{Nn+j}}{(Nn+j)!}.$$

The forced factor t^j may be removed without changing the residue address.

Historical definition D.20 (Congruential reciprocal packet). Define

$$\boxed{\mathcal{C}_{N,j}(t) = \frac{t^j/j!}{\mathcal{E}_{N,j}(t)} = \sum_{n \geq 0} C_{N,j}(n) \frac{t^{Nn}}{(Nn)!}.}$$

Historical theorem D.21 (Triangular inverse system). The coefficients satisfy $C_{N,j}(0) = 1$ and

$$\boxed{\sum_{q=0}^{n} \binom{Nn+j}{Nq} C_{N,j}(q) = \delta_{n0}.}$$

For $n \geq 1$,

$$\boxed{C_{N,j}(n) = -\frac{\sum_{q=0}^{n-1} \binom{Nn+j}{Nq} C_{N,j}(q)}{\binom{Nn+j}{Nn}}.}$$

Proof. Multiply $\mathcal{C}_{N,j}(t)$ by $\mathcal{E}_{N,j}(t)$ and compare the coefficient of $t^{Nn+j}/(Nn+j)!$. The last summand is the diagonal term, which is nonzero over $\mathbb{Q}$. □

Historical proposition D.22 (A denominator bound). Let

$$D_{N,j}(n) = \prod_{r=1}^{n} \binom{Nr+j}{Nr}, \qquad D_{N,j}(0) = 1.$$

Then

$$\boxed{D_{N,j}(n) C_{N,j}(n) \in \mathbb{Z}.}$$

For every prime p,

$$v_p(C_{N,j}(n)) \geq -\sum_{r=1}^{n} v_p\binom{Nr+j}{Nr}.$$

Proof. Use induction on n in the recurrence of historical theorem D.21. Every earlier denominator divides $D_{N,j}(n-1)$, while the new diagonal denominator is the final factor of $D_{N,j}(n)$. The valuation statement follows. □

Historical example D.23 (Euler and Bernoulli boundaries). For $(N, j) = (2, 0)$,

$$\mathcal{C}_{2,0}(t) = \operatorname{sech} t,$$

so

$$C_{2,0}(n) = E_{2n}.$$

For $(N, j) = (2, 1)$,

$$\mathcal{C}_{2,1}(t) = \frac{t}{\sinh t},$$

and

$$C_{2,1}(n) = 2\left(1 - 2^{2n-1}\right) B_{2n}.$$

The usual convention $B_1 = -1/2$ is used. Euler and Bernoulli number conventions are reviewed in [10, 23].

D.4.2 Standard windows

Fix a prime p, an integer $m \geq 1$, and an address $0 \leq j < p$. Put

$$N = mp.$$

For arbitrary m, let $\widetilde{e}_n^{(m,p)} \in \mathbb{F}_p$ be the unique sequence satisfying

$$\boxed{\sum_{q=0}^{n} \binom{mn}{mq} \widetilde{e}_q^{(m,p)} = \delta_{n0}.}$$

When $m \mid p-1$, this sequence agrees with the compressed Hurwitz inverse in historical proposition D.11.

Historical standard fact D.24 (Lucas congruence). If $a = a_0 + a_1 p + \cdots$ and $b = b_0 + b_1 p + \cdots$ are base p expansions, then

$$\binom{a}{b} \equiv \prod_r \binom{a_r}{b_r} \pmod{p}.$$

This is the classical theorem of Lucas [18].

Historical theorem D.25 (Standard window reduction). For $0 \leq q \leq n$,

$$\boxed{\binom{mpn+j}{mpq} \equiv \binom{mn}{mq} \pmod{p}.}$$

The diagonal coefficient is a unit modulo p:

$$\binom{mpn+j}{mpn} \equiv 1 \pmod{p}.$$

Proof. Write

$$mpn + j = p(mn) + j, \qquad mpq = p(mq).$$

Lucas reduction separates the lowest base p digit and gives

$$\binom{mpn+j}{mpq} \equiv \binom{j}{0}\binom{mn}{mq} \pmod{p}.$$

The diagonal case is obtained by setting $q = n$. □

Historical corollary D.26 (Standard window integrality). Every coefficient $C_{mp,j}(n)$ belongs to $\mathbb{Z}_p$.

Proof. The recurrence in historical theorem D.21 starts with a p integral constant term. The binomial coefficients are integers, and every diagonal term is a p adic unit by historical theorem D.25. Induction gives the result. □

Historical corollary D.27 (Address independence modulo the prime). Let $\widetilde{e}_n^{(m,p)}$ be the reduced standard inverse above. Then

$$\boxed{C_{mp,j}(n) \equiv \widetilde{e}_n^{(m,p)} \pmod{p}}$$

for every $0 \leq j < p$.

Proof. Modulo p, the recurrence in historical theorem D.21 becomes

$$\sum_{q=0}^{n} \binom{mn}{mq} C_{mp,j}(q) = \delta_{n0}.$$

This is the defining recurrence for $\widetilde{e}_n^{(m,p)}$. The diagonal is one, so the solution is unique. □

Historical theorem D.28 (Uniform standard window period). Assume that p is odd and $m \mid p-1$. Put

$$h = \frac{p-1}{m}.$$

Then for every standard window address $0 \le j < p$,

$$\boxed{C_{mp,j}(n+h) \equiv C_{mp,j}(n) \pmod{p} \qquad (n \ge 1).}$$

Proof. By historical corollary D.27, every address reduces to the same reduced standard inverse. Under the divisibility assumption, that inverse agrees with the compressed Hurwitz inverse by historical proposition D.11. Its positive tail has period h by historical corollary D.12. □

Historical corollary D.29 (Rational residual generating function). Under the hypotheses of historical theorem D.28, for every standard address j,

$$\sum_{n \ge 0} \left(C_{mp,j}(n) \bmod p\right) z^n = 1 + \frac{\sum_{r=1}^{h} \widetilde{e}_r^{(m,p)} z^r}{1 - z^h}.$$

Proof. Combine historical corollaries D.13 and D.27. □

Historical theorem D.30 (Fourier invariance of the standard window operator). Let p be odd, let $m \mid p-1$, and let $h = (p-1)/m$. Define

$$(T_{m,p} f)_n = \sum_{q=0}^{n} \binom{mn}{mq} f_q \quad \text{in } \mathbb{F}_p.$$

Let V_h be the vector space of sequences satisfying

$$f_{n+h} = f_n \qquad (n \ge 1).$$

Then

$$\boxed{T_{m,p}(V_h) = V_h.}$$

The restriction of $T_{m,p}$ to V_h is an automorphism.

Proof. Choose $d_0 = f_h$ and $d_n = f_n$ for $n \ge 1$. Then d is fully h periodic and $f = d + c\delta_0$ for $c = f_0 - f_h$. Since $T_{m,p}\delta_0$ is the constant sequence one, it is enough to treat d. Expand

$$d_q = \sum_{\chi \in \mu_h} c_\chi \chi^q.$$

The mth power map on $\mathbb{F}_p^\times$ has image μ_h. Choose y with $y^m = \chi$. The root filter gives

$$T_{m,p}(\chi^\bullet)_n = \frac{1}{m} \sum_{\zeta \in \mu_m} (1 + \zeta y)^{mn}.$$

p = 7, m = 2, h = 3	root filter	mode at row n
$\chi = 2,\ y = 3$	input character	2^n
$(1 + y)^2 = 2$	first output mode	$4\,2^n$
$(1 - y)^2 = 4$	second output mode	$4\,4^n$
$T(2^\bullet)_n$	sum modulo 7	$4(2^n + 4^n)$

Figure 4: New reconstruction. Finite field Fourier transfer for a standard window. Each character becomes a finite sum of positive index modes with period dividing h.

If $1 + \zeta y = 0$, the corresponding term vanishes for positive n. Otherwise

$$((1 + \zeta y)^m)^h = (1 + \zeta y)^{p-1} = 1.$$

Every summand is therefore h periodic for positive n. Thus $T_{m,p}$ preserves V_h. It is unit lower triangular on every finite initial segment. Its restriction to the finite dimensional space determined by $f_0, f_1, \ldots, f_h$ is injective and hence bijective. □

Historical corollary D.31 (A second proof of the standard window period). The reduced inverse $e^{(m,p)}$ and every standard address satisfy the period in historical theorem D.28.

Proof. The sequence $\delta = (1, 0, 0, \ldots)$ belongs to V_h. Its inverse image under the automorphism in historical theorem D.30 also belongs to V_h. Address independence modulo p gives the statement for every standard address. □

Historical remark D.32. The theorem gives a period modulo p. A period modulo higher powers requires additional control of the triangular inverse and cannot be inferred from the finite field argument alone.

D.5 One local Euler source and kernel calculus

The odd and dyadic reconstructions have one convolutional form. Their sources, kernels, and valuation profiles are different. We first construct each chart and then state the common theorem.

D.5.1 The odd source and its inverse

Let q be a positive odd integer. Define

$$S_n(q) = \sum_{a=0}^{q-1} (-1)^a (2a + 1)^n.$$

n / q	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0
2	1	6	1	0	0	0	0	0	0
3	1	1	1	1	0	0	0	0	0
4	1	0	0	0	1	0	0	0	0
5	1	3	0	0	3	1	0	0	0
6	1	3	5	0	5	3	1	0	0
7	1	0	0	0	0	0	0	1	0
8	1	1	0	0	4	0	0	1	1

Figure 5: New reconstruction. The standard window matrix for $p = 7$, $m = 2$, and $0 \le j < 7$. Reduction removes the address and retains diagonal one.

Let the signed Euler numbers and Euler zigzag numbers be defined by

$$\operatorname{sech} t = \sum_{n \ge 0} E_n \frac{t^n}{n!}$$

and

$$\sec t + \tan t = \sum_{n \ge 0} Z_n \frac{t^n}{n!}.$$

Historical lemma D.33 (Odd source generating function). For every positive odd q,

$$\boxed{\sum_{n \ge 0} S_n(q) \frac{t^n}{n!} = \mathrm{e}^{qt} \cosh(qt) \operatorname{sech} t.}$$

Its even part is

$$\boxed{\sum_{n \ge 0} S_{2n}(q) \frac{t^{2n}}{(2n)!} = \cosh^2(qt) \operatorname{sech} t.}$$

Proof. The finite geometric sum gives

$$\sum_{a=0}^{q-1} (-1)^a \mathrm{e}^{(2a+1)t} = \mathrm{e}^t \frac{1 + \mathrm{e}^{2qt}}{1 + \mathrm{e}^{2t}} = \mathrm{e}^{qt} \cosh(qt) \operatorname{sech} t.$$

Averaging at t and $-t$ gives the even part. □

Historical lemma D.34 (Zigzag inverse kernel).

$$\boxed{\operatorname{sech}^2 t = \sum_{r \ge 0} (-1)^r Z_{2r+1} \frac{t^{2r}}{(2r)!}.}$$

Proof. The identity $\tanh t = -i\tan(it)$ gives the odd zigzag expansion of $\tanh t$. Differentiate. □

Historical theorem D.35 (Odd modulus Euler coordinate pair). For every positive odd q and every even $n \geq 0$,

$$\boxed{E_n = \sum_{r=0}^{n/2} (-1)^r Z_{2r+1} \binom{n}{2r} q^{2r} S_{n-2r}(q).}$$

The inverse transform is

$$\boxed{S_n(q) = E_n + \sum_{r=1}^{n/2} 2^{2r-1} \binom{n}{2r} q^{2r} E_{n-2r}.}$$

Both transforms are lower triangular with diagonal one.

Proof. The even source identity gives

$$\operatorname{sech} t = \operatorname{sech}^2(qt) \sum_{n\geq 0} S_{2n}(q) \frac{t^{2n}}{(2n)!}.$$

Insert historical lemma D.34 and compare coefficients. For the inverse, multiply $\operatorname{sech} t$ by

$$\cosh^2(qt) = 1 + \sum_{r\geq 1} 2^{2r-1} q^{2r} \frac{t^{2r}}{(2r)!}.$$

□

Historical corollary D.36 (Odd modulus truncation). For every $M \geq 0$, every positive odd q, and every even n,

$$\boxed{E_n \equiv \sum_{r=0}^{\min(M,n/2)} (-1)^r Z_{2r+1} \binom{n}{2r} q^{2r} S_{n-2r}(q) \pmod{q^{2M+2}}.}$$

If $p \mid q$ and $a = v_p(q)$, the same truncation is valid modulo $p^{2a(M+1)}$.

Proof. Every omitted term contains q^{2r} with $r \geq M+1$. The local statement follows by taking the p adic valuation. □

Historical proposition D.37 (Odd row weight). For an odd prime p, the rth nonzero term in the reconstruction has valuation

$$\boxed{W_p(n,r) = 2r + v_p(Z_{2r+1}) + v_p\binom{n}{2r} + v_p\big(S_{n-2r}(p)\big).}$$

The binomial contribution is

$$\boxed{v_p\binom{n}{2r} = \frac{s_p(2r) + s_p(n-2r) - s_p(n)}{p-1},}$$

where s_p is the base p digit sum.

Proof. The first identity separates the valuation of every factor. The second is Legendre's formula, equivalently Kummer's carry theorem [14]. □

D.5.2 The dyadic source and its inverse

For even $n \geq 0$, define

$$\boxed{R_n = \frac{3^{n+2} - 1}{4(n+1)(n+2)}.}$$

Its even exponential generating function is

$$\boxed{\sum_{n \text{ even}} R_n \frac{t^n}{n!} = \frac{\sinh(2t)\sinh t}{2t^2}.}$$

Define β_r by

$$\boxed{\frac{z^2}{\sinh^2 z} = \sum_{r \geq 0} \beta_r z^{2r}.}$$

Historical proposition D.38 (Dyadic inverse coefficients). For $r \geq 1$,

$$\boxed{\beta_r = -\frac{2^{2r}(2r-1)B_{2r}}{(2r)!}.}$$

Consequently,

$$\boxed{v_2(\beta_r) = s_2(r) - 1.}$$

Every R_n is a dyadic unit.

Proof. Differentiate the Bernoulli expansion of $\coth z$ and use $-(\coth z)' = \sinh(z)^{-2}$. The von Staudt and Clausen theorem gives $v_2(B_{2r}) = -1$. Legendre's formula gives

$$v_2((2r)!) = 2r - s_2(r).$$

For R_n, put $k = n + 2$. Since k is even,

$$v_2(3^k - 1) = 2 + v_2(k).$$

The denominator has the same valuation because $n + 1$ is odd. □

Historical theorem D.39 (Dyadic Euler coordinate pair). For every even $n \geq 0$,

$$\boxed{E_n = \sum_{r=0}^{n/2} \beta_r 2^{2r} (n)_{2r} R_{n-2r}.}$$

The inverse transform is

$$\boxed{R_n = \sum_{r=0}^{n/2} \frac{2^{4r+1}}{(2r+2)!} (n)_{2r} E_{n-2r}.}$$

Proof. Substitute $z = 2t$ in the inverse series and multiply

$$\frac{4t^2}{\sinh^2(2t)} \frac{\sinh(2t)\sinh t}{2t^2} = \operatorname{sech} t.$$

Coefficient extraction gives the forward transform. The inverse follows from

$$\left(\frac{\sinh(2t)}{2t}\right)^2 = \sum_{r \geq 0} \frac{2^{4r+1}}{(2r+2)!} t^{2r}.$$

□

D.5.3 The common source and kernel law

For a prime p, define

$$A_{p,n} = \begin{cases} S_n(p), & p \text{ odd}, \\ R_n, & p = 2 \text{ and } n \text{ even}, \\ 0, & p = 2 \text{ and } n \text{ odd}, \end{cases}$$

and

$$\kappa_{p,r} = \begin{cases} (-1)^r Z_{2r+1} p^{2r}, & p \text{ odd}, \\ \beta_r 2^{2r} (2r)!, & p = 2. \end{cases}$$

Historical theorem D.40 (Unified local Euler transfer). For every prime p and every even $n \geq 0$,

$$\boxed{E_n = \sum_{r=0}^{n/2} \binom{n}{2r} \kappa_{p,r} A_{p,n-2r}.}$$

If

$$H_p(t) = \sum_{m \geq 0} A_{p,2m} \frac{t^{2m}}{(2m)!}$$

and

$$K_p(t) = \sum_{r \geq 0} \kappa_{p,r} \frac{t^{2r}}{(2r)!},$$

then

$$\boxed{K_p(t) H_p(t) = \operatorname{sech} t.}$$

For odd p,

$$H_p(t) = \cosh^2(pt) \operatorname{sech} t, \qquad K_p(t) = \operatorname{sech}^2(pt).$$

For $p = 2$,

$$H_2(t) = \frac{\sinh(2t) \sinh t}{2t^2}, \qquad K_2(t) = \frac{4t^2}{\sinh^2(2t)}.$$

Proof. The two source and kernel products were proved in historical theorems D.35 and D.39. Their divided power coefficient laws have the displayed common form. □

Historical theorem D.41 (Unified inverse local transfer). Define

$$\lambda_{p,r} = \begin{cases} 1, & p \text{ odd and } r = 0, \\ 2^{2r-1} p^{2r}, & p \text{ odd and } r \geq 1, \\ \dfrac{2^{4r+1}}{(2r+1)(2r+2)}, & p = 2. \end{cases}$$

Then for every prime p and every even $n \geq 0$,

$$\boxed{A_{p,n} = \sum_{r=0}^{n/2} \binom{n}{2r} \lambda_{p,r} E_{n-2r}.}$$

Equivalently, if

$$L_p(t) = K_p(t)^{-1},$$

then

$$\boxed{H_p(t) = L_p(t)\operatorname{sech} t,}$$

where

$$L_p(t) = \cosh^2(pt)$$

for odd p, and

$$L_2(t) = \left(\frac{\sinh(2t)}{2t}\right)^2$$

for $p = 2$.

Proof. For odd p, this is the inverse identity in historical theorem D.35. For $p = 2$, rewrite the inverse identity in historical theorem D.39 by using

$$(n)_{2r} = (2r)!\binom{n}{2r}.$$

The coefficient becomes

$$\frac{2^{4r+1}(2r)!}{(2r+2)!} = \frac{2^{4r+1}}{(2r+1)(2r+2)}.$$

The generating functions are the inverses of the kernels in historical theorem D.40. □

Historical corollary D.42 (Exact change between local source charts). **Editorial range correction.** This comparison holds for the even source subsequences, equivalently the evenized generating functions H_p. It does not identify their full odd-degree source values. For $p = 3$, the original odd source has $A_{3,1} = S_1(3) = 3$, whereas every odd dyadic source coefficient is zero; the even kernel L_3K_2 cannot change that odd coefficient. Read the coefficient consequence below only at even indices. The displayed identity $H_p = L_pK_\ell H_\ell$ and its proof remain valid.

For any two primes p and ℓ, the source sequences $A_{p,n}$ and $A_{\ell,n}$ are related by an exact unit lower triangular transformation obtained by composing the inverse kernel L_p with the reconstruction kernel K_ℓ.

Proof. Both source series reconstruct $\operatorname{sech} t$. Hence

$$H_p(t) = L_p(t)K_\ell(t)H_\ell(t).$$

The multiplier has constant coefficient one, so historical theorem D.16 gives an exact unit lower triangular coordinate change. □

Historical example D.43 (The first odd reconstruction). For $p = 3$ and $n = 2$,

$$S_2(3) = 1 - 9 + 25 = 17.$$

Since $Z_1 = 1$, $Z_3 = 2$, and $S_0(3) = 1$, the odd coordinate formula gives

$$E_2 = 17 - 2 \cdot 9 = -1.$$

The large source value and the depth one correction are both necessary.

Table 3: The common local Euler architecture and the parts that remain prime dependent.

Feature	Odd prime chart	Dyadic chart
Finite torsion address	Modulo $p-1$	Sign address modulo two
Principal units	$1+p\mathbb{Z}_p$	$1+4\mathbb{Z}_2$
Source	Alternating odd power sum $S_n(p)$	Weighted source R_n
Kernel	$\operatorname{sech}^2(pt)$	$4t^2/\sinh^2(2t)$
Positive depth weight	At least $2r$, with zigzag and row factors	Exactly $4r-1$, before the binomial carry
Analytic branch	Depleted unit source	Direct integral source
Natural refined step	$(p-1)p^a$	2^{a+1}

Historical example D.44 (The first nontrivial dyadic row). For $n=4$,

$$R_4=\frac{91}{15},\qquad R_2=\frac{5}{3},\qquad R_0=1.$$

The exact kernel coefficients are

$$\kappa_{2,1}=-\frac{8}{3},\qquad \kappa_{2,2}=\frac{128}{5}.$$

Therefore

$$E_4=\frac{91}{15}-\frac{8}{3}\binom{4}{2}\frac{5}{3}+\frac{128}{5}=5.$$

The rational source values are dyadic units, while the transfer depth is carried by the kernel.

Historical remark D.45. The common convolution does not identify the integral charts. At an odd prime the finite character coordinate has order $p-1$. At the dyadic prime it begins with the sign coordinate, and the principal unit subgroup begins at $1+4\mathbb{Z}_2$.

D.5.4 Exact dyadic transfer depth

For $r\geq 1$, write

$$\kappa_{2,r}=\beta_r 2^{2r}(2r)!=-2^{4r}(2r-1)B_{2r}.$$

Historical theorem D.46 (Exact dyadic kernel weight). **Editorial index conventions.** The standalone kernel valuation holds for all $r\geq 1$. Its reconstruction-term formula requires even $n\geq 0$ and $1\leq r\leq n/2$. In the following truncation formulas only terms with $r\leq n/2$ occur; terms that would use R_{n-2r} at a negative index are omitted. An empty tail is exactly zero, and the minimum over an empty set of term valuations is $+\infty$. These conventions include $M\geq n/2$.

For every $r\geq 1$,

$$\boxed{v_2(\kappa_{2,r})=4r-1.}$$

The rth reconstruction term has exact valuation

$$\boxed{v_2\left(\kappa_{2,r}\binom{n}{2r}R_{n-2r}\right)=4r-1+v_2\binom{n}{2r}.}$$

Equivalently,

$$\boxed{W_2(n,r)=4r-1+s_2(r)+s_2(n-2r)-s_2(n).}$$

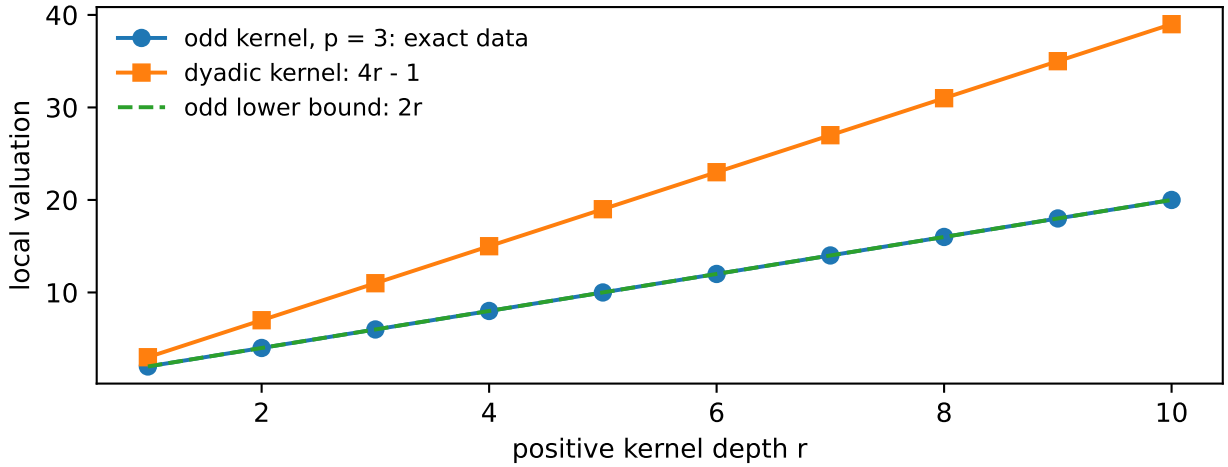


Figure 6: New reconstruction. Odd and dyadic kernels share one convolution law but have different valuation profiles. The positive depth dyadic kernel has exact slope four and offset minus one.

Proof. The Bernoulli formula gives $v_2(B_{2r}) = -1$. The source value is a unit. Kummer's binary carry formula gives the last expression. □

Historical theorem D.47 (Optimal dyadic truncation). Let

$$T_{2,M}(n) = \sum_{r=0}^{\min(M,n/2)} \kappa_{2,r} \binom{n}{2r} R_{n-2r}.$$

Then

$$\boxed{E_n \equiv T_{2,M}(n) \pmod{2^{4M+3}}.}$$

More precisely,

$$\boxed{v_2\big(E_n - T_{2,M}(n)\big) \geq \min_{M+1 \leq r \leq n/2} \left(4r - 1 + v_2\binom{n}{2r}\right).}$$

The uniform exponent $4M+3$ is best possible.

Proof. Every omitted term has valuation at least $4(M+1)-1$. For optimality, take $n = 2M+2$. There is one omitted term. Its binomial coefficient and source value are one, so its valuation is exactly $4M+3$. □

The first three sharp truncations are

$$\begin{aligned} E_n &\equiv R_n && \pmod 8, \\ E_n &\equiv R_n - \frac{8}{3}\binom{n}{2} R_{n-2} && \pmod{128}, \\ E_n &\equiv R_n - \frac{8}{3}\binom{n}{2} R_{n-2} + \frac{128}{5}\binom{n}{4} R_{n-4} && \pmod{2048}. \end{aligned}$$

D.6 Mahler transfer and local analytic branches

The finite character coordinate resolves the torsion part of a local unit. A sufficiently long arithmetic jump moves the remaining unit into a principal unit neighborhood. The coarse variable then has a convergent Mahler expansion with an explicit valuation slope.

local coordinate	odd p	p = 2
finite torsion	modulo p - 1	sign modulo 2
principal units	$1 + p\mathbb{Z}_p$	$1 + 4\mathbb{Z}_2$
source choice	depleted unit branch	direct dyadic source
kernel floor	$2r$	$4r - 1$

Figure 7: New reconstruction. The local Euler charts share source and kernel reconstruction. Their torsion addresses, principal unit depths, source corrections, and precision growth remain different.

D.6.1 Finite Mahler coordinates

For a function f on the nonnegative integers, define

$$\Delta f(n) = f(n+1) - f(n).$$

Historical theorem D.48 (Finite Mahler reconstruction). Given $f(0), \dots, f(N)$, put $c_r = \Delta^r f(0)$. Then

$$\boxed{f(n) = \sum_{r=0}^{n} c_r \binom{n}{r} \qquad (0 \le n \le N).}$$

For every s with $0 \le s \le N$,

$$\boxed{\Delta^s f(n) = \sum_{r=s}^{N} c_r \binom{n}{r-s} \qquad (0 \le n \le N - s).}$$

Proof. Let

$$P(x) = \sum_{r=0}^{N} c_r \binom{x}{r}.$$

Newton interpolation gives $P(n) = f(n)$ for $0 \le n \le N$. Iterating

$$\Delta \binom{x}{r} = \binom{x}{r-1}$$

gives

$$\Delta^s P(n) = \sum_{r=s}^{N} c_r \binom{n}{r-s}.$$

When $0 \le n \le N - s$, every value used in $\Delta^s f(n)$ lies among the prescribed data, so $\Delta^s f(n) = \Delta^s P(n)$. □

Historical corollary D.49 (Mahler valuation ledger). If $c_r \in \mathbb{Z}_p$ and $v_p(c_r) \geq w(r)$, then

$$\boxed{v_p(\Delta^s f(n)) \geq \min_{r \geq s} \left(w(r) + v_p\binom{n}{r-s} \right).}$$

Proof. Apply the nonarchimedean triangle inequality to historical theorem D.48. □

The infinite interpolation theorem states that every continuous function on $\mathbb{Z}_p$ has a unique convergent Mahler expansion [19, 20]. We use that theorem as a standard input when the local variable is allowed to range over all of $\mathbb{Z}_p$.

D.6.2 Carmichael and Mahler transfer for unit modes

Historical standard fact D.50 (Local unit decomposition). For odd p, every unit has a unique factorization

$$u = \omega(u)\langle u \rangle, \qquad \omega(u)^{p-1} = 1, \qquad \langle u \rangle \in 1 + p\mathbb{Z}_p.$$

Every dyadic unit has a unique factorization

$$u = (-1)^{\epsilon(u)}\langle u \rangle, \qquad \epsilon(u) \in \{0, 1\}, \qquad \langle u \rangle \in 1 + 4\mathbb{Z}_2.$$

The first factor gives the finite character address. The second factor admits logarithmic and Mahler coordinates.

For odd p and $L \geq 1$, put

$$\Lambda_{p,L} = (p-1)p^{L-1}.$$

For $p = 2$ and $L \geq 3$, put

$$\Lambda_{2,L} = 2^{L-2}.$$

These are exponents of the unit groups modulo the declared prime powers.

Historical theorem D.51 (One unit mode). Let $u \in \mathbb{Z}_p^\times$, let $H = \Lambda_{p,L}$, and put

$$q_{u,L} = u^H - 1 \in p^L\mathbb{Z}_p.$$

For every integer address j and every $x \in \mathbb{Z}_p$,

$$\boxed{u^{Hx+j} = u^j \sum_{r \geq 0} q_{u,L}^r \binom{x}{r}.}$$

The rth Mahler coefficient has valuation at least Lr.

Proof. Write $u^{Hx+j} = u^j(1 + q_{u,L})^x$ and use the convergent binomial series on $\mathbb{Z}_p$. □

Historical example D.52 (Paired principal unit expansions). At the odd prime three,

$$2^{2x+j} = 2^j 4^x = 2^j \sum_{r \geq 0} 3^r \binom{x}{r}.$$

At the dyadic prime,

$$3^{2x+j} = 3^j 9^x = 3^j \sum_{r \geq 0} 8^r \binom{x}{r}.$$

The formal expansions are identical. Their first local slopes are one and three because $4 - 1 = 3$ while $9 - 1 = 8$.

Historical corollary D.53 (Basic odd and dyadic difference gains). For odd p,

$$\boxed{\Delta^r_{(p-1)p^a} u^n = u^n \left(u^{(p-1)p^a} - 1\right)^r, \qquad v_p\left(\Delta^r_{(p-1)p^a} u^n\right) \geq (a+1)r.}$$

For odd $u \in \mathbb{Z}_2^\times$,

$$\boxed{\Delta^r_{2^{a+1}} u^n = u^n \left(u^{2^{a+1}} - 1\right)^r, \qquad v_2\left(\Delta^r_{2^{a+1}} u^n\right) \geq (a+3)r.}$$

Proof. One step difference multiplies a unit mode by $u^H - 1$. Fermat reduction and lifting the exponent give the odd estimate. For odd u,

$$v_2\left(u^{2^{a+1}} - 1\right) = v_2(u^2 - 1) + a \geq a + 3.$$

□

D.6.3 Polynomial exponential modes

Historical theorem D.54 (Polynomial exponential Mahler transfer). Let

$$f(n) = P(n)u^n$$

with $P \in \mathbb{Z}_p[n]$ of degree d. Let $H = \Lambda_{p,L}$, fix an address j, and write

$$Q_j(x) = P(Hx + j), \qquad y = u^H, \qquad q = y - 1.$$

Then

$$\boxed{f(Hx + j) = \sum_{r \geq 0} c_{j,r} \binom{x}{r},}$$

where

$$\boxed{c_{j,r} = u^j \sum_{k=0}^{\min(d,r)} \binom{r}{k} \Delta^k Q_j(0) y^k q^{r-k}.}$$

Consequently,

$$\boxed{v_p(c_{j,r}) \geq L \max(r - d, 0).}$$

For $R \geq d$, truncation at depth R is valid modulo $p^{L(R+1-d)}$ uniformly on $\mathbb{Z}_p$.

Proof. On the polynomial factor the difference operator acts as $q + y\Delta$. The two terms commute. Expand $(q + y\Delta)^r$ and evaluate at zero. Since $\Delta^k Q_j = 0$ for $k > d$, every surviving term contains at least q^{r-d}. □

D.6.4 Matrix state systems and finite adelic synchronization

Let

$$v_{n+1} = Cv_n, \qquad a_n = w^T C^n v_0,$$

with $C \in M_d(\mathbb{Z}_p)$ invertible modulo p.

Historical theorem D.55 (Matrix Carmichael and Mahler transfer). **Editorial hypothesis correction.** The original displayed statement leaves the vectors untyped. Its integrality conclusion requires $v_0, w \in \mathbb{Z}_p^d$, $L \geq 1$, and $H, j, n \in \mathbb{N}$ with $H > 0$; the value at $Hx + j$ for $x \in \mathbb{Z}_p$ means the matrix-binomial interpolation. With $p = 3$, $C = 4$, $L = 1$, $w = 1$ and $v_0 = 1/3$, one has $\Delta_1 a_0 = 1$, which is not in $3\mathbb{Z}_3$. The printed claim without these restrictions is therefore not an as-written certificate. The full original proof follows under the corrected hypotheses.

Assume

$$C^H = I + p^L B$$

with $B \in M_d(\mathbb{Z}_p)$. For every address j and every $x \in \mathbb{Z}_p$,

$$\boxed{a_{Hx+j} = \sum_{r \geq 0} p^{Lr} \binom{x}{r} w^T C^j B^r v_0.}$$

The series converges on $\mathbb{Z}_p$, and

$$\boxed{\Delta_H^r a_n \equiv 0 \pmod{p^{Lr}}.}$$

Proof. Use

$$C^{Hx+j} = C^j (I + p^L B)^x$$

and the matrix binomial series. One difference in the coarse variable multiplies the series by $p^L B$. □

Historical theorem D.56 (Finite adelic transfer). **Editorial hypothesis correction.** Read this statement with $v_0, w \in \mathbb{Z}^d$, $H > 0$ and $j, n, r \in \mathbb{N}$, and positive integer L_p. These restrictions make the integral congruences and powers well typed. Allowing $v_0 = 1/3$ in the one-prime example above loses the advertised digit; invertibility of C alone does not restore it. The original proof is retained under these hypotheses.

Let

$$Q = \prod_{p \in S} p^{L_p}$$

for a finite set of primes. Let $C \in M_d(\mathbb{Z})$ be invertible modulo Q, and choose H such that

$$C^H = I + QB$$

with $B \in M_d(\mathbb{Z})$. Then for every nonnegative integer x,

$$\boxed{a_{Hx+j} = \sum_{r=0}^{x} Q^r \binom{x}{r} w^T C^j B^r v_0.}$$

For every r,

$$\boxed{\Delta_H^r a_n \equiv 0 \pmod{Q^r}.}$$

Proof. Apply the integer matrix binomial theorem and reduce at every selected prime. □

Historical example D.57 (A simultaneous Fibonacci jump). Let

$$C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

The matrix C is invertible modulo twenty four, and

$$C^{24} - I = 24 \begin{pmatrix} 3126 & 1932 \\ 1932 & 1194 \end{pmatrix}.$$

For every scalar coordinate of the state sequence $C^n v_0$,

$$\boxed{\Delta_{24}^{r} a_n \equiv 0 \pmod{24^r}.}$$

One integral jump therefore gives simultaneous congruence ladders at the primes two and three.

D.6.5 Amice pullback

Historical definition D.58 (Amice pullback). For a formal exponential generating series

$$F(z) = \sum_{n \geq 0} a_n \frac{z^n}{n!},$$

define

$$\widehat{F}(T) = F(\log(1+T)).$$

Historical theorem D.59 (Derivative conjugacy).

$$\boxed{\widehat{F}'(T) = (1+T)\frac{d}{dT}\widehat{F}(T).}$$

Proof. Differentiate the composition $F(\log(1+T))$. □

Historical proposition D.60 (Weighted convergence). If

$$\widehat{F}(T) = \sum_{r \geq 0} c_r T^r$$

and

$$v_p(c_r) + r\alpha \longrightarrow +\infty,$$

then the series converges whenever $v_p(T) \geq \alpha$.

Proof. The valuation of the rth term is $v_p(c_r) + r v_p(T)$, which tends to positive infinity. □

D.6.6 Odd prime depletion

For an odd prime p, the source $S_n(p)$ contains one nonunit base. Put

$$\sigma_p = (-1)^{(p-1)/2}$$

and

$$S_n^{\times}(p) = S_n(p) - \sigma_p p^n.$$

Historical theorem D.61 (Depleted odd prime reconstruction). For every even n,

$$\boxed{\left(1 - \sigma_p p^n\right) E_n = \sum_{r=0}^{n/2} (-1)^r Z_{2r+1} \binom{n}{2r} p^{2r} S_{n-2r}^{\times}(p).}$$

Proof. The removed source mode contributes $\sigma_p \cosh(pt)$. Multiplication by $\operatorname{sech}^2(pt)$ gives $\sigma_p \operatorname{sech}(pt)$. Subtract it from the full source and kernel product. □

Fix an even address $j \in \{0, 2, \dots, p-3\}$. For $x \in \mathbb{Z}_p$, define

$$\mathscr{E}^{\circ\times}_{p,j}(x) = \sum_{r\geq 0} (-1)^r Z_{2r+1} p^{2r} \binom{(p-1)x+j}{2r} \times \sum_{\substack{0\leq a<p \\ 2a+1\neq p}} (-1)^a (2a+1)^{j-2r} \left((2a+1)^{p-1}\right)^x.$$

Historical theorem D.62 (Odd prime analytic Euler branch). The series defining $\mathscr{E}^{\circ\times}_{p,j}$ converges in the Tate algebra on $\mathbb{Z}_p$. For every nonnegative integer x,

$$\boxed{\mathscr{E}^{\circ\times}_{p,j}(x) = \left(1 - \sigma_p p^{(p-1)x+j}\right) E_{(p-1)x+j}.}$$

Proof. Every retained base is a unit, and its $(p-1)$st power lies in $1 + p\mathbb{Z}_p$. The polynomial

$$p^{2r} \binom{(p-1)x+j}{2r}$$

has Gauss norm at most

$$p^{-2r+v_p((2r)!)}.$$

This tends to zero because p is odd. The remaining factors have bounded Gauss norm. Thus the series converges in the Tate algebra. At a nonnegative integer x, the binomial factor vanishes for large r, and historical theorem D.61 gives the value. □

Historical remark D.63. The analytic function is defined by its unit source series. The expression $p^{(p-1)x+j}$ in the interpolation identity is evaluated only at nonnegative integers. No analytic function $x \mapsto p^x$ on all of $\mathbb{Z}_p$ is used.

D.6.7 Direct dyadic interpolation

For $x \in \mathbb{Z}_2$, define

$$\boxed{\mathscr{R}(x) = \frac{9^{x+1} - 1}{8(x+1)(2x+1)}.}$$

The apparent singularity at $x = -1$ is removable, with value $-\log(9)/8$.

Historical lemma D.64 (Analytic dyadic source). The function $\mathscr{R}$ belongs to the dyadic Tate algebra and takes values in $\mathbb{Z}_2$. For every nonnegative integer n,

$$\boxed{\mathscr{R}(n) = R_{2n}.}$$

Proof. Put $y = x + 1$ and let $\ell = \log 9$. Since $v_2(\ell) = 3$,

$$\frac{9^y - 1}{8y} = \frac{\ell}{8} \sum_{m\geq 0} \frac{\ell^m}{(m+1)!} y^m.$$

The coefficient of y^m has valuation

$$3m - v_2((m+1)!),$$

which is nonnegative and tends to positive infinity. The quotient therefore belongs to $\mathbb{Z}_2\langle y\rangle$. The factor $2x+1 = 1+2x$ is a unit in $\mathbb{Z}_2\langle x\rangle$, with inverse $\sum_{m\geq 0}(-2x)^m$. Hence $\mathscr{R}$ is integral and analytic. The integer values follow from the definition of R_{2n}. □

Define

$$\boxed{\mathscr{E}_2(x) = \sum_{r\geq 0} \kappa_{2,r} \binom{2x}{2r} \mathscr{R}(x-r).}$$

Historical theorem D.65 (Direct dyadic Euler interpolation). The series defining $\mathscr{E}_2$ converges in the dyadic Tate algebra. For every nonnegative integer n,

$$\boxed{\mathscr{E}_2(n) = E_{2n}.}$$

Truncation after depth M is uniformly valid modulo 2^{4M+3} on $\mathbb{Z}_2$.

Proof. The monomial Gauss norm of $\binom{2x}{2r}$ is at most $2^{v_2((2r)!)}$. Since

$$v_2(\kappa_{2,r}) = 4r - 1,$$

the product tends to zero in Gauss norm. Translation by the integer r preserves the unit disc, so $\mathscr{R}(x-r)$ remains an integral analytic function with uniformly bounded supremum norm. The series therefore converges in the Tate algebra.

At a nonnegative integer n, every term with $r > n$ vanishes and historical theorem D.39 remains. For arbitrary $x \in \mathbb{Z}_2$, the integer valued polynomial $\binom{2x}{2r}$ and the source value $\mathscr{R}(x-r)$ lie in $\mathbb{Z}_2$. Every omitted term after depth M is therefore divisible by 2^{4M+3}. This proves the uniform truncation statement. □

Historical theorem D.66 (Unified local interpolation statement). The odd branch $\mathscr{E}_{p,j}^{\times}$ and the dyadic branch $\mathscr{E}_2$ are analytic local realizations of the source and kernel law in historical theorem D.40. At integer arguments they recover, respectively,

$$\left(1 - \sigma_p p^n\right) E_n$$

on each even odd prime character branch and

$$E_{2n}$$

on the dyadic even branch. The two realizations have distinct source corrections and distinct kernel valuation slopes.

Proof. Combine historical theorems D.62 and D.65 with the source and kernel definitions in historical theorem D.40. □

D.7 Polyphase refinement of local Mahler coordinates

Polyphase refinement replaces one local branch by finitely many digit branches while preserving exact synthesis. For sequences it is residue analysis. For local functions it is the digit tree of $\mathbb{Z}_p$. In Mahler coordinates it becomes a finite triangular kernel.

D.7.1 Discrete analysis and exact synthesis

For a sequence $a = (a_n)_{n\geq 0}$ and $M \geq 2$, define

$$a_r^{[M]}(q) = a_{Mq+r} \qquad (0 \leq r < M).$$

The analysis map is

$$\mathcal{P}_M a = \left(a_0^{[M]}, \ldots, a_{M-1}^{[M]}\right).$$

Synthesis interleaves the branches by the rule

$$a_{Mq+r} = a_r^{[M]}(q).$$

Historical theorem D.67 (Exact discrete polyphase reconstruction). The analysis map $\mathcal{P}_M$ is a bijection between sequences and ordered families of M sequences. If

$$A(z) = \sum_{n\geq 0} a_n z^n$$

and

$$A_r^{[M]}(w) = \sum_{q\geq 0} a_{Mq+r} w^q,$$

then

$$\boxed{A(z) = \sum_{r=0}^{M-1} z^r A_r^{[M]}(z^M).}$$

Proof. Every nonnegative integer has a unique expression $Mq + r$ with $0 \leq r < M$. Group the terms of the ordinary generating function by this residue. □

Historical theorem D.68 (Nested discrete refinement). For positive integers M, N,

$$\boxed{\left(a_r^{[M]}\right)_s^{[N]} = a_{r+Ms}^{[MN]}.}$$

For a mixed radix path $(M_1, \ldots, M_\ell)$ with digits $0 \leq s_i < M_i$, the final address is

$$\boxed{r(\boldsymbol{s}) = s_1 + M_1 s_2 + \cdots + \left(\prod_{i=1}^{\ell-1} M_i\right) s_\ell.}$$

Only the product radix and accumulated address determine the final branch.

Proof. Direct substitution gives

$$\left(a_r^{[M]}\right)_s^{[N]}(q) = a_{M(Nq+s)+r}.$$

Iterate this identity along the mixed radix path. □

D.7.2 Carry convolution and the shift matrix

Let $c = a * b$ be the ordinary Cauchy convolution. Write A_r, B_r, and C_r for the phase ordinary generating functions.

Historical proposition D.69 (One carry convolution law). For every residue r,

$$\boxed{C_r(w) = \sum_{u+v=r} A_u(w)B_v(w) + w \sum_{u+v=r+M} A_u(w)B_v(w).}$$

The first sum has no phase carry. The second has one coarse index carry.

Proof. Write the input indices as $Mq + u$ and $Mq' + v$. Since $0 \le u, v < M$, their sum has residue $u + v$ when $u + v < M$, and residue $u + v - M$ with one additional coarse index when $u + v \ge M$. □

Historical theorem D.70 (Polyphase shift matrix). Let E be the scalar shift $(Ea)_n = a_{n+1}$, and let E_q shift the coarse index. Then

$$\boxed{\mathcal{P}_M(Ea) = \mathsf{S}_M(E_q)\mathcal{P}_M a,}$$

where

$$\mathsf{S}_M(E_q) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ E_q & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

It also satisfies

$$\boxed{\mathsf{S}_M(E_q)^M = E_q I_M.}$$

Proof. For $r < M - 1$,

$$(Ea)_{Mq+r} = a_{Mq+r+1} = a_{r+1}^{[M]}(q).$$

For $r = M - 1$,

$$(Ea)_{Mq+M-1} = a_{M(q+1)} = E_q a_0^{[M]}(q).$$

After M shifts every phase has advanced by one coarse index. □

D.7.3 Local branches and digit reconstruction

For $a \ge 1$ and $0 \le s < p^a$, define

$$f_{a,s}(x) = f(p^a x + s).$$

Historical theorem D.71 (Exact local branch reconstruction). The family $\{f_{a,s}\}_{0 \le s < p^a}$ determines f uniquely on $\mathbb{Z}_p$. More precisely, if $y \in s + p^a\mathbb{Z}_p$, then

$$\boxed{f(y) = f_{a,s}\left(\frac{y-s}{p^a}\right).}$$

For $a, b \ge 1$,

$$\boxed{(f_{a,s})_{b,t} = f_{a+b,s+p^a t}.}$$

Thus local refinement is exactly the p adic digit tree.

Proof. The cosets $s + p^a\mathbb{Z}_p$ form a disjoint partition of $\mathbb{Z}_p$. The first formula is the inverse affine change of variables on one coset. The second follows from

$$p^a(p^b x + t) + s = p^{a+b}x + p^a t + s.$$

□

D.7.4 The affine Mahler kernel

Let

$$\Phi_M(z) = (1+z)^M - 1.$$

For $M \geq 1$, $s \geq 0$, and $r, k \geq 0$, define

$$\boxed{\Theta_{M,s}(r,k) = [z^r](1+z)^s \Phi_M(z)^k.}$$

Equivalently,

$$\boxed{\Theta_{M,s}(r,k) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \binom{Mj+s}{r}.}$$

It vanishes unless

$$k \leq r \leq Mk + s.$$

Historical proposition D.72 (Edge coefficients of the affine kernel). For every $M \geq 1$, $s \geq 0$, and $k \geq 0$,

$$\boxed{\Theta_{M,s}(k,k) = M^k, \qquad \Theta_{M,s}(Mk+s,k) = 1.}$$

For binary radix and zero address,

$$\boxed{\Theta_{2,0}(k+j,k) = 2^{k-j}\binom{k}{j} \qquad (0 \leq j \leq k).}$$

Hence

$$\boxed{v_2\left(\Theta_{2,0}(k+j,k)\right) = k - j + v_2\binom{k}{j}.}$$

Proof. The lowest degree term of $\Phi_M(z)$ is Mz, and its highest degree term is z^M. Multiplication by $(1+z)^s$ gives the two edge coefficients. When $M = 2$ and $s = 0$,

$$\Phi_2(z)^k = (2z + z^2)^k = z^k(2+z)^k.$$

Taking the coefficient of z^{k+j} gives the binary formula. □

Historical theorem D.73 (Mahler polyphase transfer). Let

$$f(x) = \sum_{r \geq 0} c_r \binom{x}{r}$$

be a convergent Mahler expansion. For

$$g_s(q) = f(Mq + s),$$

one has

$$\boxed{g_s(q) = \sum_{k\geq 0} d_{s,k} \binom{q}{k},}$$

where

$$\boxed{d_{s,k} = \sum_{r=k}^{Mk+s} c_r \Theta_{M,s}(r,k).}$$

Every refined coefficient uses a finite window of the original Mahler coefficients.

Proof. Use

$$(1+z)^{Mq+s} = (1+z)^s \left(1+\Phi_M(z)\right)^q = \sum_{k\geq 0} \binom{q}{k} (1+z)^s \Phi_M(z)^k.$$

Taking the coefficient of z^r gives the affine expansion of $\binom{Mq+s}{r}$. Since $c_r \to 0$ and every $d_{s,k}$ uses indices $r \geq k$, the transformed coefficients also tend to zero. □

Historical theorem D.74 (Kernel composition). For positive integers M, N,

$$\boxed{\sum_{k\geq 0} \Theta_{M,s}(r,k)\Theta_{N,t}(k,\ell) = \Theta_{MN,Mt+s}(r,\ell).}$$

The sum is finite.

Proof. Apply historical theorem D.73 first to $x = Mq + s$ and then to $q = Ny + t$. The composite substitution is $x = MNy + Mt + s$. Uniqueness of Mahler coefficients gives the identity. □

D.7.5 Valuation transport

Historical theorem D.75 (Polyphase valuation ledger). For every refined coefficient,

$$\boxed{v_p(d_{s,k}) \geq \min_{k\leq r\leq Mk+s} \left(v_p(c_r) + v_p(\Theta_{M,s}(r,k))\right).}$$

If $v_p(c_r) \geq w(r)$ for a nondecreasing function w, then

$$\boxed{v_p(d_{s,k}) \geq w(k).}$$

Thus every monotone Mahler precision ladder survives arbitrary polyphase refinement.

Proof. Apply the nonarchimedean triangle inequality to the finite coefficient formula. Since $r \geq k$, monotonicity gives $w(r) \geq w(k)$, and the affine kernel is integral. □

Historical theorem D.76 (Prime power sparsification). **Editorial hypothesis correction.** The affine-kernel congruence below needs no change, but its consequence for transformed Mahler coefficients requires $c_r \in \mathbb{Z}_p$ for every r. Convergence of a $\mathbb{Q}_p$-valued Mahler expansion alone is insufficient. For $p = 2$, $M = 2$, $s = 0$ and $f(x) = x/2$, one has $c_1 = 1/2$, $c_2 = 0$, while $g(q) = q$ has $d_1 = 1$; thus $d_1 \not\equiv c_2 \pmod 2$. The coefficient consequence is read with integral original coefficients.

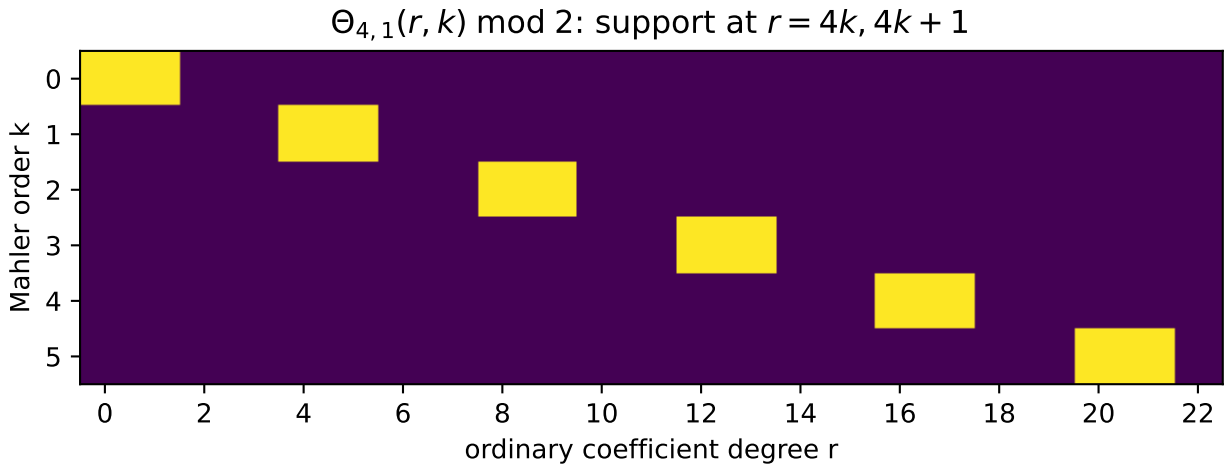


Figure 8: New reconstruction. The affine Mahler kernel has finite triangular support. At prime power radix its reduction is confined to a short Frobenius window.

Let $M = p^a$. Then

$$\boxed{\Theta_{p^a,s}(r,k) \equiv \binom{s}{r - p^a k} \pmod{p},}$$

where the binomial coefficient is zero outside its natural range. Consequently,

$$\boxed{d_{s,k} \equiv \sum_{u=0}^{s} \binom{s}{u} c_{p^a k+u} \pmod{p}.}$$

In particular,

$$\boxed{d_{0,k} \equiv c_{p^a k} \pmod{p}.}$$

Proof. In characteristic p,

$$(1+z)^{p^a} - 1 \equiv z^{p^a}.$$

Substitute this congruence in the definition of $\Theta_{p^a,s}$. □

D.7.6 Precision amplification

Historical theorem D.77 (Scalar polyphase precision amplification). Let $u \in \mathbb{Z}_p^\times$, let $a, r \geq 0$, and put

$$H_a = \begin{cases} (p-1)p^a, & p \text{ odd}, \\ 2^{a+1}, & p = 2. \end{cases}$$

Then

$$\boxed{\Delta_{H_a}^r u^n = u^n \left(u^{H_a} - 1\right)^r.}$$

For odd p,

$$\boxed{v_p\left(\Delta_{H_a}^r u^n\right) \geq (a+1)r.}$$

For $p = 2$ and odd u,

$$\boxed{v_2\left(\Delta_{H_a}^r u^n\right) \geq (a+3)r.}$$

Equivalently, every additional prime power digit increases the certified Mahler slope by one.

Proof. One difference with step H_a multiplies the unit mode by $u^{H_a} - 1$. Iterate. The valuation estimates are those of historical corollary D.53. □

Historical lemma D.78 (Matrix power amplification). **Editorial hypothesis correction.** Require $L \geq 1$. At $L = 0$ the standalone claim is false: for $p = 3$, $d = 1$ and $X = 2 = 1 + B$, one has $X^3 - 1 = 7 \notin 3\mathbb{Z}_3$. With $L \geq 1$, every higher binomial term in the retained proof has the required valuation, including $p = 2$.

If

$$X = I + p^L B$$

with $B \in M_d(\mathbb{Z}_p)$, then for every $a \geq 0$ there is $B_a \in M_d(\mathbb{Z}_p)$ such that

$$\boxed{X^{p^a} = I + p^{L+a} B_a.}$$

Proof. Induct on a. Expand the pth power of $I + p^{L+a} B_a$. The linear term has valuation at least $L + a + 1$, and every higher term has at least that valuation. □

Historical theorem D.79 (Matrix polyphase precision amplification). **Editorial hypothesis correction.** Require $L \geq 1$, $C, B \in M_d(\mathbb{Z}_p)$, $v_0, w \in \mathbb{Z}_p^d$, $H > 0$, and $a, j, n, r \in \mathbb{N}$. These are the hypotheses of the two corrected matrix statements used by the retained proof. Without integral observation vectors or positive L, the preceding counterexamples invalidate the claimed precision.

Assume

$$C^H = I + p^L B.$$

Put $H_a = Hp^a$. Then

$$C^{H_a} = I + p^{L+a} B_a$$

for an integral matrix B_a. Every refined packet

$$b_{a,j}(x) = a_{H_a x + j}$$

has Mahler coefficients satisfying

$$\boxed{v_p(c_{a,j,r}) \geq (L+a)r.}$$

Equivalently,

$$\boxed{\Delta_{H_a}^r a_n \equiv 0 \pmod{p^{(L+a)r}}.}$$

Proof. Combine historical lemma D.78 and historical theorem D.55. □

Historical example D.80 (Refined dyadic Fibonacci ladder). For the Fibonacci companion matrix

$$C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

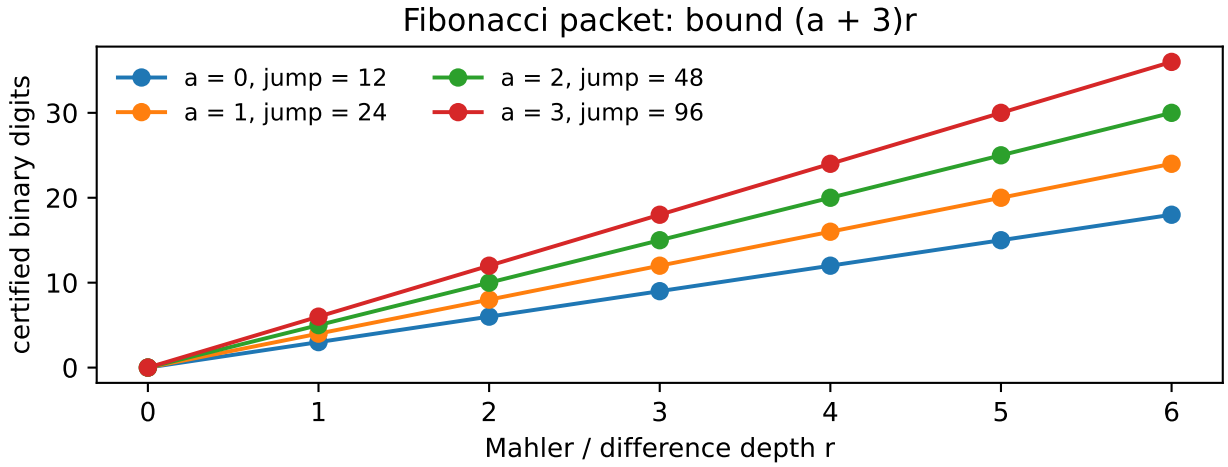


Figure 9: New reconstruction. Every additional prime power refinement digit increases the certified Mahler slope of a principal unit matrix packet by one.

one has

$$C^{12} = I + 8\begin{pmatrix} 29 & 18 \\ 18 & 11 \end{pmatrix}.$$

Therefore

$$\boxed{\Delta^{r}_{12\cdot 2^{a}} F_n \equiv 0 \pmod{2^{(a+3)r}}.}$$

Every dyadic refinement doubles the step and adds one guaranteed binary digit at each Mahler depth.

D.7.7 Character and polyphase coordinates

For odd p, write

$$\boxed{n = j + (p-1)(s + p^{a}q),}$$

with $0 \le j < p-1$ and $0 \le s < p^{a}$. The address j records the finite character. The address s records the principal unit digit. The variable q carries the remaining Mahler expansion.

At the dyadic prime, write

$$\boxed{n = j + 2(s + 2^{a}q),}$$

with $j \in \{0, 1\}$ and $0 \le s < 2^{a}$. The sign coordinate replaces the odd prime character coordinate.

Historical proposition D.81 (Local phase lattice). For a unit mode u^{n}, the odd prime coordinate gives

$$\boxed{u^{n} = u^{j+(p-1)s}\left(u^{(p-1)p^{a}}\right)^{q}.}$$

The dyadic coordinate gives

$$\boxed{u^{n} = u^{j+2s}\left(u^{2^{a+1}}\right)^{q}.}$$

Thus finite character phase, local digit address, and Mahler depth are separate coordinates.

Proof. Substitute the two displayed decompositions of n. □

Historical corollary D.82 (Polyphase refinement of the Euler branches). The analytic branches in historical theorems D.62 and D.65 admit exact refinements under $x = p^a q + s$. Their refined Mahler coefficients are obtained from $\Theta_{p^a,s}$. The original monotone valuation ladders are preserved, and unit mode or invertible matrix representations gain one certified digit per refinement level.

Proof. Apply historical theorems D.73, D.75 and D.79 and historical corollary D.53. □

Historical remark D.83. Exact refinement does not imply that every branch is simpler. A simplification theorem requires an additional invariant, such as Frobenius sparsity, valuation gain, spectral collision, or a vanishing transfer block.

D.8 Arithmetic lift obstructions

A congruence class may solve an equation to one precision and fail at the next. The correct obstruction is not an arbitrary residual vector. It is the residual vector modulo the image of the declared correction map.

D.8.1 Linear lifting

Let p be a prime. Let

$$M \in M_{r,d}(\mathbb{Z}_p), \qquad b \in \mathbb{Z}_p^r.$$

Suppose that $x_L \in \mathbb{Z}_p^d$ satisfies

$$Mx_L \equiv b \pmod{p^L}$$

for some $L \geq 1$. Define the reduced residual

$$\rho_L(x_L) = \frac{Mx_L - b}{p^L} \pmod{p} \in \mathbb{F}_p^r.$$

Write $\overline{M}$ for the reduction of M modulo p.

Historical definition D.84 (Linear lift obstruction). The linear obstruction at depth L is

$$\boxed{\mathrm{Ob}_L^{\mathrm{lin}}(x_L) = [\rho_L(x_L)] \in \mathrm{coker}(\overline{M}).}$$

Historical theorem D.85 (One level linear lifting). A class represented by x_L lifts to a solution modulo p^{L+1} if and only if

$$\boxed{\mathrm{Ob}_L^{\mathrm{lin}}(x_L) = 0.}$$

More precisely, a correction $y \in \mathbb{F}_p^d$ gives a lift

$$x_{L+1} = x_L + p^L y \pmod{p^{L+1}}$$

if and only if

$$\boxed{\overline{M}y = -\rho_L(x_L).}$$

When one correction exists, the set of all correction classes is a coset of $\ker \overline{M}$.

Proof. Modulo p^{L+1},

$$M(x_L + p^L y) - b = Mx_L - b + p^L My.$$

After division by p^L and reduction modulo p, the lift condition becomes

$$\rho_L(x_L) + \overline{M}y = 0.$$

This equation is solvable exactly when the residual lies in the image of $\overline{M}$. If y_0 is one solution, every solution is $y_0 + z$ with $z \in \ker \overline{M}$. □

Historical proposition D.86 (Representative independence). The class $\mathrm{Ob}_L^{\mathrm{lin}}(x_L)$ depends only on the solution class of x_L modulo p^L.

Proof. Replace x_L by $x_L + p^L z$. The reduced residual changes by $\overline{M}z$. Its class in the cokernel is unchanged. □

Historical corollary D.87 (Exact number of linear children). If $\rho = \mathrm{rank}_{\mathbb{F}_p} \overline{M}$, then a solvable depth L class has exactly

$$\boxed{p^{d-\rho}}$$

children modulo p^{L+1}.

Proof. The correction set is a coset of a vector space of dimension $d - \rho$. □

Historical example D.88 (A visible linear obstruction). Take $p = 3$, $L = 1$, and

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

The class $x_1 = (0, 0)$ solves

$$Mx_1 \equiv b \pmod 3$$

for both

$$b_1 = (-3, 0) \qquad \text{and} \qquad b_2 = (0, -3).$$

For b_1, the reduced residual is $(1, 0)$, which belongs to the image of $\overline{M}$. There are three correction classes, given by

$$y_1 + y_2 = 2 \pmod 3.$$

For b_2, the residual is $(0, 1)$, whose cokernel class is nonzero. No lift modulo nine exists because the second row of M remains zero.

D.8.2 Polynomial lifting

Let

$$F = (F_1, \ldots, F_r) : \mathbb{Z}_p^d \longrightarrow \mathbb{Z}_p^r$$

be a polynomial map.

Editorial coefficient clarification. Here polynomial means $F_i \in \mathbb{Z}_p[X_1, \ldots, X_d]$, not merely an integer-valued polynomial with coefficients in $\mathbb{Q}_p$. This makes the Jacobian and Taylor remainder integral and their reduction well typed.

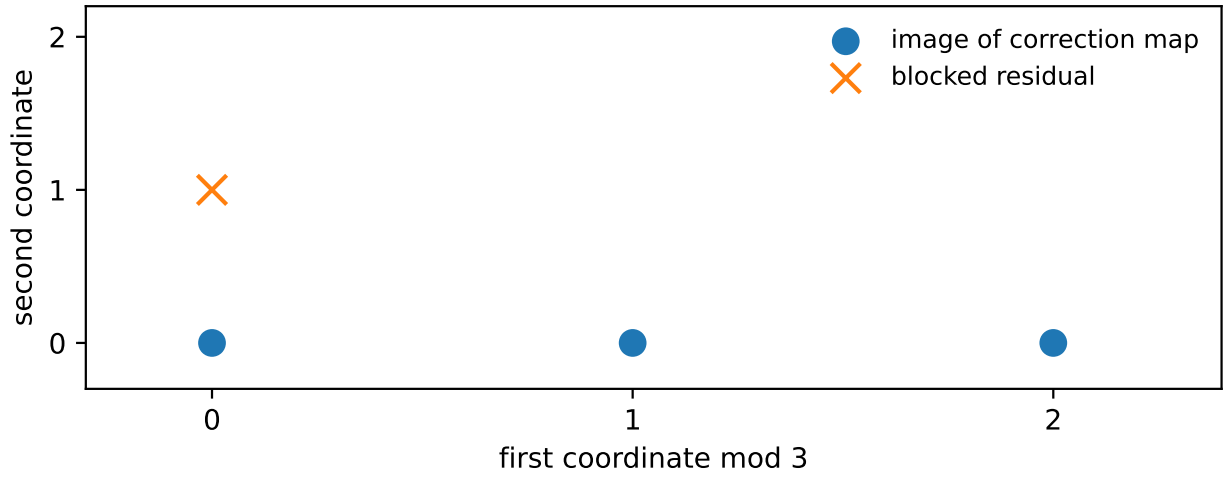


Figure 10: New reconstruction. A residual vector lifts precisely when its class vanishes in the cokernel of the reduced correction map.

Suppose

$$F(x_L) \in p^L \mathbb{Z}_p^r.$$

Define

$$\rho_L^F(x_L) = \frac{F(x_L)}{p^L} \pmod{p}.$$

Let $J_F(x_L)$ denote the Jacobian matrix and write $\bar{J}_F(x_L)$ for its reduction modulo p.

Historical definition D.89 (Polynomial lift obstruction). The polynomial obstruction at depth L is

$$\boxed{\mathrm{Ob}_L^F(x_L) = [\rho_L^F(x_L)] \in \mathrm{coker}\big(\bar{J}_F(x_L)\big).}$$

Historical theorem D.90 (One level polynomial lifting). Let $L \geq 1$. A class represented by x_L lifts to a zero of F modulo p^{L+1} if and only if

$$\boxed{\mathrm{Ob}_L^F(x_L) = 0.}$$

A correction $y \in \mathbb{F}_p^d$ gives a lift if and only if

$$\boxed{\bar{J}_F(x_L) y = -\rho_L^F(x_L).}$$

When a correction exists, the correction classes form a coset of

$$\ker \bar{J}_F(x_L).$$

Proof. The polynomial Taylor expansion gives

$$F(x_L + p^L y) \equiv F(x_L) + p^L J_F(x_L) y \pmod{p^{L+1}}.$$

Every term of degree at least two in the increment contains p^{2L}, which is divisible by p^{L+1}. Divide by p^L, reduce modulo p, and apply the same cokernel criterion as in historical theorem D.85. □

Historical proposition D.91 (Polynomial representative independence). The class $\mathrm{Ob}_L^F(x_L)$ depends only on the class of x_L modulo p^L.

Proof. If x_L is replaced by $x_L + p^L z$, Taylor expansion shows that the reduced residual changes by an element of the image of $\bar{J}_F(x_L)$. The reduced Jacobian itself is unchanged because $L \geq 1$. □

Historical corollary D.92 (Simple root lifting). Assume $r = d$ and

$$\det J_F(x_L) \not\equiv 0 \pmod{p}.$$

Then the depth L solution has exactly one child modulo p^{L+1}. Iterating gives a unique p adic zero above the initial class.

Proof. The reduced Jacobian is invertible, so every residual has one correction. The resulting compatible sequence is Cauchy in $\mathbb{Z}_p^d$, and continuity of F gives a zero in the limit. Uniqueness holds at every depth. □

Historical example D.93 (A singular polynomial obstruction). Let $p = 3$ and consider the class $x_1 = 0$ modulo three. For

$$F_0(x) = x^2,$$

the residual and the reduced derivative both vanish. Every correction $y \in \mathbb{F}_3$ gives

$$F_0(3y) \equiv 0 \pmod{9},$$

so the class has three children. For

$$F_1(x) = x^2 + 3,$$

the reduced derivative is still zero, but the reduced residual is one. Its class in the cokernel is nonzero, so no lift modulo nine exists.

This is the usual simple root mechanism behind Hensel lifting. The obstruction formulation also covers singular Jacobians, where some residual classes lift and others do not [22, 12, 20].

D.8.3 Packetwise obstruction splitting

Historical theorem D.94 (Packet decomposition of a lift obstruction). Suppose

$$\mathbb{F}_p^d = \bigoplus_{a \in A} U_a, \qquad \mathbb{F}_p^r = \bigoplus_{a \in A} V_a,$$

and suppose a linear correction map $D : \mathbb{F}_p^d \to \mathbb{F}_p^r$ satisfies

$$D(U_a) \subseteq V_a$$

for every a. Write $D_a = D|_{U_a}$. Then

$$\boxed{\operatorname{coker} D \cong \bigoplus_{a \in A} \operatorname{coker} D_a.}$$

A residual lifts if and only if every packet obstruction vanishes.

Proof. The image of D is the direct sum of the images of the maps D_a. Taking the quotient of the codomain packet decomposition gives the stated cokernel decomposition. □

Historical remark D.95. The obstruction theorems solve one declared local lifting step. They do not prove that a compatible branch exists across unrelated primes, that a global Diophantine point exists, or that an arithmetic representation is automorphic.

address j	k = 0	k = 1	k = 2	k = 3	k = 4
0	0	2	8	34	144
1	1	3	13	55	233
2	1	5	21	89	377

Figure 11: New reconstruction. Fibonacci residues F_{3k+j}, with $F_0 = 0$ and $F_1 = 1$. Each packet evolves by the same jump matrix C^3 and has a different initial address.

D.9 Recurrence packets and a common integral spine

Linear recurrences provide a setting where residue subsequences, local completions, and a common global source can all be described exactly. Background on recurrence sequences may be found in [11, 23].

D.9.1 Companion jumps

Let R be a commutative ring. Let

$$v_{n+1} = Cv_n, \qquad C \in M_d(R).$$

For a positive integer N and an address $j \in \{0, \dots, N-1\}$, define the residue state packet

$$u_k^{(j)} = v_{Nk+j}.$$

Historical theorem D.96 (Companion jump recurrence). Every residue state packet satisfies

$$\boxed{u_{k+1}^{(j)} = C^N u_k^{(j)}.}$$

More explicitly,

$$\boxed{u_k^{(j)} = C^j (C^N)^k v_0.}$$

Proof. The state relation gives $v_n = C^n v_0$. Substitute $n = Nk + j$. □

We use the resultant convention

$$\mathrm{Res}_y(f, g) = \prod_{f(\alpha)=0} g(\alpha)$$

when f is monic, with roots counted with multiplicity.

Historical theorem D.97 (Characteristic polynomial of the jump). Let

$$\chi_C(y) = \det(yI - C).$$

Then

$$\boxed{\chi_{C^N}(x) = \mathrm{Res}_y(\chi_C(y), x - y^N).}$$

The identity holds over the coefficient ring after interpreting the resultant universally.

Fibonacci state matrix	characteristic polynomial	roots
C = [[1, 1], [1, 0]]	$x^2 - x - 1$	α, β
C^3 = [[3, 2], [2, 1]]	$x^2 - 4x - 1$	α^3, β^3

Figure 12: New reconstruction. Passing to every Nth term raises each characteristic root to its Nth power.

Proof. After passage to an algebraic closure, let $\alpha_1, \dots, \alpha_d$ be the eigenvalues of C, counted with algebraic multiplicity. The resultant is

$$\prod_{i=1}^{d} (x - \alpha_i^N),$$

which is the characteristic polynomial of C^N. Both sides are polynomial expressions with integer coefficients in the entries of C, so the identity descends to every commutative coefficient ring. □

D.9.2 Second order recurrences

Historical theorem D.98 (Residue subsequences of a second order recurrence). Suppose

$$x_{n+2} = P x_{n+1} - Q x_n$$

over a commutative ring. Define $V_0 = 2$, $V_1 = P$, and

$$V_{n+2} = P V_{n+1} - Q V_n.$$

For every positive integer N and every address j, the subsequence

$$u_k = x_{Nk+j}$$

satisfies

$$\boxed{u_{k+2} = V_N u_{k+1} - Q^N u_k.}$$

Proof. The companion matrix

$$C = \begin{pmatrix} P & -Q \\ 1 & 0 \end{pmatrix}$$

has trace P and determinant Q. The trace of C^N is V_N, while its determinant is Q^N. Cayley Hamilton applied to C^N gives

$$(C^N)^2 - V_N C^N + Q^N I = 0.$$

Apply this identity to the residue state packet and take its scalar coordinate. □

Historical corollary D.99 (Fibonacci residue packets). Let $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$. Let L_N denote the Lucas number. For every $N \geq 1$ and every address j,

$$\boxed{F_{N(k+2)+j} = L_N F_{N(k+1)+j} - (-1)^N F_{Nk+j}.}$$

Proof. Apply historical theorem D.98 with $P = 1$ and $Q = -1$. Then $V_N = L_N$. □

D.9.3 Finite and local periodicity

Historical theorem D.100 (Periodicity of an invertible recurrence modulo a prime power). Let

$$C \in M_d(\mathbb{Z}), \qquad p \nmid \det C.$$

For every $L \geq 1$, every state orbit modulo p^L is purely periodic. Its period divides the order of C in

$$\mathrm{GL}_d(\mathbb{Z}/p^L\mathbb{Z}).$$

If $p \mid \det C$, every orbit is still eventually periodic.

Proof. When $p \nmid \det C$, multiplication by C is a permutation of the finite state set $(\mathbb{Z}/p^L\mathbb{Z})^d$. Every orbit of a permutation is periodic, and applying the group order relation for C gives the divisibility statement. Without invertibility, repeated application of a map on a finite set eventually repeats a state, after which the orbit is periodic. □

Historical corollary D.101 (Residue packet periods). Under the invertibility hypothesis of historical theorem D.100, every residue packet with jump N has period dividing the order of C^N modulo p^L.

Proof. Use the jump matrix from historical theorem D.96. □

D.9.4 The common integral spine

Historical theorem D.102 (Base change of an integral recurrence). Let

$$C \in M_d(\mathbb{Z}), \qquad v_0 \in \mathbb{Z}^d.$$

The integral orbit

$$v_n = C^n v_0$$

determines, by base change, compatible orbits in every ring R equipped with a homomorphism $\mathbb{Z} \to R$. In particular, the reductions modulo every positive integer and the orbits in every $\mathbb{Z}_p^d$ share the same companion matrix, characteristic polynomial, and integer recurrence relations.

Proof. Every coordinate of $C^n v_0$ is an integer polynomial in the entries of C and v_0. Applying a ring homomorphism commutes with matrix multiplication and preserves every polynomial relation with integer coefficients. □

Historical corollary D.103 (Prime independent recurrence spine). An integral companion system gives a canonical common source for all of its local realizations. No choice of coefficient prime is required to define the recurrence or its residue packets.

Proof. Apply historical theorem D.102 to the companion matrix and its initial state. □

Historical remark D.104. This theorem supplies a prime independent spine for integral linear recurrences. It does not prove that every unrelated family of local arithmetic packets arises from one global object.

D.10 Chinese remainder packet refinement

The Chinese remainder theorem gives an exact factorization of residue addresses. We state it at the level of sequence projectors so that it can be combined directly with the earlier coordinate systems.

Historical definition D.105 (Residue projector). For a sequence $a = (a_n)_{n\ge 0}$, a positive integer N, and an address r modulo N, define

$$(P_{N,r}a)_n = \begin{cases} a_n, & n \equiv r \pmod N, \\ 0, & n \not\equiv r \pmod N. \end{cases}$$

Historical proposition D.106 (Complete residue packet system). For fixed N, the projectors $P_{N,r}$ satisfy

$$P_{N,r}P_{N,s} = \delta_{rs}P_{N,r}$$

and

$$\boxed{\sum_{r \bmod N} P_{N,r} = I.}$$

Proof. Every nonnegative integer belongs to exactly one residue class modulo N. □

Historical theorem D.107 (Coprime packet factorization). Let A and B be coprime positive integers. Let r_A be an address modulo A and r_B an address modulo B. Let r be the unique address modulo AB satisfying

$$r \equiv r_A \pmod A, \qquad r \equiv r_B \pmod B.$$

Then

$$\boxed{P_{AB,r} = P_{A,r_A}P_{B,r_B} = P_{B,r_B}P_{A,r_A}.}$$

Proof. The product retains exactly the indices satisfying both congruences. By the Chinese remainder theorem, those indices form one residue class modulo AB [14]. □

Historical corollary D.108 (Several coprime moduli). Let $N_1, \dots, N_s$ be pairwise coprime and put

$$N = \prod_{j=1}^{s} N_j.$$

For compatible addresses r_j,

$$\boxed{P_{N,r} = \prod_{j=1}^{s} P_{N_j,r_j},}$$

where r is the unique combined address modulo N.

Proof. Iterate historical theorem D.107. □

Historical standard fact D.109 (Finite local reconstruction). Let

$$N = \prod_{j=1}^{s} p_j^{a_j}$$

be the prime power factorization of a positive integer. The map

$$\mathbb{Z}/N\mathbb{Z} \longrightarrow \prod_{j=1}^{s} \mathbb{Z}/p_j^{a_j}\mathbb{Z}$$

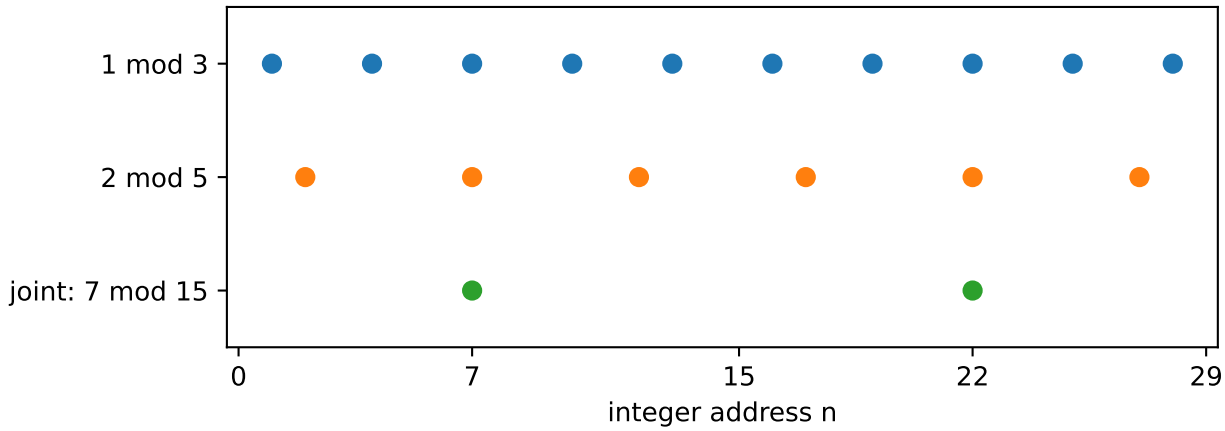


Figure 13: New reconstruction. A residue modulo a product of coprime moduli is the exact intersection of its local residue packets.

is a ring isomorphism. Thus a compatible collection of prime power residues reconstructs one residue modulo N [14].

Historical proposition D.110 (Dyadic and odd residue refinement). Let $N = 2^a m$ with m odd. Every residue packet modulo N factors uniquely as the product of one dyadic packet modulo 2^a and one odd packet modulo m.

Proof. The moduli 2^a and m are coprime. Apply historical theorem D.107. □

Historical remark D.111. The residue address factors exactly, but valuation weights do not become identical across primes. The odd coordinate system of appendix D.5 and the dyadic coordinate system of appendix D.5 remain genuinely different integral charts.

D.11 Verification and exact boundaries

The proofs in the article are symbolic. Exact computation tests formulas, finite ranges, and the source implementation. It does not replace a uniform proof.

D.11.1 The remaining congruential difference problem

Let $C_{mp,j}(n)$ be a congruential reciprocal coefficient and let

$$h = \frac{p-1}{m}.$$

The unit mode calculations suggest bounds for

$$v_p\left(\Delta_h^s C_{mp,j}(n)\right).$$

The theorems of appendices D.6 and D.7 determine the gain before the congruential triangular inverse is applied. They do not yet determine every valuation loss introduced by that inverse. The full family bound therefore remains open.

Table 4: Status of the principal local arithmetic blocks.

Block	Status	Main boundary
Standard window period	Proved modulo the residue prime	Higher prime powers require inverse estimates
Unified local Euler transfer	Proved for every prime	Odd and dyadic integral charts remain different
Dyadic kernel and truncation	Exact weight and optimal uniform modulus proved	Row cancellation may give stronger values
Carmichael and Mahler transfer	Proved for scalar, polynomial exponential, matrix, and finite adelic systems	A jump congruence must first be available
Local Euler interpolation	Proved for depleted odd branches and the direct dyadic branch	The odd nonunit factor is only evaluated at integer interpolation points
Affine polyphase transfer	Exact analysis, synthesis, carry law, kernel composition, residual sparsity, and valuation preservation proved	General exact kernel valuations remain open
Lift obstruction and recurrence spine	Proved for the declared local problems and integral recurrences	Unrestricted global gluing is not proved

D.11.2 Higher valuations of the affine kernel

The residual formula

$$\Theta_{p^a,s}(r,k) \equiv \binom{s}{r-p^a k} \pmod{p}$$

is exact. A closed formula for

$$v_p\left(\Theta_{p^a,s}(r,k)\right)$$

in terms of base p carries and the three digit addresses is not proved here. Such a formula would sharpen the min plus valuation ledger and could distinguish optimal refinement trees.

D.11.3 The remaining global gluing problem

The finite adelic theorem synchronizes a finite set of primes when one integral state system is already fixed. The recurrence theorem gives a common local source under the same assumption. Chinese remainder projectors reconstruct finite residue addresses. None of these results proves that an arbitrary collection of local analytic branches has a common global arithmetic origin.

scope	conclusion or boundary
standard windows	finite periods and inverse coordinates
declared correction map	one-step cokernel obstruction
fixed integral recurrence	common source for its local images
arbitrary local families	global source NOT supplied
affine kernel valuations	general sharp higher laws remain open

Figure 14: New reconstruction. The proved local coordinate theory and the remaining inverse valuation, affine kernel, and global gluing problems.

D.11.4 Historical scope

The article uses classical work on Euler congruences, finite differences, local fields, Hurwitz series, Mahler interpolation, Lucas congruences, and recurrence sequences [8, 26, 27, 24, 9, 18, 17, 11]. The exact formulas and local transfer statements should be compared individually with that literature before any historical priority claim is made.

D.12 Conclusion

The local arithmetic theory now has one common architecture. The finite character coordinate identifies the torsion phase. The source and kernel product reconstructs the Euler coefficient. The kernel valuation records certified precision. Carmichael and Mahler coordinates describe the remaining principal unit motion. Polyphase analysis resolves additional local digits, records carries and shifts exactly, and transports Mahler coefficients through a finite affine kernel.

The odd and dyadic cases agree at the level of convolution and differ at the level of integral geometry. At odd primes the source must be depleted before it becomes a unit mode interpolation. At the dyadic prime the source extends directly to an analytic integral function. The dyadic kernel has exact weight $4r - 1$, giving the optimal uniform modulus 2^{4M+3} after depth M.

Three tasks remain outside the proved region. The first is a complete finite difference estimate after the congruential triangular inverse. The second is an exact higher valuation theory for the affine Mahler kernel. The third is a global existence theorem for local systems that do not already arise from one integral source.

Current status. This dated statement is superseded for the stated divisor-branch family by Theorem 3.7, with the $j!$ normalization and factorial valuation loss explicitly retained. It makes no claim for unrestricted families.

E Complete mathematical checkpoint supplements

These eleven supplements reproduce the complete mathematical bodies of their successive checkpoints. The numerical inventories, coverage fractions and then-pending statements belong to each original checkpoint. They are not current total counts. In particular the later all-residue and Nishibuchi completion supersede earlier pending labels. The final arithmetic and polynomial scopes are the ones stated in the main text. All derivations, examples, formal-interface explanations and original qualification paragraphs remain available below.

E.1 First Lean Block Supplement

Checkpoint scope. This complete source body retains the status of its dated original supplement. Earlier pending or partial-coverage labels are historical; the final core records their resolution.

Scope and result

This block uses the 28-page *status-corrected* Paper IX manuscript. That PDF is preserved unchanged. The release supplies a first formalization companion, not a completed or rewritten arithmetic paper.

Seven of 24 named manuscript-proved items now have Lean coverage in the stated scopes. Seventeen remain pending. Three classical inputs and two conjectures are registered separately. The development contains **89 explicit theorem declarations and 22 definitions across eight modules**. Supporting lemmas and concrete instances are included; these are not 89 independent discoveries.

Item	Covered mathematical content
1.2	Residue reconstruction and divisor refinement
2.1	Actual unit-EGF coordinate equivalence and transported projectors
2.2	Filtered finite coefficient reconstruction, including $\mathbb{Z}_p$
3.2	Constructed unique triangular inverse and original section identity
3.3	Universal denominator clearing
8.2	Lucas standard-window reduction and the diagonal unit
13.1	Exact normalization counterexample and sharp factorial loss

The correction is preserved

The normalized packet is not the same object as the packet multiplied by $j!$. The checked recurrence gives

$$C_{6,3}(1) = -\frac{1}{84}, \qquad C_{6,3}(2) = \frac{703}{5460},$$
$$C_{6,3}(2) - C_{6,3}(1) = \frac{64}{455}, \qquad v_3\left(\frac{64}{455}\right) = 0.$$

Thus the unrestricted normalized no-loss claim fails already at a valid one-step parameter instance. Multiplication by 3! restores one factor of three in this instance. The general corrected finite-difference theorem is **not** inferred from this calculation and is not yet formalized here.

Verification standard

All eight modules passed a fresh cumulative build with compiler warnings treated as errors. The audit resolved all 89 explicit theorems and all 22 explicit definitions. Their transitive axiom dependencies are confined to `propext`, `Classical.choice` and `Quot.sound`, or subsets. Three invalid-proof controls were rejected. The proof, scope, and numerical-regression records remain distinct.

1. Residue packets and actual EGF coordinates

Source items 1.2 and 2.1

For a sequence $a : \mathbb{N} \to R$, define the actual selector

$$(P_{N,j}a)(n) = \begin{cases} a(n), & n \bmod N = j, \\ 0, & \text{otherwise.} \end{cases}$$

The source proves idempotence for every selector, orthogonality for distinct standard residue labels, and, when $N > 0$,

$$\sum_{j=0}^{N-1} P_{N,j}a = a.$$

If $N \mid M$ and $M > 0$, it also proves

$$P_{N,j}a = \sum_{\substack{0 \le r < M \\ r \bmod N = j}} P_{M,r}a.$$

The formal power-series versions are proved coefficientwise for actual `PowerSeries` objects. There is no finite truncation of the sequence domain in these theorems.

A constructed inverse, not an inverse hypothesis

Let K be a characteristic-zero field. Define

$$\mathcal{E}(a) = \sum_{n \ge 0} \frac{a_n}{n!} t^n, \qquad \mathcal{D}(A)_n = n![t^n]A.$$

Both inverse laws $\mathcal{D}\mathcal{E} = I$ and $\mathcal{E}\mathcal{D} = I$ are proved. The ordinary power-series product then yields the genuine binomial convolution

$$\mathcal{D}\big(\mathcal{E}(h)\mathcal{E}(a)\big)_n = \sum_{r=0}^{n} \binom{n}{r} h_r a_{n-r}.$$

For $h_0 = 1$, the formal inverse of $H = \mathcal{E}(h)$ is used to construct a sequence equivalence

$$U_H a = \mathcal{D}(H\mathcal{E}(a)), \qquad U_H^{-1} a = \mathcal{D}(H^{-1}\mathcal{E}(a)).$$

The equivalence has proved left and right inverse laws. The inverse's constant coefficient is proved to be one. The associated lower-triangular entries have diagonal one and vanish above the diagonal.

For the actual conjugated selectors

$$\widehat{P}_{N,j} = U_H P_{N,j} U_H^{-1},$$

idempotence, orthogonality, and finite reconstruction are proved from the constructed maps. No projector identity for the conjugates is inserted as an assumption.

Formal boundary. These are formal series and coefficient identities. They do not establish convergence for an analytic EGF. The characteristic-zero field assumption is explicit; unrestricted factorial division in an arbitrary ring is not asserted.

Principal targets.

`ResiduePackets.packet_reconstruction`

`ResiduePackets.packet_refinement`

`EGF.coordinateEquiv,EGF.coordinate_forward,EGF.coordinate_inverse`

`EGF.transported_reconstruction`

2. Triangular inversion and denominator control

Source items 3.2 and 3.3

For a general lower-triangular coefficient array T over a field with nonzero diagonal, the code defines forward substitution by well-founded recursion:

$$a_n = \frac{b_n - \sum_{q<n} T_{nq} a_q}{T_{nn}}.$$

It proves that this constructed sequence solves every row and that every sequence solving those rows equals it. No existence or uniqueness premise is used.

The congruential coefficients

Over Q, specialize to

$$T_{nq} = \binom{Nn+j}{Nq}, \qquad b_n = \delta_{n0}.$$

The diagonal is positive as an integer, hence nonzero. The resulting sequence $C_{N,j}(n)$ satisfies

$$C_{N,j}(0) = 1, \qquad \sum_{q=0}^{n} \binom{Nn+j}{Nq} C_{N,j}(q) = \delta_{n0}.$$

The displayed recurrence in the manuscript and the unique-solution theorem are consequences of this definition.

The formalization does not stop at a recurrence. In the variable $z = t^N$, construct

$$K_{N,j}(z) = \sum_{n\geq 0} \frac{j!}{(Nn+j)!} z^n, \qquad F_{N,j}(z) = \sum_{n\geq 0} \frac{C_{N,j}(n)}{(Nn)!} z^n.$$

It proves the factorial coefficient identity, then $F_{N,j} K_{N,j} = 1$ and therefore $F_{N,j} = K_{N,j}^{-1}$.

For $N > 0$ and $0 \leq j < N$, the independently defined original exponential section is

$$E_{N,j}(t) = \sum_{d \equiv j \pmod N} \frac{t^d}{d!}.$$

Using the actual power-series expansion homomorphism $z \mapsto t^N$, the code proves

$$E_{N,j}(t) = \frac{t^j}{j!} K_{N,j}(t^N), \qquad \boxed{F_{N,j}(t^N) E_{N,j}(t) = \frac{t^j}{j!}.}$$

Thus the recurrence is connected back to the original normalized EGF definition. The section vanishes at zero when $j > 0$; no inverse of that nonunit section is assumed.

Universal clearing

Define

$$D_{N,j}(n) = \prod_{r=1}^{n} \binom{Nr+j}{Nr}, \qquad D_{N,j}(0) = 1.$$

For every n, the code proves $D_{N,j}(n)C_{N,j}(n) \in \mathbb{Z}$. The general proof promotes each earlier clearing denominator to the common product, then cancels the new nonzero diagonal. This is a denominator bound, not a claim that every normalized coefficient is integral.

3. Filtered tails and the standard window

Source items 2.2 and 8.2

Filtration at a finite coefficient row

Let R be any commutative ring. For $w \in \mathbb{N} \cup \{\infty\}$ define

$$p^{[w]} = \begin{cases} p^w, & w < \infty, \\ 0, & w = \infty. \end{cases}$$

The infinity convention expresses a vanishing coefficient, not an infinite exponent operation. For any finite index set, every term with weight at least L is divisible by p^L. A finite-sum proof yields

$$\boxed{p^L \mid \sum_{r=0}^{n} \binom{n}{r} c_r p^{[w(r)]} S_{n-r} - \sum_{\substack{0 \le r \le n \\ w(r) < L}} \binom{n}{r} c_r p^{[w(r)]} S_{n-r}.}$$

The result is specialized to actual p-adic integers $R = \mathbb{Z}_p$ for prime p. Divisibility there is precisely the asserted membership in $p^L\mathbb{Z}_p$.

For an individual row the sum is finite, so the proof does not require $w(r) \to \infty$. This is a stronger coefficient-row scope, not a theorem about an infinite limiting process. A supporting lemma also proves a p^{2M+2} tail bound for a sum with visible factors p^{2m}. It is not counted as either Euler supercongruence theorem: their generating-function reconstruction identities remain pending.

Actual Lucas reduction

For prime p, all $m, n, q \in \mathbb{N}$, and $0 \le j < p$, the checked binomial congruence is

$$\boxed{\binom{mpn + j}{mpq} \equiv \binom{mn}{mq} \pmod{p}.}$$

The proof applies Mathlib's Lucas low-digit reduction to the actual natural binomial coefficients. It does not assume a pre-reduced row. The statement also permits $q > n$. If $m > 0$, both binomial coefficients then vanish.

Taking $q = n$ gives

$$\binom{mpn + j}{mpn} \equiv 1 \pmod{p}.$$

Nondivisibility by p and invertibility of that diagonal in $\mathbb{Z}/p\mathbb{Z}$ are proved. The separate supporting lemma $p \nmid j!$ for $j < p$ establishes the standard-window factorial-unit condition.

Dependency boundary. Lucas' theorem is a named classical input in the manuscript and is reused from the pinned Mathlib library. This block does not independently reconstruct Lucas' full digit theorem. The residue-independence result (8.3), uniform standard-window period (8.5), and the global finite-difference theorem (7.5) remain pending.

Principal targets.

`Filtered.padic_filtered_reconstruction`

`StandardWindow.lucas_row_reduction`

`StandardWindow.standard_diagonal_unit`

`StandardWindow.factorial_valuation_zero`

4. Sharp normalization boundary

Source item 13.1

The construction fixes the two coefficient conventions:

$$C_{N,j}(n), \qquad \widetilde{C}_{N,j}(n) = j!\, C_{N,j}(n).$$

The witness uses $p = 3$, $m = 2$, $N = mp = 6$, $j = 3$, $n = 1$ and difference step $h = (p-1)/m = 1$. The parameter conditions are checked explicitly, including odd primality, $m \mid p-1$ and $j < N$.

The first row gives $84C_{6,3}(1) + 1 = 0$. The second row, evaluated by the same recursive definition, gives $C_{6,3}(2) = 703/5460$. Therefore

$$\Delta_1 C_{6,3}(1) = \frac{64}{455} \neq 0.$$

Neither numerator nor denominator is divisible by three, so its actual rational 3-adic valuation is zero. A further theorem negates the universal one-step no-loss statement by applying it to this valid tuple and deriving $1 \leq 0$.

For the unnormalized convention,

$$\Delta_1 \widetilde{C}_{6,3}(1) = 3!\,\frac{64}{455} = \frac{384}{455}, \qquad v_3\left(\frac{384}{455}\right) = 1.$$

The factorial's valuation is also proved:

$$v_3(3!) = 1, \qquad v_3\big(\Delta_1 C_{6,3}(1)\big) = 1 - v_3(3!) = 0.$$

This proves sharpness at the displayed normalization boundary. It does not establish the corrected global inequality for all parameters, and does not assert that every nonunit-factorial parameter tuple must violate every no-loss specialization. All valuation calculations here concern nonzero rationals.

Work remaining in the named proved register

Items	Pending mathematical content
4.1–4.3	Odd-power generating function, Euler reconstruction, odd-prime ladder
5.1–5.4	Dyadic inverse and source integrality, reconstruction, weighted ladder
6.2	Finite unit-decomposition certificates
7.2–7.5	Finite Mahler reconstruction, weighted differences, unit modes, global odd-prime differences
8.3, 8.5	Standard-window residue independence and uniform period
9.1, 9.2	Companion jumps and Fibonacci packet recurrence
10.1	Arithmetic CRT packet factorization

The two conjectures, 13.2 and 13.3, remain unproved and outside the 24-item register. The 17 pending entries are manuscript proof claims awaiting this project's Lean verification, not conjectures newly created by the formalization process.

5. Verification and reproducibility

Exact source pins, observed checks and retained failures

Module	Theorems	Definitions
ResiduePackets	8	2
Triangular	14	3
EGFCoordinates	21	6
FilteredReconstruction	7	1
CongruentialEuler	16	7
ExponentialSection	5	3
StandardWindow	6	0
NormalizationBoundary	12	0
Total	89	22

Lean run. All eight modules were compiled into a new empty output directory under Lean 4.34.0-rc2. All nine dependency revisions matched the pinned manifest. Mathlib's revision is

`aeb72f1e9981b6d35b426b16dca7759be3e7ba06`.

The 111 explicit-declaration axiom reports matched the inventory. A false natural equality, an admitted placeholder, and the false normalized bound were all rejected. The placeholder control exercises the warnings-as-errors policy; `sorryAx` is also excluded by the separate axiom whitelist. The 21 infrastructure and release tests passed.

Independent exact regressions. Python integer and rational mirrors passed 120 triangular systems, 45 congruential parameter pairs (495 rows), 120 EGF-coordinate cases, 45 residue-refinement cases, 240 filtered tails, 10,824 Lucas rows, the normalization witness and eight invalid-input controls. These are not executions extracted from Lean or reruns of the original historical numerical package.

Failures retained. Of 29 captured development checks, 15 passed and 14 failed before repair. A malformed generic-field draft triggered a release-candidate compiler panic after proof errors. The repaired sources passed the final clean run. Development products were not reused in that run. An earlier cumulative invocation was interrupted before completion and is retained separately. One initial Python scanner-test assertion had the wrong expected whitespace count and was repaired; its failure log is retained.

Trust and source boundary. Mathlib's precompiled dependency cache was reused, not rebuilt from source. No earlier paper was rebuilt. Only the status-corrected primary PDF was recovered for this manuscript baseline, not its original TeX, experiment package, or complete global finite-difference supplement. Formal-to-prose correspondence is a reviewed crosswalk, not an automated semantic-equivalence theorem. Native Windows was not tested.

Reproduce. With Python 3.10+ and the separate extracted Linux portable kit:

```
bash Run-Portable.sh /path/to/Komorebi_Portable_Lean_4.34.0-rc2
```

The ZIP includes sources, inventories, the unchanged manuscript, this supplement, exact-regression scripts, tests, and successful and failed logs. The large runtime and local build products are not included.

E.2 Second Lean Block Supplement

Checkpoint scope. This complete source body retains the status of its dated original supplement. Earlier pending or partial-coverage labels are historical; the final core records their resolution.

Scope and result

This release continues the 28-page *status-corrected* Paper IX manuscript. It completes formal coverage of Lemma 4.1, Theorem 4.2 and Corollary 4.3. The original manuscript and all eight first-block proof modules remain unchanged.

Ten of 24 named manuscript-proved items are now covered in their recorded scopes. Fourteen remain pending. Three classical inputs and two conjectures are tracked separately. The cumulative corpus has **163 named theorem declarations and 37 explicit definitions in 13 modules**. The 74 new theorems include auxiliary identities and concrete examples; they are not 74 independent discoveries.

Item	New formal coverage	Theorems
4.1	Literal odd-power EGF and its even part	18
4.2	Integral Euler and signed zigzag coefficient bridges	29
4.2, 4.3	Exact reconstruction, matrix rows and truncation	16
Example	Exact scalar anchors and sharpness controls	11
	Total added across five modules	74

The five added modules are:

`OddPowerSeries`, `EulerCoefficients`, `ZigzagCoefficients`, `OddEulerReconstruction` and `OddEulerExamples`.

A useful scope strengthening

The algebraic proof needs *oddness*, not primality. The exact reconstruction and the integer divisibility statement are therefore checked for every positive odd length p, including $p = 1$ and odd composites. The manuscript's odd-prime statement follows as a specialization. This is a sharpening of assumptions in this development, not a historical-priority claim. Even lengths are not included; a checked row-zero counterexample shows why.

What is not completed

The weighted dyadic theory and the general corrected odd-prime finite-difference theorem remain pending. No continuum convergence or analytic summability theorem is inferred from the formal power-series equalities in this block. The historical numerical study is not being relabelled as a fresh experiment.

1. The actual odd-power generating function

Lemma 4.1; manuscript page 10

Work in the rational formal power-series ring $\mathbb{Q}[[t]]$. Define

$$S_n(p) = \sum_{a=0}^{p-1} (-1)^a (2a+1)^n \in \mathbb{Z}, \qquad F_p(t) = \sum_{n \geq 0} S_n(p) \frac{t^n}{n!}.$$

The formal exponential is Mathlib's `PowerSeries.exp`; scalar substitution is its actual `rescale` homomorphism. We define

$$\exp_a(t) = e^{at}, \qquad \cosh_a(t) = \frac{e^{at} + e^{-at}}{2}, \qquad \operatorname{sech}(t) = \cosh_1(t)^{-1}.$$

The constant term of $\cosh_a$ is proved to be one, so the reciprocal used here is a genuine unit inverse.

From finite sums to a series identity

Coefficient extraction proves

$$F_p(t) = \sum_{a=0}^{p-1} (-1)^a e^{(2a+1)t}.$$

An induction on the actual finite summation length establishes the telescoping identity

$$F_p(t) \cosh t = \frac{1 - (-1)^p e^{2pt}}{2}.$$

For odd p this gives, without a geometric-series convergence assumption,

$$\boxed{F_p(t) = \frac{1 + e^{2pt}}{2} \operatorname{sech} t.}$$

This is an equality of complete formal series, not a fixed-degree polynomial check.

Even projection is identified coefficientwise

For any formal series f, let $\operatorname{EvenPart}(f) = (f(t) + f(-t))/2$. Its nth coefficient is proved to be the coefficient of f when n is even and zero otherwise. Rescaling the inverse law proves $\operatorname{sech}(-t) = \operatorname{sech}(t)$. The exponential multiplication identity then gives

$$\cosh^2(pt) = \frac{2 + e^{2pt} + e^{-2pt}}{4},$$

and hence

$$\boxed{\sum_{r \geq 0} S_{2r}(p) \frac{t^{2r}}{(2r)!} = \cosh^2(pt) \operatorname{sech} t.}$$

Both the full and even EGF formulas are derived from the original integer sum. Neither is supplied as a hypothesis.

Principal targets.

`OddPower.source_as_exponentials`

`OddPower.source_times_cosh`

`OddPower.odd_source_egf`, `OddPower.odd_source_even_egf`

2. Integral Euler coefficients and the zigzag bridge

Coefficient foundations for Theorem 4.2

The first block already constructed $C_{N,j}(r)$ by the manuscript's triangular recurrence. At $(N, j) = (2, 0)$, every diagonal binomial coefficient equals one. Its denominator-clearing product is therefore one, and $C_{2,0}(r)$ is an integer.

The source defines E_{2r} as the integer numerator of that rational coefficient and proves that its rational cast equals $C_{2,0}(r)$. Set $E_{2r+1} = 0$. The actual exponential-section identity from the first block proves

$$\boxed{\sum_{n\geq 0} E_n \frac{t^n}{n!} = \operatorname{sech} t.}$$

Thus the integer sequence is connected to the intended signed Euler family, rather than merely assigned the name Euler.

The inverse coefficients are integral by construction

Define the integer binomial convolution

$$B_n = \sum_{r=0}^{n} \binom{n}{r} E_r E_{n-r}, \qquad b_m = B_{2m}.$$

The product formula for actual EGFs gives

$$\sum_{n\geq 0} B_n \frac{t^n}{n!} = \operatorname{sech}^2 t.$$

Odd coefficients vanish by parity. In particular, every b_m is an integer; this is the integrality required by the later congruence.

Connecting to the paper's Z_n convention

The paper defines Z_n through $\sec t + \tan t$. The Lean construction uses Mathlib's actual formal sine and cosine series and defines

$$\sec t = (\cos t)^{-1}, \quad \tan t = (\sin t)\sec t, \quad Z_n = n![t^n](\sec t + \tan t).$$

It proves the sine and cosine derivative identities, $\sin^2 t + \cos^2 t = 1$, and

$$(\tan t)' = \sec^2 t.$$

Sign substitution in the squared variable carries the hyperbolic cosine kernel to the actual cosine kernel. The same homomorphism carries their inverses and their squares. Extracting the even coefficient in the derivative of tangent yields

$$\boxed{b_m = (-1)^m Z_{2m+1}.}$$

This formula is *proved*, not assumed as an input to reconstruction. The formalization follows the generating-function definition of zigzag numbers; it does not additionally formalize their interpretation as counts of alternating permutations.

Principal targets.

`EulerCoefficients.euler_egf,EulerCoefficients.inverse_coeff_cast`

`Zigzag.derivative_tan,Zigzag.inverse_is_signed_zigzag`

3. Exact reconstruction and the triangular matrix

Theorem 4.2; manuscript page 11

For odd p, rescaling the already proved inverse law gives

$$\operatorname{sech}(pt)\cosh(pt) = 1.$$

Multiplying the even-source identity by $\operatorname{sech}^2(pt)$ therefore proves

$$\operatorname{sech}^2(pt)\left(\sum_{r\geq 0} S_{2r}(p)\frac{t^{2r}}{(2r)!}\right) = \operatorname{sech} t.$$

Actual EGF multiplication now gives the integer coefficient formula

$$\boxed{E_{2r} = \sum_{m=0}^{r}\binom{2r}{2m} b_m p^{2m} S_{2(r-m)}(p).}$$

The formal proof first extracts the sum over all indices and then proves that the odd summands vanish. A checked even-index summation identity supplies the displayed change of variables.

Substituting the independently proved signed-zigzag identity recovers the manuscript's notation. For every even degree n,

$$\boxed{E_n = \sum_{m=0}^{n/2}(-1)^m Z_{2m+1}\binom{n}{2m} p^{2m} S_{n-2m}(p).}$$

The Lean version uses $n = 2r$ to avoid ambiguous natural-number division. Every even natural degree has that form. Primality is not used in this equality.

The row action is connected to the matrix entries

Reversing the finite sum and using binomial symmetry gives the actual source-index row action

$$E_{2r} = \sum_{q=0}^{r} U_{r,q}(p) S_{2q}(p),$$

where

$$U_{r,q}(p) = \begin{cases}\binom{2r}{2q} b_{r-q} p^{2(r-q)}, & q \leq r,\\ 0, & q > r.\end{cases}$$

The row-action equality, the diagonal identity $U_{r,r} = 1$, and vanishing above the diagonal are all checked. The arithmetic weight is also explicit:

$$p^{2(r-q)} \mid U_{r,q}(p) \quad \text{in } \mathbb{Z}, \qquad 0 \leq q \leq r.$$

No inverse-coordinate identity, unit diagonal or weight bound is assumed for these rows.

Principal targets.

`OddEuler.odd_reconstruction_series`

`OddEuler.odd_euler_reconstruction`

`OddEuler.odd_euler_reconstruction_zigzag`

`OddEuler.row_action,OddEuler.row_diagonal,OddEuler.row_weight`

4. Uniform truncation and a sharp exact example

Corollary 4.3 and Example 4.4

For row $2r$ and depth M, the actual truncation is

$$T_M(p,2r) = \sum_{m=0}^{\min(M,r)} \binom{2r}{2m} b_m p^{2m} S_{2(r-m)}(p).$$

Every omitted term has $m \geq M+1$. Its other factors are integers, so the first-block omitted-tail theorem, now applied to the *proved Euler identity*, gives

$$\boxed{p^{2M+2} \mid E_{2r} - T_M(p,2r) \quad \text{in } \mathbb{Z}.}$$

This is not divisibility in $\mathbb{Q}$, where a nonzero divisor would make the assertion vacuous. The statement holds for every row and depth. For an odd prime it is precisely the claimed supercongruence. If $M \geq r$, complete retention is proved exact.

The displayed manuscript row is kernel-checked

The scalar anchors are derived from the same general coefficient recurrence:

$$(E_0, E_2, E_4, E_6, E_8) = (1, -1, 5, -61, 1385),$$

$$(b_0, b_1, b_2, b_3, b_4) = (1, -2, 16, -272, 7936).$$

The literal source sums at $p = 5$ give

$$(S_0, S_2, S_4, S_6, S_8) = (1, 49, 4705, 428689, 37665985).$$

Consequently,

$$T_2(5,8) = 2\,731\,001\,385, \qquad E_8 - T_2(5,8) = -2\,731\,000\,000.$$

The source proves both

$$5^6 \mid E_8 - T_2(5,8), \qquad 5^7 \nmid E_8 - T_2(5,8).$$

Thus the guaranteed power is attained in this example. No stronger uniform exponent is inferred.

Oddness cannot be silently dropped

For length two, $S_0(2) = 1 - 1 = 0$, whereas $E_0 = 1$. The row-zero reconstruction already fails. The source proves this failure, and the release verifier separately attempts and rejects an invalid proof of the even-length identity.

Principal targets.

`OddEuler.odd_euler_supercongruence`

`OddEuler.odd_prime_supercongruence`, `OddEuler.full_depth_exact`

`OddEulerExamples.manuscript_error_sharp`

`OddEulerExamples.even_length_reconstruction_fails`

5. Verification, reproducibility and remaining work

Cumulative release evidence

Check	Result
Fresh cumulative proof-module compilation	13/13
Named theorem declarations audited	163/163
Explicit definitions audited	37/37
Exact dependency-revision checks	9/9
Invalid proof controls rejected as intended	5/5
Infrastructure and release-contract unit tests	33 passed
New captured development attempts	15
Development attempts failing before correction	8

The pinned runtime is Lean 4.34.0-rc2 and Mathlib revision

`aeb72f1e9981b6d35b426b16dca7759be3e7ba06`.

Every final module compiled with warnings treated as errors. All 200 expected axiom reports were resolved and used only `propext`, `Classical.choice` and `Quot.sound`, or subsets. The audit is of explicit project declarations. Mathlib's compiled cache was reused, not independently rebuilt.

The negative controls cover a false equality, a proof placeholder, the old false normalized finite-difference bound, the false even-length extension, and a false extra digit of precision. Missing imports and timeouts do not count as successful rejection. The placeholder is rejected by warnings-as-errors, and `sorryAx` is separately excluded from the axiom audit.

Independent finite arithmetic checks

Exact Python mirrors passed 234 reconstruction rows and 2,340 truncation-depth cases for odd lengths 1 through 35 and even degrees 0 through 24. They also checked 468 coefficients of each generating-function identity, 13 signed-zigzag bridges and seven invalid-input controls. These are rational/integer calculations, not executions extracted from Lean. The first-block mirror suite was rerun separately. The original historical numerical suite was not recovered or rerun.

Provenance and limits

All eight upstream proof files, the original 28-page corrected manuscript and the first supplement are preserved. The initial clean baseline and a preliminary successful second-block build are separate from the final build that includes the explicit matrix row-action theorem. Eight failed development drafts and their diagnostics are retained, not counted as successful proofs. An unsupported interactive container invocation was replaced by successful noninteractive runtime recovery; every archive-part hash and the combined hash matched.

Fourteen manuscript-proved items remain pending. The next coherent target is the separately weighted dyadic reconstruction. The global finite-difference theorem, three classical inputs and two conjectures have not been promoted by this block.

Reproduce. Run `bashRun-Portable.sh/path/to/extracted/kit`. This performs unit tests, both independent regression suites and a fresh Lean build. The kit is separate; native Windows is not tested. Precise commands, source hashes, raw compiler logs, scope entries and failures accompany the release. Manuscript-to-Lean correspondence remains a mathematical review, not an automated semantic-equivalence theorem.

E.3 Third Lean Block Supplement

Checkpoint scope. This complete source body retains the status of its dated original supplement. Earlier pending or partial-coverage labels are historical; the final core records their resolution.

Scope and result

This release continues the unchanged 28-page *status-corrected* Paper IX manuscript and the checked second block. It completes the recorded scopes of Lemmas 5.1 and 5.2, Theorem 5.3 and Corollary 5.4. All 13 upstream proof modules and both earlier mathematical supplements remain byte-identical.

Fourteen of 24 named manuscript-proved items now have formal coverage; ten remain pending. Three classical inputs and two conjectures are tracked separately. The cumulative corpus has **219 named theorem declarations and 53 explicit definitions in 17 modules**. The 56 added theorems include supporting identities and exact examples, not 56 independent discoveries.

New module	Scope	Theorems	Defs.
`DyadicIntegrality`	5.1, 5.2	21	8
`DyadicSeries`	5.1, 5.3	11	4
`DyadicReconstruction`	5.3, 5.4	10	4
`DyadicExamples`	Exact controls	14	0
Total added	Four named items	56	16

The completed mathematical chain

The inverse coefficients are defined from a genuine unit power series and connected to $z^2/\sinh^2 z$. The source is the paper's explicit rational formula, not an abstract integral input. Its valuation calculation proves the stronger statement that *every even source coefficient is a* 2*-adic unit*. Formal-series multiplication and coefficient extraction then yield the Euler identity with the correct falling factorial. The omitted-tail theorem gives divisibility in the actual completed integer ring $\mathbb{Z}_2$.

The clean cumulative run compiled all 17 modules, matched all 272 expected transitive axiom reports and rejected eight invalid-proof controls. All 49 infrastructure and release tests passed. Exact-rational regression checks are recorded separately from the universal proofs.

Completion boundary

This is a formalization companion, not a revised primary manuscript. The unnumbered Bernoulli closed form for the inverse coefficients is not universally formalized here. Formal-series identities do not establish analytic convergence. The corrected global odd-prime finite-difference theorem remains pending. Historical numerical evidence is not presented as a fresh reproduction.

1. Integral coefficients and an exact unit source

Lemmas 5.1 and 5.2; manuscript page 14

Define the rational subring

$$\mathcal{O}_2 = \{q \in \mathbb{Q} : |q|_2 \leq 1\}.$$

Its ring laws follow from the actual nonarchimedean norm. For every $q \in \mathcal{O}_2$, the code constructs its element in Mathlib's completed ring $\mathbb{Z}_2$, and proves that the underlying value in $\mathbb{Q}_2$ is the rational cast of q. No integrality claim is hidden in a coercion.

The unit kernel and its reciprocal

Use a squared formal variable u and define

$$k_m = \frac{2^{2m+1}}{(2m+2)!}, \qquad K(u) = \sum_{m \geq 0} k_m u^m, \qquad \beta_m = [u^m]K(u)^{-1}.$$

The constant term is proved to be one. The factorial valuation bound gives

$$v_2(k_m) = 2m + 1 - v_2((2m+2)!) \geq 0.$$

Thus every k_m belongs to $\mathcal{O}_2$. A general coefficientwise subring theorem is proved by strong induction using the actual unit-inverse coefficient formula. Applied to K, it establishes

$$\boxed{\beta_m \in \mathcal{O}_2 \quad \text{for every } m \geq 0.}$$

The reciprocal's coefficients are therefore proved integral; the proof does not assume them integral in order to obtain the final congruence. Their connection to the intended hyperbolic function is proved on the next page.

The source has valuation exactly zero

At even degree $2r$, define the literal rational source

$$R_{2r} = \frac{3^{2r+2} - 1}{4(2r+1)(2r+2)}.$$

Both numerator and denominator are nonzero. The pinned Mathlib lifting-the-exponent theorem gives

$$v_2(3^{2r+2} - 1) = 2 + v_2(2r+2).$$

Because $2r+1$ is odd, the denominator has the same valuation. Consequently,

$$\boxed{v_2(R_{2r}) = 0, \qquad |R_{2r}|_2 = 1.}$$

The code proves actual `IsUnit` status for the constructed $\mathbb{Z}_2$ element. This strengthens the manuscript's integrality conclusion without changing its source. The argument applies at $r = 0$, where $R_0 = 1$, as well as at every later row.

Principal targets.

`Dyadic.kernel_coefficient_integral`,`Dyadic.beta_integral`

`Dyadic.source_valuation`,`Dyadic.sourcePadic_isUnit`

2. Connecting the definitions to actual formal series

Generating-function foundations for Theorem 5.3

Work in $\mathbb{Q}[[t]]$. The formal exponential and rescaling homomorphism are the same checked objects used in the second block. Define

$$\sinh_a(t) = \frac{e^{at} - e^{-at}}{2}, \qquad \cosh_a(t) = \frac{e^{at} + e^{-at}}{2}.$$

Let $B(t) = K(t^2)^{-1} = \sum_{m\geq 0} \beta_m t^{2m}$ and let

$$F(t) = \sum_{r\geq 0} R_{2r} \frac{t^{2r}}{(2r)!}.$$

These series are constructed independently from the definitions on the preceding page.

No inverse of a zero-constant series is assumed

Coefficient extraction proves the cleared identity

$$t^2 K(t^2) = \frac{\cosh(2t) - 1}{2}.$$

A separate identity from the formal exponential gives

$$\sinh^2 t = \frac{\cosh(2t) - 1}{2}.$$

Since $K(t^2)$ has constant term one, its reciprocal exists as a unit. Multiplication then proves

$$\boxed{B(t)\sinh^2 t = t^2.}$$

This is the precise formal meaning of $B(t) = t^2/\sinh^2 t$. The forced quadratic zero is removed first; the proof does not claim that $\sinh t$ itself is an invertible formal power series.

The explicit rational source supplies its own identity

From the formula for R_{2r}, coefficient arithmetic proves

$$\boxed{t^2 F(t) = \frac{\cosh(3t) - \cosh t}{4}.}$$

The product relation for the formal hyperbolic cosines yields

$$F(t)\cosh t = K(4t^2).$$

This equality is proved after multiplication by t^2 and cancellation of that monomial in the formal-series ring. It is not an assumed source-transform relation.

Rescale the reciprocal by $t \mapsto 2t$. The unit inverse law and the preceding identity now give

$$\boxed{B(2t)F(t) = \operatorname{sech} t.}$$

The right side is connected to the same integral signed Euler sequence constructed in the previous release. These are equalities of full formal series, not fixed-degree polynomial experiments.

Principal targets.

`DyadicSeries.beta_original_generating_function`

`DyadicSeries.source_shift_identity,DyadicSeries.source_times_cosh`

`DyadicSeries.dyadic_reconstruction_series`

3. Exact reconstruction and genuine dyadic congruence

Theorem 5.3 and Corollary 5.4; manuscript pages 14–15

The reciprocal coefficients β_m use an *ordinary* power expansion. In contrast, the source and Euler series use exponential coefficients. Extracting degree $2r$ from $B(2t)F(t) = \operatorname{sech} t$ therefore gives

$$\boxed{E_{2r} = \sum_{m=0}^{r} \beta_m 2^{2m} (2r)_{2m} R_{2(r-m)},} \qquad (2r)_{2m} = \frac{(2r)!}{(2r-2m)!}.$$

The formal proof uses the actual natural-number descending factorial and proves its factorial-quotient identity for $2m \le 2r$. The odd coefficient terms vanish by a proved parity argument. The binomial EGF product then supplies the falling factorial, rather than treating it as an extra assumed factor.

Both row indexing by r and the literal even-degree version with hypothesis $\mathrm{Even}(n)$ are checked. All rows include degree zero. The exact identity is a rational equality with the Euler coefficient's integer-to-rational cast made explicit.

The finite truncation

For $M \ge 0$, define

$$T_M(2r) = \sum_{m=0}^{\min(M,r)} \beta_m 2^{2m} (2r)_{2m} R_{2(r-m)}.$$

The integral rational factors are embedded into $\mathbb{Z}_2$ using their proved witnesses. The cast of this $\mathbb{Z}_2$ truncation into $\mathbb{Q}_2$ is proved to equal the cast of the literal rational truncation.

Each omitted term has $m \ge M+1$. Its remaining factors lie in $\mathbb{Z}_2$, so the checked weighted-tail theorem proves

$$\boxed{2^{2M+2} \mid E_{2r} - T_M(2r) \quad \text{in } \mathbb{Z}_2.}$$

This is not divisibility in $\mathbb{Q}$ or $\mathbb{Q}_2$, where division by any nonzero scalar would make the claim vacuous. A second theorem supplies the equivalent rational statement

$$\boxed{\frac{E_{2r} - T_M(2r)}{2^{2M+2}} \in \mathcal{O}_2.}$$

It follows from actual $\mathbb{Z}_2$ divisibility and the proved rational/completed-field bridge. No precision assertion is passed in as a hypothesis.

Endpoints and scope

When $M \ge r$, the sum retains every term and $T_M(2r) = E_{2r}$ exactly, in both rational and dyadic coordinates. The zero-error case is included; no finite valuation of zero is used. This is a separate dyadic construction, not substitution of $p = 2$ into the odd-length theorem.

Principal targets.

`DyadicReconstruction.dyadic_reconstruction`

`DyadicReconstruction.dyadic_supercongruence`

`DyadicReconstruction.normalized_error_integral`

4. Exact examples and adversarial checks

Example 5.5 and controls against incorrect normalizations

The actual reciprocal recurrence gives

$$(\beta_0, \beta_1, \beta_2, \beta_3, \beta_4) = \left(1, -\frac{1}{3}, \frac{1}{15}, -\frac{2}{189}, \frac{1}{675}\right).$$

These values are obtained from the general formal-series definition. They are not substituted as independent input data for the reconstruction theorem. The source formula gives

$$(R_0, R_2, R_4, R_6, R_8) = \left(1, \frac{5}{3}, \frac{91}{15}, \frac{205}{7}, \frac{7381}{45}\right).$$

Using the retained Euler anchor $E_8 = 1385$, Lean checks the literal decomposition

$$E_8 = \frac{7381}{45} - \frac{6560}{3} + \frac{163072}{15} - \frac{204800}{9} + \frac{229376}{15} = 1385.$$

Truncations and actual valuations

Depth M	$T_M(8)$	Guaranteed exponent $2M+2$
0	$7381/45$	2
1	$-91019/45$	4
2	$398197/45$	6
3	$-208601/15$	8
4	1385	10

The checked nonzero error valuations are

$$E_8 - T_0(8) = \frac{54944}{45}, \qquad v_2(E_8 - T_0(8)) = 5,$$

$$E_8 - T_3(8) = \frac{229376}{15}, \qquad v_2(E_8 - T_3(8)) = 15.$$

They exceed the uniform guarantees; these two examples do not establish a universally stronger exponent.

What the negative controls rule out

Omitting the rescaling 2^{2m} already breaks the degree-two identity: the altered sum is 1, whereas $E_2 = -1$. Confusing ordinary beta coefficients with EGF coefficients would replace $\beta_1 = -1/3$ by $-2/3$; a checked inequality excludes that normalization. A deliberately false claim that the depth-zero degree-eight error has valuation at least six is rejected.

These three negative proof files are separate from the active mathematical modules. The verifier also retains the five earlier invalid controls, including the normalized all-residue counterexample. Rejection of an attempted proof is an engineering control; the actual counterexample theorems establish the mathematical failures.

Principal targets.

`DyadicExamples.beta_one`, `DyadicExamples.source_anchor_values`

`DyadicExamples.manuscript_error_zero_valuation`

`DyadicExamples.manuscript_error_three_valuation`

5. Verification, provenance and remaining work

Actual results for the v0.3.0 source snapshot

Check	Recorded result
Fresh local Lean modules	17 / 17
Explicit theorem declarations	219 / 219
Explicit definitions	53 / 53
Matched transitive axiom reports	272 / 272
Exact dependency revisions	9 / 9
Invalid-proof controls rejected	8 / 8
Infrastructure and release tests	49 / 49
Independent exact reconstruction rows	49
Independent truncation-depth cases	2,548

The clean cumulative run used **Lean 4.34.0-rc2**, with compiler warnings treated as errors. Mathlib is pinned to

`aeb72f1e9981b6d35b426b16dca7759be3e7ba06`.

Every expected axiom report uses only `propext`, `Classical.choice` and `Quot.sound`, or a subset. No additional project axiom or admitted proof is permitted. The verifier audits the explicit declaration inventory, not every generated declaration or all imported Mathlib proofs.

Independent regressions. Exact Python integers and rational fractions check even degrees through 96. The 2,548 truncation cases include 1,176 nonzero errors and 1,372 exact zeros. Zero is handled separately. Kernel and reciprocal integrality and source unit status are checked in 49 cases each. Ninety-seven series coefficients agree. Forty-eight finite comparisons with the Bernoulli closed form pass, but do not prove that unnumbered formula universally. Both older independent Python suites were also rerun. None of these mirrors is Lean-extracted code or a reproduction of the original historical experiments.

Development failures. Eight of 15 captured development checks failed before repair. Their exact inactive source snapshots and diagnostics are retained separately. The seven successful development checks do not substitute for the clean cumulative release run. All 13 upstream proof files, the original corrected PDF and both earlier supplements remain unchanged. Previous metadata and original test copies are preserved under `provenance/second_block/`.

Ten pending named items. These are 6.2 (finite units), 7.2–7.5 (Mahler and finite differences), 8.3 and 8.5 (residue independence and period), 9.1 and 9.2 (recurrence jumps), and 10.1 (CRT factorization). Three

classical inputs and both conjectures remain separate. The corrected global finite-difference claim is not promoted by the dyadic proof.

Trust and reproduction. The pinned compiled Mathlib cache was reused, not independently rebuilt. The integral-source proof invokes the pinned library's LTE theorem; this project does not reprove LTE. Papers I–VIII and the historical numerical suite were not rerun. The source hashes, commands, axiom output and raw logs accompany the release. The external release audit separately records archive integrity and post-packaging replay. Manuscript correspondence remains a mathematical review, not an automated semantic-equivalence theorem.

Run `bashRun-Portable.sh/path/to/extracted/kit`. The runtime is separate from this ZIP. The full source-level scope and target names are in `coverage.json` and the third-block ledger.

E.4 Fourth Lean Block Supplement

Checkpoint scope. This complete source body retains the status of its dated original supplement. Earlier pending or partial-coverage labels are historical; the final core records their resolution.

Scope and result

This release continues the unchanged 28-page status-corrected manuscript and the checked third block. It completes the recorded scopes of Theorem 7.2, Corollary 7.3 and Theorem 7.4. All 17 upstream proof files and the three earlier mathematical supplements remain unchanged.

Seventeen of 24 named manuscript-proved items now have formal coverage; seven remain pending. Three classical inputs and two conjectures are tracked separately. The cumulative project contains **272 named theorem declarations and 60 explicit definitions across 21 modules**.

New module	Scope	Theorems	Defs.
`FiniteMahler`	7.2	19	3
`PadicDifferenceBounds`	7.3	12	2
`UnitModeDifferences`	7.4	9	0
`MahlerExamples`	Exact controls	13	2
Total added	Three items	53	7

The 53 additions include supporting lemmas and examples, not 53 independent discoveries. The classical Gregory–Newton and binomial-difference results are reused from pinned Mathlib. The new modules connect those results to the actual Paper IX coefficient definitions, interpolation scopes, valuation bounds and unit modes.

Two explicit specification boundaries

A finite sample through N determines $\Delta^s f(n)$ only when $n + s \leq N$. For arbitrary orders, the finite-coefficient identity describes the constructed interpolant, not every unknown continuation of the sample. All three versions are separately checked.

The additive valuation is extended to $\mathrm{ord}_p(0) = \infty$. This avoids incorrectly applying Mathlib's finite-valued default at zero to an inequality that must also cover cancellation and identically zero differences.

Verification status

The fresh cumulative Lean run compiled all 21 modules with warnings as errors, matched all 332 explicit declaration axiom reports, and rejected 12 invalid-proof controls. All 74 infrastructure and release tests passed. Independent exact-rational mirrors are recorded separately from the universal proofs. The archive is also subjected to a fresh-extraction rebuild and integrity audit; its separate report records that outcome.

1. Reconstruction and finite-data scope

Theorem 7.2; manuscript page 17

Let R be any commutative ring. The implementation uses the actual forward-difference operator

$$\Delta_h f(n) = f(n+h) - f(n), \qquad c_r = \Delta_1^r f(0).$$

For a full sequence $f : \mathbb{N} \to R$, pinned Mathlib's Gregory–Newton formula gives

$$\boxed{f(n) = \sum_{r=0}^{n} c_r \binom{n}{r}.}$$

The source applies the theorem to these exact definitions. A separate explicit formula relates an s-th difference to its original sampled values, so the data dependence is inspectable:

$$\Delta_h^s f(n) = \sum_{j=0}^{s} (-1)^{s-j} \binom{s}{j} f(n+jh).$$

Applying Newton reconstruction to $\Delta_1^s f$ and composing the iterated operators proves

$$\boxed{\Delta_1^s f(n) = \sum_{j=0}^{n} c_{s+j} \binom{n}{j}.}$$

The apparently infinite coefficient sum is finite at each natural argument. Neither continuity nor convergence is needed for these identities.

An independently defined interpolant

For a finite coefficient list, define

$$I_N(c)(n) = \sum_{r=0}^{N} c_r \binom{n}{r}.$$

The source proves exact coefficient extraction:

$$\Delta_1^s I_N(c)(0) = \begin{cases} c_s, & s \le N, \\ 0, & s > N. \end{cases}$$

It follows that $I_N(c)$ has the stated shifted-coefficient formula at every order, with zero coefficients beyond N. Every difference of order greater than N vanishes. Its coefficients are uniquely determined by its values at $0, \ldots, N$.

What the finite sample does and does not determine

For $c_r = \Delta_1^r f(0)$ with $r \le N$, the source proves

$$I_N(c)(n) = f(n) \quad (n \le N), \qquad \Delta_1^s I_N(c)(n) = \Delta_1^s f(n) \quad (n + s \le N).$$

The condition is essential. The sequences $f(n) = n$ and $g(n) = 0$ agree on the sample $\{0\}$, but $\Delta f(0) = 1$ and $\Delta g(0) = 0$. This is a checked counterexample to unrestricted inference outside the observed stencil.

Principal targets.

`Mahler.finite_mahler_reconstruction`,`Mahler.coefficient_interpolant`

`Mahler.difference_from_coefficients`,`Mahler.finite_sample_difference`

2. Weighted bounds in the actual ring $\mathbb{Z}_p$

Corollary 7.3; manuscript page 18

Let p be prime. For $x \in \mathbb{Z}_p$, define

$$\operatorname{ord}_p(x) = \begin{cases} \infty, & x = 0, \\ v_p(x), & x \neq 0. \end{cases}$$

The nonzero branch uses Mathlib's actual p-adic valuation. The implementation proves its connection to genuine integral divisibility:

$$\boxed{L \leq \operatorname{ord}_p(x) \quad \Longleftrightarrow \quad p^L \mid x \text{ in } \mathbb{Z}_p, \qquad L \in \mathbb{N}.}$$

This is not divisibility in $\mathbb{Q}_p$, where a nonzero scalar can divide any field element. The zero case is proved separately, so zero does not spuriously violate a positive lower bound.

The source also proves multiplicativity of this extended order, the nonarchimedean lower bound for addition, and its finite-sum consequence. The natural-number cast is connected to the actual integer binomial valuation.

The binomial valuation is retained

Let $f : \mathbb{N} \to \mathbb{Z}_p$, let $c_r = \Delta_1^r f(0)$, and suppose

$$w(s+j) \leq \operatorname{ord}_p(c_{s+j}) \quad (0 \leq j \leq n), \qquad w : \mathbb{N} \to \mathbb{N} \cup \{\infty\}.$$

Set

$$B_p(w; n, s) = \min_{0 \leq j \leq n} \left\{ w(s+j) + v_p\binom{n}{j} \right\}.$$

Every binomial in this range is nonzero. The code proves

$$\boxed{B_p(w; n, s) \leq \operatorname{ord}_p(\Delta_1^s f(n)).}$$

This is the manuscript minimum after the substitution $r = s + j$. Coefficients outside this range have zero binomial multiplier. Weights equal to infinity are allowed, and a completely vanishing difference has order infinity.

For a uniform coefficient hypothesis, the simpler conclusion follows:

$$\boxed{\inf_{j \geq 0} w(s+j) \leq \operatorname{ord}_p(\Delta_1^s f(n)).}$$

For each finite $L \leq B_p(w; n, s)$, a separate theorem yields $p^L \mid \Delta_1^s f(n)$ inside $\mathbb{Z}_p$. Another specialization handles arbitrary finite interpolants with infinite weights beyond their degree.

Principal targets.

`PadicBounds.nat_le_order_iff`, `PadicBounds.weighted_difference_bound`

`PadicBounds.weighted_difference_divisibility`

`PadicBounds.coefficient_tail_bound`, `PadicBounds.interpolant_weighted_bound`

3. Unit modes and residue-field reduction

Theorem 7.4; manuscript page 18

For u in any commutative ring, and every natural step h, iteration proves

$$\boxed{\Delta_h^s(u^n) = u^n(u^h - 1)^s.}$$

The proof unfolds the actual forward-difference operator and uses the power multiplication law. No valuation premise or geometric-difference identity is assumed. More generally, a proved divisibility $p^k \mid u^h - 1$ implies

$$p^{ks} \mid \Delta_h^s(u^n).$$

Why a p-adic unit supplies one factor of p

Now let $u \in \mathbb{Z}_p^\times$. The actual ring homomorphism

$$\pi : \mathbb{Z}_p \longrightarrow \mathbb{F}_p$$

sends u to a nonzero unit. Fermat's finite-field theorem therefore gives

$$\pi(u)^{p-1} = 1, \qquad \pi(u^{p-1} - 1) = 0.$$

Using the proved kernel identity $\ker \pi = p\mathbb{Z}_p$, the source derives

$$\boxed{p \mid u^{p-1} - 1 \quad \text{inside } \mathbb{Z}_p.}$$

The conclusion is not supplied as an abstract certificate. It follows from the unit hypothesis and the standard residue-field construction.

Combining it with the geometric identity establishes

$$\boxed{p^s \mid \Delta_{p-1}^s(u^n), \qquad s \le \operatorname{ord}_p(\Delta_{p-1}^s(u^n)).}$$

These statements hold for every prime, including $p = 2$ with step one. Zero differences and order zero are included. Every multiple of the Fermat step satisfies the same lower bound.

Finite integral linear combinations

For any finite index set S, coefficients $a_i \in \mathbb{Z}_p$ and units $u_i \in \mathbb{Z}_p^\times$, linearity of the actual iterated operator gives

$$\Delta_h^s\left(\sum_{i\in S} a_i u_i^n\right) = \sum_{i\in S} a_i u_i^n (u_i^h - 1)^s.$$

At $h = p - 1$, the result is divisible by p^s. The empty sum is permitted. This does not prove that the general congruential Euler packets of Theorem 7.5 admit such a representation.

Principal targets.

`UnitModes.geometric_difference,UnitModes.unit_fermat_divisibility`

`UnitModes.unit_mode_valuation_gain`

`UnitModes.finite_unit_modes_valuation_gain`

4. Exact examples and adversarial checks

The examples use the same definitions as the general results

A polynomial anchor

The actual difference table for $f(n) = n^2$ yields

$$(c_0, c_1, c_2, c_3) = (0, 1, 2, 0).$$

The finite interpolant therefore reconstructs $I_2(c)(7) = 49$. This checks the sign and indexing conventions, not just the names of the defined objects.

A sharp binomial-weight improvement

Let

$$F(n) = 81\binom{n}{1} + 3\binom{n}{2} + 3\binom{n}{3} + 81\binom{n}{4}.$$

At $n = 3$ and $s = 1$, the four coefficient valuations are $(4, 1, 1, 4)$, while the binomial valuations are $(0, 1, 1, 0)$. Thus the refined bound is

$$B_3(w; 3, 1) = \min(4, 2, 2, 4) = 2,$$

stronger than the coefficient-only lower bound one. The source checks both the minimum and the actual result:

$$\boxed{\Delta F(3) = 180, \qquad \operatorname{ord}_3(180) = 2.}$$

The valuation equality is proved in the actual completed ring $\mathbb{Z}_3$, with an independent exact integer divisibility check as well.

A sharp unit-mode gain

For $p = 5$ and $u = 2$,

$$\Delta_4^2(u^n)\big|_{n=0} = (2^4 - 1)^2 = 225.$$

The source proves $5^2 \mid 225$ and $5^3 \nmid 225$. This example attains the guaranteed gain; no additional uniform power of p is claimed.

What fails when hypotheses or conventions are dropped

The nonunit $u = 3$ at $p = 3$ gives $\Delta_2(u^n)|_{n=0} = 8$, with no factor of three. A constant unit has every positive-order difference zero, hence infinite extended valuation. The finite-sample counterexample on page 2 distinguishes a true interpolant identity from an unsupported claim about unknown continuation.

The release verifier adds four negative Lean controls: an unrestricted finite-sample inference, zero assigned finite extended valuation, nonunit valuation gain, and an extra unit-mode precision claim. All are required to fail with substantive proof diagnostics. Import errors or timeouts do not qualify as successful rejection tests.

Source. The module `MahlerExamples` contains these exact certificates. Principal target names are listed in the fourth-block ledger.

5. Verification record and remaining work

Evidence is retained separately from conclusions

Cumulative proof checks

Lean 4.34.0-rc2 rebuilt all 21 local modules into a fresh directory with warnings treated as errors. All 272 theorem declarations and 60 explicit definitions matched their expected transitive axiom reports. Only `propext`, `Classical.choice` and `Quot.sound`, or subsets thereof, occurred. All nine dependency pins matched. Twelve invalid-proof controls were rejected.

All 74 infrastructure and release tests passed. Historical version-specific tests use immutable metadata snapshots. Current source counts, preserved upstream hashes, exact newly covered manuscript items, remaining items and substantive mathematical boundary cases are checked by the active suite.

Ten targeted development runs preceded the cumulative build. Six failed before repair, due to elaboration, coercions, finite-sum normalization, an inferred index type and tactic or linter diagnostics. Those drafts and compiler outputs remain in a separate inactive evidence directory. No failed draft is counted as a passing theorem.

Independent exact-rational mirrors

The new standard-library Python program uses repeated difference tables and independently evaluated binomial sums. It passed 2,079 reconstructed values, 15,939 in-range sample differences, 2,600 arbitrary-interpolant differences, 900 weighted cases, 966 unit-mode cases and 350 finite-mode-sum cases. Among the weighted cases, 522 differences were zero. It also rejected 189 out-of-range stencils and 15 nonunit inputs.

These are finite tests on rational examples, not executable extraction of Lean functions, a substitute for a universal proof, or a new run of the manuscript's historical experiments.

Remaining named manuscript-proved items

Item	Remaining formalization
6.2	Finite unit decomposition certificates
7.5	Global odd-prime Euler finite-difference theorem
8.3	Standard-window residue independence
8.5	Uniform standard-window period
9.1	Companion-matrix jump theorem
9.2	Fibonacci packet recurrence
10.1	Arithmetic CRT packet factorization

The three classical inputs and two conjectures remain separate. Mathlib's pinned compiled cache was reused, not independently rebuilt. No infinite Mahler convergence theorem or prime-independent arithmetic spine is proved here. The original manuscript and three earlier supplements remain unchanged; the finite-data and zero-valuation clarifications accompany them in this release.

Reproduce. Run `bashRun-Portable.sh/path/to/kit`. The report in `verification/fourth_release/` records the cumulative build. The separately distributed release audit records archive integrity and fresh-extraction replay.

E.5 Fifth Lean Block Supplement

Checkpoint scope. This complete source body retains the status of its dated original supplement. Earlier pending or partial-coverage labels are historical; the final core records their resolution.

Scope and result

This release continues the unchanged 28-page status-corrected manuscript and the checked fourth block. It completes Proposition 6.2, Corollary 8.3 and Theorem 10.1 in the scopes stated below. All 21 upstream proof files and the four earlier mathematical supplements remain unchanged.

Twenty of 24 named manuscript-proved items now have formal coverage; four remain pending. Three classical inputs and two conjectures are tracked separately. The cumulative project contains **333 named theorem declarations and 77 explicit definitions across 28 modules**.

New module	Scope	Theorems	Defs.
`UnitTriangular`	Integral solver	5	1
`WindowIndependence`	8.3	11	3
`FiniteUnits`	6.2, prime powers	13	4
`DyadicUnits`	6.2, sign branch	10	4
`UnitRefinement`	Finite compatibility	4	2
`CRTPackets`	10.1	5	1
`FiniteArithmeticExamples`	Exact controls	13	2
Total added	Three items	61	17

The 61 new theorems include supporting lemmas and examples, not 61 independent discoveries. Pinned Mathlib supplies finite-field, unit-group, ideal-power, p-adic residue and Chinese-remainder results. The new modules connect these to the actual packet coefficients, explicit finite lifts and existing selectors.

The specification matters

Rational coefficients are not reduced through an imaginary map $\mathbb{Q} \to \mathbb{F}_p$. The proof first constructs their integral lifts in $\mathbb{Z}_p$ and establishes equality in $\mathbb{Q}_p$. Root representatives are actual units at the requested precision, not unchanged residues that need not satisfy the root equation modulo p^ℓ.

Verification status

The fresh cumulative Lean build compiled all 28 modules with warnings treated as errors, matched 410 declaration axiom reports, and rejected all 17 invalid-proof controls. All 103 infrastructure and release tests passed. Independent finite regressions are recorded separately from the proofs. A separate release audit records the archive's fresh-extraction rebuild and integrity checks.

1. Integral reduction of the original coefficients

Corollary 8.3; manuscript pages 19–20

Fix a prime p, an integer $m \geq 1$ and a window address $0 \leq j < p$. The earlier project defines the rational packet coefficients by the actual recurrence

$$\sum_{q=0}^{n} \binom{mpn+j}{mpq} C_{mp,j}(q) = \delta_{n0}.$$

The previous Lucas theorem proves

$$\binom{mpn+j}{mpq} \equiv \binom{mn}{mq} \pmod{p}, \qquad \binom{mpn+j}{mpn} \equiv 1 \pmod{p}.$$

The last congruence makes each diagonal entry a unit in the actual completed ring $\mathbb{Z}_p$.

First construct the integral solution

For a ring R, a triangular row system T, right-hand side b, and proved unit pivots $d_n = T_{nn}$, define

$$s_n = d_n^{-1} \left(b_n - \sum_{q<n} T_{nq} s_q \right).$$

The new ring-level solver uses well-founded recursion. Its row identities and unique-solution property are proved. The source also proves compatibility of this solver with ring homomorphisms into fields.

Specializing to $R = \mathbb{Z}_p$ constructs $c_{m,j}^{(p)}(n)$. Mapping to $\mathbb{Q}_p$ and comparing with the original rational forward substitution gives

$$\boxed{c_{m,j}^{(p)}(n) = C_{mp,j}(n) \quad \text{in } \mathbb{Q}_p.}$$

Thus every original standard-window coefficient is p-integral, for every row. Integrality is a conclusion, not an additional hypothesis on the coefficients.

Then apply the residue map

Independently define $e^{(m,p)}$ over $\mathbb{F}_p$ by the unit-diagonal system

$$\sum_{q=0}^{n} \binom{mn}{mq} e_q^{(m,p)} = \delta_{n0}.$$

Its recurrence and uniqueness are checked. Applying the actual homomorphism $\pi : \mathbb{Z}_p \to \mathbb{F}_p$ to the integral row, and then Lucas reduction, proves

$$\boxed{\pi(c_{m,j}^{(p)}(n)) = e_n^{(m,p)}.}$$

For $j, k < p$, a separate ideal statement gives

$$p \mid c_{m,j}^{(p)}(n) - c_{m,k}^{(p)}(n) \quad \text{inside } \mathbb{Z}_p.$$

The formal solver also covers $m = 0$; the manuscript uses $m \geq 1$. No global finite-difference theorem or period assertion enters this proof.

Principal targets.

`WindowIndependence.coeffPadic_coe,.standard_window_integral`

`WindowIndependence.residue_independence,.reduced_unique`

`WindowIndependence.two_window_addresses_congruent`

2. Explicit finite prime-power decomposition

Proposition 6.2; odd-prime branch

Let p be prime, $\ell \geq 1$, and

$$a \in \left(\mathbb{Z}/p^{\ell}\mathbb{Z}\right)^{\times}.$$

Define actual units in the same ring by

$$\boxed{\omega_{\ell}(a) = a^{p^{\ell-1}}, \qquad \langle a \rangle_{\ell} = a\,\omega_{\ell}(a)^{-1}.}$$

The Lean source writes $\ell = L + 1$, so zero precision is excluded without truncated natural-number subtraction.

The root equation and residue are derived

Euler's unit-group exponent gives

$$\omega_{\ell}(a)^{p-1} = a^{p^{\ell-1}(p-1)} = a^{\varphi(p^{\ell})} = 1.$$

Reduction to the actual field $\mathbb{F}_p$ commutes with powers. Its Frobenius identity gives

$$\overline{\omega_{\ell}(a)} = \bar{a}^{p^{\ell-1}} = \bar{a}.$$

Consequently,

$$\boxed{a = \omega_{\ell}(a)\langle a \rangle_{\ell}, \quad \omega_{\ell}(a)^{p-1} = 1, \quad \overline{\langle a \rangle_{\ell}} = 1.}$$

The manuscript's finite existence conclusion is obtained directly. The proof does not assume a Hensel lift or an already supplied root representative.

Uniqueness requires a lifting argument

The source proves the actual kernel statement

$$\bar{x} = 0 \quad \Longleftrightarrow \quad p \mid x \quad \text{in } \mathbb{Z}/p^{\ell}\mathbb{Z}.$$

For $x \equiv y \pmod{p}$, the checked ideal-power lifting theorem gives

$$x^{p^{\ell-1}} - y^{p^{\ell-1}} \in (p)^{\ell} = (0).$$

If $w^{p-1} = 1$, then $w^{p} = w$ and hence $w^{p^{k}} = w$ for every k. Therefore two roots with the same residue are equal. The decomposition is unique whenever its second factor reduces to one. Multiplicativity of both factors is also proved.

For $p = 3, \ell = 2, a = 2$, the checked values are

$$\omega_2(2) = 8, \qquad \langle 2 \rangle_2 = 7, \qquad 8^2 = 1, \quad 8 \cdot 7 = 2 \quad \text{in } \mathbb{Z}/9\mathbb{Z}.$$

The unlifted representative 2 is not a square root of one modulo nine.

Principal targets.

`FiniteUnits.finite_unit_decomposition, .power_lift_congruence`

`FiniteUnits.omega_unique, .decomposition_unique`

`FiniteUnits.omega_mul, .principal_mul`

3. Precision compatibility and the dyadic sign

Finite refinements and Proposition 6.2′s dyadic branch

For $1 \le k \le \ell$, let

$$\rho_{\ell,k} : \mathbb{Z}/p^{\ell}\mathbb{Z} \longrightarrow \mathbb{Z}/p^{k}\mathbb{Z}$$

be the actual reduction homomorphism. It induces a map on units. The explicit construction satisfies

$$\boxed{\rho_{\ell,k}(\omega_\ell(a)) = \omega_k(\rho_{\ell,k}(a)),}$$

$$\boxed{\rho_{\ell,k}(\langle a\rangle_\ell) = \langle \rho_{\ell,k}(a)\rangle_k.}$$

The proof uses the exponent factorization $p^{\ell-1} = p^{k-1}p^{\ell-k}$ and the already proved Frobenius invariance of a lifted root. Principal factors also satisfy

$$\langle a\rangle_\ell^{\,p^{\ell-1}} = 1.$$

At precision one the principal factor is exactly one. These are coherent finite certificates; no infinite inverse limit or topological unit-group equivalence is constructed in this release.

Why the mod-four branch is separate

Now let $\ell \ge 2$ and $a \in (\mathbb{Z}/2^{\ell}\mathbb{Z})^{\times}$. The actual units of $\mathbb{Z}/4\mathbb{Z}$ are exhausted by 1 and -1. Define

$$\varepsilon_\ell(a) = \begin{cases} 1, & a \equiv 1 \pmod 4, \\ -1, & a \equiv 3 \pmod 4, \end{cases} \qquad \langle a\rangle_{2,\ell} = \varepsilon_\ell(a)a.$$

The source uses $\ell = L + 2$. It proves

$$\boxed{a = \varepsilon_\ell(a)\langle a\rangle_{2,\ell}, \quad \varepsilon_\ell(a) \in \{1,-1\}, \quad \langle a\rangle_{2,\ell} \equiv 1 \pmod 4.}$$

The sign is uniquely determined by the residue modulo four, and cancellation gives uniqueness of the principal factor.

The checked example $a = 3$ modulo eight has sign $7 = -1$ and principal factor 5:

$$7 \cdot 5 = 3 \pmod 8, \qquad 5 = 1 \pmod 4.$$

The code separately proves $1 = -1$ in $\mathbb{Z}/2\mathbb{Z}$ but $1 \ne -1$ in $\mathbb{Z}/4\mathbb{Z}$. Modulo two alone cannot identify the sign. The generic prime-power construction at $p = 2$ only asks for a principal residue modulo two and is not silently substituted for this stronger mod-four factorization.

Principal targets.

`UnitRefinement.omega_compatible, .principal_compatible, .principal_power`

`DyadicUnits.dyadic_unit_decomposition, .decomposition_unique`

4. CRT addresses and the actual packet selectors

Theorem 10.1; manuscript page 22

Let $a \geq 0$, let M be odd, and put $N = 2^a M$. Coprimality is derived from oddness:

$$\gcd(2^a, M) = 1.$$

The construction uses the actual Chinese-remainder ring equivalence

$$\boxed{\mathbb{Z}/N\mathbb{Z} \;\simeq\; \mathbb{Z}/2^a\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z}.}$$

Its bijectivity and component formula are checked. On a natural representative r, the components are precisely its reductions modulo the two factors, not arbitrary coordinate labels.

The algebraic bridge to projectors

For natural indices n, r, the source proves

$$n \equiv r \pmod{2^a M} \quad\Longleftrightarrow\quad \big(n \equiv r \pmod{2^a} \text{ and } n \equiv r \pmod{M}\big).$$

For the original coefficient selector

$$(P_{N,r} f)(n) = \begin{cases} f(n), & n \bmod N = r, \\ 0, & \text{otherwise}, \end{cases}$$

take the canonical label $0 \leq r < N$. The logical equivalence yields the actual equality of sequence operators

$$\boxed{P_{N,r} f = P_{2^a,\, r \bmod 2^a}\big(P_{M,\, r \bmod M} f\big).}$$

This statement holds for coefficients in any type with zero. A separate theorem lifts the equality coefficientwise to the existing formal power-series selectors over any semiring.

The cases $a = 0$ and $M = 1$ are included. The corresponding one-element residue ring is not excluded. Oddness of M already implies positivity.

What is not asserted

CRT factors addresses, not the visibility weights or valuation filtrations associated with those addresses. Nothing in this proof gives an equal-weight decomposition at every prime or an additional analytic convergence theorem.

Coprimality cannot be discarded. The distinct classes 0 and 2 modulo four both map to $(0, 0)$ when both components are reduced modulo two. The source checks this counterexample. It also checks a composite-modulus failure of the Fermat exponent and a coefficient outside the standard window that is not 3-adically integral.

Principal targets.

`CRTPackets.address_bijective,.address_components,.residue_equivalence`

`CRTPackets.packet_factorization,.series_packet_factorization`

`FiniteArithmeticExamples.outside_window_not_integral`

5. Verification, regressions and remaining work

Cumulative release evidence

The fresh build started with an empty output directory under Lean 4.34.0-rc2 and Mathlib revision

`aeb72f1e9981b6d35b426b16dca7759be3e7ba06`.

All nine dependency revisions matched the runtime manifest and repository heads.

Check	Result
Modules compiled, warnings treated as errors	28/28
Named theorem declarations	333
Explicit definitions audited separately	77
Expected axiom reports matched	410/410
Invalid-proof controls rejected	17/17
Infrastructure and release tests	103 passed

Dependencies are among `propext`, `Classical.choice` and `Quot.sound`, or subsets thereof. The source scan excludes admitted proofs, new axioms and selected trust bypasses. The five new rejection controls cover root lifting, primality, dyadic signs, integrality and coprimality.

Independent finite checks

The exact Python mirrors passed 2,132 standard-window coefficient rows, 1,476 prime-power unit cases, 555 exhaustive root-uniqueness checks, 4,594 precision comparisons and 576 multiplicativity comparisons. The dyadic suite checked 454 decompositions and their uniqueness. CRT covered 1,240 addresses and 6,200 selector cases. Thirteen invalid inputs were rejected.

These are independently implemented finite arithmetic checks, not code extracted from Lean or replacements for its universal proofs. Their deterministic mathematical payload is replayed separately from elapsed time in the fresh-extraction release audit.

Failures and preservation

Eleven of 18 development checks failed before repair, on lemma interfaces, coercions, finite cases and simplification. Their inactive snapshots and diagnostics are retained. All 21 upstream proof files and four earlier supplements are unchanged. Edition-specific test updates retain archived originals.

Four manuscript-proved items remain pending

7.5 Global odd-prime Euler finite-difference theorem
8.5 Uniform standard-window period
9.1 Companion jump theorem
9.2 Fibonacci packet recurrence

Three classical inputs and two conjectures remain separate. Mathlib's cache was reused, not independently rebuilt. Earlier papers and historical experiments were not rerun. The preserved manuscript must be read with the cumulative supplements and amendments.

E.6 Sixth Lean Block Supplement

Checkpoint scope. This complete source body retains the status of its dated original supplement. Earlier pending or partial-coverage labels are historical; the final core records their resolution.

Scope and result

This release completes Theorem 9.1 and Corollary 9.2 of the unchanged 28-page status-corrected manuscript. The preceding 28 proof modules and five supplements remain unchanged.

Twenty-two of 24 named manuscript-proved items now have formal coverage; two remain pending. Three classical inputs and two conjectures are tracked separately. The cumulative corpus has **382 named theorem declarations and 84 explicit definitions across 31 modules**.

New module	Scope	Theorems	Defs.
`CompanionPackets`	General jump and recurrence	17	2
`FibonacciPackets`	Actual Fibonacci specialization	18	4
`RecurrenceExamples`	Examples and boundary cases	14	1
Total added	Two named items	49	7

These counts include supporting lemmas and examples, not 49 independent mathematical discoveries. The principal statements are universal over their explicit inputs, not finite numerical observations.

The main mathematical connection

A sampled state $u_k = v_{Nk+j}$ evolves by the actual jump matrix C^N. Its recurrence is derived from the characteristic polynomial of that matrix. No scalar recurrence or spectral decomposition is supplied as a substitute for the desired conclusion.

For Fibonacci numbers, the state is independently defined from Mathlib's actual sequence. The Lucas sequence is independently defined by its initial values and recurrence. The trace and determinant of C^N are then proved to supply the sampled recurrence coefficients.

Important qualification

A d-dimensional state system supplies a recurrence of **order at most** d. Its minimal recurrence need not have order d. The original statement is read as the characteristic-polynomial recurrence, not a minimal-order claim. The formal equations also handle singular matrices, zero-dimensional spaces and zero step.

1. Actual matrix orbits and arithmetic jumps

Theorem 9.1; manuscript page 22

Let R be a commutative ring, I a finite coordinate type and $d = |I|$. For $C \in M_I(R)$ and $v_0 \in R^I$, define

$$\mathrm{orbit}(C, v_0, n) = C^n v_0, \qquad u_k = \mathrm{packet}(C, v_0, N, j, k) = C^{Nk+j} v_0.$$

The implementation uses actual matrix powers and matrix–vector multiplication. A separately supplied sequence v with

$$v_{n+1} = C v_n$$

is proved by induction to equal this power orbit. Thus the source-sequence formulation is connected to the construction.

The jump formula

Associativity and the laws for powers of one matrix give

$$\boxed{u_k = C^j (C^N)^k v_0 = (C^N)^k (C^j v_0).}$$

Only powers of the same matrix are interchanged. The argument does not assume that arbitrary matrices commute. In particular,

$$u_{k+1} = C^N u_k, \qquad u_{k+a} = (C^N)^a u_k.$$

The step matrix is C^N, not C. A checked example with the Fibonacci matrix and $N = 2$ rejects replacing the jump with a single original step.

Refinement of an arithmetic address

Taking another subsequence with step M and offset s yields

$$u_{Mk+s} = v_{(NM)k+(Ns+j)}.$$

The formal proof establishes this identity for the actual packet function by checking the index arithmetic. It is not inferred from a diagram or a naming convention.

Endpoints and interpretation

The algebraic identities allow $N = 0$, in which case $u_k = v_j$ is constant. Positive-modulus residue packets are the special case $N > 0$ and $0 \le j < N$. The formula also permits offsets beyond that standard range. No inverse matrix, diagonalization, eigenvalue, or characteristic-zero assumption is needed.

Principal targets.

`Companion.orbit_of_recurrence`

`Companion.companion_jump,Companion.source_companion_jump`

`Companion.packet_step,Companion.packet_advance`

`Companion.packet_refinement,Companion.packet_zero_step`

2. From the characteristic polynomial to a recurrence

Theorem 9.1, continued

Put $D = C^N$ and let

$$\chi_D(X) = \det(XI - D) = X^d + \sum_{a=0}^{d-1} c_a X^a$$

be its actual characteristic polynomial. The implementation reuses the pinned Mathlib Cayley–Hamilton theorem, expands polynomial evaluation into a finite sum and proves

$$\sum_{a=0}^{d} \chi_D[a] D^a = 0, \qquad \chi_D[d] = 1.$$

Here $\chi_D[a]$ denotes the coefficient at index a. The identities hold over any commutative coefficient ring; the zero-ring case is handled explicitly by the proof's subsingleton branch.

The complete vector recurrence

Apply the matrix identity to the actual state u_k. The already proved advance identity identifies each $D^a u_k$ with u_{k+a}. Consequently,

$$\sum_{a=0}^{d} \chi_D[a] u_{k+a} = 0, \qquad \boxed{u_{k+d} = -\sum_{a=0}^{d-1} c_a u_{k+a}.}$$

This is the desired recurrence, with its monic leading coefficient and all earlier coefficients supplied by χ_D. It is not an assumption on the packet sequence.

Every coordinate and every R-linear scalar observation $\lambda : R^I \to R$ satisfies the corresponding equation:

$$\lambda(u_{k+d}) = -\sum_{a=0}^{d-1} c_a \lambda(u_{k+a}).$$

A separate source-sequence theorem transports the recurrence back to the original v_{Nk+j} notation.

Order versus minimal order

The characteristic polynomial gives a recurrence of order at most d. For example, $C = I_2$ gives a degree-two characteristic polynomial, but every orbit is constant and obeys $u_{k+1} = u_k$. Both statements are checked. No minimal-polynomial classification is claimed.

For $d = 0$, the unique empty vector is zero and the formula remains valid. For singular matrices, Cayley–Hamilton still applies: invertibility is irrelevant. The nilpotent example $C = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ has $C^2 = 0$ and $\chi_C = X^2$.

Principal targets.

`Companion.charpoly_sum_powers`, `Companion.charpoly_coeff_dimension`

`Companion.packet_charpoly_annihilation`

`Companion.packet_charpoly_recurrence`

`Companion.packet_coordinate_recurrence`

`Companion.packet_observable_recurrence`

3. Fibonacci and Lucas coefficients are derived

Corollary 9.2; manuscript page 22

Define the actual integer matrix and state

$$C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad v_n = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}, \qquad v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The values F_n are Mathlib's `Nat.fib`, cast into $\mathbb{Z}$. Its basic recurrence proves $v_{n+1} = Cv_n$. The second coordinate of the packet state is therefore proved to be the intended F_{Nk+j}, not a newly assumed sequence.

An independent Lucas sequence

Define

$$L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_n + L_{n+1}.$$

Direct finite matrix arithmetic proves $C^2 = C + I$. Multiplication by C^n gives $C^{n+2} = C^n + C^{n+1}$. Taking traces shows that $\operatorname{tr}(C^n)$ satisfies the same recurrence and initial values as L_n. Induction yields

$$\boxed{\operatorname{tr}(C^N) = L_N.}$$

The implementation separately proves

$$\det C = -1, \qquad \det(C^N) = (-1)^N, \qquad L_N = 2F_{N+1} - F_N.$$

No Binet roots or diagonalizability argument is needed.

The sampled Fibonacci recurrence

The actual two-by-two characteristic-polynomial formula gives

$$\chi_{C^N}(X) = X^2 - L_N X + (-1)^N.$$

Insert these coefficients into the general packet recurrence and take the second coordinate:

$$\boxed{F_{N(k+2)+j} = L_N F_{N(k+1)+j} - (-1)^N F_{Nk+j}.}$$

The full two-coordinate state satisfies the same equation. All subtraction is in $\mathbb{Z}$, not truncated natural-number subtraction. The result holds for all natural N, j, k, including zero step and zero initial values.

Principal targets.

`FibonacciPackets.state_is_orbit,FibonacciPackets.packet_is_fibonacci`

`FibonacciPackets.trace_power_is_lucas,FibonacciPackets.det_power`

`FibonacciPackets.jump_charpoly`

`FibonacciPackets.fibonacci_packet_recurrence`

4. Exact examples and meaningful boundary checks

Example 9.3 and supporting results

The independently defined Lucas recurrence gives

$$(L_0, L_1, L_2, L_3, L_4, L_5) = (2, 1, 3, 4, 7, 11).$$

For $N = 3$, the general theorem therefore specializes to

$$u_{k+2} = 4u_{k+1} + u_k.$$

The implementation proves this for every offset j and every k, not only the displayed sample. At $j = 2$, direct evaluation of the actual Fibonacci sequence gives

$$(F_2, F_5, F_8, F_{11}, F_{14}, F_{17}) = (1, 5, 21, 89, 377, 1597).$$

This is the exact manuscript example.

The parity sign cannot be removed

For $N = 2$, the recurrence is

$$u_{k+2} = 3u_{k+1} - u_k.$$

At $j = 1, k = 0$, the correct equation is $5 = 3 \cdot 2 - 1$. Replacing the minus sign with a plus would assert $5 = 7$. The source proves the inequality, and a separate invalid-proof control attempts and rejects the false equation.

At $N = 1$ the ordinary Fibonacci recurrence is recovered. At $N = 0$, $L_0 = 2$ and the packet is constant, so the same formula becomes $u = 2u - u$.

Uniqueness from the first two packet values

For a fixed N, j, any integer sequence satisfying the proved sampled recurrence and starting at F_j, F_{N+j} agrees with F_{Nk+j} for all k. This is proved by two-step induction. It is a consequence of the established coefficients and recurrence, not a premise used to identify Fibonacci values initially.

Independent finite mirrors

A pure-Python exact-arithmetic implementation computes characteristic polynomials by the permutation determinant and states by repeated matrix application. It checks 280 matrix systems over integers, rationals and several modular rings, including composite moduli and the zero ring. Separately it computes sampled Fibonacci sequences recursively and compares them with direct Fibonacci evaluation. These are useful regression checks, not Lean extraction or universal proofs.

Principal targets.

`RecurrenceExamples.manuscript_sequence`

`RecurrenceExamples.manuscript_step_three,RecurrenceExamples.step_two`

`FibonacciPackets.fibonacci_packet_unique`

`RecurrenceExamples.even_step_wrong_sign`

`RecurrenceExamples.wrong_jump_matrix`

5. Verification, provenance and remaining work

Cumulative checks and exact scope

Lean. The fresh cumulative run uses Lean 4.34.0-rc2 and Mathlib revision

`aeb72f1e9981b6d35b426b16dca7759be3e7ba06`.

All 31 modules compile with warnings treated as errors. The inventory contains 382 theorem declarations and 84 explicit definitions; all 466 expected axiom reports must match. Allowed dependencies are `propext`, `Classical.choice` and `Quot.sound`, or subsets. The source-policy check excludes admitted proofs, added axioms and selected trust bypasses. All nine package revision pins are checked.

Controls and tests. Twenty invalid-proof controls are expected to fail for mathematical reasons, not missing imports or timeouts. The three new controls test the Fibonacci sign, the jump matrix and the trace coefficient. All 131 infrastructure and release tests pass. These include 28 new sixth-block tests and preserved earlier arithmetic tests.

Independent exact finite check	Cases
Matrix systems across seven coefficient domains	280
Jump identities and characteristic recurrences	1,680 each
One-step packet transitions	1,400
Nested packet addresses	280
Fibonacci packet sequences	357
Fibonacci scalar recurrence steps	11,067
Lucas trace, determinant and Fibonacci identities	81
Invalid numerical inputs rejected	10

Development failures. Four Lean development checks failed before correction. The errors concerned a matrix-associativity step, recursive simplification, coefficient simplification and an ineffective coordinate for a counterexample. Their source snapshots and compiler logs are retained. An initial unit-test run failed two stale baseline-count assertions; those expected counts were updated, not the mathematical tests weakened.

Trust boundary. Pinned Mathlib supplies Cayley–Hamilton, characteristic-polynomial facts, matrix algebra and the basic Fibonacci recurrence. Its compiled cache is reused, not independently rebuilt. Python mirrors do not execute extracted Lean definitions. The historical numerical suite and earlier papers are not rerun. The original manuscript and all 28 upstream proof files remain unchanged.

Remaining pair. Theorem 7.5, the global odd-prime Euler finite-difference estimate, and Corollary 8.5, the uniform standard-window period, remain pending. The two conjectures are still unproved. This block does not use those claims as assumptions.

Detailed compiler logs, inventories, amendments, the completed item crosswalk, and fresh-extraction integrity evidence accompany the release. Manuscript-to-Lean correspondence is a mathematical review, not an automatically certified semantic equivalence.

E.7 Seventh Lean Block Supplement

Checkpoint scope. This complete source body retains the status of its dated original supplement. Earlier pending or partial-coverage labels are historical; the final core records their resolution.

Scope and result

This block completes Corollary 8.5 of the unchanged status-corrected manuscript by an independent finite-field argument. It does **not** assume the still-pending global finite-difference theorem, item 7.5.

Twenty-three of 24 named manuscript-proved items now have formal coverage. The cumulative corpus contains **439 named theorem declarations and 94 explicit definitions across 38 modules**. Three classical inputs and two conjectures remain separately tracked. All 31 upstream proof modules and the six earlier supplements remain unchanged.

New module	Theorems	Definitions
`HurwitzConvolution`	13	3
`HurwitzInverse`	6	1
`StandardWindowPeriod`	8	1
`DifferenceFiltration`	4	0
`GlobalEulerBoundary`	7	1
`DworkCriterion`	7	4
`PeriodExamples`	12	0
Total added	57	10

The counts include supporting lemmas, conditional implications and exact examples. They are not counts of independent discoveries. The conditional implications do not promote their antecedents to proved facts.

The independent route

In characteristic p, both the first and pth coefficient shifts obey a Leibniz rule for Hurwitz convolution. A section supported at multiples of m, with $m \mid p-1$, has equal first and pth shifts. Its constructed reciprocal inherits this equality. Identifying that reciprocal with the existing reduced Euler sequence gives its period on the positive-index tail.

What remains

Theorem 7.5 asks for powers as high as $p^{\min(k,mn)}$ and covers nonstandard residue classes. A period modulo p does not establish that stronger result. This release proves a genuine standard-window mod-p fragment and precise conditional interfaces for the remaining argument, while keeping item 7.5 pending.

1. Coefficient products and reciprocal differentiation

A characteristic-p argument without factorial division

For sequences $f, g : \mathbb{N} \to K$, where K is a field, define

$$(f \star g)(d) = \sum_{i=0}^{d} \binom{d}{i} f(i) g(d-i), \qquad (U^a f)(d) = f(d+a).$$

These are Hurwitz coefficients. The notation does not divide by $d!$ in $\mathbb{F}_p$. In particular, the implementation is not ordinary power-series multiplication with undefined inverse factorials in positive characteristic.

The identity sequence is $\mathbf{1}(d) = \delta_{d0}$. The new source proves the actual higher product rule

$$U^a(f \star g) = \sum_{i=0}^{a} \binom{a}{i} (U^i f) \star (U^{a-i} g).$$

The argument starts with Pascal's identity for the first shift, then inducts on a. All sums are finite at each coefficient.

The prime-shift cancellation

For prime p and characteristic p, every interior binomial coefficient in row p vanishes. Hence

$$U^p(f \star g) = (U^p f) \star g + f \star (U^p g).$$

The first-shift rule has exactly the same form. Both are checked for the literal coefficient product.

Suppose $g(0) = 1$. The source constructs f by the full triangular recurrence

$$f(d) = \delta_{d0} - \sum_{i<d} \binom{d}{i} f(i) g(d-i)$$

and proves $f \star g = \mathbf{1}$. It also proves cancellation: if $a \star g = 0$, then $a = 0$, by induction on the first nonzero coefficient.

If $U^p g = Ug$, differentiate the inverse equation with both operators and subtract. The terms involving $f \star (Ug)$ cancel, leaving

$$(U^p f - Uf) \star g = 0.$$

The proved cancellation law yields

$$\boxed{U^p f = Uf.}$$

The section identity is an explicit premise of this general reciprocal lemma; the next page proves that premise for the specific section used by the Euler sequence.

Principal targets.

`Hurwitz.shift_convolution`, `Hurwitz.prime_shift_convolution`

`Hurwitz.convolution_left_eq_zero`

`HurwitzInverse.inverse_convolution`

`HurwitzInverse.inverse_prime_difference`

2. Identification with the reduced Euler sequence

Corollary 8.5; manuscript page 21

Work over $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. For positive $m \mid p-1$, take the literal section

$$g_m(d) = \begin{cases} 1, & m \mid d, \\ 0, & m \nmid d. \end{cases}$$

It satisfies $g_m(0) = 1$. Since $p \equiv 1 \pmod{m}$, the conditions $m \mid d+p$ and $m \mid d+1$ are equivalent. The source therefore proves $U^p g_m = U g_m$.

Let f_m be the reciprocal constructed on the previous page. A separate strong-induction proof establishes its actual support:

$$m \nmid d \quad \Longrightarrow \quad f_m(d) = 0.$$

This support is not included as an assumption of the construction.

The same residual sequence, not a renamed substitute

Evaluate $f_m \star g_m = \mathbf{1}$ at $d = mn$. Reindex the surviving summation indices by $i = mq$ using the proved support. The result is

$$\sum_{q=0}^{n} \binom{mn}{mq} f_m(mq) = \delta_{n0}.$$

The fifth block independently constructed the unique sequence $e_n^{(m,p)}$ satisfying this triangular system. Applying that uniqueness theorem gives

$$\boxed{f_m(mn) = e_n^{(m,p)}.}$$

Thus the reciprocal really is connected to the manuscript's Lucas-reduced inverse.

The positive-tail period

Write $h = (p-1)/m$. For $n \geq 1$, evaluate $U^p f_m = U f_m$ at the nonnegative degree $mn - 1$:

$$f_m(mn + p - 1) = f_m(mn).$$

Since $mn + p - 1 = m(n+h)$, the identified sequence satisfies

$$\boxed{e_{n+h}^{(m,p)} = e_n^{(m,p)} \quad (n \geq 1).}$$

Induction gives $e_{n+uh}^{(m,p)} = e_n^{(m,p)}$ for every $u \geq 0$. This proves a period, not necessarily the least positive period. The proof also includes $p = 2$, where $m = 1$ and $h = 1$.

Principal targets.

`WindowPeriod.section_prime_shift`

`WindowPeriod.inverse_off_support`, `WindowPeriod.inverse_on_support`

`WindowPeriod.uniform_standard_window_period`

`WindowPeriod.multiple_period`

3. Actual packet congruences and sharp boundaries

Consequences for the standard window; not the full Theorem 7.5

The fifth block constructed integral lifts $\widehat{C}_{mp,j}(n) \in \mathbb{Z}_p$ of the original rational coefficients when $0 \le j < p$, and proved that their residues equal $e_n^{(m,p)}$.

Applying the actual residue map and its kernel ideal to the new period gives

$$\boxed{p \mid \widehat{C}_{mp,j}(n+uh) - \widehat{C}_{mp,j}(n) \quad \text{in } \mathbb{Z}_p}$$

for $n \ge 1$ and $u \ge 0$. Every higher positive difference is a difference of terms already divisible by p. Consequently,

$$p \mid \Delta_{uh}^k \widehat{C}_{mp,j}(n) \qquad (k \ge 1,\ n \ge 1).$$

A ring-homomorphism lemma identifies the image of these differences in $\mathbb{Q}_p$ with the differences of the original rational coefficient function. This is not vacuous field divisibility. The valuation version uses order infinity at zero.

Three checked boundaries

The initial row. For $(p, m) = (3, 2)$, the same reduced recurrence gives $e_0 = 1$ and $e_1 = -1 = 2$ in $\mathbb{F}_3$. Its proposed period is one, so the positive-tail result cannot in general include row zero.

The divisor hypothesis. At $(p, m) = (5, 3)$, where $3 \nmid 4$, the independently computed reduced row is

$$(e_0, e_1, e_2, e_3) = (1, 4, 4, 2).$$

Replacing h by the integer quotient $\lfloor 4/3 \rfloor = 1$ does not give a period: $e_3 \ne e_2$.

No automatic extra power. The original rational triangular coefficients satisfy

$$C_{6,0}(1) = -1, \quad C_{6,0}(2) = 923, \quad v_3\big(C_{6,0}(2) - C_{6,0}(1)\big) = v_3(924) = 1.$$

A period modulo 3 therefore cannot simply be reported as a period modulo 9. This is consistent with the global theorem's proposed first-difference bound; it is not a counterexample to Theorem 7.5.

Principal targets.

`WindowPeriod.standard_window_period_divisibility`

`GlobalEulerBoundary.standard_window_all_orders_mod_p`

`GlobalEulerBoundary.lifted_difference_coe`

`PeriodExamples.period_can_fail_at_zero`

`PeriodExamples.first_difference_precision_is_one`

4. Exact interfaces for the remaining global argument

Conditional results are not counted as completion of item 7.5

Let p be prime and m a positive divisor of $p - 1$. For a sequence e in a commutative coefficient ring, define its actual packed Hurwitz sequence

$$X(d) = \begin{cases} e(d/N), & N \mid d, \\ 0, & N \nmid d, \end{cases} \qquad N = mp.$$

Let $L = p(p-1)$ and $\delta = U^{p+L} - U^p = U^{p^2} - U^p$. The source proves the operator dictionary

$$(\delta^s X)(d) = \Delta_L^s X(d + ps).$$

At the valid nonnegative coefficient degree $d = mpn - ps$, with $s \leq mn$, this becomes

$$(\delta^s X)(mpn - ps) = \Delta_h^s e(n), \qquad h = (p-1)/m.$$

The packed sequence and difference operations are definitions. Their coefficient correspondence is proved before any divisibility assumption is introduced.

A sufficient condition, explicitly conditional

Define flatness by the actual coefficientwise divisibility assertion

$$\delta^s X(d) \in p^s R \qquad (s, d \geq 0).$$

The new theorem proves that this hypothesis implies

$$\boxed{p^{\min(k,mn)} \mid \Delta_h^k e(n).}$$

First, coefficient extraction proves the triangular region $k \leq mn$. A separate induction on difference order proves that this region is equivalent to the complete saturated estimate. This implication is useful over $\mathbb{Z}_p$; interpreting it over a field would not supply the intended integral content.

Normalization is also an explicit implication

Using the existing definition $E_{Nn}^{(N,j)} = j! C_{N,j}(n)$, the source proves that multiplication by $j!$ commutes with every iterated difference. If the unnormalized difference has precision A, the normalized difference has precision $A - v_p(j!)$. Precision is defined as

$$x = 0 \quad \text{or} \quad v_p(x) \geq A,$$

so zero and negative factorial-loss thresholds are handled correctly. For $j < p$, the loss is zero.

Not established here. The general unnormalized Euler family has not been proved to possess the required flat integral lift, especially across singular residue intervals. The retrieved 18-page supporting prose proof uses weighted Amice/Mahler bounds, singular ratios and nonsingular propagation. Those obligations remain, and the full arbitrary-multiple global estimate is not promoted by these interfaces.

5. Verification record and reproducibility boundary

Evidence accompanies the source, not only the summary

The release verifier compiles the cumulative 38-module project into an initially empty output directory, with warnings treated as errors. It checks all **439 theorem declarations and 94 explicit definitions**, requiring all **533 expected axiom reports** and allowing only `propext`, `Classical.choice` and `Quot.sound`, or subsets thereof.

The pinned runtime is Lean 4.34.0-rc2. The Mathlib revision is

`aeb72f1e9981b6d35b426b16dca7759be3e7ba06`.

Nine dependency heads are compared with their recorded manifest pins. The compiled Mathlib cache is reused, not independently rebuilt. The source checker rejects admitted proofs and selected trust-bypass constructs; the transitive axiom audit is a separate check.

Negative controls and infrastructure

The verifier retains its earlier 20 invalid-proof controls and adds three: a false row-zero period, a false nondivisor period, and an unjustified extra power of the modulus. Missing imports, timeouts and compiler panics do not count as successful rejections.

The infrastructure suite contains **162 tests**. New tests verify the one-item coverage transition, preservation of the 31 upstream sources, the explicit conditional flatness premise, resolution of partial targets, and continued pending status of 7.5. Historical census tests use their frozen release snapshots. Current census assertions use the new inventory.

Independent exact finite checks

The new rational and finite-field mirrors passed 4,848 reciprocal coefficient rows, 2,592 higher-Leibniz cases, 4,520 reciprocal prime-shift cases, 3,720 period comparisons, and 1,476 standard-window rational rows. They also checked 1,104 rational period differences, 620 higher differences modulo p, and 586 Dwork extraction interfaces. Eleven invalid input configurations were rejected.

A further 4,096 finite tests of the remaining global bound passed. **These do not prove that bound and do not change the coverage count.** All mirrors are independent Python implementations, not execution extracted from Lean. The original historical numerical suite was not rerun.

Scope, preservation and failures

The original manuscript and all six earlier supplements remain unchanged. The new amendment replaces the period's dependence on the still-pending global theorem with the direct proof. Development failures, source snapshots and logs are kept separately from the release run. A second clean build from a freshly extracted archive is recorded in the companion release audit.

Run `bashRun-Portable.sh/path/to/kit` to repeat the current unit tests, cumulative finite mirrors and Lean verification. Run `pythonscripts/check_manifest.py` to check the delivered file snapshot. The manifest does not claim that every prose sentence is equivalent to its Lean type; that correspondence remains a mathematical review.

E.8 Eighth Lean Block Supplement

Checkpoint scope. This complete source body retains the status of its dated original supplement. Earlier pending or partial-coverage labels are historical; the final core records their resolution.

Scope and result

The all-residue Theorem 7.5 remains pending. This release proves its **full saturated higher-power estimate throughout the standard window**, rather than only a period modulo p. For every prime p, positive $m \mid p-1$, $0 \le j < p$, and natural n, k, u, put $h = (p-1)/m$. Then

$$\boxed{\Delta_{uh}^{k} C_{mp,j}(n) = 0 \quad \text{or} \quad v_p\Big(\Delta_{uh}^{k} C_{mp,j}(n)\Big) \ge \min\{k, mn\}.}$$

The unnormalized family $j!C_{mp,j}(n)$ satisfies the same bound. Both statements concern the original rational coefficient function, identified with its actual integral p-adic lift. They include $p = 2$, $k = 0$, $n = 0$, and the additional harmless case $u = 0$.

Coverage remains 23 of 24 named manuscript-proved items. Three classical inputs and two conjectures remain separate. The new result covers every permissible residue when $m = 1$, but not the range $p \le j < mp$ when $m > 1$.

New module	Theorems	Definitions
`DworkAlgebra`	7	3
`DworkClosure`	20	2
`HurwitzDworkClosure`	15	2
`ResidueZeroDwork`	9	2
`DifferenceStepMultiples`	5	1
`EulerDefect`	4	2
`StandardWindowDwork`	14	1
`StandardWindowPrecision`	5	0
`DworkWindowExamples`	8	0
Total added	87	13

The cumulative corpus has **526 explicit theorem declarations and 107 explicit definitions in 47 modules**. Counts include support lemmas and examples, not independent discoveries. All 38 upstream proof modules and seven earlier supplements are preserved.

1. The characteristic-zero product rule

The normalized defect is not an ordinary derivation

Let A be a commutative algebra over a characteristic-zero field K, and let U be an actual K-derivation. The source defines

$$B = \frac{U^{p^2} - U^{p}}{p}, \qquad M_q(f,g) = \sum_{a=1}^{q-1} \frac{1}{p}\binom{q}{a}(U^a f)(U^{q-a} g), \quad q = p, p^2.$$

The coefficients of each M_q are proved integral: the middle entries of a prime-power binomial row are divisible by p. In Lean they are implemented as natural quotients, with exact division proved before the product identity is used. Induction on derivative order establishes the full Leibniz formula and hence

$$\boxed{B(fg) = B(f)g + fB(g) + M_{p^2}(f,g) - M_p(f,g).}$$

The middle terms are essential. At $p = 3$, $f = t$ and $g = t^2$, the constant coefficient of $B(fg)$ is -2, whereas that of $B(f)g + fB(g)$ is zero. This inequality is itself Lean-checked.

All orders, without a circular closure assumption

Let $I \subset A$ be a subring stable under U. Define

$$\mathrm{Bounded}_r(f) \iff B^s f \in I \ (0 \le s \le r), \qquad \mathrm{Flat}(f) \iff B^s f \in I \ (s \ge 0).$$

The definitions contain no product or inverse closure requirement. Since B commutes with U, derivatives of a bounded element remain bounded. Induction on r in the displayed product rule then proves closure under multiplication. The finite middle sums use the lower-order induction hypothesis, not the desired next-order conclusion.

For an actual inverse $fg = 1$, applying the product rule and multiplying by g gives

$$B(g) = -g^2 B(f) - gM_{p^2}(f,g) + gM_p(f,g).$$

If $g \in I$, induction on r proves boundedness of g from boundedness of f. At each successor stage the preceding stage supplies boundedness of g needed on the right. Passing to all finite orders proves reciprocal closure for an integral inverse.

Principal targets.

`DworkAlgebra.B_mul`

`DworkClosure.bounded_mul,DworkClosure.bounded_inverse`

`DworkClosure.flat_mul,DworkClosure.flat_inverse`

2. The actual integral Hurwitz lattice

Connecting differential expressions to coefficient divisibility

The specialization works inside $\mathbb{Q}_p[[t]]$ and uses factorial-scaled coefficients

$$a_f(d) = d![t^d]f, \qquad I_p = \{f \in \mathbb{Q}_p[[t]] : a_f(d) \in \mathbb{Z}_p \text{ for every } d\}.$$

The source constructs I_p as a subring. Actual exponential-series multiplication gives

$$a_{fg}(d) = \sum_{r=0}^{d} \binom{d}{r} a_f(r) a_g(d-r), \qquad a_{Uf}(d) = a_f(d+1).$$

These identities prove multiplication and derivative stability of the lattice. Membership does not require ordinary power-series coefficients to lie in $\mathbb{Z}_p$.

The reciprocal is genuinely integral

For $f \in I_p$ with constant coefficient one, the usual power-series reciprocal is used. Its inverse equation yields the exact coefficient recurrence

$$a_{f^{-1}}(d) = \delta_{d0} - \sum_{r=0}^{d-1} \binom{d}{r} a_{f^{-1}}(r) a_f(d-r).$$

Strong induction proves that all of these coefficients are integral. Combining this construction with the preceding all-order theorem proves flatness of the *actual reciprocal*. No flatness or integrality of that reciprocal is assumed in the normalized-series specialization.

The earlier coefficient criterion is recovered exactly

Define the literal coefficient defect

$$(\delta a)(d) = a(d + p^2) - a(d + p).$$

The new source proves, for every s, d,

$$a_{B^s f}(d) = p^{-s}(\delta^s a_f)(d).$$

For an integral coefficient x, it proves the bridge

$$\boxed{p^{-s}x \in \mathbb{Z}_p \quad \Longleftrightarrow \quad p^s \mid x \text{ in } \mathbb{Z}_p.}$$

Thus the differential definition is equivalent to the existing coefficient-level Dwork criterion. Division occurs in $\mathbb{Q}_p$, while the assertion after division is membership in $\mathbb{Z}_p$. It is not divisibility in the field. Zero coefficients satisfy every finite precision condition.

Principal targets.

`HurwitzDworkClosure.integral_inverse`

`HurwitzDworkClosure.flat_product`, `HurwitzDworkClosure.flat_reciprocal`

`HurwitzDworkClosure.decode_B_iterate`

`HurwitzDworkClosure.normalized_mem_iff`, `HurwitzDworkClosure.flat_encode_iff`

3. A nonsingular Euler-operator argument

The standard interval without an analytic power-series-in-the-index premise

On Hurwitz coefficients, the actual Euler operator plus c is

$$(\Lambda_c a)(d) = (d + c)a(d).$$

Direct induction proves its commutator with every defect iterate:

$$\delta^s \Lambda_c a = \Lambda_c \delta^s a + sp\big(pU^{p^2} - U^p\big)\delta^{s-1}a, \qquad s \geq 1.$$

Suppose a is integral and supported at degrees divisible by p, and $p \nmid c$. Every supported pivot $d + c$ is proved to be a unit of $\mathbb{Z}_p$. The defect preserves this support. Induction in the commutator identity therefore proves

$$\mathrm{Flat}(\Lambda_c a) \Longrightarrow \mathrm{Flat}(a).$$

This is a reflection theorem on the integral supported domain. It does not assume flatness of the proposed preimage, and does not assert invertibility at a singular pivot $p \mid c$.

Apply it to the explicit normalized sections

Set $N = mp$ and, for $0 \le j < p$, define integral Hurwitz coefficients

$$g_j(d) = \begin{cases} \binom{d+j}{d}^{-1}, & N \mid d, \\ 0, & N \nmid d. \end{cases}$$

The inverse binomial is constructed using the actual unit pivots already proved by Lucas reduction. Its image in $\mathbb{Q}_p$ is proved equal to the displayed reciprocal.

For $j = 0$, g_0 is the literal N-divisibility indicator. Because $N \mid p(p-1)$, the two shifts in δg_0 agree and $\delta g_0 = 0$. It is flat at every order. The actual binomial identity gives

$$\Lambda_{j+1} g_{j+1} = (j+1) g_j.$$

As $1 \le j+1 < p$, the nonsingular reflection theorem inductively proves flatness of every g_j in the standard window. There is no uniform-convergence assumption in this argument.

Principal targets.

`EulerDefect.iterated_commutator`, `EulerDefect.flat_of_euler_flat`

`StandardWindowDwork.kernel_coe`, `StandardWindowDwork.kernel_euler_step`

`StandardWindowDwork.kernel_flat`

4. From these sections to the original Euler packets

Actual reciprocal, actual rational sequence, every order and step multiple

The factorial-ratio identity is proved coefficient by coefficient:

$$\sum_{d \ge 0} g_j(d) \frac{t^d}{d!} = j! \sum_{r \ge 0} \frac{t^{Nr}}{(Nr+j)!}.$$

The right side is precisely the normalized source kernel from the first block, not a replacement family. The earlier rational identity for its reciprocal is transported to $\mathbb{Q}_p[[t]]$. A separate encoding theorem identifies the reciprocal's coefficients with the already constructed $\mathbb{Z}_p$ lifts of the original $C_{N,j}(r)$.

All-order reciprocal closure therefore gives flatness of the actual packed Euler coefficient sequence. The earlier coefficient-extraction theorem now has a *proved antecedent* for every $j < p$, and yields

$$p^{\min(k,mn)} \mid \Delta_h^k C_{mp,j}(n) \quad \text{in } \mathbb{Z}_p, \qquad h = (p-1)/m.$$

A new finite telescoping argument proves the extension to arbitrary natural multiples of h. Its first-step identity is

$$f(n+uh) - f(n) = \sum_{a=0}^{u-1} \Delta_h f(n+ah).$$

Commutation of differences and induction on their order transfer any fixed ideal bound valid on a forward tail. The required precision $\min(k, mn)$ is nondecreasing as the starting row advances. Hence

$$\boxed{p^{\min(k,mn)} \mid \Delta_{uh}^k C_{mp,j}(n) \quad (0 \le j < p).}$$

The scalar-extension bridge identifies these differences with the original rational coefficient function. The rational theorem explicitly allows zero differences, avoiding the library's finite valuation convention at zero. Since $j!$ is a unit for $j < p$, both normalized and unnormalized versions have the same precision.

Scope. The proof includes all primes, all rows and orders, and all natural step multipliers. For $m = 1$, the standard window is the complete allowed residue range. For larger m, residues $p \le j < mp$ are not covered by this theorem. The earlier one-power period is not the source of the new higher-power estimate.

Principal targets.

`StandardWindowDwork.window_is_reciprocal,StandardWindowDwork.standard_window_flat`

`StandardWindowDwork.standard_window_global_bound`

`StandardWindowPrecision.normalized_window_precision`

`StandardWindowPrecision.unnormalized_window_precision`

`StandardWindowPrecision.modulus_p_complete_family`

5. Examples, verification and the remaining gap

The coverage count is unchanged; the unresolved theorem now has a stronger proved part

The original rational recurrence gives

$$C_{6,2}(1) = -\frac{1}{28}, \quad C_{6,2}(2) = \frac{425}{364}, \quad C_{6,2}(3) = -\frac{2039759}{2660}.$$

Consequently, the new source proves

$$\Delta_1^2 C_{6,2}(1) = -\frac{949959}{1235}, \qquad v_3\big(\Delta_1^2 C_{6,2}(1)\big) = 2.$$

This attains the bound $\min(2, 2 \cdot 1) = 2$. A claim of a third power is rejected. The separate cubic example proves that the normalized differential defect is not an ordinary derivation. Controls also reject an asserted unit pivot at $c = p$ and the prime-power binomial-divisibility formula at a composite base.

Recorded checks

The cumulative verifier rebuilds all 47 modules and audits all 633 explicit declarations: 526 theorems and 107 definitions. Warnings are errors, unexpected axiom dependencies are rejected, and 27 deliberately invalid proof controls are tested. The allowed transitive axioms remain `propext`, `Classical.choice` and `Quot.sound`. Exact results, compiler commands and source hashes are in the accompanying machine-readable reports.

All 189 infrastructure and release tests passed. Independent exact-arithmetic mirrors checked 3,040 normalized and 3,040 unnormalized bounds in 95 standard-window families, 6,596 source defects and 6,596 reciprocal defects, 75 product identities, 360 inverse-closure cases, 720 Euler commutators and 150 telescoping identities. They are finite Python mirrors, not code extracted from Lean and not substitutes for the universal proofs.

Failed development snapshots and diagnostics are preserved separately. Historical source files and earlier supplements remain unchanged. Mathlib's pinned compiled cache is reused, not independently rebuilt; the original historical numerical suite is not rerun. The manuscript-to-formal correspondence remains a mathematical review.

What is still unproved

The all-residue Theorem 7.5 requires the additional range $p \leq j < mp$. Its supporting prose proof treats singular ratios at $c = p, 2p, \ldots, (m-1)p$ through weighted Amice estimates and a dual Mahler pairing, then propagates across the intervening indices. Those actual-family estimates and the full remaining propagation are not supplied by this release. A unit argument at $p \nmid c$ cannot be used at $p \mid c$.

Release boundary: 23/24 named manuscript-proved items, with Theorem 7.5 still pending. The two conjectures remain unproved. The proved standard-window result is an infinite family, but no numerical proportion of the remaining work is inferred from the width of that window.

E.9 Singular Amice Supplement

Checkpoint scope. This complete source body retains the status of its dated original supplement. Earlier pending or partial-coverage labels are historical; the final core records their resolution.

Scope and result

Let p be prime, let $m \mid p-1$, and put $N = mp$. For $1 \le a < m$, put $c = ap$ and define the original exponential section and ratio by

$$H_{N,c}(t) = \sum_{n\ge 0} \frac{t^{Nn+c}}{(Nn+c)!}, \qquad R_c(t) = \frac{tH_{N,c-1}(t)}{H_{N,c}(t)}.$$

The quotient is interpreted after removing its forced zero. This block proves, for the actual rational coefficients α_d,

$$R_{ap}(\log(1+T)) = \sum_{d\ge 0} \alpha_d T^d, \qquad \boxed{\alpha_d = 0 \text{ or } v_p(\alpha_d) \ge -\lfloor d/p \rfloor .}$$

Equivalently, every $p^{\lfloor d/p \rfloor}\alpha_d$ belongs to the actual completed ring $\mathbb{Z}_p$. The zero-safe alternative is explicit in the formal rational statement.

This is the weighted estimate of Proposition 4.7 in the supporting argument. It is **not yet the singular higher finite-difference theorem**. The full all-residue Paper IX Theorem 7.5 remains pending, and full manuscript coverage stays at **23 of 24**.

Inventory	v0.8.0	This release
Proof modules	47	57
Explicit theorem declarations	526	630
Explicit definitions audited separately	107	128
Fully covered manuscript items	23/24	23/24

The 104 added declarations include 30 declarations in four previously compiled research modules, now included in the cumulative audit, and 74 declarations in six new modules. They include supporting lemmas and examples, not independent discoveries. All 47 earlier checked proof modules are preserved byte for byte.

No hidden estimate. The final weighted-ratio theorem assumes primality, the modulus divisibility and the specified residue bounds. It does not assume weighted section membership, reciprocal closure, the ratio identity, a Mahler pairing or the desired higher-difference bound. Pages 2–5 describe the constructed objects and proved bridges.

1. Integral cyclic coefficients

Work first in an arbitrary commutative ring R. If $z^p = 1$, set

$$I = (p, z-1), \qquad P = (p).$$

Expanding $((z-1)+1)^p = 1$ and using divisibility of every interior binomial coefficient in prime row p gives

$$(z-1)^p \in PI.$$

Every other degree-p product of the two generators of I also belongs to PI. The formal ideal proof therefore establishes

$$\boxed{I^p \subseteq PI.}$$

For $q = p\lfloor q/p \rfloor + (q \bmod p)$, strong induction and the factorial valuation identity give

$$\boxed{I^q \subseteq P^{v_p(q!)}}.$$

In detail, write $b = \lfloor q/p \rfloor$. Then

$$I^q \subseteq P^b I^b \subseteq P^{b+v_p(b!)} = P^{v_p(q!)}.$$

No integral-domain hypothesis or division in a field is used here.

Now take the integral cyclic group ring

$$R_N = \mathbb{Z}[\mathbb{Z}/N\mathbb{Z}], \qquad x = [1], \qquad x^N = 1,$$

and assume $N \mid p(p-1)$. Thus $(x^{p-1})^p = 1$. Each genuine consecutive block

$$\prod_{i=0}^{p-1} \big(x - (pb+i)\big)$$

belongs to $(p, x^{p-1} - 1)$. This follows from the actual polynomial identity

$$\prod_{i \in \mathbb{F}_p} (X - i) = X^p - X$$

modulo p, not from a hypothesized valuation of the product.

For the falling factorial $(x)_d = \prod_{i=0}^{d-1}(x-i)$, grouping into consecutive blocks gives

$$(x)_d \in I^{\lfloor d/p \rfloor} \subseteq \big(p^{v_p(\lfloor d/p \rfloor !)}\big).$$

Every integral residue coordinate therefore satisfies

$$\boxed{p^{v_p(\lfloor d/p \rfloor !)} \mid [(r)](x)_d \quad \text{in } \mathbb{Z}.}$$

Dividing a coordinate by $d!$ is done only afterward, using $v_p(d!) = \lfloor d/p \rfloor + v_p(\lfloor d/p \rfloor !)$. This establishes the weighted denominator estimate before any field-valued divisibility could make it vacuous.

Modules. `CyclotomicFiltration`, `PrimeBlockPolynomial`, and `CyclicAmiceCoefficients`.

2. Connection to the actual section

An integral coordinate estimate becomes relevant only after connecting it to the manuscript's generating functions. For a residue $r \in \mathbb{Z}/N\mathbb{Z}$, define

$$H_r(t) = \sum_{d \geq 0} \mathbf{1}_{d \equiv r \ (N)} \frac{t^d}{d!}, \qquad F_r(T) = H_r(\log(1+T)).$$

The implementation uses the actual formal logarithm and substitution in $\mathbb{Q}[[T]]$. It proves

$$H_r' = H_{r-1}, \qquad (1+T)F_r' = F_{r-1},$$

and hence the coefficient recurrence

$$(d+1)[T^{d+1}]F_r = [T^d]F_{r-1} - d[T^d]F_r.$$

The initial values are $F_r(0) = \mathbf{1}_{r=0}$.

Independently, let $b_{d,r}$ be the residue-r coordinate of $(x)_d$ in R_N. Actual group-ring multiplication gives

$$b_{0,r} = \mathbf{1}_{r=0}, \qquad b_{d+1,r} = b_{d,r-1} - d\, b_{d,r}.$$

Induction comparing these two recurrences proves

$$\boxed{d![T^d]H_r(\log(1+T)) = b_{d,r}.}$$

Thus the cyclic coefficient is not an unrelated stand-in for a section coefficient. For the canonical label $0 \leq c < N$, the formal section is also proved equal to the original `ExponentialSection.originalSection` used in the existing Euler project.

Weighted membership

The class used in the formalization is

$$\mathcal{W}_p = \left\{ F \in \mathbb{Q}_p[[T]] : p^{\lfloor d/p \rfloor}[T^d]F \in \mathbb{Z}_p \text{ for every } d \right\}.$$

The integral-coordinate estimate from page 2 now proves

$$\boxed{H_r(\log(1+T)) \in \mathcal{W}_p}$$

for every residue r, including the singular ones. Rational coefficients are mapped into $\mathbb{Q}_p$ by the actual coefficient ring homomorphism.

The same source package retains the earlier independent formal identity

$$d![T^d]\exp(x\log(1+T)) = (x)_d$$

over a commutative $\mathbb{Q}$-algebra. The section bridge above does not rely on selecting or averaging roots in a cyclotomic extension.

Principal targets.

`factorial_coeff_section`; `section_weighted_integral`;

`weighted_section`.

3. Weighted closure and the normalized denominator

Let $w(d) = \lfloor d/p \rfloor$. Since $w(i) + w(d-i) \leq w(d)$, the coefficient formula for a product proves closure of $\mathcal{W}_p$ under multiplication. Additive closure is immediate. For $U \in \mathcal{W}_p$ with $U(0) = 1$, strong induction in the actual reciprocal coefficient recurrence proves

$$\boxed{U^{-1} \in \mathcal{W}_p.}$$

This inverse property is a theorem, not an additional field in a structure encoding weighted series.

Differentiation and the Euler field

The derivative coefficient is $(d+1)a_{d+1}$. If $p \nmid d+1$, the adjacent weights agree. If $p \mid d+1$, the integer factor $d+1$ compensates for the one-unit weight increase. Therefore ordinary differentiation preserves $\mathcal{W}_p$.

For the actual logarithm, its coefficient denominator is d, and the proved inequality

$$v_p(d) \le \lfloor d/p \rfloor \qquad (d > 0)$$

implies $\log(1+T) \in \mathcal{W}_p$. Consequently

$$L_T F = \log(1+T)(1+T)F'$$

also preserves $\mathcal{W}_p$.

Removing a singular forced zero

Put $c = ap < N$ and define the actual normalized kernel

$$K_c(t) = c! \sum_{n \ge 0} \frac{t^{Nn}}{(Nn+c)!}, \qquad K_c(0) = 1.$$

Let $\ell(T) = \log(1+T)/T$, defined by shifting logarithm coefficients, not by inverting T. Set

$$U(T) = \ell(T)^c K_c(\log(1+T)).$$

The formal proof establishes $U(0) = 1$ and the exact factorization

$$H_{N,c}(\log(1+T)) = \frac{T^c}{c!} U(T).$$

It follows coefficientwise that

$$[T^d]U = c![T^{d+c}]H_{N,c}(\log(1+T)).$$

Because $p^a \mid (ap)!$ and $\lfloor (d+ap)/p \rfloor = a + \lfloor d/p \rfloor$, the already proved section estimate gives $U \in \mathcal{W}_p$. Thus $U^{-1} \in \mathcal{W}_p$ as well. The factorial divisibility is derived from the pinned Legendre valuation theorem.

Boundary. The original positive-address section has zero constant coefficient. It is not the series to which unit inverse closure is applied.

4. The singular ratio estimate

For a positive address c, the rational ratio is independently constructed from the original kernels:

$$R_c(t) = c\, K_{c-1}(t) K_c(t)^{-1}.$$

The scalar factor c is necessary. The source proves the cleared identity

$$\boxed{R_c(t) H_{N,c}(t) = t H_{N,c-1}(t).}$$

This connects the constructed object to the original unnormalized ratio A_{c-1}/A_c. It does not assign the desired transformed formula as its definition.

After actual logarithmic substitution and the chain rule,

$$R_c(\log(1+T))H_{N,c}(\log(1+T)) = L_T H_{N,c}(\log(1+T)).$$

Insert the proved factorization $H_{N,c}(\log(1+T)) = T^c U/c!$. Leibniz differentiation and cancellation of that nonzero section give

$$\boxed{R_c(\log(1+T)) = c(1+T)\ell(T) + (L_T U)U^{-1}.}$$

All differentiation, substitution, nonzeroness and cancellation obligations are proved in the source.

For $c = ap$, the second summand belongs to $\mathcal{W}_p$ by the closure results on page 4. The first does too: shifting the logarithm coefficient loses at most one unit of the floor weight, so $p\ell(T) \in \mathcal{W}_p$, and the integer a supplies the remaining factor. Therefore

$$\boxed{R_{ap}(\log(1+T)) \in \mathcal{W}_p.}$$

The final rational coefficient theorem states the bound with an explicit zero alternative. No finite-valued default valuation at zero is used as the meaning of unlimited precision.

Checks that prevent a normalization mistake

The source proves that the actual ratio has constant coefficient c, not one. In the concrete case $N = 6, c = 3$, the universal theorem gives weighted membership and the constant coefficient is proved to be 3.

It also derives $[T^3]H_{3,0}(\log(1+T)) = 1/6$ from the actual differential recurrence and checks its valuation $v_3(1/6) = -1$. Thus weighted membership does not mean ordinary coefficient integrality. This value attains the allowed floor weight at degree three.

Principal target. `SingularRatioAmice.paper_singular_amice_estimate`. The theorem's assumptions contain only the declared arithmetic parameter conditions.

5. Verification and the remaining theorem

The cumulative inventory consists of 57 modules, 630 explicit theorem declarations and 128 explicit definitions. The release verifier compiles every module in dependency order into a new directory, treats warnings as errors, and requests a transitive axiom report for all 758 listed declarations. The allowed axioms remain `propext`, `Classical.choice` and `Quot.sound`.

The verifier retains the previous 27 invalid-proof controls and adds two: incorrectly promoting a weighted coefficient to an integral one, and incorrectly normalizing the singular ratio constant to one. Missing imports, panics and timeouts do not qualify as successful negative controls. The preserved source checker rejects proof placeholders, added axioms and selected kernel-bypass constructs.

The Python infrastructure suite now contains 209 tests. Four old cumulative-count assertions initially failed against the extended inventory. Their original files and failed test output are preserved; the historical tests now inspect the frozen eighth-block inventory where appropriate, while current-corpus tests enforce the new counts.

Independent finite regressions

The new exact-rational regression program is a separate implementation, not Lean extraction. Its checked scope is:

Finite check	Count
Integral cyclic-coordinate numerator tests	17,775
Corresponding section-weight tests	17,775
Direct logarithmic composition comparisons	24
Singular ratio families	13
Singular ratio coefficient comparisons	597
Weighted product, inverse and derivative cases	120

For $p = 3, N = 6, c = 3$, this independent computation obtains the degree-six ratio coefficient $1/10080$, with valuation -2, attaining the allowed weight. This particular coefficient calculation is recorded as a finite regression, not as a new Lean theorem. A violating modulus $p = 3, N = 4$ yields a section coefficient of valuation -4 at degree nine, showing that the modulus divisibility hypothesis cannot simply be discarded.

What has not been proved by this block

To obtain the singular higher-difference theorem, the weighted coefficient estimate must still be connected to the dual Mahler polynomial filtration and its integral pairing. The argument must then establish the required ratio flatness, propagate between singular indices, and assemble the original all-residue Euler family. None of these obligations is replaced by the finite regression tests.

The full manuscript register remains 23/24. Neither conjecture is resolved. The original manuscript is unchanged. Mathlib's pinned compiled cache is reused, not independently rebuilt. Frozen compiler logs, source hashes, failed drafts and fresh-extraction results accompany the release; its machine-readable audit is authoritative about the runs actually performed.

E.10 All Residue Completion Supplement

Checkpoint scope. This complete source body retains the status of its dated original supplement. Earlier pending or partial-coverage labels are historical; the final core records their resolution.

Scope and final statement

Let p be a prime, let $m > 0$ divide $p-1$, put $N = mp$ and $h = (p-1)/m$, and let $0 \le j < N$. The original rational packet is defined by

$$\sum_{q=0}^{n} \binom{Nn+j}{Nq} C_{N,j}(q) = \delta_{n0}, \qquad E_j(n) = j!\, C_{N,j}(n).$$

For every $n, k, u \in \mathbb{N}$, the new formal result establishes

$$\boxed{E_j(n) \in \mathbb{Z}_p, \qquad \Delta_{uh}^k E_j(n) \in p^{\min(k,mn)}\mathbb{Z}_p.}$$

Here $\Delta_h f(n) = f(n+h) - f(n)$. The corresponding rational valuation statement is zero-safe:

$$\Delta_{uh}^k E_j(n) = 0 \quad \text{or} \quad v_p\big(\Delta_{uh}^k E_j(n)\big) \ge \min(k, mn).$$

For the normalized coefficients, the exact correction is retained:

$$\boxed{\Delta_{uh}^k C_{N,j}(n) = 0 \quad \text{or} \quad v_p\big(\Delta_{uh}^k C_{N,j}(n)\big) \ge \min(k, mn) - v_p(j!).}$$

The manuscript's odd-prime, positive-u theorem is a specialization. The formal statement also includes zero row, zero order, zero multiplier, and the $p = 2, m = 1$ subcase. This does not assert a different or broader external dyadic conjecture.

Inventory	Prior v0.9.0	Current source corpus
Named manuscript-proved items covered	23/24	24/24
Active mathematical modules	57	70
Explicit theorem declarations	630	753
Explicit definitions audited separately	128	148

The 123 added theorem declarations include supporting lemmas and examples. They are not 123 independent discoveries. Three classical inputs and two conjectures remain separately tracked. The original manuscript and all 57 upstream proof modules are preserved unchanged.

Principal declaration. `KomorebiPaperIX.Theorem75.global_finite_difference` states the integrality and both precision bounds together, for the original coefficient functions. Its hypotheses are primality, $m > 0$, $m \mid p-1$ and $j < mp$, not an assumed flatness or weightedness certificate.

1. Stability of the weighted differential ring

The recovered checkpoint used the actual weighted subring

$$\mathcal{W}_p = \left\{ F(T) = \sum_{d \ge 0} a_d T^d : \; p^{\lfloor d/p \rfloor} a_d \in \mathbb{Z}_p \text{ for every } d \right\}.$$

Its coefficient condition is identified with the one in the previously checked singular-Amice development. The failed `ThetaWeighted` draft is retained separately; the repaired active source checks the complete formal-series statement below.

Put

$$\Theta = (1+T)\frac{d}{dT}, \qquad B_\Theta = \frac{\Theta^{p^2} - \Theta^p}{p}.$$

The coefficient formula

$$[T^d]\Theta F = (d+1)a_{d+1} + d a_d$$

proves $\Theta\mathcal{W}_p \subseteq \mathcal{W}_p$. If the floor weight increases between d and $d+1$, then $p \mid d+1$, which compensates for that increase.

The normalized defect is not assumed to be a derivation. Its proved product rule contains the two middle Leibniz sums:

$$B_\Theta(FG) = (B_\Theta F)G + F(B_\Theta G) + \mathcal{B}_p(F,G).$$

The coefficients of $\mathcal{B}_p$ are the actual integral quotients $\binom{p^2}{a}/p$ and $\binom{p}{a}/p$. Derivative stability and this correction formula show that the elements $F \in \mathcal{W}_p$ satisfying $B_\Theta F \in \mathcal{W}_p$ are closed under the finite ring operations.

The scalar input is proved for every natural b, including nonunits:

$$\boxed{p^2 \mid b^{p^2} - b^p \quad \text{in } \mathbb{Z}.}$$

Because $\Theta(1+T)^b = b(1+T)^b$, the binomial expansion of $T^p = ((1+T)-1)^p$ then proves stability of the divided generator T^p/p. For $d = pq + r$, $0 \le r < p$, every weighted monomial has the exact factorization

$$a_d T^d = (p^q a_d)\left(\frac{T^p}{p}\right)^q T^r.$$

The scalar $p^q a_d$ is integral by the actual weight condition. Thus all weighted monomials, and all finite weighted polynomials, satisfy the defect condition.

Finally, $[T^d]B_\Theta F$ depends only on coefficients through $d + p^2$. Replacing F by this finite prefix proves the result for the entire formal series, without an analytic convergence assumption:

$$\boxed{B_\Theta\mathcal{W}_p \subseteq \mathcal{W}_p, \qquad B_\Theta^s\mathcal{W}_p \subseteq \mathcal{W}_p \ (s \ge 0).}$$

Modules. `WeightedAmiceRing`, `FrobeniusSquare`, `ThetaWeighted`.

2. Actual substitution and singular higher differences

Let

$$\mathcal{E}(F)(t) = F(e^t - 1), \qquad D = \frac{d}{dt}, \qquad B_D = \frac{D^{p^2} - D^p}{p}.$$

All substitutions use Mathlib's actual formal exponential, logarithm and substitution maps. The chain rule gives

$$D\mathcal{E}(F) = \mathcal{E}(\Theta F), \qquad B_D^s\mathcal{E}(F) = \mathcal{E}(B_\Theta^s F).$$

The implementation also proves the inverse-substitution identity

$$\mathcal{E}\big(f(\log(1+T))\big) = f(t).$$

The relevant integral subring is the *Hurwitz* subring

$$\mathcal{H}(\mathbb{Z}_p) = \{f \in \mathbb{Q}_p[[t]] : \ n![t^n]f \in \mathbb{Z}_p \text{ for every } n\}.$$

It is not coefficientwise integrality of ordinary power-series coefficients. The precise connection is

$$n![t^n]\mathcal{E}(F) = [T^0]\Theta^n F.$$

Since every iterate of Θ preserves $\mathcal{W}_p$, the right side is integral. Therefore $\mathcal{E}(\mathcal{W}_p) \subseteq \mathcal{H}(\mathbb{Z}_p)$. Applying the same statement to every $B^s_\Theta F$ proves

$$\boxed{f(\log(1+T)) \in \mathcal{W}_p \quad \Longrightarrow \quad B^s_D f \in \mathcal{H}(\mathbb{Z}_p) \text{ for every } s.}$$

This is the proved transfer that was missing in the earlier attempt. No weighted Mahler pairing is assumed; the formal differential route supplies an alternative bridge.

Application to the actual singular ratios

The preceding v0.9.0 source independently constructs

$$H_{N,c}(t) = \sum_{n\geq 0} \frac{t^{Nn+c}}{(Nn+c)!}, \quad A_c(t) = t^{-c}H_{N,c}(t), \quad R_c = \frac{A_{c-1}}{A_c}.$$

The notation $t^{-c}H_{N,c}$ means removing its proved initial zero coefficients. For each $c = ap$, $1 \leq a < m$, that source proves $R_{ap}(\log(1+T)) \in \mathcal{W}_p$. The new transfer consequently proves flatness of the *same* rational ratio after scalar extension to $\mathbb{Q}_p$.

The actual ratio is also proved to be an expansion in t^N, so its coefficients vanish outside multiples of N. Constructed $\mathbb{Z}_p$ lifts and the established coefficient criterion now give, for its literal Hurwitz coefficients $r_a(n)$,

$$\boxed{\Delta^k_{uh} r_a(n) \in p^{\min(k,mn)}\mathbb{Z}_p.}$$

The separate rational precision theorem is obtained through the coefficient-lift equality. Divisibility in the field $\mathbb{Q}_p$ is never substituted for this assertion.

Modules. `WeightedDworkTransfer`, `SeriesDifferenceBridge`, `RatioSeriesSupport`.

3. Nonsingular propagation without an assumed limit

For consecutive original sections, the already defined ratios satisfy

$$R_c H_{N,c} = tH_{N,c-1}, \qquad H'_{N,c} = H_{N,c-1}.$$

Differentiating the first relation, using the relation for R_{c-1}, and cancelling a section whose leading coefficient is proved nonzero gives the actual Riccati equation

$$\boxed{tR'_c = R_c(1 + R_{c-1} - R_c), \qquad 2 \leq c < N.}$$

It is not an equation imposed on a replacement ratio. Write

$$q = R_{c-1} - (c-1), \qquad y = R_c - c.$$

Both have zero constant coefficient. With $\Lambda_c = tD + c$, the equation becomes

$$\boxed{\Lambda_c y = cq + yq - y^2.}$$

When $p \nmid c$, define the actual inverse coefficientwise:

$$[t^d]\Lambda_c^{-1} f = \frac{[t^d]f}{d+c}.$$

On p-supported degrees, $d + c$ is a genuine p-adic unit. The proof establishes Hurwitz integrality of this solution, its support, both inverse identities, and preservation of full flatness using the earlier all-order Euler commutator theorem. The singular pivots $c = ap$ are never declared to be units.

Now define explicit iterations

$$y_0 = 0, \qquad y_{r+1} = \Lambda_c^{-1}(cq + y_r q - y_r^2).$$

If q is flat and p-supported, every iterate has those properties. The crucial link to the actual ratio is an exact finite-prefix theorem:

$$\boxed{[t^d]y_r = [t^d]y \qquad (d \le r).}$$

Indeed,

$$(cq + zq - z^2) - (cq + yq - y^2) = (z - y)(q - z - y).$$

The second factor has zero constant term. Multiplication therefore raises the established agreement prefix by one; coefficientwise division by $d + c$ preserves that prefix.

For fixed s, d, the coefficient $d![t^d]B_D^s y$ depends only on the coefficients of y through $d + p^2 s$. Choosing the matching finite iterate proves its integrality. This establishes flatness of the actual y, and hence of R_c, without postulating flatness of the solution or invoking an unproved analytic fixed-point theorem.

Modules. `EulerSeriesInverse`, `RiccatiFlatPropagation`, `RatioRiccati`.

4. All-residue assembly and original coefficients

The standard-window source and inverse results supply R_1 and $F_{N,0}$. Induction over $1 \le c < N$ now has two proved cases. At $p \mid c$, the singular weighted estimate and the new substitution theorem give flatness. At $p \nmid c$, the preceding Riccati propagation applies. Thus every actual ratio R_c is flat.

Let $K_j(t)$ be the normalized section kernel with constant coefficient one. The constructed unnormalized Euler series is

$$F_{N,j}(t) = j!\, K_j(t)^{-1}.$$

The source/inverse identities and the factorial successor identity prove

$$F_{N,j+1} = F_{N,j} R_{j+1}, \qquad F_{N,j} = F_{N,0} \prod_{c=1}^{j} R_c.$$

The product formula is not an assembly hypothesis. Proved flatness of all factors and the original product-closure theorem establish flatness of the full family.

Coefficient extraction identifies, literally,

$$(Nn)![t^{Nn}]F_{N,j} = j!\, C_{N,j}(n).$$

The support outside multiples of N is proved separately. The constructed integral lift therefore represents the original rational coefficient function. The existing saturation criterion and the proved step-multiple argument give the ideal bound at every order, row and multiplier. Multiplication by $j!$ commutes with finite differences, giving precisely the displayed normalization loss.

Exact examples outside the standard window

For $(p, m, j) = (3, 2, 3)$, the source derives

$$C_{6,3}(1) = -\frac{1}{84}, \quad C_{6,3}(2) = \frac{703}{5460}, \quad C_{6,3}(3) = -\frac{223199}{7980}.$$

Consequently,

$$\Delta_1^2\big(3!C_{6,3}\big)(1) = -\frac{1464768}{8645}, \qquad v_3\left(-\frac{1464768}{8645}\right) = 2.$$

This attains $\min(2, 2 \cdot 1) = 2$. At the regular address $j = 4$ beyond the singular boundary, the source likewise proves

$$\Delta_1^2\big(4!C_{6,4}\big)(1) = -\frac{102745512}{2377375},$$

and applies the universal exponent-two bound to this actual coefficient function.

The earlier no-loss counterexample is retained: $v_3(C_{6,3}(2) - C_{6,3}(1)) = 0$. Proving the corrected theorem does not make that false normalized claim true.

Modules. `AllRatiosDwork`, `AllResidueEuler`, `Theorem75`, `AllResidueExamples`.

5. Verification record and claim boundary

Each local module is rebuilt in a fresh output directory, with warnings treated as errors. A live `#printaxioms` audit checks every registered declaration, permitting only `propext`, `Classical.choice` and `Quot.sound`, or subsets.

Required release check	Inventory
Local proof modules	70
Named theorem declarations	753
Explicit definitions	148
Expected axiom reports	901
Pinned dependency revisions	9
Deliberate invalid-proof controls	33
Infrastructure and release tests	235

The accompanying JSON verification record, not the existence of this table, determines whether the actual run passed. The final package also includes a fresh-extraction replay record, checksums, the declaration inventory and the updated manuscript coverage ledger.

The negative controls preserve the earlier rejected false equalities, placeholders, invalid normalizations and unit pivots. Four new controls demand an extra power in the sharp singular example, remove the factorial loss, misuse a composite base, or center a ratio at the wrong constant. An infrastructure timeout or missing import is not an accepted negative result.

Independent finite checks

The standard-library Python mirror checks 328 complete residue families for primes $2, 3, 5, 7, 11$, including 233 higher-residue families. It checks 4,264 integral unnormalized coefficients, 22,137 normalized and 22,137 unnormalized difference bounds, and 1,640 comparisons with a separately computed ordinary reciprocal. Further checks cover 108 weighted differential iterations, 205 integer Frobenius-square cases and 180 exact Picard agreement prefixes.

These are finite exact-arithmetic checks, not extracted Lean code and not substitutes for the universal formal proofs. The old normalization counterexample and the new sharp exponent-two example remain visible.

Preservation and limitations

All 57 earlier proof modules and the original 28-page status-corrected manuscript are retained unchanged. The new supplement and amendments give the current proof route. Historical failed drafts and interrupted recovery attempts are retained separately from accepted builds. Historical tests that assert an older unfinished status now read that frozen historical status; current tests require the actual unconditional capstone theorem before accepting completion.

The portable Lean and Mathlib versions are pinned. Mathlib's compiled dependency cache is reused, not independently rebuilt. The historical numerical experiment suite and Papers I–VIII are not rerun. The two native conjectures remain unproved. No independent historical-priority assessment is made. Manuscript-to-Lean correspondence remains mathematical review, not a theorem equating natural-language prose with formal types.

E.11 Nishibuchi conjecture completion

Checkpoint scope. This complete source body retains the status of its dated original supplement. Earlier pending or partial-coverage labels are historical; the final core records their resolution.

1. The complete external statement

For $N > 0$ and $0 \le j < N$, use the original unnormalized convention

$$\sum_{d\ge 0} \mathcal{E}_d^{(N,j)} \frac{t^d}{d!} = \left(\sum_{n\ge 0} \frac{t^{Nn}}{(Nn+j)!} \right)^{-1}.$$

The existing packet convention satisfies $\mathcal{E}_{Nn}^{(N,j)} = j! C_{N,j}(n)$. The formalization defines the degree-indexed coefficients through the actual factorial-scaled series coefficients, proves their packet identification, and checks the cleared generating identity with the original exponential section.

Theorem E.1. *Let p be prime, $m > 0$, and assume $m \mid p-1$ or $m = 2$. Put $N = mp$ and $q = \operatorname{lcm}(2, p-1)$. Every $\mathcal{E}_{Nn}^{(N,j)}$ is p-adically integral. For every $r \ge 1$,*

$$\boxed{\mathcal{E}_{Nn+qp^r}^{(N,j)} \equiv \mathcal{E}_{Nn}^{(N,j)} \pmod{p^r} \qquad (0 \le j < N,\ n \ge r).}$$

In particular, the positive sufficient threshold $n_0 = r$ works uniformly in all allowed parameters.

This is the parameter range and displacement of Nishibuchi's Conjecture 3.9 in arXiv:2605.10677v2. The explicit threshold also establishes the suggested bound $n_0 \le r$. The original paper already proves the $p = 2, m = 2, j = 0$ period as a special case, with all starting rows; this extension does not claim that special case as a new discovery.

Principal declaration

`KomorebiPaperIX.NishibuchiCompletion.conjecture_3_9`

The theorem type includes only the stated arithmetic hypotheses. There is no assumed congruence, weighted estimate, or flatness certificate. Precision is the zero-safe predicate $x = 0$ or $v_p(x) \ge r$. Its connection to genuine ideal divisibility in $\mathbb{Z}_p$ is used in the proof.

Claim boundary. This establishes a sufficient threshold and a period, not the earliest starting index, the least period, or a pointwise optimal valuation formula. The correspondence with the externally written statement is a mathematical review of definitions and quantifiers, not an automated semantic-equivalence theorem. The two native conjectures in Paper IX remain unchanged and unproved.

2. The missing dyadic branch

For odd primes, the extra option $m = 2$ already divides $p - 1$. Therefore the only allowed family outside the preceding all-residue theorem is $p = 2, m = 2$, with $N = 4$.

The new proof establishes a higher-difference estimate for every one of its four addresses:

$$\boxed{v_2\Big(\Delta_u^k C_{4,j}(n)\Big) \ge \min(k, n) \qquad (0 \le j < 4).}$$

All natural u, k, n are allowed, including zero step, zero order and zero differences. The same bound holds for $j!C_{4,j}(n)$. Both a genuine $\mathbb{Z}_2$ divisibility theorem and a valuation theorem for the original rational sequence are included.

An explicit source instead of an assumed regularity condition

Let $g_j(n) = \binom{4n+j}{4n}^{-1}$. Its four formulas are

$$\begin{aligned} g_0(n) &= 1, \\ g_1(n) &= (4n+1)^{-1}, \\ g_2(n) &= ((4n+1)(2n+1))^{-1}, \\ g_3(n) &= 3((4n+1)(2n+1)(4n+3))^{-1}. \end{aligned}$$

Every denominator is odd. The code constructs the actual unit inverses in $\mathbb{Z}_2$ and proves these identities against the original binomial coefficients, including $n = 0$.

Define a Kummer sequence over a commutative ring by

$$\mathcal{P}_a(f) \quad \Longleftrightarrow \quad a^k \mid \Delta_1^k f(n) \text{ for all } k, n.$$

The exact discrete product rule

$$\Delta_1^s(fg)(n) = \sum_{i=0}^{s} \binom{s}{i} \Delta_1^i f(n)\, \Delta_1^{s-i} g(n+i)$$

is proved by induction. It proves closure under products. Applied to $fg = 1$, the same formula isolates $\Delta_1^s g(n) f(n+s)$; induction and multiplication by $g(n+s)$ prove closure under actual pointwise inverses.

An affine sequence whose slope is divisible by a satisfies $\mathcal{P}_a$. Consequently the four displayed source sequences satisfy $\mathcal{P}_2$. No numerical interpolation or finite set of samples is used in that deduction.

Source modules: `KummerSequences`, `DyadicFourKernel`.

3. The dyadic differential operator and reciprocity

The operator required here is

$$\boxed{B_4 = \frac{D^8 - D^4}{2}, \qquad D = \frac{d}{dt}.}$$

It is not the preceding $p = 2$ specialization $(D^4 - D^2)/2$. It is also not an ordinary derivation. The new algebra module proves the actual product formula, including the middle derivative sums whose coefficients are $\binom{8}{i}/2$ and $\binom{4}{i}/2$.

Write $\mathcal{H}_{\mathrm{int}}$ for the subring of $\mathbb{Q}_2[[t]]$ whose factorial-scaled coefficients belong to $\mathbb{Z}_2$. Flatness means $B_4^s F \in \mathcal{H}_{\mathrm{int}}$ for every s. The proof of multiplicative and reciprocal closure is induction on s, using the full product correction. It separately proves that the actual reciprocal has integral Hurwitz coefficients when $F(0) = 1$.

Pack g_j into degrees divisible by four and form its actual EGF. The formal coefficient identity is

$$G_j(t) = \sum_{n \geq 0} g_j(n) \frac{t^{4n}}{(4n)!} = j! \sum_{n \geq 0} \frac{t^{4n}}{(4n+j)!}.$$

Its inverse is therefore the original normalized packet EGF. The source Kummer property gives flatness of G_j. The proved reciprocal closure gives flatness of G_j^{-1}.

For a packed coefficient sequence e, the exact extraction formula is

$$\left(U^8 - U^4\right)^k \operatorname{pack}_4(e)(4n - 4k) = \Delta_1^k e(n), \qquad k \le n.$$

Here U is the coefficient shift. The nonnegative extraction index is why the first estimate covers $k \le n$. The existing saturation lemma extends it to all k with exponent $\min(k, n)$, and the step-multiple lemma extends step one to every natural u.

Identity with the original rational coefficients

The integral lift is constructed from the decoded coefficients of the actual reciprocal. The code proves

$$\operatorname{coe}(e_j(n)) = C_{4,j}(n) \quad \text{in } \mathbb{Q}_2,$$

and proves that the coefficient map commutes with every difference stencil. Multiplying by $j!$ yields the unnormalized family. The valuation result is not divisibility in a field.

Source modules: `CubicDworkAlgebra`, `CubicDworkClosure`; `DyadicFourDwork` and `DyadicFourCongruences`.

4. From differences to the exact external period

Let f take values in a commutative ring and suppose

$$p^{\min(k,r)} \mid \Delta_h^k f(n) \qquad \text{for every } k.$$

The proved finite Newton expansion is

$$f(n + p^{r-1}h) - f(n) = \sum_{k=1}^{p^{r-1}} \binom{p^{r-1}}{k} \Delta_h^k f(n).$$

For $k \ge r$, the difference already supplies p^r. For $1 \le k < r$, the code proves the integer divisibility

$$p^r \mid \binom{p^{r-1}}{k} p^k.$$

It follows from $k\binom{p^{r-1}}{k} = p^{r-1}\binom{p^{r-1}-1}{k-1}$ and $v_p(k) < k$. Thus every summand is divisible by p^r, including the endpoint $r = 1$.

Divisor branch

For $m \mid p - 1$, the preceding all-residue theorem provides the needed differences as soon as $r \le mn$. With $q = \operatorname{lcm}(2, p - 1)$, the required packet step q/m is proved to be an integer multiple of $(p - 1)/m$. The degree conversion is checked exactly:

$$mp\left(n + p^{r-1}\frac{q}{m}\right) = mpn + qp^r.$$

This retains the stronger sufficient condition $r \le mn$ on the divisor branch. The uniform choice $n \ge r$ implies it for every positive m.

Exceptional branch

For $p = 2, m = 2$, $q = 2$ and the packet displacement is 2^{r-1}. The new dyadic bound supplies the hypothesis when $n \geq r$, and

$$4\left(n + 2^{r-1}\right) = 4n + 2^{r+1}.$$

The $p = 2, m = 1$ branch is also checked using $q = 2$, not an incorrectly substituted $q = p - 1 = 1$.

An exact arithmetic check, not an optimality claim

The independent rational recurrence gives $C_{4,1}(1) = -1/5$ and $C_{4,1}(2) = 121/45$. Their difference is $26/9$, of 2-adic valuation one. This agrees with the new bound at $n = k = u = 1$. The source also contains universal instantiations of the degree-indexed theorem at all four dyadic addresses and at an odd singular address.

Source modules: `PeriodFromDifferences`, `NishibuchiCompletion`, `ConjectureExamples`.

5. Verification, provenance and boundary

The release has 79 active modules: 70 unchanged baseline modules and nine new modules. Its inventory contains 836 named theorem declarations, 162 explicit definitions and two abbreviations. The transitive audit includes all 1,000 of these declarations. The increase of 83 named theorems includes supporting lemmas and examples, not 83 independent discoveries.

A clean cumulative run compiled all **79 modules**, matched all **1,000 axiom reports**, verified all **nine dependency revisions**, and correctly rejected all **37 invalid-proof controls**. The statement-contract file separately checked the external parameter disjunction, integrality, positive threshold, and exact degree displacement. Warnings were treated as errors.

The capstone depends only on `propext`, `Classical.choice`, and `Quot.sound`. The 25 extension-infrastructure tests passed. Exact-rational checks passed 6,500 higher-difference cases in each dyadic normalization, 588 dyadic periods and 1,690 divisor-branch periods. These finite checks are not the universal proof. A fresh-extraction replay, when recorded in the separate release audit, is additional evidence beyond the source build.

Reproduction

The portable runtime is the pinned Linux x86-64 Lean 4.34.0-rc2 kit. It is not duplicated inside this source archive. From the extracted release, run

```
bash Run-Portable.sh /path/to/Komorebi_Portable_Lean_4.34.0-rc2
```

The launcher checks file hashes, runs 25 extension-infrastructure tests and the new exact-arithmetic regressions, then builds every module in a new output directory and audits every registered declaration. It also compiles an explicit statement contract and negative controls. The final check verifies the released snapshot again.

The source-policy scan excludes proof placeholders and additional project axioms. Compiler errors, timeouts, missing imports and invalid negative-control preludes are not accepted as successful theorem verification. Failed development snapshots are retained separately from active proof sources.

Limits of the completion claim

The pinned Mathlib compiled cache is reused, not rebuilt from source. No independent third-party checker or peer review was run. The original numerical experiment suite and Papers I–VIII were not rerun. Prior verification evidence is retained as historical evidence and is not relabeled as fresh testing. The original Paper IX manuscript still requires cumulative supplements and amendments; this extension is not a rewritten primary manuscript.

The external Conjecture 3.9 is distinct from Paper IX's native Conjectures 13.2 and 13.3. Those two remain unproved. The present result does not assert least periods, earliest thresholds, or an optimal exponent for every individual parameter tuple. It makes no historical-priority claim relative to other unpublished work.

Reference

Yuta Nishibuchi. *p-adic Congruences of Generalized Euler Numbers and Relations to Even Zeta Values*. arXiv:2605.10677v2, revised 1 June 2026. Definition 2.1, Conjecture 3.9 and Proposition 3.10. The version identifier is fixed for the statement comparison.

F Worked arithmetic and lifting atlas

These examples connect the coefficient calculus with the correction machinery. They instantiate the statements in the main text rather than add new universal claims. Each displayed rational value, residue, and finite reconstruction is checked by the accompanying exact script.

F.1 Compute the reciprocal before estimating it

For width N and address j, the normalized source and its reciprocal are

$$G_{N,j}(t) = j! \sum_{n\geq 0} \frac{t^{Nn}}{(Nn+j)!}, \qquad G_{N,j}(t)^{-1} = \sum_{n\geq 0} C_{N,j}(n) \frac{t^{Nn}}{(Nn)!}.$$

Multiplying the two series and clearing the factorials gives an exact triangular recurrence:

$$C_{N,j}(0) = 1, \qquad C_{N,j}(n) = -\frac{\sum_{r=0}^{n-1} \binom{Nn+j}{Nr} C_{N,j}(r)}{\binom{Nn+j}{Nn}}.$$

address j	C(0)	C(1)	C(2)
0	1	-1	69
1	1	-1/5	121/45
2	1	-1/15	13/45
3	1	-1/35	59/1155

The table is for $N = 4$. The first two address-zero rows illustrate the computation without any valuation estimate:

$$C_{4,0}(1) = -1, \qquad C_{4,0}(2) = -\left(1 + \binom{8}{4}(-1)\right) = 69.$$

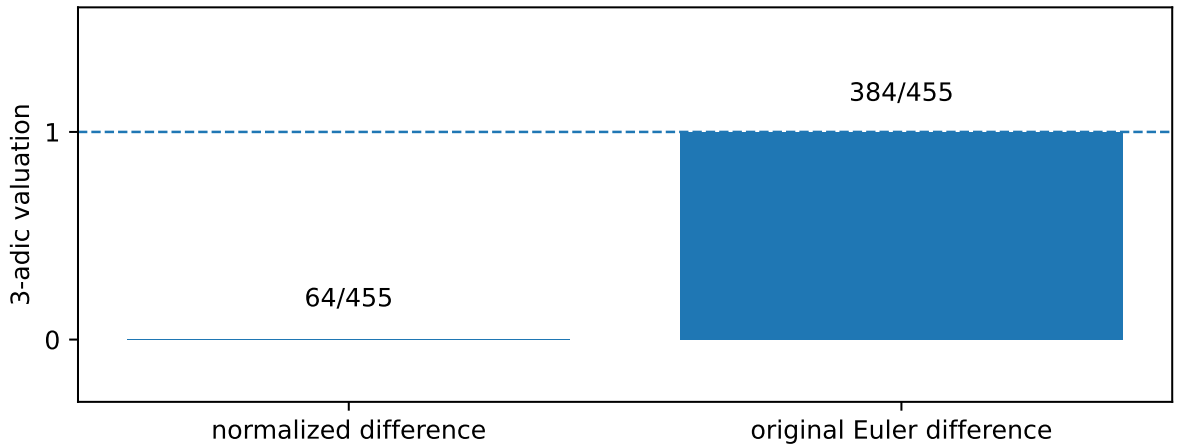
At address one, the same recurrence yields $C_{4,1}(1) = -1/5$ and $C_{4,1}(2) = 121/45$. These are exact rational coefficients, not rounded approximations. The second implementation in the supplied script inverts the ordinary power series first, then multiplies by $(Nn)!$. Agreement between the two constructions checks the factorial placement.

F.2 The normalization loss is a real boundary

The original Euler coefficient is $E_{Nn}^{(N,j)} = j!C_{N,j}(n)$. Multiplication by $j!$ changes the available p-adic precision. That change cannot be discarded outside the unit-factorial window.

Take $p = 3$, $N = 6$, $j = 3$, and compare rows one and two. Exact inversion gives

$$C_{6,3}(1) = -\frac{1}{84}, \quad C_{6,3}(2) = \frac{703}{5460}, \quad C_{6,3}(2) - C_{6,3}(1) = \frac{64}{455}.$$



Neither 64 nor 455 is divisible by three, so the normalized difference has valuation zero. The original coefficient difference is $3! \cdot 64/455 = 384/455$, of valuation one.

This is the counterexample behind the corrected factorial-loss statement. A bound for the original Euler normalization must lose $v_p(j!)$ when translated to $C_{N,j}$. For $0 \leq j < p$, that loss is zero. The distinction concerns which sequence the theorem measures, not a numerical weakness of the calculation.

F.3 Odd primes require the sign twist

The prime-power transfer controls a sign-twisted reciprocal sequence at odd primes. It must not be read as an ordinary-difference theorem for the untwisted coefficients.

At $N = p = 3$ and $j = 0$, put $A(n) = (-1)^n C_{3,0}(n)$. The first three rows are

row n	C at width 3, address 0	(-1)^n C(n)
0	1	1
1	-1	1
2	19	19

The ordinary first difference at row zero is $-1 - 1 = -2$. It is a 3-adic unit, so an asserted lower bound of one for that difference would be false. The twisted first difference is zero, whose mathematical valuation is $+\infty$. At the next row, the twisted difference is $19 - 1 = 18$, of valuation two.

Equivalently, the untwisted sequence is acted on by $T_H = S^H - (-1)^H$, where S advances one row. At odd step one this is $S + 1$, not $S - 1$. The zero in the example is handled by a zero-safe divisibility statement rather than a finite valuation equality. The explicit sign and the zero convention are both part of the theorem's interface.

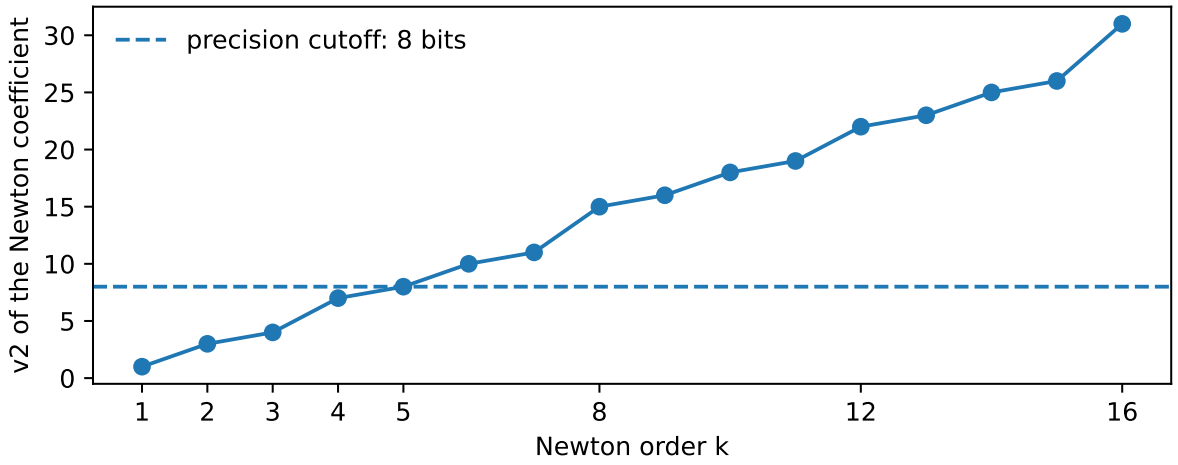
F.4 Newton coordinates turn precision into a finite computation

Let $b_k = \Delta_1^k C_{4,0}(0)$. The first coefficients obtained from the difference triangle are

$$(b_0, b_1, b_2, b_3, b_4) = (1, -2, 72, -33872, 60511872).$$

The exact Newton identity recovers each nonnegative integer row:

$$C_{4,0}(n) = \sum_{k=0}^{n} b_k \binom{n}{k}.$$



For this address the proved law is $v_2(b_k) = 2k - s_2(k)$ for $k \geq 1$. Every $k \geq 5$ therefore has $v_2(b_k) \geq 8$. Since $\binom{n}{k}$ is integral, those terms vanish modulo 256. Thus, for every $n \geq 0$,

$$C_{4,0}(n) \equiv 1 + 254\binom{n}{1} + 72\binom{n}{2} + 176\binom{n}{3} + 128\binom{n}{4} \pmod{256}.$$

The uniform truncation follows from the coefficient theorem. The script separately checks the exact Newton reconstruction and this reduction for rows 0 through 32. Those finite checks validate the worked arithmetic without replacing the theorem.

F.5 A compatible root is built one digit at a time

Take $F(x) = x^2 - 7$ over the 3-adic integers and start with $x_1 = 1$ modulo three. If $F(x_L) = 3^L r_L$, the next representative has the form $x_{L+1} = x_L + 3^L t_L$. Its correcting digit satisfies

$$r_L + 2x_L t_L \equiv 0 \pmod 3.$$

L	3^L	current x	next digit	next x
1	3	1	1	4
2	9	4	1	13
3	27	13	0	13
4	81	13	2	175
5	243	175	0	175

For example, $F(1)/3 = -2$ and $2x_1 = 2$, so the next digit is one and $x_2 = 4$. Then $F(4)/9 = 1$, again giving digit one and $x_3 = 13$. At the next stage the residual is divisible by three, so digit zero is correct. A zero digit is not a stalled construction.

Every center remains congruent to one modulo three. Hence $2x_L$ stays a unit, and the correction equation has a unique digit at every stage. The representatives are compatible and their increments are divisible by 3^L. Completeness gives a limit, and continuity of the polynomial gives a root of F. The finite table exhibits the construction. The unit derivative is what supports indefinite continuation.

F.6 A singular residual can stop a lift

Now take $F(x) = x^2 - 3$ at the residue $x = 0$ modulo three. The residual is $r = F(0)/3 = -1$ and the Jacobian is zero modulo three. The next-digit equation would require

$$-1 + 0 \cdot t \equiv 0 \pmod 3,$$

which has no solution.

digit t	candidate y	y^2 - 3	residue mod 9
0	0	-3	6
1	3	6	6
2	6	33	6

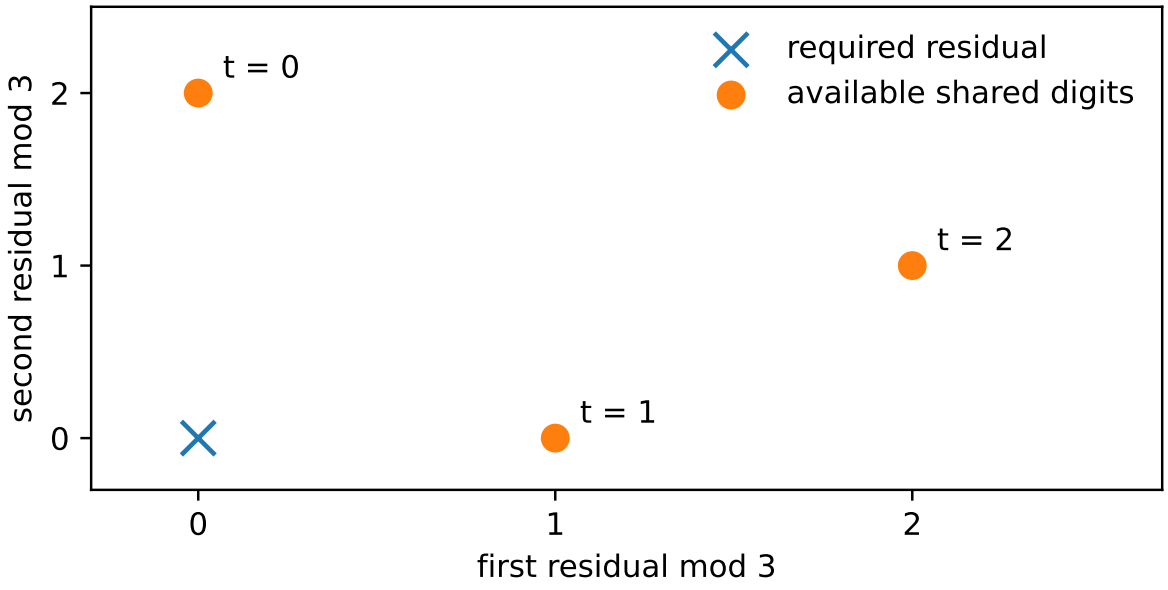
The three representatives of the old root class modulo nine all leave residual six modulo nine. Thus the root class exists at precision three but does not lift to precision nine. This is an exact obstruction, not failure of an iterative search.

A singular Jacobian alone does not imply failure. For $F(x) = x^2$ at the same center, the residual is zero and all three candidate digits lift to roots modulo nine. The obstruction is the residual class in the cokernel of the actual Jacobian. It is this class, rather than the rank by itself, that decides whether a next lift exists.

F.7 Separate corrections need not be one shared correction

Consider the two-component system $F(x) = (x, x-3)$ at $x = 0$, with $q = 3$ and $L = 1$. Both components vanish modulo three. After writing a candidate as $y = 3t$, its residual vector modulo three is

$$\bar{r} + J\bar{t} = (0,2) + (t,t) = (t, t-1).$$



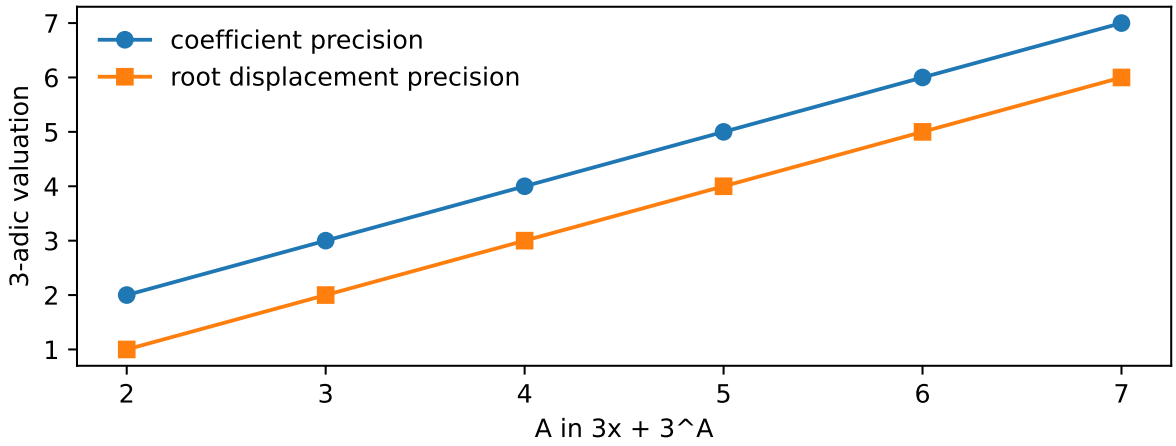
The first coordinate can be corrected by $t = 0$. The second can be corrected by $t = 1$. No digit corrects both. The available residuals are $(0,2)$, $(1,0)$, and $(2,1)$, none of which is $(0,0)$.

The combined Jacobian maps t to the diagonal vector (t,t). Its image is smaller than the product of its coordinate images. Checking the two projected obstructions independently therefore loses the compatibility condition. The common-correction theorem retains the full residual vector and the shared digit space, which is exactly the information needed to detect this failure.

F.8 A derivative weight can cost one digit of root precision

Compare the linear equations $F(x) = 3x = 0$ and $G_A(x) = 3x + 3^A = 0$, where $A \geq 2$. Their coefficient difference has 3-adic valuation A, but their roots differ by

$$x_A - 0 = -3^{A-1}, \qquad v_3(x_A) = A - 1.$$



The derivative has valuation one. Solving the correction equation divides by three, so one digit of coefficient precision is lost in root displacement. The calculation is exact for every displayed A and also follows immediately from the two linear equations.

This example explains why weighted lifting and deformation estimates track a Jacobian loss rather than silently treating every invertible derivative over $\mathbb{Q}_p$ as a unit over $\mathbb{Z}_p$. A valuation-zero derivative would not incur this particular loss. Nonlinear systems additionally require their stated neighborhood, residual, and right-inverse hypotheses.

Reproducibility. The eight examples use exact rational and integer arithmetic in `tools/worked_examples.py`. The release record states the executed finite ranges. This atlas adds no Lean endpoint and changes no formal claim in the companion.

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Recovered historical references. The bibliography underlying Appendix D has been recovered from the original source package. Its 22 citation keys are retained below, with bibliographic identities checked against publisher and author records. This repairs the source attributions; it does not enlarge the historical chapter's stated proof or formalization scope.

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