

CORE SERIES IX

Euler Congruences

Completion, factorial strengthening, and sharp dyadic geometry

Visual guide

Komorebi P.

Monograph reading edition / 10 October 2026

Reading scope

This guide explains the indexing, valuation gains, exact periods, and remaining boundaries through exact arithmetic and theorem-derived figures. The main paper supplies the proofs; the Lean companion supplies the formal source and evidence map.

Contents

1 The object: rows, degrees, and normalization 1

2 What the original conjecture asked 2

3 Removing the starting-row threshold 3

4 Each step contributes its own precision 4

5 The final rank-four congruence 5

6 Why address two has a different slope 6

7 When mixed steps attain the baseline 7

8 First differences determine an exact metric 8

9 Least periods, rather than sufficient periods 9

10 A full period visits every allowed residue 10

11 Seeing the scaled isometry 11

12 How far the simple metric law extends 12

13 From factorial gain to analytic radius 13

14 Which evidence does which job 14

15 Reading the conclusion at the right strength 15

16 Compute the reciprocal before estimating it 16

17 The normalization loss is a real boundary 17

18 Odd primes require the sign twist 18

19 Newton coordinates turn precision into a finite computation 19

20 A compatible root is built one digit at a time 20

21 A singular residual can stop a lift 21

22 Separate corrections need not be one shared correction 22

23 A derivative weight can cost one digit of root precision 23

1 The object: rows, degrees, and normalization

The visual guide accompanies *Euler Congruences: Completion, Factorial Strengthening, and Sharp Dyadic Geometry*. It makes the indexing, precision gains, and boundaries of the results visible. The main paper contains the proofs. The separate Lean guide records which statements are already formalized and which belong to the new mathematical extension.

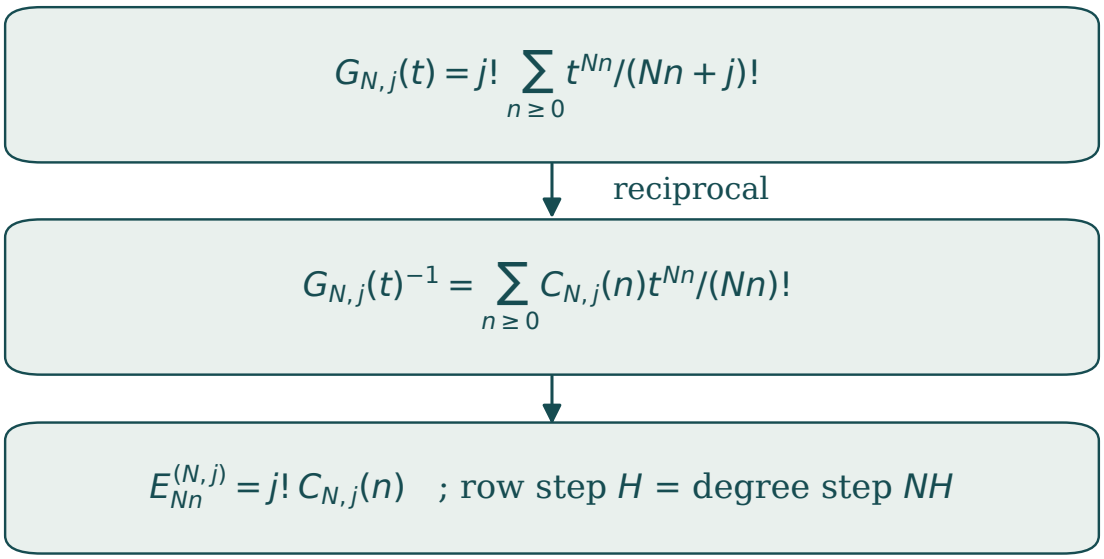


Figure 1: The actual reciprocal series is the source of every coefficient in this guide. The factorial normalization changes valuations as well as the constant term.

There are two coefficient conventions, and they must stay distinct:

$$C_{N,j}(0) = 1, \qquad E_0^{(N,j)} = j!, \qquad v_2(E_{Nn}^{(N,j)}) = v_2(C_{N,j}(n)) + v_2(j!).$$

The last equality uses $v_2(0) = +\infty$. Congruences of dyadic rational numbers mean membership in the corresponding ideal of $\mathbb{Z}_2$; a displayed rational denominator is invertible modulo 2^r when it is odd.

Symbol	Meaning	At rank four
n	packed row index	Euler degree $4n$
H	positive row step	Euler degree step $4H$
k	number of differences	order k, independent of n
j	address within a section	$0, 1, 2, 3$

For example, $C_{4,1}(1) = -1/5$ and $C_{4,1}(2) = 121/45$. Their difference is $26/9$, of valuation one. The row difference joins Euler degrees 4 and 8.

2 What the original conjecture asked

Nishibuchi's Conjecture 3.9 concerns a prime p, a positive integer m with $m \mid p-1$ or $m = 2$, and every address $0 \le j < mp$. Put

$$N = mp, \qquad q = \operatorname{lcm}(2, p-1).$$

It asserts p-adic integrality of the actual generalized Euler coefficients and eventual periodicity at every precision:

$$\text{for every } r \ge 1, \quad E^{(mp,j)}_{mpn+qp^r} \equiv E^{(mp,j)}_{mpn} \pmod{p^r} \quad \text{for all sufficiently large } n.$$

The completed theorem gives the explicit sufficient threshold $n \ge r$ in every permitted branch. Its formal capstone is the theorem named `NishibuchiCompletion.conjecture_3_9`. No conjectural flatness premise is supplied to that theorem.

Prime branch	Allowed m	q	Row step for the original shift
$p = 2$	1	2	2^r
$p = 2$	2	2	2^{r-1}
p odd	$m \mid p-1$	$p-1$	$\dfrac{p-1}{m}p^{r-1}$

For odd primes, $m = 2$ is already in the divisor branch. The conversion is exact:

$$N\left(\frac{q}{m}p^{r-1}\right) = qp^r.$$

At $p = 2, m = 2$, the original conjectured Euler-degree shift 2^{r+1} is the row step 2^{r-1} for the rank-four sequence.

The source already contains special cases. In particular, Nishibuchi's Proposition 3.10 treats the rank-four address-zero dyadic period from row zero and an eventual cubic case. The completion covers the full quantified conjecture; the new dyadic work then removes starting-row restrictions from higher differences and determines substantially more precise laws.

The conjecture's primary source is Yuta Nishibuchi, *p-adic Congruences of Generalized Euler Numbers and Relations to Even Zeta Values*, arXiv:2605.10677v2, revised 1 June 2026. The statement above uses that version's normalization and actual Euler degrees.

3 Removing the starting-row threshold

At rank four the earlier completed estimate was $v_2(\Delta_H^k C_j(n)) \geq \min(k, n)$. The strengthened theorem reaches every starting row, including $n = 0$, and every order, including $k > n$.

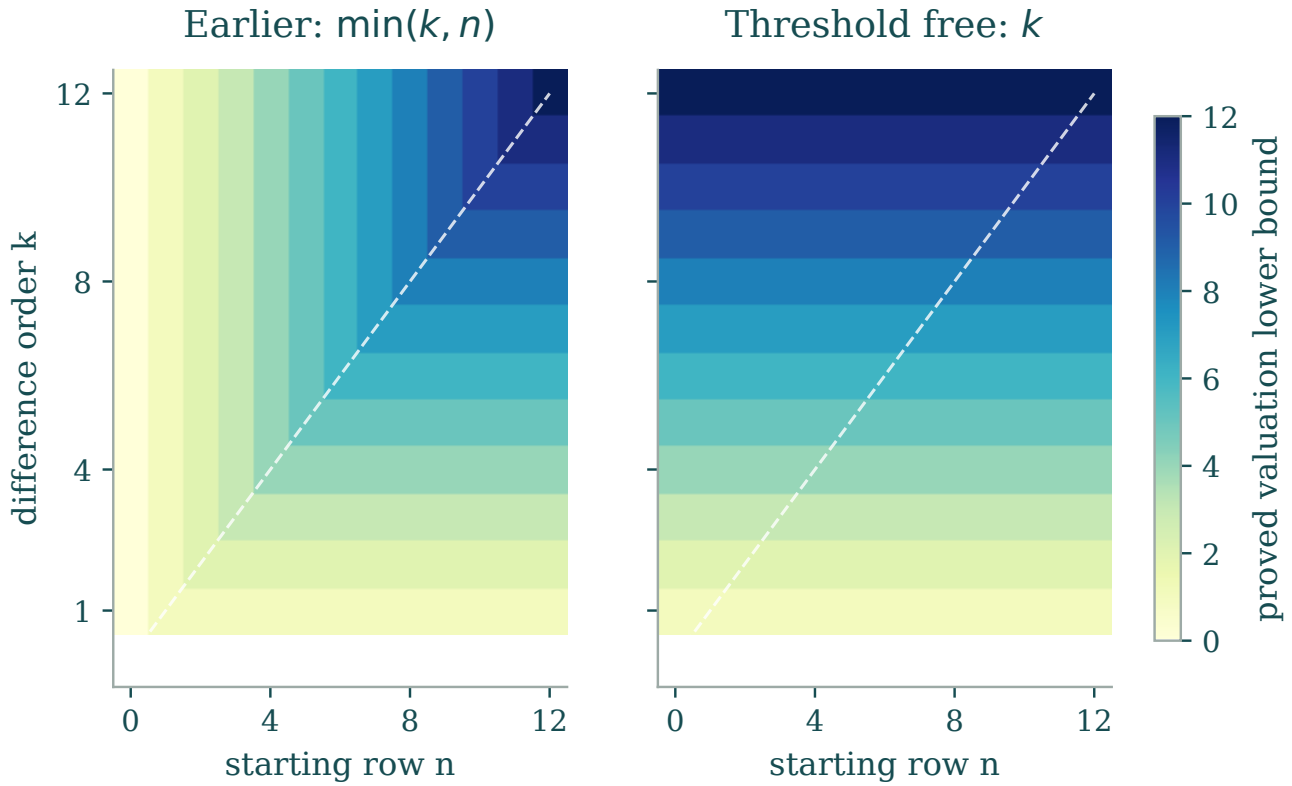


Figure 2: Unit-step lower bounds on a finite display window. Each colored cell is obtained from a proved formula, rather than from sampled coefficients. The dashed line indicates $k = n$.

For *every* dyadic section size $N = 2^a$, $a \geq 0$, every permitted address, and all positive steps,

$$v_2\big(\Delta_{H_1} \cdots \Delta_{H_k} C_{N,j}(n)\big) \geq k + \sum_{\nu=1}^{k} v_2(H_\nu), \qquad n \geq 0.$$

The corresponding statement for $E_{Nn}^{(N,j)}$ adds $v_2(j!)$.

The proof acts on the full factorial-scaled power series before it extracts a packed row. This avoids the old extraction constraint $k \leq n$. Reciprocal induction controls the unique term with the highest inverse order in the commuting product expansion; all other terms are covered by the preceding induction stages.

The figure displays the gain at unit step. An even step contributes additional precision, and the theorem permits different steps in the same difference.

4 Each step contributes its own precision

Write $Sf(n) = f(n+1)$ and $\Delta_H = S^H - 1$. Mixed differences commute. Their exact finite expression is

$$\Delta_{H_1} \cdots \Delta_{H_k} f(n) = \sum_{A \subseteq \{1,\dots,k\}} (-1)^{k-|A|} f\left(n + \sum_{i \in A} H_i\right).$$

Repeated subset offsets are combined with their signed multiplicities.

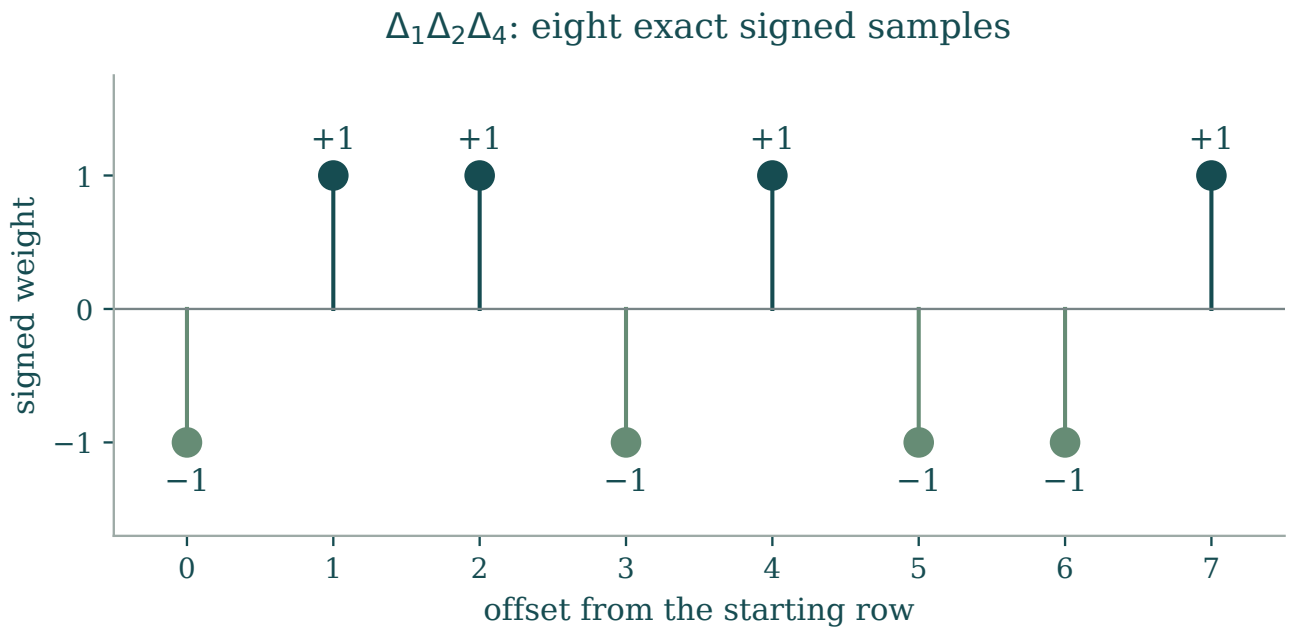


Figure 3: For steps $(1,2,4)$ all eight offsets are distinct. This is an exact signed stencil, rather than an approximate derivative.

For this stencil, the general dyadic bound is

$$3 + v_2(1) + v_2(2) + v_2(4) = 6.$$

At rank four the factorial theorem improves it to seven for addresses zero and one. Exact rational evaluation gives $v_2(\Delta_1\Delta_2\Delta_4 C_0(0)) = 7$. The order is odd, so the mixed-step equality criterion also proves this valuation at *every* starting row.

There is a prime-power extension. For $N = p^a$, $a \geq 1$, the sign-corrected operator is

$$T_H = S^H - (-1)^H, \qquad v_p\Big(T_{H_1} \cdots T_{H_k} E^{(N,j)}(n)\Big) \geq v_p(j!) + \sum_{i=1}^{k} \big(1 + v_p(H_i)\big).$$

Here $E^{(N,j)}(n) = E^{(N,j)}_{Nn}$. For an odd prime, an ordinary first difference need not have positive valuation: at address zero its first two normalized values are 1 and -1, with difference -2. The sign correction is therefore essential to this prime-power theorem.

5 The final rank-four congruence

For $n \geq 0$, $k \geq 1$, and $H \geq 1$, the actual Euler coefficients satisfy

$$\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} E^{(4,j)}_{4(n+iH)} \equiv 0 \pmod{2^{\,v_2(k!)+k v_2(H)+A_j(k)}},$$

where

$$\begin{aligned} A_0(k) = A_1(k) &= k, \\ A_2(k) &= \max\{k+4,\ 2k+\lfloor k/2 \rfloor + 1\}, \\ A_3(k) &= k+2. \end{aligned}$$

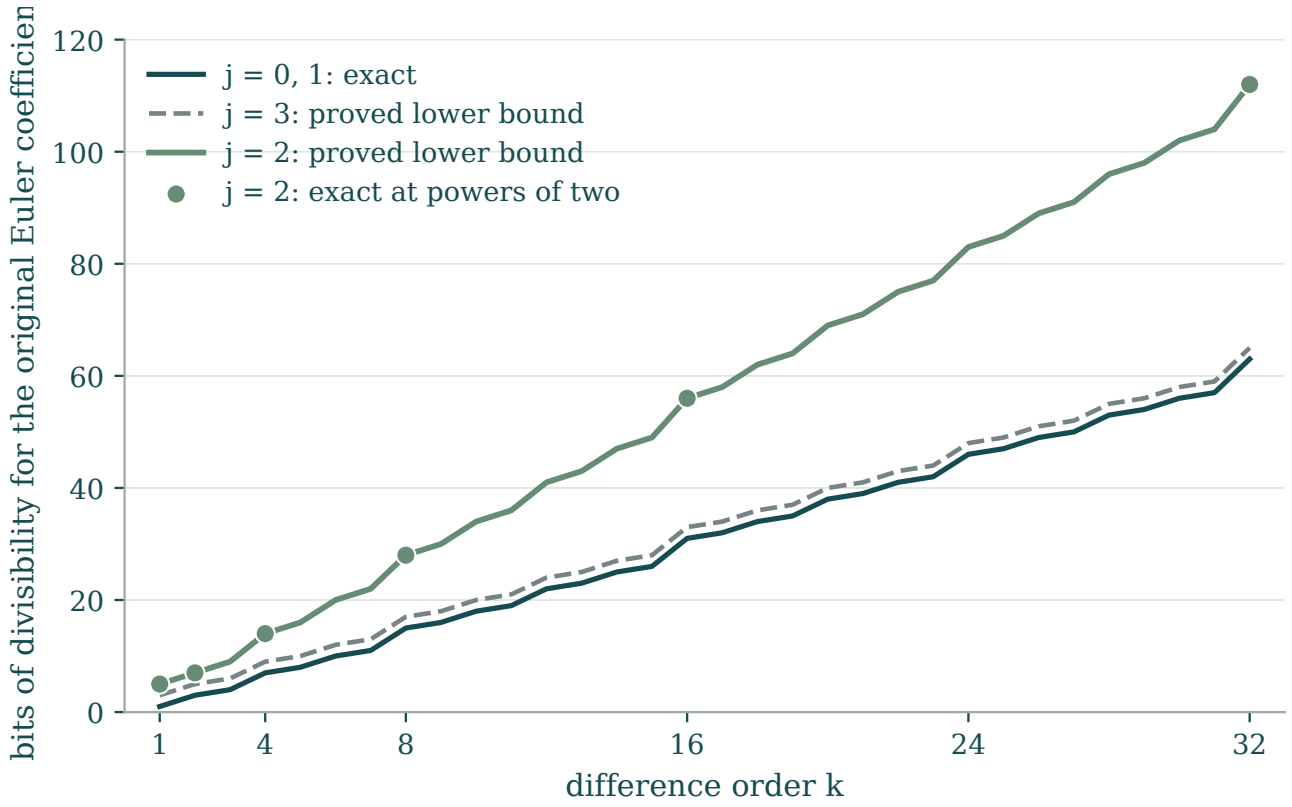


Figure 4: The exponent of the final Euler congruence at unit step. The address zero and one curve is exact at every order. Marked address-two orders are universally exact; the other displayed address-two and address-three values are proved lower bounds.

At a common step with $v_2(H) = s$, add ks to every displayed exponent. Thus a doubling of the step gains one additional bit *per difference*, rather than one bit for the whole expression.

This statement combines two different proved bounds at address two. A lower bound is not automatically an exact valuation: divisibility may be stronger at unmarked orders or at particular starting rows.

6 Why address two has a different slope

The normalized address-two sequence obeys both

$$v_2(\Delta_{H_1}\cdots\Delta_{H_k}C_2(n)) \geq k + v_2(k!) + 3 + \sum_i v_2(H_i),$$

$$v_2(\Delta_{H_1}\cdots\Delta_{H_k}C_2(n)) \geq 2k + v_2(k!) + \lfloor k/2 \rfloor + \sum_i v_2(H_i).$$

Their maximum supplies the useful envelope.



Figure 5: Normalized address-two precision. The paired filtration controls the large-order slope. At order one the constant-surplus bound is stronger.

For every base row and every positive step list, equality is proved when k is a power of two:

$$v_2(\Delta_{H_1}\cdots\Delta_{H_k}C_2(n)) = V_2(k) + \sum_i v_2(H_i),$$

with $V_2(1) = 4$, $V_2(2) = 6$, and $V_2(k) = 2k + v_2(k!) + k/2$ for $k = 2^r \geq 4$. The infinite marked family establishes sharpness at unbounded orders.

For original Euler coefficients add one, since $v_2(2!) = 1$. The extra $\lfloor k/2 \rfloor$ comes from a paired divided-power filtration in the Bernoulli transformation. It does not arise by fitting the plotted points.

7 When mixed steps attain the baseline

For addresses $j = 0, 1$, put

$$V = k + v_2(k!) + \sum_i v_2(H_i), \qquad q = \#\{i : H_i \text{ is even}\}.$$

The valuation equals V exactly when k is odd or q is even.

Order parity	Number of even steps	Conclusion
k odd	either parity	exact valuation V
k even	q even	exact valuation V
k even	q odd	valuation at least $V + 1$

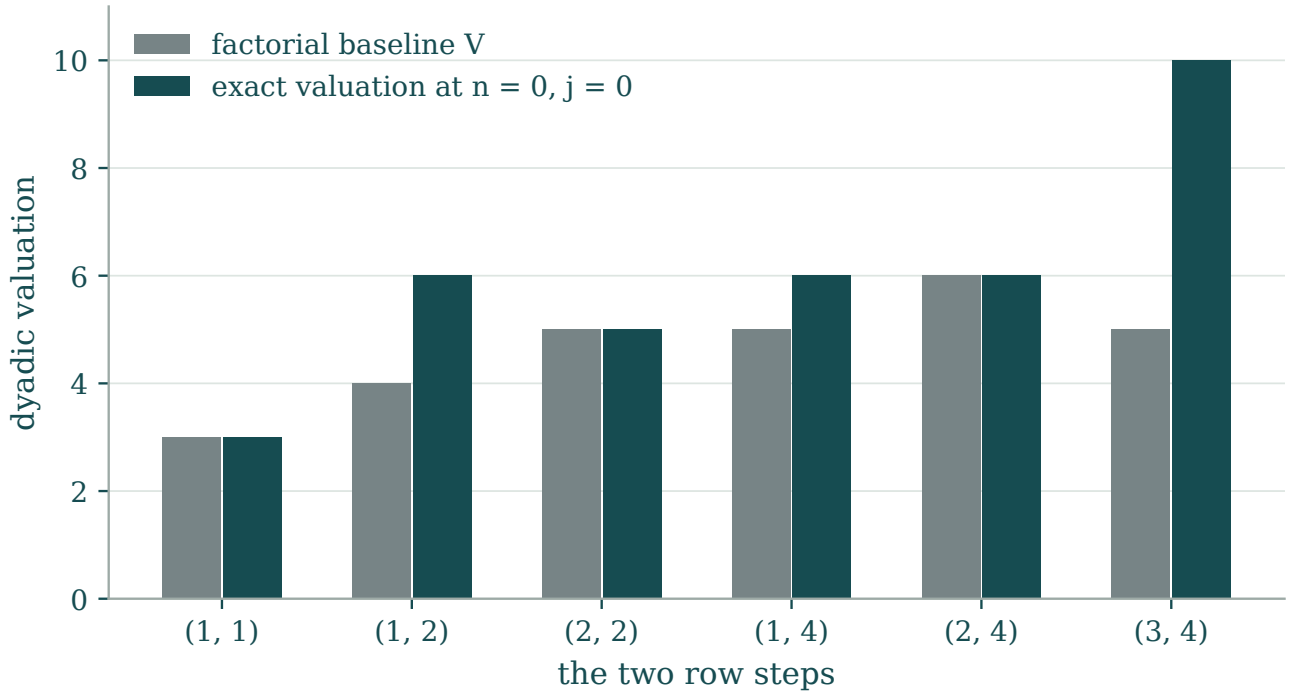


Figure 6: Exact rational examples at order two and base zero. Strict cases show why the factorial baseline cannot always be called an equality for mixed steps. The universal parity criterion is proved separately.

For instance,

$$\Delta_1\Delta_2 C_0(0) = -33728, \qquad v_2(-33728) = 6, \qquad V = 4.$$

Cancellation removes the leading term, and here it removes two bits beyond the baseline. The parity theorem proves strictness in this case; it does not give a complete formula for the further depth.

With a common step, q is either zero or k. Consequently the exceptional parity never occurs, and the homogeneous address-zero/one formula is always exact.

8 First differences determine an exact metric

All four rank-four sequences satisfy the sharp first-difference law

$$v_2(C_j(n+H) - C_j(n)) = v_2(H) + b_j, \qquad b = (1,1,4,2).$$

For the actual Euler coefficients this becomes

$$v_2(E^{(4,j)}_{4(n+H)} - E^{(4,j)}_{4n}) = v_2(H) + B_j, \qquad B = (1,1,5,3).$$

Both laws hold at every starting row and every positive integer step.

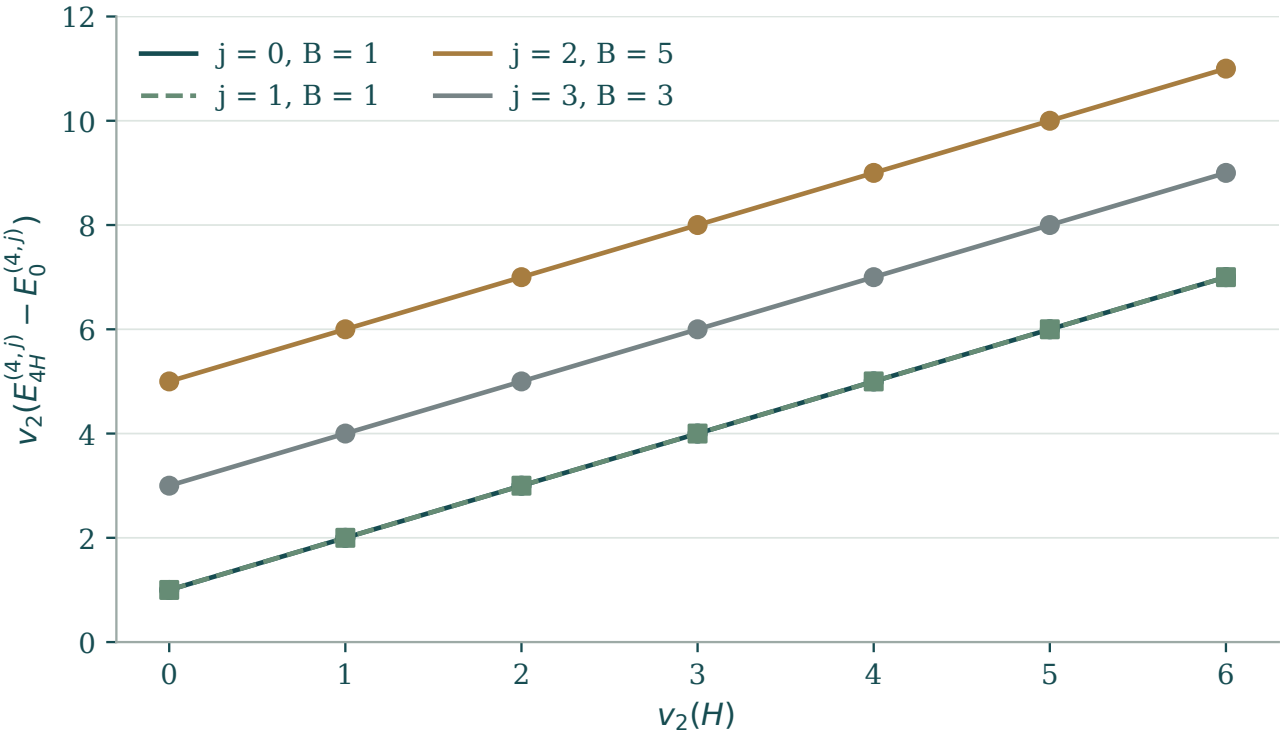


Figure 7: The proved Euler metric laws, with exact modular sample points at base zero and steps $H = 2^s$. Addresses zero and one have the same scale; their circle and square marks overlap.

In dyadic distance, $|x|_2 = 2^{-v_2(x)}$. After continuous interpolation to $\mathbb{Z}_2$, the law reads

$$|E^{(4,j)}(x) - E^{(4,j)}(y)|_2 = 2^{-B_j}|x-y|_2.$$

It says exactly how much the map shrinks distances. It also prevents any hidden collision at finer precision: distinct inputs remain distinct after the known scale is removed.

The scale of address two is 2^{-5} in the original normalization and 2^{-4} in the normalized sequence. Omitting its factor 2! would change the period claimed for the original coefficients.

9 Least periods, rather than sufficient periods

At precision 2^r the least positive *row* period is

$$P_j(r) = 2^{\max(0,r-B_j)}, \qquad B = (1,1,5,3).$$

The exact valuation law proves both periodicity and minimality. A step H is a period precisely when $v_2(H) + B_j \geq r$.

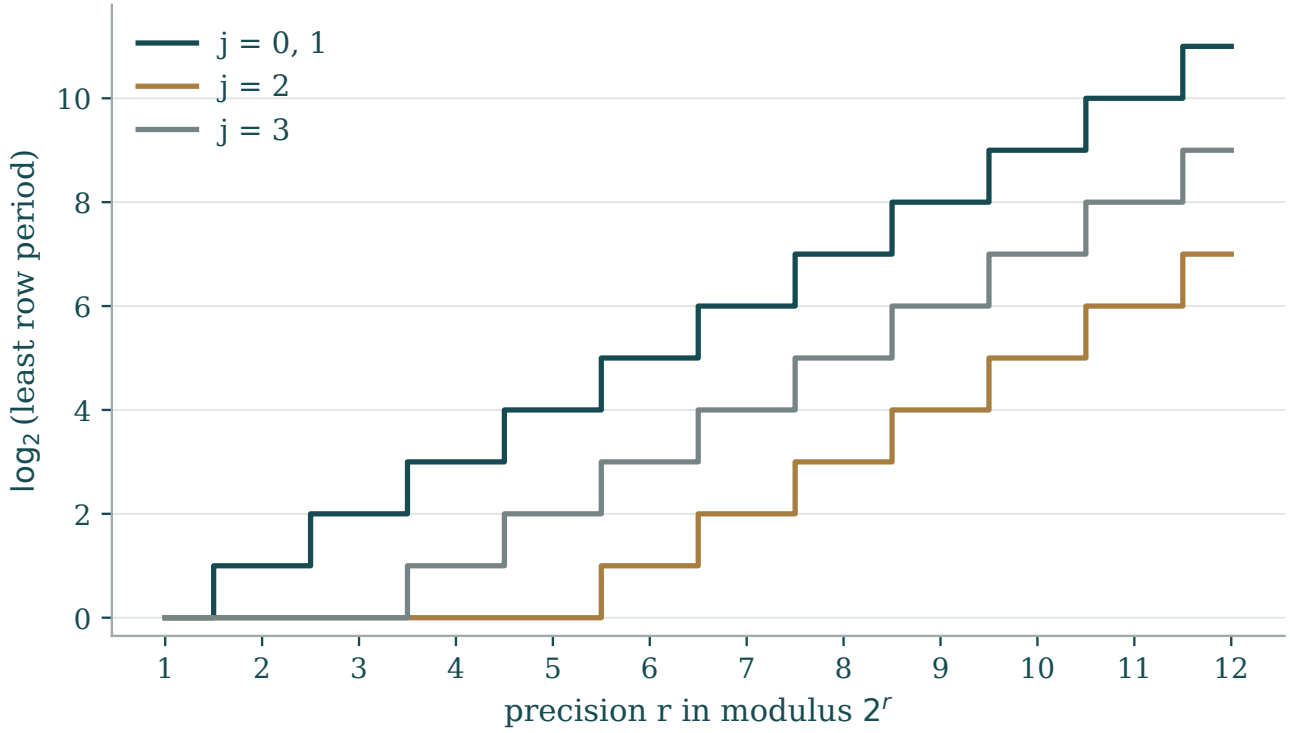


Figure 8: Binary logarithms of the least row periods. A height of zero means period one. These are theorem-derived values, valid at every precision.

Modulo $256 = 2^8$, the least periods are

$$(P_0(8), P_1(8), P_2(8), P_3(8)) = (128, 128, 8, 32).$$

In actual Euler degree, multiply by four:

$$(512, 512, 32, 128).$$

For address two, precision at most 2^5 already has row period one. Its Euler coefficients all lie in $2 + 32\mathbb{Z}_2$; further precision resolves additional dyadic digits within that residue class.

The original rank-four conjectural shift has row step 2^{r-1}. It remains a valid period for every address, but addresses two and three usually admit shorter periods. The sharp theorem identifies the shortest one.

10 A full period visits every allowed residue

For each address the exact interpolation image is

$$E^{(4,j)}(\mathbb{Z}_2) = j! + 2^{B_j}\mathbb{Z}_2.$$

When $r > B_j$, a least period lists every residue in this coset modulo 2^r exactly once. The following figure checks this assertion by exact arithmetic at precision 256.

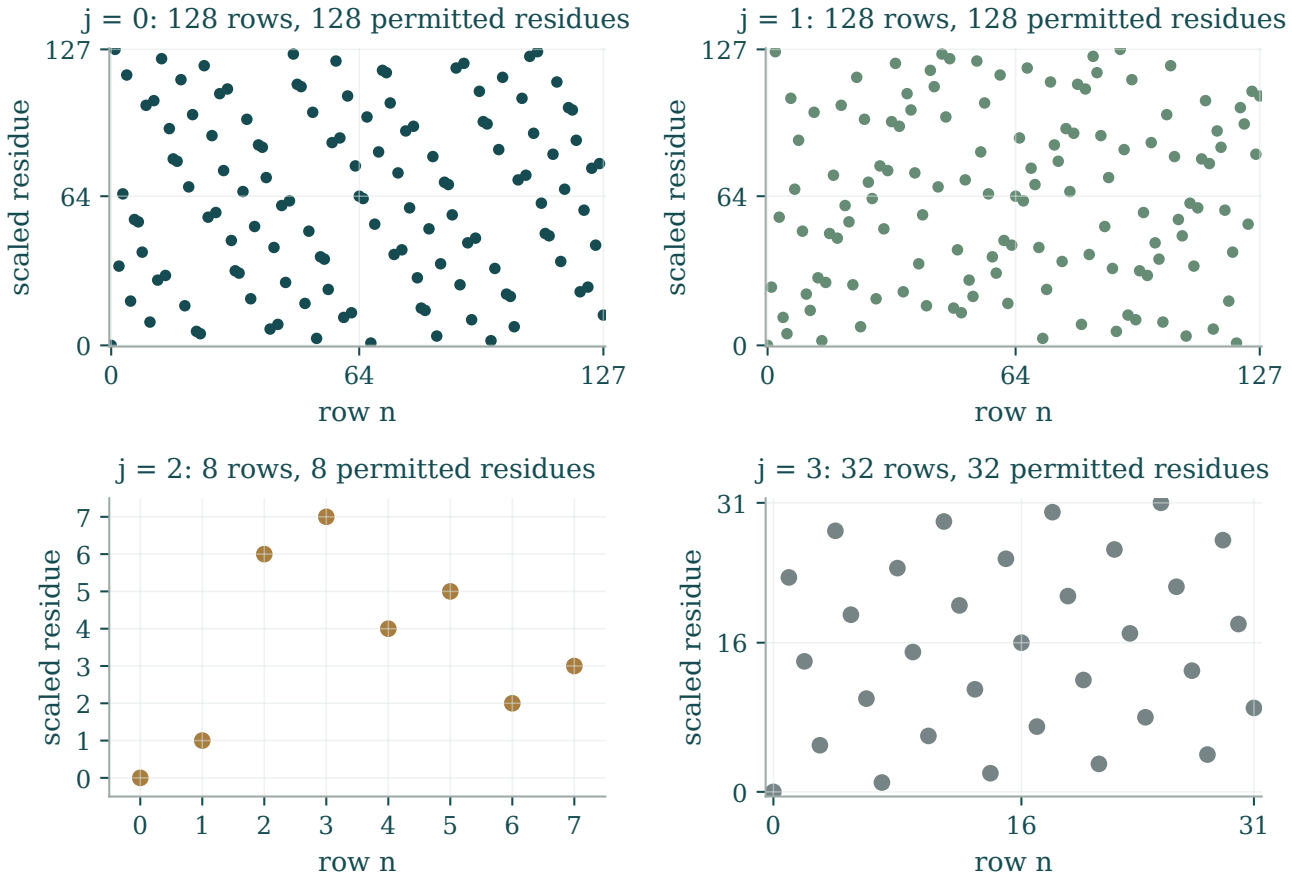


Figure 9: Exact residue permutations modulo 256. The vertical coordinate is $y_n = (E_{4n}^{(4,j)} - j!)/2^{B_j}$ modulo $P_j(8)$. Each panel contains one full least period; every vertical residue occurs once.

No decimal approximation is used to form these residues. The generating series is inverted modulo 2^{256} using odd denominators, then the exact result is reduced modulo 256. An independent rational inversion checks the initial coefficients.

The shapes are permutations, not monotone real functions. Their organizing property is dyadic distance: agreement in input low bits forces the same agreement in the scaled output low bits.

11 Seeing the scaled isometry

The map

$$x \longmapsto \frac{E^{(4,j)}(x) - j!}{2^{B_j}}$$

is a bijective isometry of $\mathbb{Z}_2$. Its finite reductions preserve the number of common low binary digits.

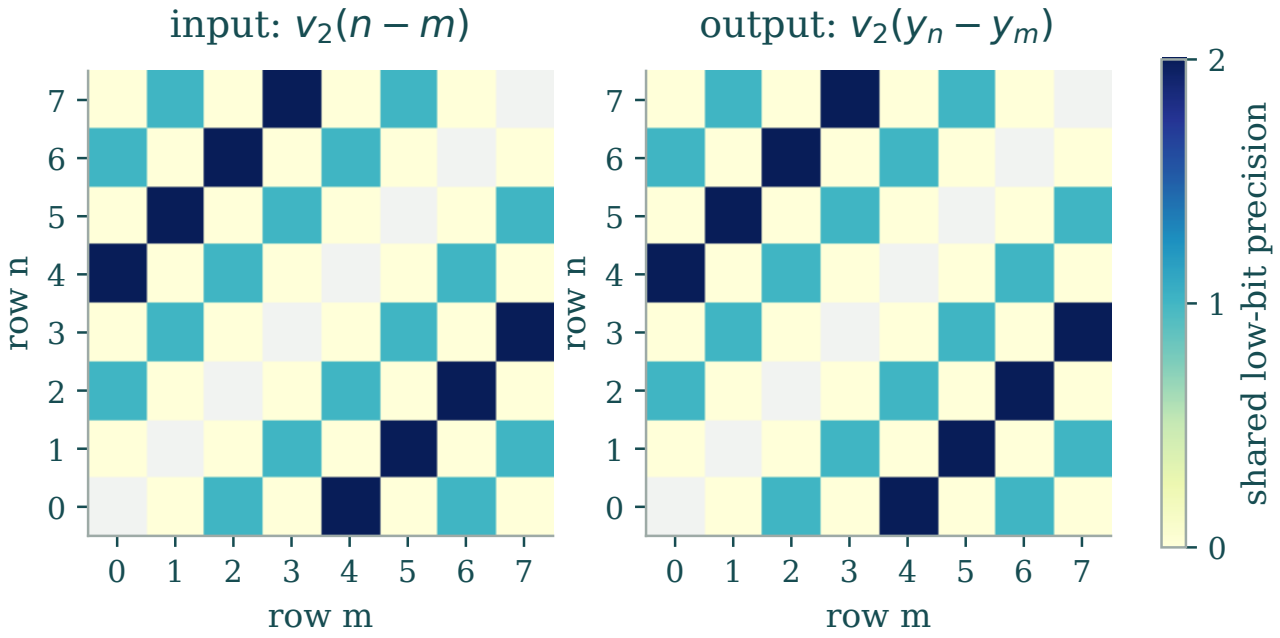


Figure 10: Address two modulo 256 gives an eight-element quotient. The two exact matrices coincide: every pair of distinct rows has the same dyadic valuation before and after the scale is removed. The diagonal represents $v_2(0) = +\infty$ and is left blank.

If $n - m$ is odd, the rows disagree in their first binary digit and the matrix entry is zero. If it is divisible by two but not four, they share exactly one low digit. An entry two means agreement through two low digits.

This structure explains the residue permutation in the preceding section. At each new precision, every input residue has exactly two lifts. An isometry distinguishes those lifts at the corresponding output precision. Starting from the single residue at the scale threshold, induction gives surjectivity and a unique inverse residue at every finite level.

The finite matrix is an illustration of that proof. Its eight rows alone would not establish a statement about the whole of $\mathbb{Z}_2$.

12 How far the simple metric law extends

Sharp first-difference laws hold for every address at ranks 2, 4, and 8. The arrays below give their normalized scales $b_{N,j}$ and original Euler scales $B_{N,j} = b_{N,j} + v_2(j!)$.

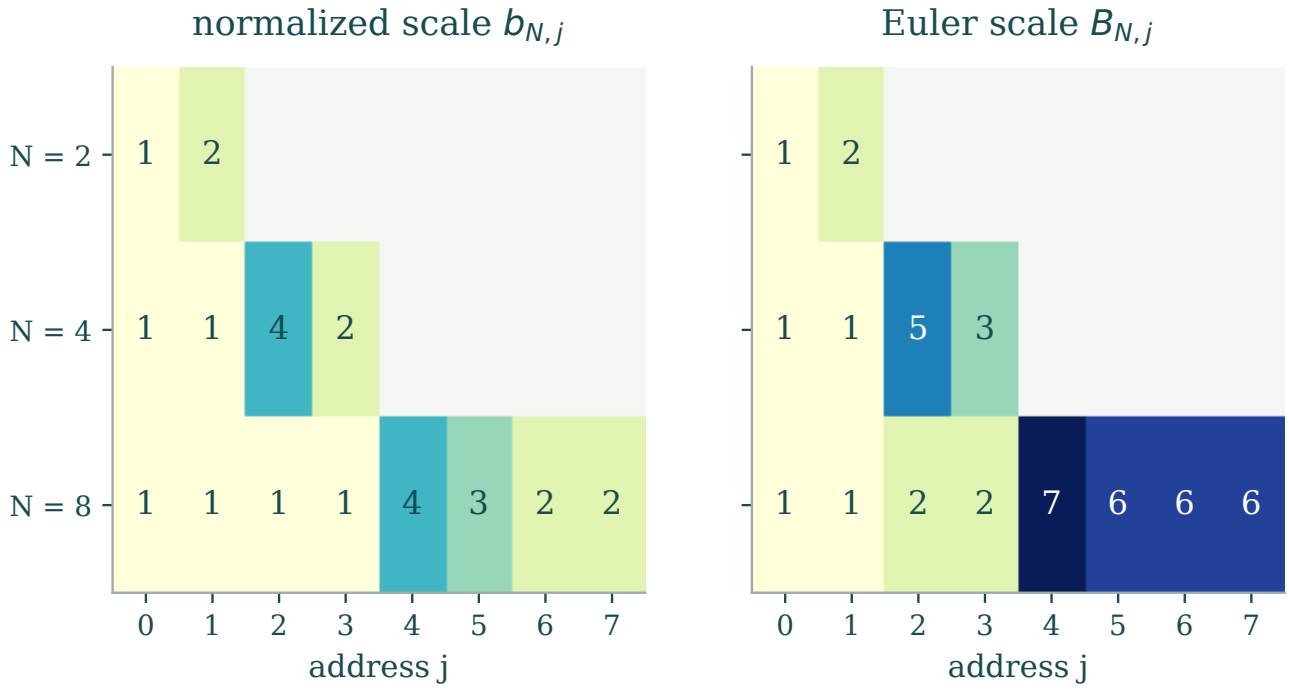


Figure 11: Proved scale arrays for the three small ranks. Blank cells are outside the address range. The Euler normalization contributes the factorial valuation, which is already substantial at the larger addresses of rank eight.

There is also an infinite family: at every $N = 2^a \geq 4$, all lower-half addresses $j < N/2$ have normalized scale one. Their Euler scale is $1 + v_2(j!)$.

The same constant-scale proposal fails at some upper-half addresses:

(N, j)	$v_2(C(1) - C(0))$	Step-two prediction	Actual step-two valuation
$(16, 9)$	5	6	7
$(32, 18)$	6	7	6
$(32, 19)$	5	6	7

These are exact rational counterexamples to the additional metric proposal. They do not contradict the universal higher-difference *lower* bound, which continues to hold at every address of every dyadic rank.

13 From factorial gain to analytic radius

At every center $x_0 \in \mathbb{Z}_2$, the interpolations admit ordinary Taylor series $C_j(x_0 + z) = \sum_{d \geq 0} a_{j,d}(x_0) z^d$. For positive degrees,

$$\begin{aligned} v_2(a_{j,d}(x_0)) &= d && (j = 0, 1), \\ v_2(a_{2,d}(x_0)) &\geq \max\{2d + \lfloor d/2 \rfloor,\ d + 3\}, \\ v_2(a_{3,d}(x_0)) &\geq d. \end{aligned}$$

At address two equality with $5d/2$ holds along every power-of-two degree $d \geq 4$.

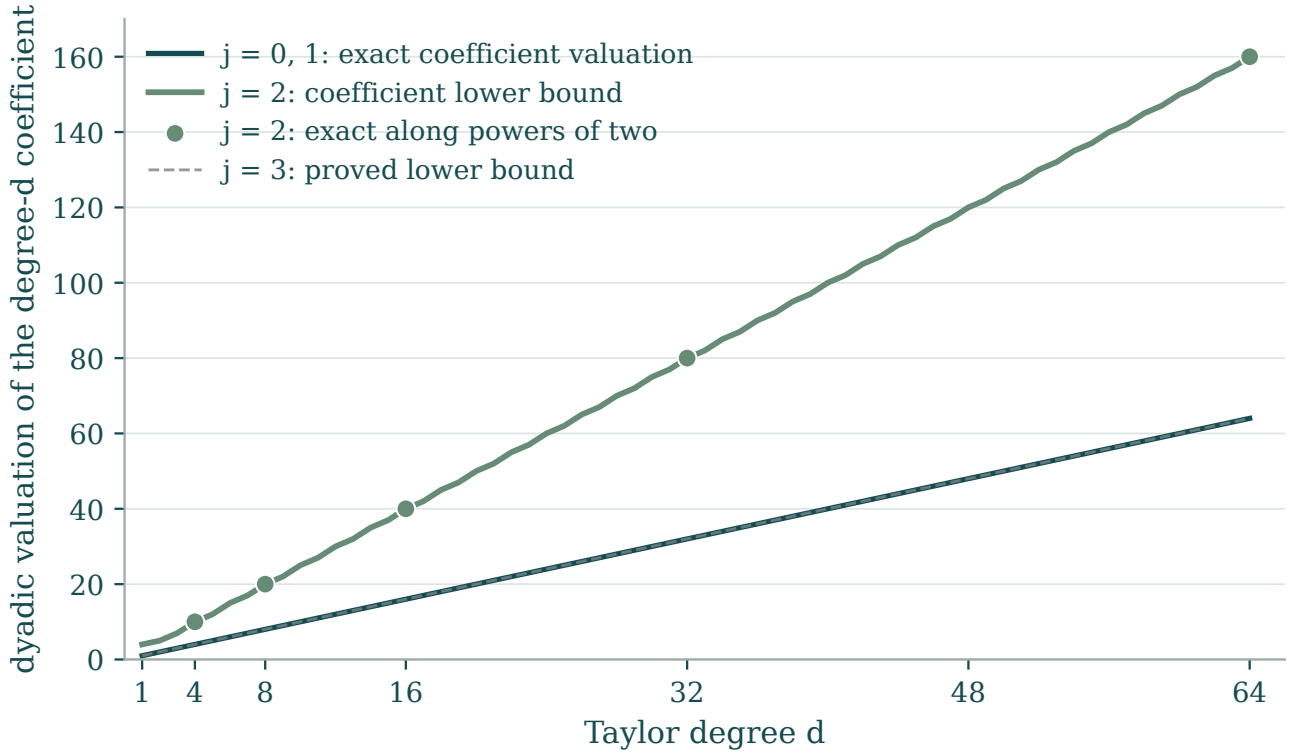


Figure 12: The Taylor coefficient precision slopes. The degree-zero coefficient is excluded. Marked address-two degrees belong to an unbounded proved equality family, rather than a finite observed pattern.

The radius is $2^{\liminf_d v_2(a_d)/d}$ in the normalized dyadic absolute value. Hence

$$R_0 = R_1 = 2, \qquad R_2 = 2^{5/2}, \qquad R_3 \geq 2.$$

The exact radius of address three is not asserted. For address two, the lower bound gives convergence inside the claimed radius and the unbounded equality family prevents a larger one.

The proof converts the factorial Newton estimates into ordinary Taylor coefficients with the Stirling relations. Generic continuity or first-order Lipschitz bounds alone do not supply this analytic conclusion.

14 Which evidence does which job

The package has three complementary levels of evidence. They answer different questions, and the final claims retain those distinctions.

Evidence	What it establishes	Scope of the final release
Lean theorem	The precise formal statement follows from its recorded dependencies and the accepted foundational axioms.	The completed Nishibuchi conjecture, earlier Paper IX theory, and the inherited rank-four ordinary-step, sharp-period, and isometry results; see the Lean guide for build provenance.
Algebraic proof and audit	A universal mathematical result, subject to the written proof and its stated inputs.	The new general prime-power transfer, mixed-step and factorial gains, exact higher-order families, broader dyadic metric laws, and radius results.
Exact computation	The implemented assertion holds in its recorded finite domain, or an exact counterexample refutes an extension.	Coefficient regressions, period illustrations, and sharpness or cancellation checks.

The generic theorem uses the integral Hurwitz-series product and reciprocal induction. The factorial theorem uses divided powers and explicit transforms of the ordinary Euler and Bernoulli series. The ordinary Euler arithmetic input is the classical Frobenius congruence, with normalization checked against Gessel's primary exposition. Address two has a self-contained central-factorial Bernoulli argument; addresses one and three use the proved tangent and cotangent transfers.

The broad exact regression through row 512 contains more than 1.8 million bounded difference assertions. Smaller later runs test the strengthened address-two filtration and the exact mixed parity law. Their manifests pin precision, coefficient definitions, input versions, and executed ranges.

The figures in this guide have a separate reproducibility record: `sources/generate_figures.py`, `figures/exact_figure_data.json`, and `figures/figure_manifest.json`. Its exact assertions validate the illustration data; theorem-derived curves are explicitly recorded as such.

The new general transfer, mixed-step, factorial, and analytic extension has not been newly formalized in Lean. The rank-four ordinary-step and sharp-period results have inherited Lean coverage, as mapped in the companion. Mathematical audits in this package are internal reviews, not claims of external peer review.

15 Reading the conclusion at the right strength

The original quantified Nishibuchi conjecture is completed. The stronger dyadic result applies without a starting-row threshold, and the rank-four theory now has factorial precision, exact common-step laws for addresses zero and one, an infinite exact address-two family, optimal periods, and exact analytic radii for three addresses.

For a common step, the final congruence to use is

$$\Delta_H^k E^{(4,j)}(n) \equiv 0 \pmod{2^{v_2(k!)+kv_2(H)+A_j(k)}}, \quad n \geq 0,\ k, H \geq 1,$$

with

$$(A_0, A_1, A_2, A_3) = \big(k, k, \max\{k+4, 2k+\lfloor k/2 \rfloor + 1\}, k+2\big).$$

For different steps, replace $kv_2(H)$ by $\sum_i v_2(H_i)$ in the lower bound. The address-zero/one equality rule then depends on order parity and the number of even steps.

Statement	Final status
Nishibuchi Conjecture 3.9, all allowed branches	completed; formal capstone
All-row dyadic mixed-difference bound	proved and audited
Rank-four factorial and paired bounds	proved and audited
All common-step exact valuations, $j = 0, 1$	proved at all rows and orders
Address-two exact valuations at power-of-two orders	proved for all positive step lists
Complete exact address-two formula at other orders	open
Complete exact higher-order address-three formula	open
Exact address-three analytic radius	open
Simple constant metric at every upper-half address	false; exact witnesses retained
Separate Euler-tail universal simplicity problem	open; a distinct research program

The remaining questions do not weaken the completed conjecture. They specify the next boundary of the stronger theory. The main paper gives the proofs and precise dependency statements, while the Lean guide supplies the formal coverage map and a reproducible route to its recorded checks.

16 Compute the reciprocal before estimating it

For width N and address j, the normalized source and its reciprocal are

$$G_{N,j}(t) = j! \sum_{n \ge 0} \frac{t^{Nn}}{(Nn+j)!}, \qquad G_{N,j}(t)^{-1} = \sum_{n \ge 0} C_{N,j}(n) \frac{t^{Nn}}{(Nn)!}.$$

Multiplying the two series and clearing the factorials gives an exact triangular recurrence:

$$C_{N,j}(0) = 1, \qquad C_{N,j}(n) = -\frac{\sum_{r=0}^{n-1} \binom{Nn+j}{Nr} C_{N,j}(r)}{\binom{Nn+j}{Nn}}.$$

address j	C(0)	C(1)	C(2)
0	1	-1	69
1	1	-1/5	121/45
2	1	-1/15	13/45
3	1	-1/35	59/1155

The table is for $N = 4$. The first two address-zero rows illustrate the computation without any valuation estimate:

$$C_{4,0}(1) = -1, \qquad C_{4,0}(2) = -\left(1 + \binom{8}{4}(-1)\right) = 69.$$

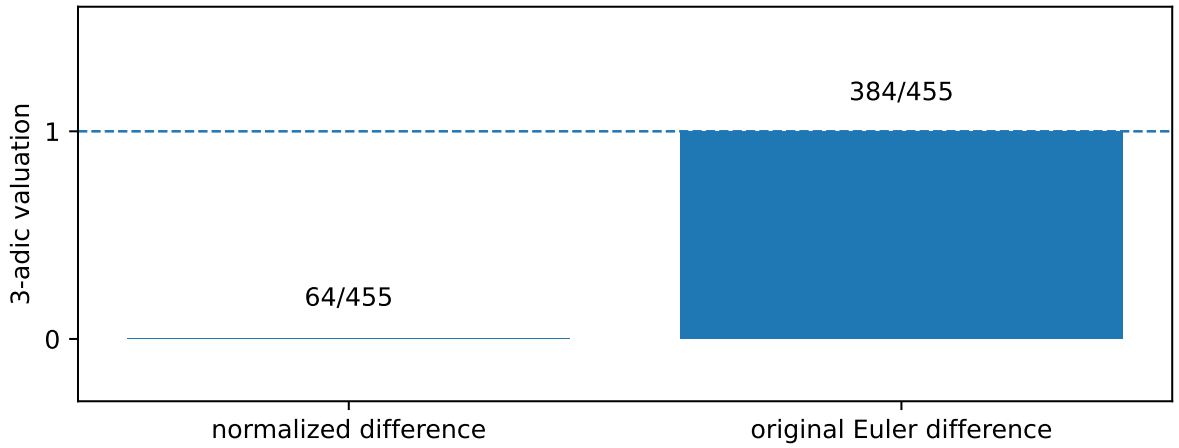
At address one, the same recurrence yields $C_{4,1}(1) = -1/5$ and $C_{4,1}(2) = 121/45$. These are exact rational coefficients, not rounded approximations. The second implementation in the supplied script inverts the ordinary power series first, then multiplies by $(Nn)!$. Agreement between the two constructions checks the factorial placement.

17 The normalization loss is a real boundary

The original Euler coefficient is $E_{Nn}^{(N,j)} = j!C_{N,j}(n)$. Multiplication by $j!$ changes the available p-adic precision. That change cannot be discarded outside the unit-factorial window.

Take $p = 3$, $N = 6$, $j = 3$, and compare rows one and two. Exact inversion gives

$$C_{6,3}(1) = -\frac{1}{84}, \quad C_{6,3}(2) = \frac{703}{5460}, \quad C_{6,3}(2) - C_{6,3}(1) = \frac{64}{455}.$$



Neither 64 nor 455 is divisible by three, so the normalized difference has valuation zero. The original coefficient difference is $3! \cdot 64/455 = 384/455$, of valuation one.

This is the counterexample behind the corrected factorial-loss statement. A bound for the original Euler normalization must lose $v_p(j!)$ when translated to $C_{N,j}$. For $0 \leq j < p$, that loss is zero. The distinction concerns which sequence the theorem measures, not a numerical weakness of the calculation.

18 Odd primes require the sign twist

The prime-power transfer controls a sign-twisted reciprocal sequence at odd primes. It must not be read as an ordinary-difference theorem for the untwisted coefficients.

At $N = p = 3$ and $j = 0$, put $A(n) = (-1)^n C_{3,0}(n)$. The first three rows are

row n	C at width 3, address 0	(-1)^n C(n)
0	1	1
1	-1	1
2	19	19

The ordinary first difference at row zero is $-1 - 1 = -2$. It is a 3-adic unit, so an asserted lower bound of one for that difference would be false. The twisted first difference is zero, whose mathematical valuation is $+\infty$. At the next row, the twisted difference is $19 - 1 = 18$, of valuation two.

Equivalently, the untwisted sequence is acted on by $T_H = S^H - (-1)^H$, where S advances one row. At odd step one this is $S + 1$, not $S - 1$. The zero in the example is handled by a zero-safe divisibility statement rather than a finite valuation equality. The explicit sign and the zero convention are both part of the theorem's interface.

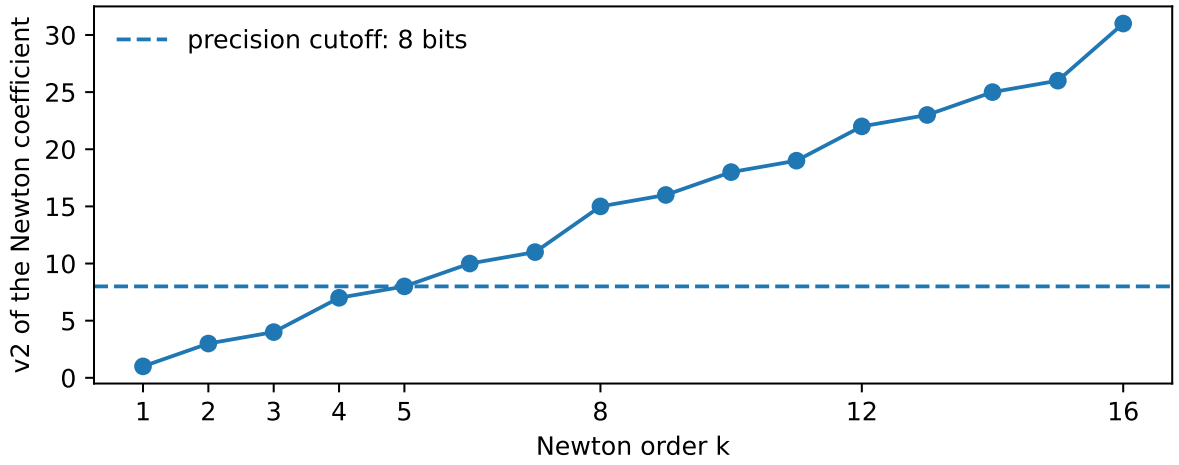
19 Newton coordinates turn precision into a finite computation

Let $b_k = \Delta_1^k C_{4,0}(0)$. The first coefficients obtained from the difference triangle are

$$(b_0, b_1, b_2, b_3, b_4) = (1, -2, 72, -33872, 60511872).$$

The exact Newton identity recovers each nonnegative integer row:

$$C_{4,0}(n) = \sum_{k=0}^{n} b_k \binom{n}{k}.$$



For this address the proved law is $v_2(b_k) = 2k - s_2(k)$ for $k \geq 1$. Every $k \geq 5$ therefore has $v_2(b_k) \geq 8$. Since $\binom{n}{k}$ is integral, those terms vanish modulo 256. Thus, for every $n \geq 0$,

$$C_{4,0}(n) \equiv 1 + 254\binom{n}{1} + 72\binom{n}{2} + 176\binom{n}{3} + 128\binom{n}{4} \pmod{256}.$$

The uniform truncation follows from the coefficient theorem. The script separately checks the exact Newton reconstruction and this reduction for rows 0 through 32. Those finite checks validate the worked arithmetic without replacing the theorem.

20 A compatible root is built one digit at a time

Take $F(x) = x^2 - 7$ over the 3-adic integers and start with $x_1 = 1$ modulo three. If $F(x_L) = 3^L r_L$, the next representative has the form $x_{L+1} = x_L + 3^L t_L$. Its correcting digit satisfies

$$r_L + 2x_L t_L \equiv 0 \pmod 3.$$

L	3^L	current x	next digit	next x
1	3	1	1	4
2	9	4	1	13
3	27	13	0	13
4	81	13	2	175
5	243	175	0	175

For example, $F(1)/3 = -2$ and $2x_1 = 2$, so the next digit is one and $x_2 = 4$. Then $F(4)/9 = 1$, again giving digit one and $x_3 = 13$. At the next stage the residual is divisible by three, so digit zero is correct. A zero digit is not a stalled construction.

Every center remains congruent to one modulo three. Hence $2x_L$ stays a unit, and the correction equation has a unique digit at every stage. The representatives are compatible and their increments are divisible by 3^L. Completeness gives a limit, and continuity of the polynomial gives a root of F. The finite table exhibits the construction. The unit derivative is what supports indefinite continuation.

21 A singular residual can stop a lift

Now take $F(x) = x^2 - 3$ at the residue $x = 0$ modulo three. The residual is $r = F(0)/3 = -1$ and the Jacobian is zero modulo three. The next-digit equation would require

$$-1 + 0 \cdot t \equiv 0 \pmod 3,$$

which has no solution.

digit t	candidate y	y^2 - 3	residue mod 9
0	0	-3	6
1	3	6	6
2	6	33	6

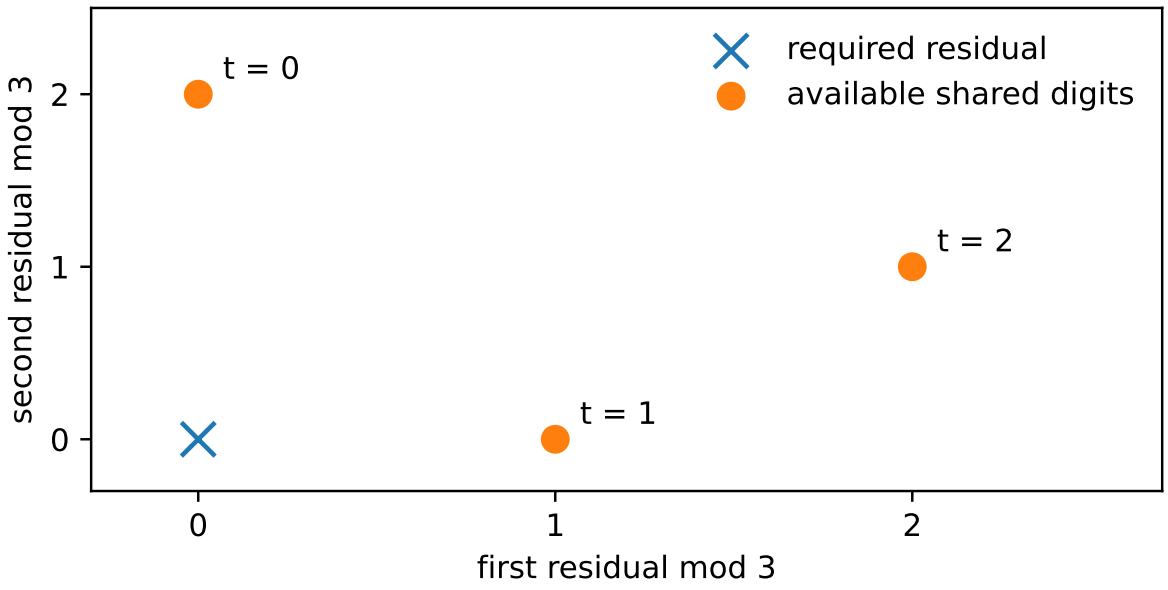
The three representatives of the old root class modulo nine all leave residual six modulo nine. Thus the root class exists at precision three but does not lift to precision nine. This is an exact obstruction, not failure of an iterative search.

A singular Jacobian alone does not imply failure. For $F(x) = x^2$ at the same center, the residual is zero and all three candidate digits lift to roots modulo nine. The obstruction is the residual class in the cokernel of the actual Jacobian. It is this class, rather than the rank by itself, that decides whether a next lift exists.

22 Separate corrections need not be one shared correction

Consider the two-component system $F(x) = (x, x-3)$ at $x = 0$, with $q = 3$ and $L = 1$. Both components vanish modulo three. After writing a candidate as $y = 3t$, its residual vector modulo three is

$$\bar{r} + J\bar{t} = (0,2) + (t,t) = (t, t-1).$$



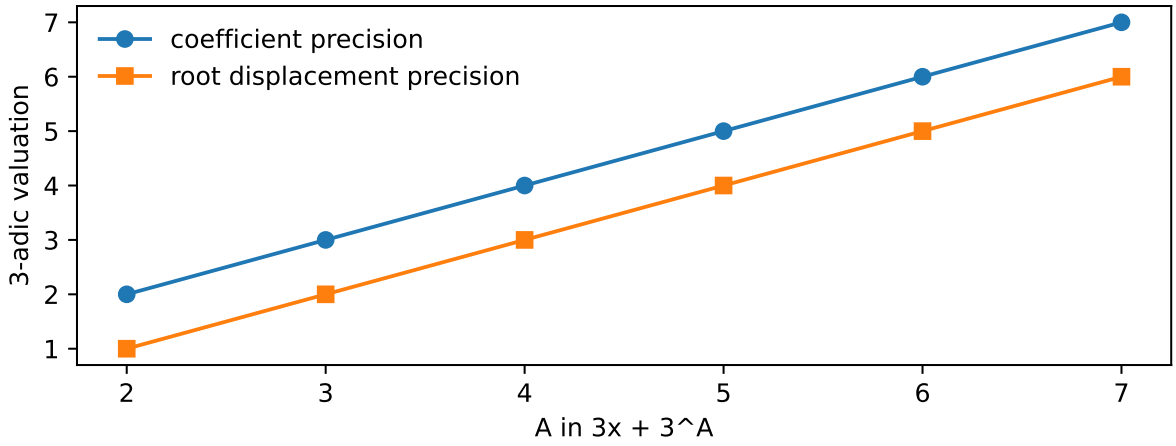
The first coordinate can be corrected by $t = 0$. The second can be corrected by $t = 1$. No digit corrects both. The available residuals are $(0,2)$, $(1,0)$, and $(2,1)$, none of which is $(0,0)$.

The combined Jacobian maps t to the diagonal vector (t,t). Its image is smaller than the product of its coordinate images. Checking the two projected obstructions independently therefore loses the compatibility condition. The common-correction theorem retains the full residual vector and the shared digit space, which is exactly the information needed to detect this failure.

23 A derivative weight can cost one digit of root precision

Compare the linear equations $F(x) = 3x = 0$ and $G_A(x) = 3x + 3^A = 0$, where $A \geq 2$. Their coefficient difference has 3-adic valuation A, but their roots differ by

$$x_A - 0 = -3^{A-1}, \qquad v_3(x_A) = A - 1.$$



The derivative has valuation one. Solving the correction equation divides by three, so one digit of coefficient precision is lost in root displacement. The calculation is exact for every displayed A and also follows immediately from the two linear equations.

This example explains why weighted lifting and deformation estimates track a Jacobian loss rather than silently treating every invertible derivative over $\mathbb{Q}_p$ as a unit over $\mathbb{Z}_p$. A valuation-zero derivative would not incur this particular loss. Nonlinear systems additionally require their stated neighborhood, residual, and right-inverse hypotheses.

Reproducibility. The eight examples use exact rational and integer arithmetic in `tools/worked_examples.py`. The release record states the executed finite ranges. This atlas adds no Lean endpoint and changes no formal claim in the companion.