PAPER II

Projector Resolved Seed Calculus II

Refinement towers, weighted transfer, and integral separation of operator packets

An exact account of how packet coordinates refine, weighted transfers accumulate, and rational splittings become visible on an integral lattice.

Komorebi P.

Modern Monograph • 19 October 2026

How to use this volume

Reading route. Exact packet coordinates and residue selectors lead to weighted inversion and rational walk series. Lattice gluing then gives separation weights and precision delays. Smith congruences and operator defects connect these formulas to finite computation.

Companions. The worked visual atlas appears in this volume and as a separate guide. The Lean guide distinguishes the current formal development from the inherited 25-row walkthrough. Each guide records its evidence and scope.

Overview

A complete family of idempotents gives exact coordinates, but a single family does not record scale or integral visibility. We study towers in which every coarse idempotent is the sum of its fine descendants. The coordinate changes are invertible and coherent, and every operator block is the sum of its refined blocks. Residue selectors give a model over arbitrary rings. When roots of unity are available they agree with character averages, and coprime moduli factor through the Chinese remainder theorem. In a complete filtered algebra, a diagonal operator plus positive weight transfers has an inverse obtained by summing admissible weighted paths. The formula truncates exactly at finite precision, while finite transfer graphs have rational path series. For a rational spectral decomposition over a discrete valuation ring, resultants give denominator bounds. The exact denominator is the largest Smith exponent of the lattice gluing module. This yields a necessary and sufficient precision delay, monotonicity under refinement, and a full congruence solving formula. We close with exact formulas for the propagation of a coarsening defect through powers, resolvents, holomorphic functions, exponentials, and multilevel towers.

This edition integrates the expanded mathematical narrative with the canonical v1.2.1 certificate level formulations. It preserves the distinction between general written proofs, formal specifications, and finite computational examples. The visual atlas and the separate Lean guide provide worked models and separate current and inherited formal crosswalks.

Contents

1 Introduction 1

2 Depth and compatibility 2

3 Packet systems and refinement towers 3
- 3.1 Typed coordinate naturality . . . 5
- 3.2 Operator blocks across refinement . . . 6
- 3.3 Common refinements . . . 7

4 Residue coordinates and coprime factorization 9
- 4.1 Finite orthogonality certificates . . . 10
- 4.2 The field specialization . . . 10

5 Complete filtered transfer algebras 12
- 5.1 Finite quotient and normed-ring realizations . . . 16
  - 5.1.1 Normalized edges and written-word convention . . . 16

6 Rational weighted transfer series 18

6.1 Automatic finite-graph construction . . . 21

7 Rational spectral projectors and resultant bounds 23
7.1 Evaluated certificates and explicit lattice action . . . 25
7.1.1 Minimal evaluated Bézout identities . . . 25
7.1.2 Certificate-level precision delay . . . 26

8 Exact lattice separation 26
8.1 Refinement and separation weight . . . 29
8.2 Exact precision delay . . . 30

9 The dyadic sign split 31
9.1 The concrete rational sign matrix . . . 32
9.1.1 Why coefficient packets remain available . . . 33

10 Smith normal form and congruence lifting 33
10.1 Concrete diagonal fibres over integer quotients . . . 36

11 Linear refinement defects 37
11.1 Inverse witnesses from one side and norm control . . . 38
11.2 Analytic functional calculus . . . 39
11.3 Several refinement levels . . . 40

12 Worked realizations in the common operator language 41
12.1 Signal polyphase coordinates . . . 41
12.2 Sparsity of transfer blocks . . . 42
12.3 A short realization in Smith chart coordinates . . . 42

13 Finite computational certificates 43

14 Scope and limits 46

A Notation table 46

B Algorithmic forms 47
B.1 Residue analysis and synthesis . . . 47
B.2 Weighted inverse at finite precision . . . 47
B.3 Walk construction for transfer zeta . . . 48

C Proof dependency map 48

D Edition and proof scope 50

E Visual atlas 51

List of Figures

1 A nested residue hierarchy. Each residue modulo 2 is the union of two residues modulo 4, and each residue modulo 4 is the union of two residues modulo 8. . . . 6
2 The incidence matrix $M_{r,s} = 1$ when the fine residue s belongs to the coarse residue r. Its rows implement exact aggregation from fine packets to coarse packets. . . . 8

3 A composite residue packet is addressed by its dyadic and odd coordinates. The displayed mod-24 system is the product of mod 8 and mod 3. 12
4 A finite packet-transfer graph. Edge weights record filtration gain rather than merely the number of operator compositions. 17
5 At precision L, weighted inversion requires only transfer words whose cumulative weight is below L. 19
6 The walk counts for one weighted graph. The exact generating function is rational, so these counts satisfy a finite linear recurrence. 22
7 The rational sign projectors for an integral involution. Their denominator has 2-adic weight one. 33
8 The denominator weight of the sign projector is nonzero only at the prime two. 34
9 A controlled defect for one step and its telescoping norm bound under repeated operator application. 39
10 Norms of packet blocks for a finite stencil. The original operator is exactly reconstructed from a sparse transfer pattern. 42
11 Nested phase orbits at powers of 2 on the Smith reflection disk. The full treatment of the Smith chart is deferred to the signal paper. 44
12 A symmetric refinement weight concentrated near the boundary points for open and short circuits. 45
13 The dependency structure of Paper II. Coordinate coherence, residue/CRT, weighted inversion, automatic finite-graph transfer rationality, and integral separation are distinct branches with explicit inputs. 49

List of Tables

1 Representative numerical and symbolic checks. 45

1 Introduction

A complete family of pairwise annihilating idempotents decomposes a module into exact coordinate pieces. Many problems require more than one such decomposition. A signal may be split first into two residue classes and then into four. A filtered operator may be resolved into transfers that appear at different valuation depths. A rational spectral splitting may exist over the fraction field while remaining invisible on the original lattice. These are all refinement problems, but they involve different notions of scale.

The first part of the article concerns exact refinement. If a fine packet family refines a coarse family, then each coarse idempotent is the sum of its fine descendants. The coordinate spaces at the two levels are canonically isomorphic because both represent the whole module. The refinement relation adds more information. It tells us which fine coordinates belong to each coarse coordinate, and it gives a compatible decomposition of every operator block. This is the algebra behind polyphase coordinates and nested filter banks. Standard treatments of those subjects may be found in [10, 2].

The second part studies inversion in a complete filtered algebra. Suppose an operator has the form

$$T = D + N,$$

where D is invertible and every packet transfer in N raises the filtration by a positive amount. The ordinary Neumann series can then be expanded into weighted paths. The weight is not the number of edges. It is the total filtration gain along the path. At precision L, every path of weight at least L vanishes. A finite transfer graph also gives a rational generating function for the number of paths of each total weight. The relevant generating function facts are classical and are discussed in [9].

The third part concerns integral separation. Let V be a vector space over the fraction field of a discrete valuation ring $\mathcal{O}$, and let Λ be an $\mathcal{O}$ lattice in V. A rational projector E need not preserve Λ. Its separation weight is the least s such that

$$\pi^{s} E\Lambda \subseteq \Lambda.$$

A Bézout construction gives an upper bound through a polynomial resultant. That bound need not be exact because it does not see the position of the lattice. The exact answer is controlled by the finite length module

$$G_{\Lambda,\mathcal{P}} = \Lambda \Big/ \bigoplus_{\alpha} (\Lambda \cap V_{\alpha}),$$

where $V = \bigoplus_{\alpha} V_{\alpha}$ is the rational packet decomposition. The largest Smith exponent of this gluing module is exactly the common denominator needed to make every packet projector integral. This interpretation also proves that separation weight cannot decrease under refinement. Background on local fields, Smith normal form, and matrix polynomials may be found in [8, 6, 3].

The last part records how a defect between fine and coarse operators propagates. If

$$\Delta = CT_{f} - T_{c}C,$$

then exact formulas describe the corresponding defect for powers, resolvents, holomorphic functions, exponentials, and a chain of several refinement levels. These formulas are elementary, but collecting them in one place makes the later analytic uses transparent. The spectral and resolvent background follows the usual operator theory in [4].

Every theorem in this paper concerns linear operators. An exact refinement retains the full object and changes its coordinates. Questions about nonlinear operation trees, nonlinear correction, and obstruction complexes require a separate theory.

2 Depth and compatibility

Packet, filtration, and separation depths describe three different kinds of scale. Compatibility is initially recorded by an operator defect. Keeping these notions separate makes the later formulas easier to read.

The first is packet depth. A finer packet family has more idempotents and a more detailed address set. The passage from residues modulo two to residues modulo four is an example. Packet depth is combinatorial. It says how many distinctions have been made in the coordinate system.

The second is filtration depth. A transfer in $F^r\mathcal{B}$ vanishes modulo $F^r\mathcal{B}$. This membership gives a lower bound on its depth. A nonzero transfer may first become visible only at a higher precision. Two paths with the same number of edges can have different filtration depths. A longer path can appear earlier when its edges have smaller weights.

The third is separation depth. A rational packet projector may require multiplication by π^s before it preserves the chosen lattice. This number depends on both the rational decomposition and the lattice. It can be zero even when the corresponding eigenvalues are close, and it can be positive even when the rational projector is simple.

The fourth notion is the compatibility defect. A fine operator and a separately chosen coarse operator need not commute with coarsening. Their discrepancy is

$$\Delta = CT_f - T_cC.$$

This operator records the failure of one model to represent another exactly. A chosen norm measures its size, while a chosen filtration can assign a depth. No integer compatibility depth is assigned without that additional structure.

The four notions meet in applications, but no one of them determines the others. An exact refinement tower may have zero coordinate error while its fine spectral packets have positive lattice weight. A filtered inverse may truncate exactly at a given depth while the coarse operator still has a nonzero compatibility defect. The paper treats each quantity with its own theorem before combining them.

> **The central idea**
>
> Packet depth changes the address of a component. Filtration depth controls when a transfer survives a precision cutoff. Separation depth clears a rational denominator on a chosen lattice. The compatibility defect records a discrepancy between two operator models. These quantities can interact, but each needs its own hypothesis and its own calculation.

A simple two packet model already contains all four notions. Let

$$X = X_0 \oplus X_1, \qquad D = \begin{pmatrix} D_0 & 0 \\ 0 & D_1 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & N_{01} \\ N_{10} & 0 \end{pmatrix}.$$

A finer family may split X_0 and X_1 into smaller summands. The blocks N_{01} and N_{10} may enter at different filtration weights. Over a fraction field, the rational packet spaces may fail to split the chosen lattice. A coarse operator on one selected summand may also fail to intertwine with the full matrix. The later sections isolate these four effects one at a time.

3 Packet systems and refinement towers

Packet coordinates split a module into pieces that can be recovered separately and then added back together. The construction uses idempotents and does not need an inner product.

Definition 3.1 (Packet system)**.** Let X be a module over a unital ring. A finite packet system on X is a family

$$\mathcal{P} = (P_\alpha)_{\alpha \in I} \subseteq \mathrm{End}(X)$$

of nonzero idempotents such that

$$P_\alpha P_\beta = 0 \quad \text{when } \alpha \neq \beta, \qquad \sum_{\alpha \in I} P_\alpha = I_X.$$

Write

$$X_\alpha = P_\alpha X, \qquad X^{\mathcal{P}} = \bigoplus_{\alpha \in I} X_\alpha.$$

The analysis and synthesis maps are

$$\mathcal{A}_{\mathcal{P}} x = (P_\alpha x)_\alpha, \qquad \mathcal{S}_{\mathcal{P}} (x_\alpha)_\alpha = \sum_\alpha x_\alpha.$$

They are inverse isomorphisms.

Definition 3.2 (Refinement)**.** Let $\mathcal{P} = (P_\alpha)_{\alpha \in I}$ and $\mathcal{Q} = (Q_\beta)_{\beta \in J}$ be packet systems on the same module. We say that $\mathcal{Q}$ refines $\mathcal{P}$ when there is a surjection

$$\rho : J \longrightarrow I$$

such that

$$P_\alpha = \sum_{\rho(\beta) = \alpha} Q_\beta$$

for every $\alpha \in I$. The index β is then called a child of $\rho(\beta)$.

Theorem 3.3 (Children, parents, and transitivity)**.** *Suppose that $\mathcal{Q}$ refines $\mathcal{P}$ through ρ. Then*

$$P_\alpha Q_\beta = Q_\beta P_\alpha = \begin{cases} Q_\beta & \rho(\beta) = \alpha, \\ 0 & \rho(\beta) \neq \alpha. \end{cases}$$

Hence $Q_\beta X \subseteq P_{\rho(\beta)} X$. If a third packet system $\mathcal{R}$ refines $\mathcal{Q}$, then $\mathcal{R}$ refines $\mathcal{P}$ through the composite parent map.

Proof. Multiply the identity

$$P_\alpha = \sum_{\rho(\gamma) = \alpha} Q_\gamma$$

by Q_β on either side. Pairwise annihilation in the fine family leaves Q_β when $\rho(\beta) = \alpha$ and leaves zero otherwise. For transitivity, substitute the child expansion of each Q_β into the expansion of P_α and group the resulting terms by the composite parent. □

A refinement tower is a sequence $(\mathcal{P}_\ell)_{\ell\geq 0}$ with parent maps

$$\rho_{m,\ell} : I_m \longrightarrow I_\ell \qquad (m \geq \ell)$$

such that $\mathcal{P}_m$ refines $\mathcal{P}_\ell$ through $\rho_{m,\ell}$ for every $m \geq \ell$, and

$$\rho_{\ell,\ell} = \mathrm{id}_{I_\ell}, \qquad \rho_{n,\ell} = \rho_{m,\ell} \circ \rho_{n,m} \qquad (n \geq m \geq \ell).$$

Thus every parent map witnesses the stated refinement and the maps compose coherently. The next result shows that the coordinate changes form a coherent system.

Definition 3.4 (Coordinate change across levels)**.** For two packet systems $\mathcal{P}$ and $\mathcal{Q}$ on X, define

$$J_{\mathcal{Q}\leftarrow\mathcal{P}} = \mathcal{A}_{\mathcal{Q}}\mathcal{S}_{\mathcal{P}} : X^{\mathcal{P}} \longrightarrow X^{\mathcal{Q}}$$

and

$$C_{\mathcal{P}\leftarrow\mathcal{Q}} = \mathcal{A}_{\mathcal{P}}\mathcal{S}_{\mathcal{Q}} : X^{\mathcal{Q}} \longrightarrow X^{\mathcal{P}}.$$

When $\mathcal{Q}$ refines $\mathcal{P}$, the first map resolves each coarse coordinate into its children and the second adds the children back to their parent.

The central idea

A coarse coordinate retains a whole vector in its coarse packet. Adding its children keeps the whole vector inside that packet. Fine analysis recovers each child by applying its projector. This explains why the two coordinate maps can be inverse even when the fine system has more packet labels.

Theorem 3.5 (Exact coordinate change and coherence)**.** *For any two packet systems on X,*

$$C_{\mathcal{P}\leftarrow\mathcal{Q}}J_{\mathcal{Q}\leftarrow\mathcal{P}} = I_{X^{\mathcal{P}}}, \qquad J_{\mathcal{Q}\leftarrow\mathcal{P}}C_{\mathcal{P}\leftarrow\mathcal{Q}} = I_{X^{\mathcal{Q}}}.$$

For three packet systems $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ on X,

$$J_{\mathcal{R}\leftarrow\mathcal{Q}}J_{\mathcal{Q}\leftarrow\mathcal{P}} = J_{\mathcal{R}\leftarrow\mathcal{P}},$$

$$C_{\mathcal{P}\leftarrow\mathcal{Q}}C_{\mathcal{Q}\leftarrow\mathcal{R}} = C_{\mathcal{P}\leftarrow\mathcal{R}}.$$

If $\mathcal{Q}$ refines $\mathcal{P}$ through ρ, then

$$\left(J_{\mathcal{Q}\leftarrow\mathcal{P}}x\right)_\beta = Q_\beta x_{\rho(\beta)}, \qquad \left(C_{\mathcal{P}\leftarrow\mathcal{Q}}y\right)_\alpha = \sum_{\rho(\beta)=\alpha} y_\beta.$$

Proof. The packet analysis and synthesis maps are inverse. Therefore

$$C_{\mathcal{P}\leftarrow\mathcal{Q}}J_{\mathcal{Q}\leftarrow\mathcal{P}} = \mathcal{A}_{\mathcal{P}}\mathcal{S}_{\mathcal{Q}}\mathcal{A}_{\mathcal{Q}}\mathcal{S}_{\mathcal{P}} = \mathcal{A}_{\mathcal{P}}\mathcal{S}_{\mathcal{P}} = I.$$

The other inverse identity is the same calculation with the two systems exchanged. The coherence identities follow by inserting

$$\mathcal{S}_{\mathcal{Q}}\mathcal{A}_{\mathcal{Q}} = I_X.$$

Now assume that $\mathcal{Q}$ refines $\mathcal{P}$. If $x = (x_\alpha)_\alpha$ with $x_\alpha \in P_\alpha X$, then

$$Q_\beta \sum_\alpha x_\alpha = Q_\beta x_{\rho(\beta)}$$

by theorem 3.3. The formula for C follows by synthesis followed by projection onto the parent packet. □

Remark 3.6 (Refinement and reduction). The maps in theorem 3.5 are isomorphisms because both coordinate spaces represent all of X. They do not reduce dimension and they do not discard packets. Approximation begins only after one chooses a truncation or compression rule.

Corollary 3.7 (Orthogonal Hilbert packets)**.** *If X is a real or complex Hilbert space and every packet system in the tower consists of orthogonal projections, equip each finite packet direct sum with the Hilbert inner product $\langle (x_\alpha), (y_\alpha) \rangle = \sum_\alpha \langle x_\alpha, y_\alpha \rangle$. Then analysis and synthesis are unitary. Every coordinate change in theorem 3.5 is unitary.*

Proof. For an orthogonal packet system,

$$\|x\|^2 = \sum_\alpha \|P_\alpha x\|^2.$$

Thus analysis is an isometry onto the full packet direct sum. Synthesis is its inverse. A coordinate change is a composition of two unitary maps. □

Every coarse packet is also uniquely reconstructed from its fine children. If vectors $x_\beta \in Q_\beta X$ sum to a coarse packet x, applying Q_γ recovers $x_\gamma = Q_\gamma x$. This is the uniqueness clause of the canonical nested-refinement statement.

3.1 Typed coordinate naturality

These coordinate changes are coherent through every finite tower. If three complete packet systems are denoted by a, b, c, then

$$J_{c,b} J_{b,a} = J_{c,a}, \qquad J_{a,a} = I, \qquad J_{b,a}^{-1} = J_{a,b}.$$

These laws hold for arbitrary exact packet systems representing the same ambient module. The parent maps supply the component formulas linking children to their parents. Coordinate coherence itself does not require them.

The same coherence applies to operators. Let P_a, P_b be packet systems on X, let Q_a, Q_b be packet systems on Y, where X and Y are modules over the same scalar ring, and let $T : X \to Y$ be a linear map over that ring. Its exact coordinate representation

$$\widehat{T}_{Q,P} = \mathcal{A}_Q T \mathcal{S}_P$$

preserves identities and typed composition, and the naturality square is

$$\boxed{J^Y_{Q_b,Q_a} \widehat{T}_{Q_a,P_a} = \widehat{T}_{Q_b,P_b} J^X_{P_b,P_a}.}$$

The identity, cocycle, composition, inverse, and naturality laws are all proved in Lean. The separate refinement hypotheses supply the local component formulas linking each child to its parent.

How to read the result

These identities concern complete coordinates of the same module. A quotient is a different operation: it identifies vectors that differ by an element of a chosen submodule. Later, finite precision will be expressed by a quotient that removes terms beyond the chosen filtration depth. This distinction explains why refinement is exactly reversible while precision truncation has a prescribed residual.

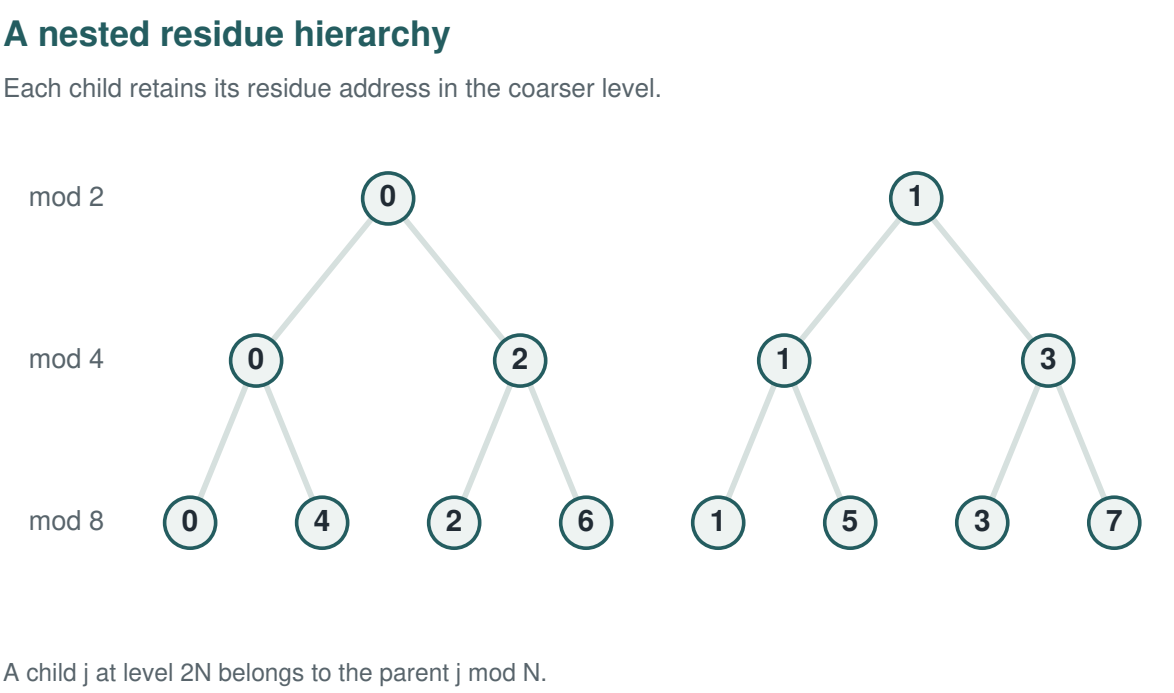


Figure 1. A nested residue hierarchy. Each residue modulo 2 is the union of two residues modulo 4, and each residue modulo 4 is the union of two residues modulo 8.

3.2 Operator blocks across refinement

For $T \in \mathrm{End}(X)$ and a packet system $\mathcal{P}$, write

$$T^{\mathcal{P}}_{\beta\alpha} = P_\beta T P_\alpha.$$

Theorem 3.8 (Refinement of operator blocks). *Suppose that $\mathcal{Q}$ refines $\mathcal{P}$ through ρ. Then*

$$T^{\mathcal{P}}_{ba} = \sum_{\substack{\rho(\beta)=b \\ \rho(\alpha)=a}} T^{\mathcal{Q}}_{\beta\alpha}.$$

The matrix of T in fine coordinates is related to the matrix in coarse coordinates by

$$[T]_{\mathcal{P}} = C_{\mathcal{P}\leftarrow\mathcal{Q}}[T]_{\mathcal{Q}}J_{\mathcal{Q}\leftarrow\mathcal{P}}.$$

Proof. Expand the two coarse projectors into their children:

$$P_b T P_a = \left(\sum_{\rho(\beta)=b} Q_\beta \right) T \left(\sum_{\rho(\alpha)=a} Q_\alpha \right).$$

Distributing the finite sums gives the first identity. For the coordinate identity, use

$$[T]_{\mathcal{P}} = \mathcal{A}_{\mathcal{P}} T \mathcal{S}_{\mathcal{P}}$$

and insert

$$I_X = \mathcal{S}_{\mathcal{Q}} \mathcal{A}_{\mathcal{Q}}.$$

□

Corollary 3.9 (Support under refinement)**.** *Under the hypotheses of theorem 3.8, a coarse block $T^{\mathcal{P}}_{ba}$ is zero if and only if every fine descendant block*

$$T^{\mathcal{Q}}_{\beta\alpha} \qquad (\rho(\beta) = b,\ \rho(\alpha) = a)$$

is zero.

Proof. If every fine block is zero, the coarse block is zero by theorem 3.8. Conversely, suppose $P_b T P_a = 0$. Multiply on the left by Q_β and on the right by Q_α. The child identities in theorem 3.3 give

$$Q_\beta T Q_\alpha = 0.$$

□

Thus refinement cannot hide an active transfer inside a zero coarse block. It can only reveal where the active transfer sits among the descendants.

3.3 Common refinements

Two packet systems need not be comparable. When their idempotents commute, their products give a common refinement.

Theorem 3.10 (Common refinement of commuting packet systems)**.** *Let $\mathcal{P} = (P_\alpha)_{\alpha \in I}$ and $\mathcal{Q} = (Q_\beta)_{\beta \in J}$ be packet systems on X. Assume*

$$P_\alpha Q_\beta = Q_\beta P_\alpha$$

for all α and β. Remove the zero operators from the family

$$R_{\alpha\beta} = P_\alpha Q_\beta.$$

The remaining operators form a packet system that refines both $\mathcal{P}$ and $\mathcal{Q}$. The same statement holds for any finite collection of pairwise commuting packet systems.

Proof. Commutativity gives

$$R^2_{\alpha\beta} = P^2_\alpha Q^2_\beta = R_{\alpha\beta}.$$

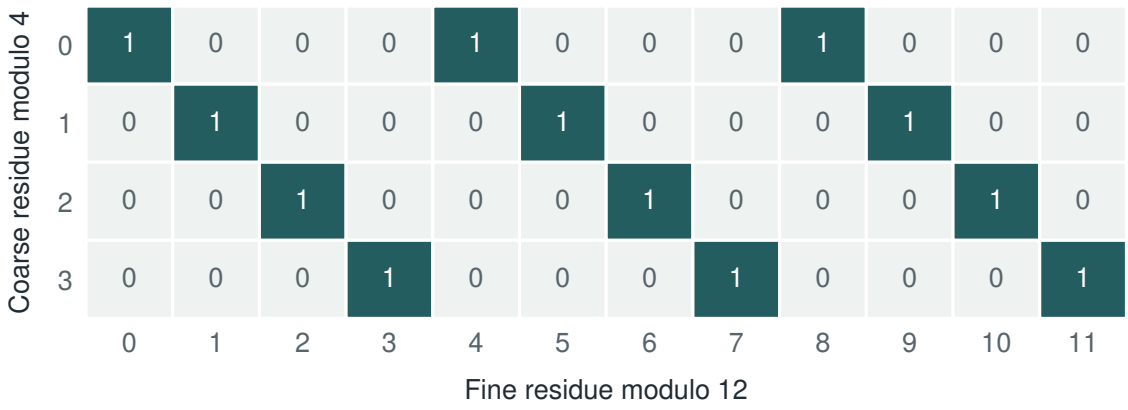


Figure 2. The incidence matrix $M_{r,s} = 1$ when the fine residue s belongs to the coarse residue r. Its rows implement exact aggregation from fine packets to coarse packets.

If $(\alpha, \beta) \neq (\alpha', \beta')$, then either $P_\alpha P_{\alpha'} = 0$ or $Q_\beta Q_{\beta'} = 0$, so the corresponding products annihilate one another. Finally,

$$\sum_{\alpha,\beta} R_{\alpha\beta} = \left(\sum_{\alpha} P_\alpha\right)\left(\sum_{\beta} Q_\beta\right) = I_X.$$

For a fixed α,

$$P_\alpha = P_\alpha \sum_{\beta} Q_\beta = \sum_{\beta} R_{\alpha\beta},$$

and the same argument works for a fixed β. Since each P_α and Q_β is nonzero, these identities ensure that every parent has a nonzero product child. The two parent maps on the supported product family are therefore surjective. Iteration proves the finite family statement. For the empty collection, use the supported identity family: (I_X) when $X \neq 0$ and the empty family when $X = 0$. □

Proposition 3.11 (Coarsest common refinement)**.** *Under the hypotheses of theorem 3.10, let $\mathcal{R}$ be a packet system that refines both $\mathcal{P}$ and $\mathcal{Q}$. Then $\mathcal{R}$ refines the nonzero product family $(P_\alpha Q_\beta)$. Hence that product family is the coarsest packet system that refines both $\mathcal{P}$ and $\mathcal{Q}$.*

Proof. Let R_γ be a packet in $\mathcal{R}$. Because $\mathcal{R}$ refines $\mathcal{P}$, there is a unique α with

$$P_\alpha R_\gamma = R_\gamma.$$

Likewise there is a unique β with

$$Q_\beta R_\gamma = R_\gamma.$$

Therefore

$$P_\alpha Q_\beta R_\gamma = R_\gamma.$$

Every packet of $\mathcal{R}$ lies in one product packet. Multiplying $\sum_\gamma R_\gamma = I_X$ by $P_\alpha Q_\beta$ and using the parent identities gives

$$P_\alpha Q_\beta = \sum_{\substack{\gamma:\ P_\alpha R_\gamma = R_\gamma \\ Q_\beta R_\gamma = R_\gamma}} R_\gamma.$$

Thus the packets with the same pair of parents reconstruct the product packet. If that product packet is nonzero, the sum has a child, so the parent map onto the supported product family is surjective. □

Example 3.12 (A two level split)**.** Let

$$X = X_{00} \oplus X_{01} \oplus X_{10} \oplus X_{11}.$$

Assume that all four summands are nonzero. The four coordinate projections form a fine packet system. The two sums

$$P_0 = Q_{00} + Q_{01}, \qquad P_1 = Q_{10} + Q_{11}$$

form a coarse packet system. If $x = (x_{00}, x_{01}, x_{10}, x_{11})$, the coarse coordinates are

$$(x_{00} + x_{01}, x_{10} + x_{11})$$

when the sums are interpreted inside the two coarse direct summands. Fine analysis recovers the four entries exactly. Nothing has been approximated.

4 Residue coordinates and coprime factorization

Residue packets sort a formal series by the congruence classes of its exponents. This construction works over every coefficient ring. The character formula describes the same selectors through an operator average and needs stronger hypotheses.

Definition 4.1 (Residue selector)**.** Let R be a commutative ring and let

$$F(z) = \sum_{n \geq 0} a_n z^n \in R[[z]].$$

For $N \geq 1$ and $0 \leq j < N$, define

$$(P_{N,j} F)(z) = \sum_{n \equiv j \pmod N} a_n z^n.$$

Proposition 4.2 (Residue packet algebra)**.** *For every commutative ring R, the operators $(P_{N,j})_{0 \leq j < N}$ satisfy the identities below. If R is nonzero, they form a packet system on $R[[z]]$:*

$$P_{N,i} P_{N,j} = \delta_{ij} P_{N,j}, \qquad \sum_{j=0}^{N-1} P_{N,j} = I.$$

Proof. Every nonnegative integer belongs to one residue class modulo N. A coefficient survives both $P_{N,i}$ and $P_{N,j}$ exactly when the two labels are equal to its residue. Summing all selectors preserves every coefficient once. □

4.1 Finite orthogonality certificates

The coefficient-mask definition of $P_{N,j}$ also makes sense over a commutative semiring: it only retains or sets coefficients to zero. We use that same definition in the following semiring statement.

Let U act on coefficient sequences by

$$(UF)(z) = F(\omega z).$$

If $\omega^N = 1$, then $U^N = I$.

Theorem 4.3 (Root-of-unity realization). *Let R be a commutative semiring, let $N > 0$, and suppose $\omega, \omega^\vee, \eta \in R$ satisfy $\omega^N = 1$ and the finite $N \times N$ Fourier orthogonality table*

$$\eta \sum_{k=0}^{N-1} (\omega^\vee)^{jk} \omega^{rk} = \begin{cases} 1, & r = j, \\ 0, & r \neq j, \end{cases} \qquad 0 \leq j, r < N.$$

Then

$$\boxed{P_{N,j} = \eta \sum_{k=0}^{N-1} (\omega^\vee)^{jk} U^k.}$$

In particular, if R is a field, $N > 0$, and ω is a primitive Nth root of unity, the certificate holds with $\omega^\vee = \omega^{-1}$ and $\eta = N^{-1}$.

Proof. On the monomial z^n, the right-hand side acts by

$$\eta \sum_{k=0}^{N-1} (\omega^\vee)^{jk} \omega^{nk}.$$

The identity $\omega^N = 1$ reduces this expression to the table entry indexed by $r = n \bmod N$. The finite certificate therefore gives exactly the coefficient mask defining $P_{N,j}$. Over a field, the primitive-root geometric sum supplies the certificate. □

The field or orthogonality condition is essential. A primitive element of order N in a ring with zero divisors does not by itself force the geometric sums away from the diagonal to vanish. For example, in $\mathbb{F}_3 \times \mathbb{F}_3$ the element $(-1, 1)$ has order two, but the second component of its nontrivial character sum is nonzero. The coefficient selector uses no division and remains valid without any Fourier certificate. This distinction becomes decisive at the dyadic prime.

4.2 The field specialization

Theorem 4.4 (Character realization). *Let $\Bbbk$ be a field, assume that N is nonzero in $\Bbbk$, and suppose that $\Bbbk$ contains a primitive Nth root of unity ω. Define*

$$(UF)(z) = F(\omega z).$$

Then

$$P_{N,j} = \frac{1}{N} \sum_{k=0}^{N-1} \omega^{-jk} U^k.$$

Proof. On z^n, the right side acts by the scalar

$$\frac{1}{N}\sum_{k=0}^{N-1}\omega^{k(n-j)}.$$

This is one when $n \equiv j \pmod N$ and zero otherwise by the geometric sum identity in the field $\Bbbk$. □

Remark 4.5 (Two distinct constructions). The coefficient selector of definition 4.1 uses no division. The character formula in theorem 4.4 divides by N. At a prime that divides N, the coefficient selector may remain integral even when the character average does not.

Theorem 4.6 (Nested residue refinement). *Let N, M be positive integers with N dividing M, and let $0 \le j < N$. Then*

$$P_{N,j} = \sum_{\substack{0 \le s < M \\ s \equiv j \pmod N}} P_{M,s}.$$

When R is nonzero, the packet system modulo M refines the packet system modulo N.

Proof. An exponent n satisfies $n \equiv j \pmod N$ exactly when its unique residue s modulo M satisfies $s \equiv j \pmod N$. □

Example 4.7 (The shift graph of residue packets). Assume that R is nonzero and $N > 0$. Multiplication by z sends the residue packet $P_{N,j}R[[z]]$ into $P_{N,j+1}R[[z]]$, where the index is read modulo N. Its packet graph is the directed cycle

$$0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow N-1 \longrightarrow 0.$$

Multiplication by a polynomial supported on a set of residues gives the corresponding finite set of phase shifts. This is the simplest transfer graph behind a polyphase representation.

Theorem 4.8 (Coprime factorization). *Let A and B be coprime positive integers. For each j modulo AB, write*

$$j_A = j \pmod A, \qquad j_B = j \pmod B.$$

Then

$$P_{AB,j} = P_{A,j_A}P_{B,j_B}.$$

Every pair (j_A, j_B) occurs for one residue j modulo AB.

Proof. A coefficient survives the product on the right exactly when its exponent satisfies both congruences. The Chinese remainder theorem gives one residue class modulo AB with those two coordinates. The needed form of the theorem is standard and may be found in [5]. □

Corollary 4.9 (Dyadic and odd coordinates). *Write $N = 2^a M$ with M odd. Every residue packet modulo N is the product of one packet modulo 2^a and one packet modulo M.*

Proof. Apply theorem 4.8 to the coprime moduli 2^a and M. □

Example 4.10 (Residues modulo twelve). The pair

$$(j \pmod 4, j \pmod 3) = (3, 1)$$

corresponds to $j = 7$ modulo 12. Hence

$$P_{12,7} = P_{4,3}P_{3,1}.$$

This factorization concerns addresses. It does not say that the two factors have the same integral spectral behavior.



Figure 3. A composite residue packet is addressed by its dyadic and odd coordinates. The displayed mod-24 system is the product of mod 8 and mod 3.

5 Complete filtered transfer algebras

A filtration records how deeply a term lies in the chosen notion of precision. The weights record guaranteed filtration gains that add under multiplication. A discrete valuation ring provides the main example.

Definition 5.1 (Complete multiplicative filtration)**.** A complete filtered algebra is a unital algebra $\mathcal{B}$ with a decreasing sequence of two-sided ideals

$$\mathcal{B} = F^0\mathcal{B} \supseteq F^1\mathcal{B} \supseteq F^2\mathcal{B} \supseteq \cdots$$

such that

$$F^r\mathcal{B}\,F^s\mathcal{B} \subseteq F^{r+s}\mathcal{B}, \qquad \bigcap_{L\geq 0} F^L\mathcal{B} = 0,$$

and the natural map

$$\mathcal{B} \longrightarrow \varprojlim_{L} \mathcal{B}/F^L\mathcal{B}$$

is an isomorphism. A transfer term N_e has a declared positive integer weight $w(e) \in \mathbb{Z}_{>0}$ when

$$N_e \in F^{w(e)}\mathcal{B}.$$

In the filtration topology, reduction to $\mathcal{B}/F^L\mathcal{B}$ is continuous when the quotient has the discrete topology. Its kernel $F^L\mathcal{B}$ is therefore closed. In particular, a convergent sum of terms in $F^L\mathcal{B}$ remains in that ideal.

If $\mathcal{O}$ is a complete discrete valuation ring with uniformizer π and $\mathcal{B}$ is a separated and complete $\mathcal{O}$ algebra for the filtration

$$F^L\mathcal{B} = \pi^L\mathcal{B},$$

then this is a standard example. Finite matrix algebras over $\mathcal{O}$ satisfy these conditions. Completeness of $\mathcal{O}$ alone does not imply completeness of every $\mathcal{O}$ algebra. Basic facts about completeness in this setting are reviewed in [8].

Theorem 5.2 (Weighted inverse). *Let $\mathcal{B}$ be a complete filtered algebra. Let $D \in \mathcal{B}$ be invertible with*

$$D, D^{-1} \in F^0\mathcal{B}.$$

Let E be a finite set and suppose

$$T = D + \sum_{e\in E} N_e, \qquad N_e \in F^{w(e)}\mathcal{B}, \qquad w(e) \geq 1.$$

Set

$$A_e = D^{-1}N_e.$$

For a word $\gamma = (e_1, \dots, e_r)$, write

$$A_\gamma = A_{e_r} \cdots A_{e_1}, \qquad W(\gamma) = \sum_{j=1}^{r} w(e_j).$$

The empty word has weight zero and contributes the identity. Then T is invertible and

$$T^{-1} = \sum_{\gamma} (-1)^{|\gamma|} A_\gamma D^{-1}.$$

The series converges in the filtration, and the term indexed by γ belongs to $F^{W(\gamma)}\mathcal{B}$.

Proof. Put

$$A = \sum_{e\in E} A_e.$$

Since left multiplication by D^{-1} preserves the filtration, $A \in F^1\mathcal{B}$. Completeness gives the convergent Neumann series

$$(I + A)^{-1} = \sum_{r\geq 0} (-A)^r.$$

Because $T = D(I + A)$,

$$T^{-1} = (I + A)^{-1}D^{-1}.$$

Expanding each power of A gives the sum over words, with the rightmost edge acting first. Multiplicativity of the filtration gives

$$A_\gamma \in F^{W(\gamma)}\mathcal{B}.$$

For each L, only finitely many words have weight below L because E is finite and every edge has positive weight. Thus the word expansion converges to the same inverse. □

The preceding formula does not yet use packet blocks. When D is packet diagonal and the terms N_e are individual transfers, only words with compatible packet endpoints can contribute. These words become directed paths.

Theorem 5.3 (Block path formula)**.** *Let $\mathcal{P} = (P_\alpha)_{\alpha\in I}$ be a finite packet system in a complete filtered algebra of operators. Suppose*

$$D = \sum_\alpha P_\alpha D P_\alpha$$

belongs to $\mathcal{B}$ and is invertible on every packet. Assume that the assembled inverse

$$D^{-1} = \sum_\alpha D_\alpha^{-1} P_\alpha \in \mathcal{B} = F^0\mathcal{B}, \qquad D_\alpha = P_\alpha D P_\alpha,$$

where each packet inverse is extended by zero to the other packets. Let the edge set be finite, assume $w(e) \geq 1$ for every edge, and write

$$N = \sum_{e:\alpha\to\beta} N_e, \qquad N_e = P_\beta N_e P_\alpha, \qquad N_e \in F^{w(e)}\mathcal{B}.$$

For an edge $e : \alpha \to \beta$, put

$$A_e = (P_\beta D P_\beta)^{-1} N_e.$$

Then the block from $P_\alpha X$ to $P_\beta X$ of $(D + N)^{-1}$ is

$$P_\beta (D+N)^{-1} P_\alpha = \sum_{\gamma:\alpha\rightsquigarrow\beta} (-1)^{|\gamma|} A_\gamma D_\alpha^{-1},$$

where $D_\alpha = P_\alpha D P_\alpha$ and the sum runs over directed paths from α to β, including the empty path when $\alpha = \beta$.

Proof. The diagonal inverse is

$$D^{-1} = \sum_\alpha D_\alpha^{-1} P_\alpha.$$

The block $P_\beta D^{-1} N P_\alpha$ is nonzero only when there is a declared edge from α to β. Repeated block multiplication shows that the rth power of $D^{-1}N$ is the sum over all directed paths of length r, with the operator factors ordered along the path. Apply theorem 5.2 and multiply on the right by the packet diagonal inverse. □

Corollary 5.4 (Minimum path weight)**.** *For packets α and β, let $d(\alpha, \beta)$ be the least total weight of a directed path from α to β, with $d = \infty$ when no path exists. Use the convention $F^\infty\mathcal{B} = \bigcap_{L\geq 0} F^L\mathcal{B} = 0$. If $\alpha \neq \beta$, then*

$$P_\beta (D+N)^{-1} P_\alpha \in F^{d(\alpha,\beta)}\mathcal{B}.$$

If there is no path from α to β, the block is zero.

Proof. Every term in the block path formula is indexed by a path from α to β and lies in the filtration level given by its total weight. If no such path exists, the sum is empty. □

Example 5.5 (A longer path that appears earlier)**.** Consider three packets α, β, and γ. Suppose there is a direct edge from α to γ of weight three, together with edges

$$\alpha \longrightarrow \beta, \qquad \beta \longrightarrow \gamma$$

of weight one each. The two edge route has total weight two, so it contributes to the inverse before the direct edge does. This is why the truncation rule uses total weight rather than path length. The lower bound in corollary 5.4 need not be an equality because several paths of the same weight may cancel.

> **The central idea**
>
> Word length counts how many transfers are composed. Weight counts how much filtration gain those transfers guarantee. The sign in the inverse expansion depends on length, while the precision cutoff depends on weight. A short word can therefore disappear at a chosen precision while a longer word still contributes.

Corollary 5.6 (Acyclic transfer graph)**.** *If the packet transfer graph is acyclic, then $D^{-1}N$ is nilpotent. The inverse in theorem 5.3 is a finite sum and does not require completeness of the filtration.*

Proof. A finite acyclic directed graph has a bound on path length. Every sufficiently long product of transfer blocks is therefore zero. The Neumann series terminates. □

Corollary 5.7 (Stability of inversion at finite precision)**.** *Let T and $\widetilde{T}$ be invertible elements of $F^0\mathcal{B}$. If*

$$T - \widetilde{T} \in F^L\mathcal{B},$$

then

$$T^{-1} - \widetilde{T}^{-1} \in F^L\mathcal{B}.$$

Proof. Use the resolvent identity at zero in the form

$$T^{-1} - \widetilde{T}^{-1} = T^{-1}(\widetilde{T} - T)\widetilde{T}^{-1}.$$

Both inverse factors belong to $F^0\mathcal{B}$, so the product lies in $F^L\mathcal{B}$. □

Corollary 5.8 (Exact finite precision truncation)**.** *Under the hypotheses of theorem 5.2, define*

$$R_L = \sum_{W(\gamma)<L} (-1)^{|\gamma|} A_\gamma D^{-1}.$$

Then

$$T^{-1} - R_L \in F^L\mathcal{B},$$

$$TR_L - I \in F^L\mathcal{B}, \qquad R_L T - I \in F^L\mathcal{B}.$$

Thus R_L is the exact inverse in $\mathcal{B}/F^L\mathcal{B}$.

Proof. Every omitted word has weight at least L, so the difference from the full inverse lies in $F^L\mathcal{B}$. Multiplication by T or by its inverse preserves membership in $F^L\mathcal{B}$ under the stated filtration assumptions. The two residual statements follow. □

Example 5.9 (A triangular transfer)**.** Let $\mathcal{O}$ be complete with uniformizer π, and consider

$$T = \begin{pmatrix} 1 & \pi^a \\ 0 & 1 \end{pmatrix}$$

with $a \geq 1$. The only nontrivial transfer has weight a, and its square is zero. Hence

$$T^{-1} = \begin{pmatrix} 1 & -\pi^a \\ 0 & 1 \end{pmatrix}.$$

At precision $L \leq a$, the transfer vanishes and the inverse is the identity modulo π^L. At depth $a+1$, the transfer between the two packets first becomes visible.

Two compatible word conventions. The path expansion above labels a word in chronological order and writes its operator product in reverse order, so the first edge acts first. The following finite-word contract uses written multiplication order. Reversing all word labels identifies the sums over all words. Individual path labels must retain their stated convention. The two conventions do not change a weight or an inverse.

5.1 Finite quotient and normed-ring realizations

5.1.1 Normalized edges and written-word convention

Let $\mathcal{O}$ be a complete discrete valuation ring with uniformizer π, fraction field K, and filtration

$$F^m\mathcal{B} = \pi^m\mathcal{B}$$

on an $\mathcal{O}$-algebra $\mathcal{B}$. The finite quotient statements below do not require $\mathcal{B}$ itself to be complete. Suppose

$$T = D + H,$$

where D is invertible and the perturbation H raises filtration. The symbol K is reserved for the fraction field.

A packet decomposition resolves $H = \sum_{e\in E} H_e$ and assigns a certified filtration gain

$$D^{-1}H_e \in F^{w(e)}\mathcal{B}, \qquad w(e) \in \mathbb{Z}_{>0}.$$

Each e is a directed transfer edge. For an ordered transfer word

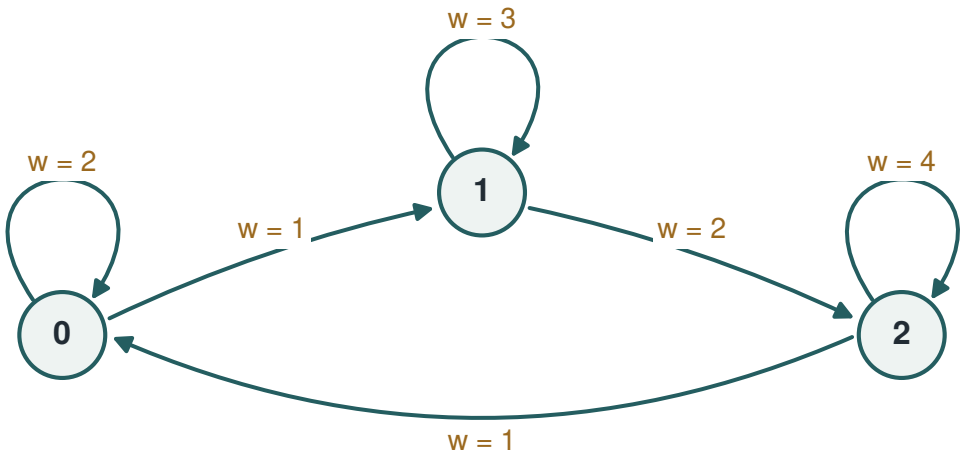
$$\gamma = (e_1, \dots, e_r),$$

define

$$W(\gamma) = \sum_{j=1}^{r} w(e_j), \qquad A_\gamma = A_{e_1}\cdots A_{e_r}, \qquad A_e = D^{-1}H_e.$$

Weighted transfer graph

Edge weights measure filtration gain. Every edge has multiplicity one.



The weighted walk series below starts at packet 0 and ends at packet 2.

Figure 4. A finite packet-transfer graph. Edge weights record filtration gain rather than merely the number of operator compositions.

Theorem 5.10 (Certified weighted inverse)**.** *Let R be a possibly noncommutative ring, let D^{-1} be a supplied two-sided inverse of D, and put $A = D^{-1}H$.*

(a) If $A^r = 0$, then $D + H$ has the two-sided inverse

$$\boxed{(D+H)^{-1} = \left(\sum_{k=0}^{r-1}(-A)^k\right) D^{-1}.}$$

(b) If R is a complete normed ring and $\|A\| < 1$, then $\sum_{k\geq 0}(-A)^k$ converges and

$$\boxed{(D+H)^{-1} = \left(\sum_{k\geq 0}(-A)^k\right) D^{-1}.}$$

Proof. Factor

$$T = D(I + A), \qquad A = D^{-1}H.$$

For a finite cutoff r, multiplication on either side gives the noncommutative telescoping identity

$$(I+A)\sum_{k=0}^{r-1}(-A)^k = \left(\sum_{k=0}^{r-1}(-A)^k\right)(I+A) = 1 - (-A)^r.$$

This proves (a). In (b), the strict norm bound makes the geometric series summable. Continuity of multiplication passes the same identities to the limit. Multiplication by D^{-1} gives the displayed inverse in both orders. □

Corollary 5.11 (Finite precision). *Let $F^\bullet R$ be a multiplicative decreasing filtration and let $q_L : R \to R_L$ be a ring homomorphism that kills $F^L R$. Suppose*

$$H = \sum_{e \in E} H_e, \qquad D^{-1} H_e \in F^{w(e)} R, \qquad w(e) \geq 1,$$

with E finite. Then $q_L(D^{-1}H)^L = 0$, the finite sum in theorem 5.10(a) is a two-sided inverse after applying q_L, and its word expansion may discard every word of weight at least L. More explicitly, for a length cutoff N, define

$$S^{\mathrm{pr}}_{L,N} = \sum_{n=0}^{N-1} (-1)^n \sum_{\substack{\gamma \in E^n \\ W(\gamma) < L}} A_\gamma.$$

Lean proves directly that $q_L(S^{\mathrm{pr}}_{L,N})$ equals the mapped geometric inverse whenever $q_L(A^N) = 0$, and hence is a two-sided inverse of $q_L(I + A)$, where $A = \sum_e A_e = D^{-1}H$. With the supplied two-sided inverse D^{-1} of D, this gives the explicit two-sided inverse

$$\boxed{q_L\left(S^{\mathrm{pr}}_{L,N} D^{-1}\right)} \quad \textit{of} \quad q_L(D + H).$$

If every edge weight is at least m and $L \leq mN$, then the nilpotence condition follows from the filtration data because $m|\gamma| \leq W(\gamma)$. For $m > 0$, the executable choice $N = \lceil L/m \rceil$ satisfies $L \leq mN$. This elementary ceiling inequality is also proved in Lean. The choice $m = 1, N = L$ gives the standard cutoff. In the DVR example $q_L : R \to R/\pi^L R$, this reads

$$q_L(T)^{-1} = q_L\left(\sum_{W(\gamma) < L} (-1)^{|\gamma|} A_\gamma D^{-1} \right).$$

Here $A_\gamma = (D^{-1}H_{e_1}) \cdots (D^{-1}H_{e_r})$ in the written order. Multiplicativity places it in $F^{W(\gamma)}R$, so the quotient kills it when $W(\gamma) \geq L$. Positive weights also imply $|\gamma| \leq W(\gamma)$, making the retained search finite.

> **How to read the result**
>
> Using the standard length cutoff, every discarded word has weight at least the chosen depth and vanishes in the quotient. The retained sum therefore satisfies both inverse identities exactly at that precision. The earlier truncation theorem states the corresponding residual bounds in the complete filtered algebra itself.

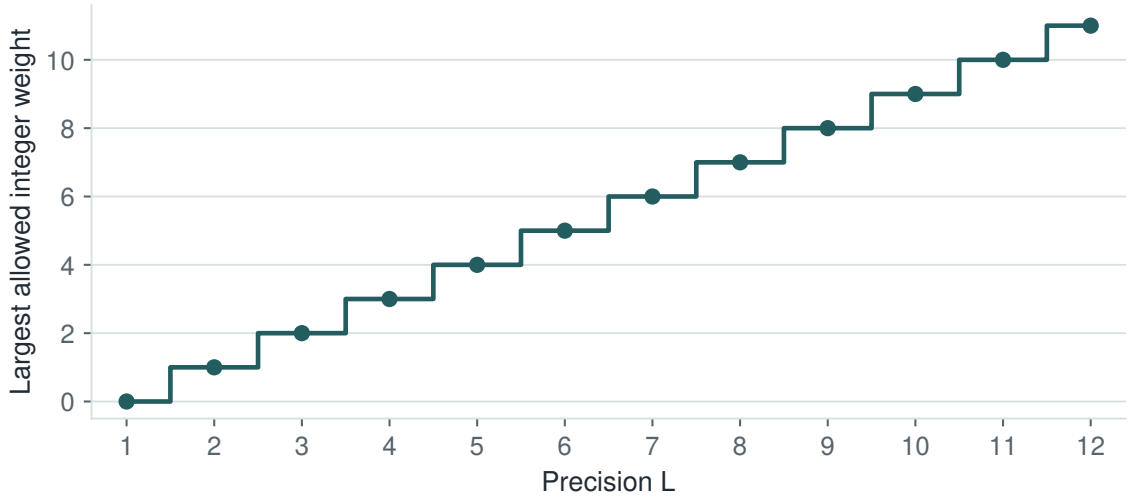
The inherited finite-word contract proves the result in the finite quotients above and in the contraction specialization for complete normed rings. The complete filtered statement has its own separation and inverse-limit completeness hypotheses. Its current Lean coverage is recorded separately in the current correspondence.

6 Rational weighted transfer series

A finite weighted graph records which path terms can occur in the inverse. Collecting paths by total weight turns this support information into a generating function. The finite adjacency matrix makes that function rational.

Finite filtration cutoff

At precision L, only transfer words with total weight less than L contribute.



This is the support cutoff L – 1. It does not count the number of surviving walks.

Figure 5. At precision L, weighted inversion requires only transfer words whose cumulative weight is below L.

Definition 6.1 (Weighted transfer series)**.** Let G be a finite directed multigraph. Give every edge e a positive integer weight $w(e)$ and a scalar multiplicity $m(e)$ in a field $\mathbb{k}$. Define the polynomial adjacency matrix

$$A(t)_{ij} = \sum_{e:i\to j} m(e)t^{w(e)}.$$

For column vectors u and v, define

$$Z_{u,v}(t) = \sum_{n\geq 0} a_n t^n,$$

where

$$a_n = \sum_{i,j} \sum_{\substack{\gamma:i\rightsquigarrow j \\ W(\gamma)=n}} u_i v_j \prod_{\nu=1}^{|\gamma|} m(e_\nu).$$

Here $\gamma = (e_1, \dots, e_{|\gamma|})$ is a directed walk. Vertices and edges may repeat, and the empty walk has weight zero and multiplicity one. Thus the endpoint vectors weight the start and end of each walk, rather than only selecting subsets of vertices.

Theorem 6.2 (Rationality of the transfer series)**.** *The weighted transfer series satisfies*

$$Z_{u,v}(t) = u^{\mathsf{T}}(I - A(t))^{-1}v.$$

It is a rational function whose denominator divides

$$\det(I - A(t)).$$

Proof. Since every entry of $A(t)$ has zero constant term, the inverse exists in the formal power series ring and

$$(I - A(t))^{-1} = \sum_{r \geq 0} A(t)^r.$$

The (i, j) entry of $A(t)^r$ is the sum of the monomials attached to all paths of length r from i to j. Summing over r records paths of every length. Multiplication by the endpoint vectors gives the stated series. The adjugate formula gives

$$(I - A(t))^{-1} = \frac{\operatorname{adj}(I - A(t))}{\det(I - A(t))},$$

so every entry is rational. □

Corollary 6.3 (Recurrence for path counts)**.** *Write*

$$Z_{u,v}(t) = \frac{P(t)}{Q(t)}, \qquad Q(t) = 1 + q_1 t + \cdots + q_d t^d.$$

Use the convention $\deg 0 = -\infty$*. Then, for every nonnegative integer* $n > \deg P$*,*

$$a_n + q_1 a_{n-1} + \cdots + q_d a_{n-d} = 0.$$

Proof. Compare coefficients in the identity

$$Q(t) Z_{u,v}(t) = P(t).$$

Beyond the degree of P, the coefficient on the right is zero. □

In the recurrence, set $a_k = 0$ when $k < 0$. This is the coefficient convention for the formal power series $Z_{u,v}(t)$ and makes the first recurrence equations meaningful even when $n < d$.

Corollary 6.4 (Several independent weights)**.** *Suppose every edge has a weight vector*

$$w(e) = (w_1(e), \dots, w_r(e)) \in \mathbb{N}^r \setminus \{0\}.$$

Replace $t^{w(e)}$ *by*

$$t_1^{w_1(e)} \cdots t_r^{w_r(e)}.$$

The resulting multigraded transfer series is rational in $t_1, \dots, t_r$*.*

Proof. The same matrix inverse proof works over the formal power series ring in several variables. The adjugate formula gives rationality. □

The multigraded form is useful when dyadic depth, odd depth, and another filtration are tracked separately rather than collapsed into one number.

Example 6.5 (A transfer series with one loop)**.** Take vertices 0, 1, and 2. Give the edge $0 \to 1$ weight one, the loop $1 \to 1$ weight two, and the edge $1 \to 2$ weight one. Every path from 0 to 2 uses the first and last edges and may traverse the loop any number of times. Hence

$$Z_{0,2}(t) = t^2(1 + t^2 + t^4 + \cdots) = \frac{t^2}{1 - t^2}.$$

There is one path at every positive even weight and no path at an odd weight. The denominator records the increase of two units produced by the loop.

6.1 Automatic finite-graph construction

Let G be a finite directed graph with positive integer edge weights. Define the polynomial adjacency matrix

$$A(t)_{ij} = \sum_{e: i \to j} t^{w(e)},$$

where parallel edges are distinct elements of the finite edge type and are therefore counted with their full multiplicity.

For fixed start and target vertices s, t, define

$$Z_{s,t}(t) = \sum_{n \geq 0} a_n(s,t) t^n,$$

where $a_n(s,t)$ counts directed walks from s to t with total weight n. Vertices and edges may repeat. Thus "path zeta" is retained as the name of the series and Lean API, but the walks are not required to visit each vertex only once.

Theorem 6.6 (Automatic finite-graph transfer-zeta rationality)**.** *Assume the vertex and edge sets are finite and every edge has positive integer weight. For every n, exact enumeration of edge tuples of lengths at most n, followed by endpoint and total-weight filtering, returns $a_n(s,t)$. These coefficients obey the recurrence obtained by removing the first edge*

$$a_n(s,t) = \mathbf{1}_{s=t,\, n=0} + \sum_{\substack{e:\, \mathrm{source}(e)=s \\ w(e) \leq n}} a_{n-w(e)}(\mathrm{target}(e), t).$$

Let

$$T(t) = I - A(t), \qquad Q(t) = \det T(t), \qquad P_{s,t}(t) = \mathrm{adj}(T(t))_{s,t}.$$

Then $Q(0) = 1$, hence $Q \neq 0$ and Q is a unit in $\mathbb{Z}[[t]]$, and the independently enumerated walk series satisfies

$$\boxed{Z_{s,t}(t) = Q(t)^{-1} P_{s,t}(t), \qquad Q(t) Z_{s,t}(t) = P_{s,t}(t)} \quad \text{in } \mathbb{Z}[[t]].$$

Proof. Every edge has weight at least one, so a word of weight n has length at most n. The enumerator therefore searches a finite set containing every contributing walk, and its filters are exactly the defining predicates for a_n. Removing the first edge gives the displayed disjoint decomposition into the empty diagonal walk and the first-edge tails. In matrix form this is

$$Z = I + AZ, \qquad (I - A)Z = I.$$

Positive weights give $A(0) = 0$, so $T(0) = I$ and $Q(0) = \det(I) = 1$. The two adjugate identities

$$T \,\mathrm{adj}(T) = QI, \qquad \mathrm{adj}(T) T = QI$$

remain valid after inclusion into $\mathbb{Z}[[t]]$. Multiplying the adjugate by the formal-series inverse of Q constructs a two-sided inverse of T. Uniqueness of the inverse identifies that construction with the independently enumerated matrix Z, yielding the entrywise formulas. □

This is the finite transfer-matrix method of [9], but here its determinant, adjugate, normalization, recurrence from the first edge, and identification with the enumerated series are all formalized. The older API accepting an independently supplied normalized coefficient identity $QZ = P$ is retained for external certificates. It is no longer needed to obtain rationality for the finite graph defined above. The Python companion performs an exact symbolic construction as an independent executable check.

Corollary 6.7 (Linear recurrence)**.** *With the canonical polynomials from theorem 6.6, write*

$$Q(t) = \sum_k q_k t^k, \qquad Z_{s,t}(t) = \sum_{n \geq 0} a_n(s,t) t^n.$$

Then the walk counts satisfy the recurrence with constant coefficients determined by Q after the finite initial range contributed by $P_{s,t}$, with $\deg 0 = -\infty$:

$$\boxed{\sum_{k=0}^{n} q_k a_{n-k}(s,t) = 0 \qquad (n > \deg P_{s,t}).}$$



Figure 6. The walk counts for one weighted graph. The exact generating function is rational, so these counts satisfy a finite linear recurrence.

Example 6.8 (A computed transfer zeta)**.** For the graph used in the numerical companion, the walk series from packet 0 to packet 2 is

$$Z_{0,2}(t) = \frac{t^3}{1 - t^2 - t^3 - 2t^4 + t^5 + t^6 + t^7 - t^9}.$$

The first nonzero coefficients are

$$1, 1, 1, 3, 1, 5, 4, 10, 7, 19, 17, 37, \ldots$$

at their corresponding weights. Direct enumeration through weight 30 agrees coefficient by coefficient with the rational expansion.

7 Rational spectral projectors and resultant bounds

Let $\mathcal{O}$ be a discrete valuation ring with fraction field K and uniformizer π. Let Λ be a finite free $\mathcal{O}$ module and put

$$V = K \otimes_{\mathcal{O}} \Lambda.$$

Suppose $A \in \mathrm{End}_{\mathcal{O}}(\Lambda)$.

Theorem 7.1 (Bézout spectral projector)**.** *Assume that*

$$\chi_A(x) = f(x)g(x)$$

with monic polynomials $f, g \in \mathcal{O}[x]$ that are coprime in $K[x]$. Choose $u, v \in \mathcal{O}[x]$ and a nonzero $r \in \mathcal{O}$ such that

$$u(x)f(x) + v(x)g(x) = r.$$

Then

$$E_f = r^{-1}v(A)g(A)$$

is the projector of V onto $\operatorname{Ker} f(A)$ *along* $\operatorname{Ker} g(A)$*. In particular,*

$$E_f^2 = E_f, \qquad rE_f \in \mathrm{End}_{\mathcal{O}}(\Lambda).$$

Proof. Cayley Hamilton gives

$$f(A)g(A) = 0.$$

If $x \in \operatorname{Ker} f(A)$, then the Bézout identity gives

$$v(A)g(A)x = rx,$$

so $E_f x = x$. If $x \in \operatorname{Ker} g(A)$, then $E_f x = 0$. The same identity gives the direct sum decomposition

$$V = \operatorname{Ker} f(A) \oplus \operatorname{Ker} g(A).$$

Indeed, every $x \in V$ is the sum of

$$r^{-1}v(A)g(A)x \in \operatorname{Ker} f(A)$$

and

$$r^{-1}u(A)f(A)x \in \operatorname{Ker} g(A).$$

Thus E_f is the stated projector and is idempotent. The formula for rE_f has coefficients in $\mathcal{O}$, so it preserves Λ. □

Theorem 7.2 (Several coprime spectral packets)**.** *Assume*

$$\chi_A(x) = \prod_{i=1}^{r} f_i(x),$$

where the monic factors $f_i \in \mathcal{O}[x]$ are pairwise coprime in $K[x]$. Put

$$g_i = \prod_{j \neq i} f_j.$$

There are complementary projectors E_i with

$$\operatorname{im} E_i = \operatorname{Ker} f_i(A), \qquad \sum_i E_i = I, \qquad E_i E_j = 0 \quad (i \neq j).$$

One may choose E_i so that

$$\operatorname{Res}(f_i, g_i) E_i \in \operatorname{End}_{\mathcal{O}}(\Lambda)$$

up to multiplication of the resultant by a unit.

Proof. Apply theorem 7.1 to the factorization $\chi_A = f_i g_i$. The resulting projector is one on $\operatorname{Ker} f_i(A)$ and zero on every $\operatorname{Ker} f_j(A)$ with $j \neq i$. The direct sum decomposition follows from the polynomial Chinese remainder theorem. The orthogonality and sum identities follow by evaluating the corresponding idempotents in $K[x]/(\chi_A)$. The Sylvester Bézout relation for (f_i, g_i) gives the stated denominator bound. □

Corollary 7.3 (Resultant denominator bound)**.** *Under the hypotheses of theorem 7.1, one may take*

$$r = \operatorname{Res}(f, g)$$

up to multiplication by a unit. Therefore

$$\pi^{\operatorname{val}_\pi(\operatorname{Res}(f,g))} E_f \Lambda \subseteq \Lambda.$$

Proof. The Sylvester matrix construction gives a Bézout relation with right side equal to the resultant. Apply theorem 7.1. Standard facts about resultants may be found in [5, 3]. □

Example 7.4 (A resultant bound that is not exact)**.** Let

$$A = \operatorname{diag}(0, \pi^s)$$

on the standard lattice $\Lambda = \mathcal{O}^2$. The factors x and $x - \pi^s$ have resultant $-\pi^s$, up to the harmless choice of resultant order, but the two spectral projectors are the integral coordinate projectors. Their exact separation weight is zero. The resultant sees the closeness of the two eigenvalues. It does not see that the chosen lattice already splits along their eigenspaces.

How to read the result

A resultant bounds the denominator through the characteristic factors. The exact separation weight also depends on how the lattice sits inside the rational packet spaces. These are two stages

of the calculation. First construct a rational projector with a known clearing denominator. Then measure the gluing of the chosen lattice to find the least denominator that actually works.

7.1 Evaluated certificates and explicit lattice action

7.1.1 Minimal evaluated Bézout identities

Let $\mathcal{O}$ be a DVR, $K = \mathrm{Frac}(\mathcal{O})$, and $A \in \mathrm{End}_{\mathcal{O}}(\Lambda)$ for a finite free lattice Λ. Suppose the characteristic polynomial factors over $\mathcal{O}$ as

$$\chi_A(x) = f(x)g(x),$$

where f and g are coprime over $K[x]$.

Theorem 7.5 (Evaluated Bézout spectral projector). *Let $F, G, U, V \in \mathrm{End}_K(\Lambda \otimes_{\mathcal{O}} K)$ and let $r \in K^\times$. Suppose*

$$UF + VG = rI, \qquad FG = 0, \qquad FG = GF, \qquad GU = UG.$$

Put $N = VG$. Then

$$\boxed{N^2 = rN.}$$

Consequently the operator

$$\boxed{E = r^{-1}VG}$$

is idempotent, acts as the identity on $\ker F$*, and acts as zero on* $\ker G$*. If V and G preserve Λ, then*

$$rE = VG \in \mathrm{End}_{\mathcal{O}}(\Lambda).$$

Proof. The Bézout identity gives $N = rI - UF$. The two stated commutation relations and $FG = 0$ give

$$N(UF) = VGUF = VUGF = VUFG = 0.$$

Therefore

$$N^2 = N(rI - UF) = rN.$$

Since $r \neq 0$, $E^2 = r^{-2}N^2 = E$. If $Fx = 0$, the evaluated Bézout identity gives $VGx = rx$, hence $Ex = x$. If $Gx = 0$, then $Ex = 0$. The lattice assertion follows directly from $rE = VG$. □

The equality $N^2 = rN$ before division uses neither $r \neq 0$ nor the other pairwise commutation relations that hold in the polynomial application. Normalizing to E uses $r \neq 0$. Its action on $\ker F$ uses only the Bézout identity and $r \neq 0$, while vanishing on $\ker G$ follows immediately from $N = VG$. The bundled Lean certificate records all pairwise commutations as convenient evaluated polynomial data, but the minimal theorems expose only the hypotheses they consume.

For the polynomial application, one supplies $F = f(A)$, $G = g(A)$, $U = u(A)$, and $V = v(A)$. A Bézout or resultant identity $uf + vg = r$ evaluates to the first certificate, while $(fg)(A) = 0$ supplies the second. When $fg = \chi_A$, this last equality is Cayley Hamilton. Standard Bézout, resultant, and DVR facts are recalled from [5, 8]. The inherited certificate theorem begins with the evaluated certificate. It does not infer exact action merely from nilpotence of $f(A)$ or $g(A)$, which would be insufficient.

Definition 7.6 (Separation weight)**.** For a rational projector E and a specified $\mathcal{O}$ lattice Λ, provided at least one clearing exponent exists, define

$$s_{\pi,\Lambda}(E) = \min\{s \geq 0 : \pi^s E(\Lambda) \subseteq \Lambda\}.$$

If the evaluated certificate arises from integral v, g and $r = u\pi^s$ with u^{-1} preserving the lattice, then theorem 7.5 gives the upper bound

$$s_{\pi,\Lambda}(E) \leq s = \mathrm{val}_\pi(r).$$

Indeed,

$$\boxed{\pi^s E = u^{-1} V G}$$

preserves Λ. The minimal Lean theorem needs only the lattice action of u^{-1}. The bundled wrapper for lattice units records both u and u^{-1}. A nonzero scalar in the fraction field need not be an integral unit. Existence of a clearing exponent is an explicit hypothesis. Once one exists, the least one is obtained from the least element property of the natural numbers. Clearing exponents are upward closed provided multiplication by π preserves the lattice.

7.1.2 Certificate-level precision delay

Theorem 7.7 (Projector precision delay)**.** *Let E be a rational projector with $\pi^s E(\Lambda) \subseteq \Lambda$. If*

$$x \equiv y \pmod{\pi^{m+s}\Lambda},$$

then

$$Ex \equiv Ey \pmod{\pi^m \Lambda}.$$

Thus $m + s$ input digits suffice to determine m output digits of the projected packet.

Proof. Write $x - y = \pi^{m+s} z$. Since $\pi^s E(z) \in \Lambda$,

$$E(x - y) = \pi^m(\pi^s E(z)) \in \pi^m \Lambda.$$

□

The theorem gives a precision budget for delayed separation. A packet with weight s can be reconstructed at the requested output precision by retaining s additional input digits.

8 Exact lattice separation

The exact denominator belongs to the pair consisting of the rational packet decomposition and the integral lattice.

Definition 8.1 (Lattice packet data)**.** Let

$$V = \bigoplus_{\alpha \in I} V_\alpha$$

be a finite direct sum over K, with packet projectors E_α. Let $\Lambda \subseteq V$ be an $\mathcal{O}$ lattice. Put

$$\Lambda_\alpha = \Lambda \cap V_\alpha, \qquad \Lambda_{\text{split}} = \bigoplus_\alpha \Lambda_\alpha,$$

and define the gluing module

$$G_{\Lambda,\mathcal{P}} = \Lambda / \Lambda_{\text{split}}.$$

The separation weight of E_α is

$$s_\Lambda(E_\alpha) = \min\{s \geq 0 : \pi^s E_\alpha \Lambda \subseteq \Lambda\}.$$

The global separation weight is

$$s_\Lambda(\mathcal{P}) = \max(\{0\} \cup \{s_\Lambda(E_\alpha) : \alpha \in I\}).$$

For a finitely generated torsion $\mathcal{O}$ module G, write $\exp_\pi(G)$ for the least s such that $\pi^s G = 0$.

The gluing module $G_{\Lambda,\mathcal{P}}$ is a finitely generated torsion $\mathcal{O}$ module, hence has finite length, because Λ_{split} and Λ are full lattices in the same vector space. To see that the split lattice is full, clear denominators of bases of the rational summands. Each $\Lambda \cap V_\alpha$ then spans V_α, so their direct sum spans V.

Theorem 8.2 (Gluing module and exact separation weight)**.** *With the notation of definition 8.1, the map*

$$\Phi : G_{\Lambda,\mathcal{P}} \longrightarrow \bigoplus_{\alpha \in I} E_\alpha \Lambda / \Lambda_\alpha, \qquad [x] \longmapsto (E_\alpha x + \Lambda_\alpha)_\alpha$$

is injective, and every coordinate projection of its image is surjective. We also have

$$\boxed{\exp_\pi(G_{\Lambda,\mathcal{P}}) = s_\Lambda(\mathcal{P}).}$$

If

$$G_{\Lambda,\mathcal{P}} \cong \bigoplus_{j=1}^{r} \mathcal{O}/\pi^{a_j}\mathcal{O}$$

with $a_1 \leq \cdots \leq a_r$, then

$$s_\Lambda(\mathcal{P}) = \max(\{0\} \cup \{a_j : 1 \leq j \leq r\}).$$

Proof. If $x \in \Lambda$ maps to zero, then $E_\alpha x \in \Lambda_\alpha$ for every α. Since

$$x = \sum_\alpha E_\alpha x,$$

we have $x \in \Lambda_{\text{split}}$. Thus Φ is injective. The coordinate map onto $E_\alpha \Lambda / \Lambda_\alpha$ is surjective by the definition of $E_\alpha \Lambda$.

Let $s = s_\Lambda(\mathcal{P})$. For every $x \in \Lambda$ and every α,

$$\pi^s E_\alpha x \in \Lambda_\alpha.$$

Hence

$$\pi^s x = \sum_\alpha \pi^s E_\alpha x \in \Lambda_{\text{split}},$$

so $\pi^s G_{\Lambda,\mathcal{P}} = 0$. This proves

$$\exp_\pi(G_{\Lambda,\mathcal{P}}) \le s.$$

If $d < s$, then for some α there is an $x \in \Lambda$ with

$$\pi^d E_\alpha x \notin \Lambda_\alpha.$$

The class $\pi^d[x]$ is therefore nonzero in the gluing module. Thus π^d does not annihilate the gluing module, and equality follows. The final formula is the usual description of the exponent through Smith normal form. □

Corollary 8.3 (Total gluing and worst precision)**.** *If*

$$G_{\Lambda,\mathcal{P}} \cong \bigoplus_{j=1}^{r} \mathcal{O}/\pi^{a_j}\mathcal{O},$$

then

$$\text{length}_{\mathcal{O}}(G_{\Lambda,\mathcal{P}}) = \sum_{j=1}^{r} a_j, \qquad s_\Lambda(\mathcal{P}) = \max(\{0\} \cup \{a_j : 1 \le j \le r\}).$$

Thus the lattice index measures the total amount of gluing, while the separation weight measures the worst precision loss in one direction.

Proof. The length and exponent of a finitely generated torsion module are read directly from its Smith decomposition. Apply theorem 8.2. □

Corollary 8.4 (Two complementary packets)**.** *Suppose*

$$V = V_1 \oplus V_2$$

with projector E onto V_1 and complementary projector $I - E$. Then

$$\Lambda/(\Lambda_1 \oplus \Lambda_2) \cong E\Lambda/\Lambda_1 \cong (I - E)\Lambda/\Lambda_2.$$

Consequently

$$s_\Lambda(E) = s_\Lambda(I - E) = \exp_\pi(\Lambda/(\Lambda_1 \oplus \Lambda_2)).$$

Proof. Map $x \in \Lambda$ to $Ex + \Lambda_1$. Its kernel consists of the x for which $Ex \in \Lambda_1$. In that case

$$(I - E)x = x - Ex \in \Lambda \cap V_2 = \Lambda_2,$$

so the kernel is $\Lambda_1 \oplus \Lambda_2$. The map is surjective by definition. The complementary map is handled in the same way. The equality of weights follows from theorem 8.2. □

Corollary 8.5 (Integral splitting criterion)**.** *Every packet projector preserves Λ if and only if*

$$\Lambda = \bigoplus_{\alpha} (\Lambda \cap V_\alpha).$$

Equivalently,

$$s_\Lambda(\mathcal{P}) = 0 \quad \Longleftrightarrow \quad G_{\Lambda,\mathcal{P}} = 0.$$

Proof. If every projector preserves Λ, then

$$x = \sum_{\alpha} E_\alpha x$$

expresses every $x \in \Lambda$ as a sum of vectors in Λ_α. The converse is immediate because each projector is the identity on one direct summand and zero on the others. The final equivalence follows from theorem 8.2. □

8.1 Refinement and separation weight

A finer rational packet system imposes more integral splitting conditions.

Theorem 8.6 (Gluing exact sequence under refinement)**.** *Let $\mathcal{Q} = (Q_\beta)$ refine $\mathcal{P} = (P_\alpha)$ on V, and let Λ be a lattice. Define*

$$\Lambda_{\mathcal{P}} = \bigoplus_{\alpha} (\Lambda \cap P_\alpha V), \qquad \Lambda_{\mathcal{Q}} = \bigoplus_{\beta} (\Lambda \cap Q_\beta V).$$

Then

$$\Lambda_{\mathcal{Q}} \subseteq \Lambda_{\mathcal{P}} \subseteq \Lambda$$

and there is a short exact sequence

$$0 \longrightarrow \Lambda_{\mathcal{P}}/\Lambda_{\mathcal{Q}} \longrightarrow \Lambda/\Lambda_{\mathcal{Q}} \longrightarrow \Lambda/\Lambda_{\mathcal{P}} \longrightarrow 0.$$

In particular,

$$\boxed{s_\Lambda(\mathcal{P}) \le s_\Lambda(\mathcal{Q}).}$$

Proof. Every fine packet space lies in its parent packet space, so every element of $\Lambda_{\mathcal{Q}}$ belongs to $\Lambda_{\mathcal{P}}$. The displayed exact sequence is the standard quotient sequence for nested submodules. The exponent of a quotient cannot exceed the exponent of the module from which it is obtained. Apply theorem 8.2 to the two gluing modules. □

Example 8.7 (Strict growth under refinement)**.** Take the standard lattice $\mathbb{Z}_2^2$ and the involution from proposition 9.1. The one packet system (I) has weight zero. Refining it into the two rational sign packets produces weight one. The coarse lattice is already split because there is only one summand. The fine eigenspaces are glued together by one dyadic digit.

Remark 8.8 (Strict growth can occur)*.* A coarse packet family may preserve the lattice even though a fine spectral split does not. Refinement can reveal rational directions that are glued together integrally. The weight records the extra depth needed to separate them.

8.2 Exact precision delay

The separation weight has an operational meaning. It is the exact number of extra input digits needed to recover a projected packet uniformly.

Theorem 8.9 (Necessary and sufficient precision delay)**.** *Let E be a rational projector on V, let Λ be a lattice, and let*

$$s = s_\Lambda(E).$$

For every $m \geq 0$, the rule

$$[x]_{m+s} \longmapsto [Ex]_m$$

from $\Lambda/\pi^{m+s}\Lambda$ to $V/\pi^m\Lambda$ is well defined. Here m is an integer. If d is an integer with $0 \leq d < s$, then for every integer $m \geq 0$ there are $x, y \in \Lambda$ such that

$$x \equiv y \pmod{\pi^{m+d}\Lambda}$$

but

$$Ex \not\equiv Ey \pmod{\pi^m\Lambda}.$$

Thus s extra digits are both sufficient and necessary for a uniform precision guarantee.

Proof. If

$$x - y = \pi^{m+s} z,$$

then

$$E(x-y) = \pi^m(\pi^s Ez) \in \pi^m\Lambda.$$

This proves that the rule is well defined. Now take $d < s$. By the minimality of s, there is a $z \in \Lambda$ with

$$\pi^d Ez \notin \Lambda.$$

Set

$$x = \pi^{m+d} z, \qquad y = 0.$$

Then $x \equiv y$ modulo $\pi^{m+d}\Lambda$, while

$$E(x-y) = \pi^m(\pi^d Ez) \notin \pi^m\Lambda.$$

□

The central idea

The necessity statement is uniform over the lattice. A particular input may need fewer extra digits, but a smaller delay cannot recover every projected packet at the prescribed precision. This is why the least clearing exponent gives an exact precision guarantee rather than only a sufficient budget.

Example 8.10 (The same rational split with two lattices)**.** Let

$$V = Ke_1 \oplus Ke_2$$

and let E project onto Ke_1. On the split lattice

$$\Lambda_0 = \mathcal{O}e_1 \oplus \mathcal{O}e_2,$$

the weight is zero. Fix $s \geq 1$ and consider

$$\Lambda_s = \mathcal{O}e_1 + \mathcal{O}(\pi^{-s}e_1 + e_2).$$

Then

$$\Lambda_s \cap Ke_1 = \mathcal{O}e_1, \qquad \Lambda_s \cap Ke_2 = \pi^s \mathcal{O}e_2.$$

The gluing module is isomorphic to $\mathcal{O}/\pi^s\mathcal{O}$, so

$$s_{\Lambda_s}(E) = s.$$

The rational decomposition has not changed. Only the lattice has changed.

The example can be realized by an integral operator. Define

$$Ae_1 = 0, \qquad Ae_2 = \pi^s e_2.$$

In the basis

$$e_1, \quad \pi^{-s}e_1 + e_2,$$

the matrix of A is

$$\begin{pmatrix} 0 & -1 \\ 0 & \pi^s \end{pmatrix}.$$

Its eigenvalue factors have resultant equal to π^s up to a unit, and the exact projector weight on Λ_s is also s.

9 The dyadic sign split

The smallest example already shows why coefficient selectors and spectral averages must be kept distinct.

Proposition 9.1 (Sign projectors over a lattice)**.** *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

act on $\mathbb{Z}_p^2$. Over $\mathbb{Q}_p$, the sign projectors are

$$E_+ = \frac{I + A}{2} = \begin{pmatrix} 1 & 1/2 \\ 0 & 0 \end{pmatrix},$$

$$E_- = \frac{I - A}{2} = \begin{pmatrix} 0 & -1/2 \\ 0 & 1 \end{pmatrix}.$$

For every odd prime p, both projectors preserve $\mathbb{Z}_p^2$. At $p = 2$,

$$s_2(E_+) = s_2(E_-) = 1.$$

The factors $x - 1$ and $x + 1$ have resultant 2.

Proof. The displayed matrices are complementary idempotents. The scalar 2 is a unit in $\mathbb{Z}_p$ for odd p, so both matrices are integral there. Over $\mathbb{Z}_2$, multiplying by 2 clears the denominators, while neither

projector preserves the standard lattice. Thus both weights are one. The resultant of the two linear factors is their root difference, which is 2. □

Modulo 2, the two eigenvalues become equal. The rational sign packets remain distinct over $\mathbb{Q}_2$, but the standard lattice does not split into their intersections until one more dyadic digit is retained.

Proposition 9.2 (Integral character averages away from the group order)**.** *Let $\mathcal{O}$ be a discrete valuation ring, let N be a unit in $\mathcal{O}$, and suppose $\mathcal{O}$ contains a primitive Nth root of unity ω. If an integral operator U satisfies $U^N = I$, then*

$$E_j = \frac{1}{N} \sum_{k=0}^{N-1} \omega^{-jk} U^k$$

preserves every U stable $\mathcal{O}$ lattice. At primes dividing N, this formula alone gives no integral projector.

Proof. Every scalar coefficient in the displayed sum belongs to $\mathcal{O}$ because N is a unit and ω is integral. Each power of U preserves the lattice, so their $\mathcal{O}$ linear combination does as well. When N is not a unit, division by N is not an operation in $\mathcal{O}$, so the same argument is unavailable. □

At a prime dividing N, the factor $1/N$ no longer guarantees that the character average preserves the lattice. Cancellation may still make a particular projector integral. Coefficient residue selectors remain exact because they never divide by the group order.

9.1 The concrete rational sign matrix

Consider the integral involution

$$A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \qquad A^2 = I.$$

Over a field of characteristic different from two, its sign projectors are

$$E_+ = \frac{I + A}{2} = \begin{pmatrix} 1 & 1/2 \\ 0 & 0 \end{pmatrix},$$

$$E_- = \frac{I - A}{2} = \begin{pmatrix} 0 & -1/2 \\ 0 & 1 \end{pmatrix}.$$

They are orthogonal idempotents and sum to I.

Proposition 9.3 (Odd-adic versus dyadic behavior)**.** *For every prime $p \neq 2$, every entry of E_+ and E_- is p-integral, and each projector has exact local denominator weight zero. At $p = 2$, each projector has exact local denominator weight one. Multiplication by 2 makes it integral, while the unscaled matrix is not integral:*

$$s_{2,\mathbb{Z}_2^2}(E_+) = s_{2,\mathbb{Z}_2^2}(E_-) = 1.$$

The characteristic factors $x - 1$ and $x + 1$ have resultant 2. Its 2-adic valuation is one, exactly the two local projector weights above, and its p-adic valuation is zero at every prime $p \neq 2$.

Proof. The reduced denominators of the displayed entries are 1 or 2, which proves the local integrality and minimality statements directly. The resultant is

$$\operatorname{Res}(x - 1, x + 1) = 2.$$



Figure 7. The rational sign projectors for an integral involution. Their denominator has 2-adic weight one.

Thus $v_2(2) = 1$ and $v_p(2) = 0$ for primes $p \neq 2$. □

The two characteristic factors $x - 1$ and $x + 1$ have the same reduction in $\mathbb{F}_2[x]$. The rational matrices E_+ and E_- themselves cannot be reduced modulo 2, because their entries contain $1/2$. Consequently the integral special fibre does not inherit their separate idempotent splitting, even though the two projectors remain distinct over $\mathbb{Q}_2$. This is the simplest model of delayed dyadic separation.

9.1.1 Why coefficient packets remain available

The failure of the character formula

$$\frac{1}{2}(I \pm A)$$

to be integral does not invalidate coefficient selectors such as even and odd residue extraction. It shows that two distinct constructions must be distinguished:

$$\boxed{\text{integral address selector} \quad \neq \quad \text{character-average spectral projector}}$$

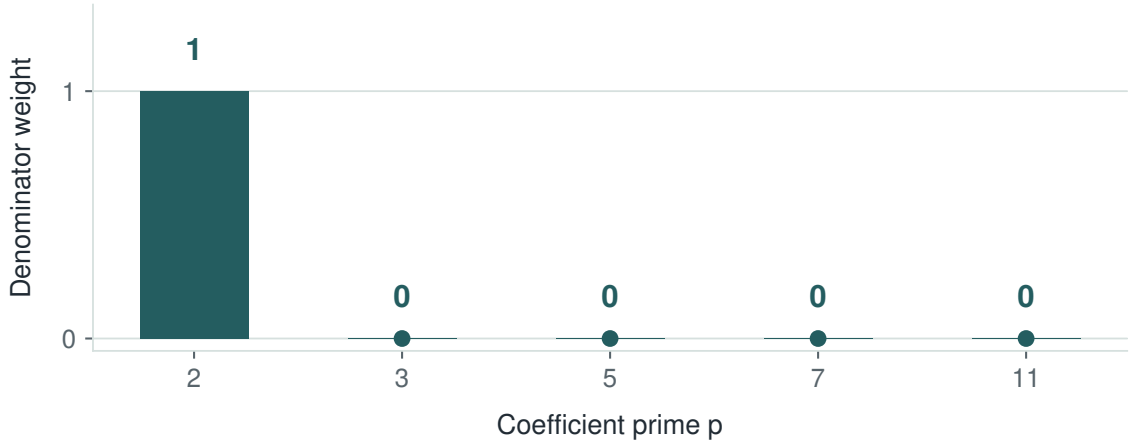
when the group order is not a unit. The first can remain exact over $\mathbb{Z}_2$. The second can acquire a denominator weight.

10 Smith normal form and congruence lifting

The gluing module gives the exact common denominator of packet projectors. Smith normal form gives the corresponding coordinate by coordinate solvability and branching rule.

The denominator singles out the prime two

The exact local weight of each sign projector is $v_p(2)$.



The denominator 2 is a unit at every odd prime.

Figure 8. The denominator weight of the sign projector is nonzero only at the prime two.

Standard Fact 10.1 (Smith normal form over a discrete valuation ring)**.** Let $M \in M_{m,n}(\mathcal{O})$. There are invertible matrices

$$U \in \mathrm{GL}_m(\mathcal{O}), \qquad V \in \mathrm{GL}_n(\mathcal{O})$$

such that

$$UMV = \mathrm{diag}(\pi^{s_1}, \dots, \pi^{s_r}, 0, \dots, 0),$$

where

$$0 \le s_1 \le \cdots \le s_r$$

and r is the rank over K. The exponents are uniquely determined. See [6].

Proposition 10.2 (Computing a packet weight from an inclusion matrix)**.** *Choose an $\mathcal{O}$ basis of Λ and an $\mathcal{O}$ basis of Λ_{split}. Let B be the matrix of the inclusion*

$$\Lambda_{\mathrm{split}} \hookrightarrow \Lambda.$$

If the Smith form of B is

$$\mathrm{diag}(\pi^{a_1}, \dots, \pi^{a_n}), \qquad 0 \le a_1 \le \cdots \le a_n,$$

then

$$G_{\Lambda,\mathcal{P}} \cong \bigoplus_{j=1}^{n} \mathcal{O}/\pi^{a_j}\mathcal{O}, \qquad s_\Lambda(\mathcal{P}) = \max(\{0\} \cup \{a_j : 1 \le j \le n\}).$$

Proof. The gluing module is the cokernel of the inclusion matrix B. Smith normal form gives its invariant factors. The weight formula follows from theorem 8.2. □

Theorem 10.3 (Smith weighted congruence solve)**.** *Let $M \in M_{m,n}(\mathcal{O})$ have Smith exponents $s_1, \dots, s_r$. Assume the residue field is finite of cardinality q, fix $L \geq 1$, and consider*

$$Mx \equiv b \pmod{\pi^L}.$$

Put

$$\tilde{b} = Ub$$

for a Smith matrix U as in standard fact 10.1. The congruence is solvable if and only if

$$\tilde{b}_i \equiv 0 \pmod{\pi^{\min(s_i, L)}} \qquad (1 \leq i \leq r)$$

and

$$\tilde{b}_i \equiv 0 \pmod{\pi^L} \qquad (r < i \leq m).$$

When it is solvable, the number of solutions in $(\mathcal{O}/\pi^L\mathcal{O})^n$ is

$$\boxed{q^{(n-r)L + \sum_{i=1}^{r} \min(s_i, L)}}.$$

Proof. Set $x = Vy$. Multiplying the congruence by U reduces it to independent equations

$$\pi^{s_i} y_i \equiv \tilde{b}_i \pmod{\pi^L} \qquad (1 \leq i \leq r)$$

and zero equations in the remaining rows. Multiplication by π^{s_i} on $\mathcal{O}/\pi^L\mathcal{O}$ has image equal to the elements divisible by $\pi^{\min(s_i, L)}$ and kernel of size

$$q^{\min(s_i, L)}.$$

Each of the $n - r$ unused variables is free and contributes q^L choices. Multiplying the independent counts gives the formula. □

Corollary 10.4 (Obstruction and branching data)**.** *The transformed entries of $\tilde{b}$ that fail the stated divisibility conditions are the complete finite congruence obstructions. When all obstructions vanish, the Smith exponents determine the number of solution branches at depth L.*

Proof. This is exactly the solvability and counting statement of theorem 10.3 written in invariant coordinates. □

Example 10.5 (Three weighted directions)**.** Take $p = 2$, $L = 6$, and the diagonal weights

$$(0, 1, 3).$$

The right side

$$(1, 2, 8)$$

is admissible. The three coordinate kernels have sizes

$$1, \quad 2, \quad 8,$$

so there are 16 solutions. Replacing the final entry 8 by 4 violates the required divisibility by 2^3 and makes the system unsolvable.

10.1 Concrete diagonal fibres over integer quotients

Let M be a matrix over a DVR. Smith normal form gives invertible matrices U, V such that

$$UMV = \operatorname{diag}(\pi^{s_1}u_1, \dots, \pi^{s_r}u_r, 0, \dots, 0),$$

where each u_i is a unit and

$$0 \le s_1 \le \cdots \le s_r.$$

The exponents s_i are directional lifting weights.

Theorem 10.6 (Concrete Smith-weighted congruence solve). *Let $p > 1$ and consider the diagonal system over $(\mathbb{Z}/(p^L)\mathbb{Z})^d$*

$$\operatorname{diag}(p^{s_1}, \dots, p^{s_d})x = b \pmod{p^L}.$$

The system is solvable if and only if

$$p^{\min(s_i,L)} \mid (b_i)_{\mathrm{val}}$$

for every i, where $(b_i)_{\mathrm{val}}$ is the canonical natural representative. Conditional on this admissibility condition, the number of solutions is

$$\boxed{p^{\sum_{i=1}^{d} \min(s_i,L)}.}$$

If any coordinate is inadmissible, the solution set is empty and its cardinality is zero.

Proof. Lean proves the one-coordinate arithmetic directly over $\mathbb{Z}/p^L\mathbb{Z}$ for every $p > 1$:

$$(\exists x \in \mathbb{Z}/p^L\mathbb{Z} : \quad p^s x = b \pmod{p^L}) \quad \Longleftrightarrow \quad p^{\min(s,L)} \mid b_{\mathrm{val}},$$

where b_{val} is the canonical natural representative, and every admissible fibre has exactly $p^{\min(s,L)}$ elements. The coordinates are independent, so the product equivalence gives the displayed global count. If one coordinate fibre is empty, the global fibre is empty. Lean proves both the global solvability equivalence and the conditional count, including cardinality zero on the inadmissible branch. □

For an arbitrary DVR $\mathcal{O}$ with residue field cardinality q, the analogous formula

$$q^{\sum_i \min(s_i,L)}$$

follows from the solvability and fibre count for each coordinate. The current revision adds direct proofs of this DVR coordinate arithmetic; the current correspondence records those targets and the rectangular transport separately. The inherited abstract product interface remains available when the coordinate facts are supplied as certificates. A Smith form and its changes of basis are inputs to the transport statement; it does not provide an extracted diagonalization routine. See [8, 5].

If a Smith form is rank deficient, a literal zero diagonal factor requires the corresponding target coordinate to vanish modulo p^L and then contributes all p^L possible coordinate values. In the concrete quotient it can equivalently be represented by any weight $s_i \ge L$, since $p^{s_i} = 0$ in $\mathbb{Z}/(p^L)\mathbb{Z}$. The theorem then gives exactly that condition and factor. The numbered theorem takes the diagonal data as its input.

Example 10.7. At $p = 2$, depth $L = 6$, and weights $(0, 1, 3)$, the right side $(1, 2, 8)$ is admissible. The coordinate kernels have sizes

$$1, 2, 8,$$

so the system has 16 solutions. Replacing 8 by 4 in the final coordinate violates the divisibility condition and makes the system unsolvable.

> **How to read the result**
>
> The gluing calculation takes the largest Smith weight because every packet must be separated to the requested precision. The congruence calculation retains the individual directional weights. A coordinate with weight s_i contributes a solvability threshold and $\min(s_i, L)$ to the branching exponent. The independent coordinate counts multiply, so these truncated weights add in the exponent.

11 Linear refinement defects

Exact coordinate refinement describes two representations of the same object. In an application, the coarse operator may instead be chosen to model coarse dynamics. An intertwining defect measures the mismatch between applying the operator before or after passing to coarse coordinates.

Let X_f and X_c be vector spaces or Banach spaces over a common field, and let

$$C : X_f \longrightarrow X_c$$

be a linear map, and let

$$T_f \in \mathrm{End}(X_f), \qquad T_c \in \mathrm{End}(X_c).$$

Define

$$\Delta = CT_f - T_cC.$$

> **The central idea**
>
> There are two routes from a fine state to a coarse result. One applies the fine operator and then maps the result to the coarse space. The other maps the state first and then applies the coarse operator. The defect is the difference between these routes. Its vanishing says that the chosen operators are compatible with the comparison map. The norm estimates below combine the size of this defect with the norms of the surrounding operators.

Theorem 11.1 (Power defect identity)**.** *For every integer $n \geq 1$,*

$$CT_f^n - T_c^nC = \sum_{k=0}^{n-1} T_c^{n-1-k}\Delta T_f^k.$$

Proof. The statement is clear for $n = 1$. For the induction step, write

$$CT_f^{n+1} - T_c^{n+1}C = (CT_f^n - T_c^nC)T_f + T_c^n(CT_f - T_cC)$$

and apply the induction hypothesis. □

Corollary 11.2 (Power norm bound)**.** *For bounded operators between Banach spaces,*

$$\|CT_f^n - T_c^nC\| \leq \|\Delta\| \sum_{k=0}^{n-1} \|T_c\|^{n-1-k}\|T_f\|^k.$$

Proof. Apply the triangle inequality and submultiplicativity to theorem 11.1. □

Corollary 11.3 (Norm bound)**.** *For a submultiplicative operator norm,*

$$\|CT_f^n - T_c^n C\| \le \|\Delta\| \sum_{k=0}^{n-1} \|T_c\|^{n-1-k} \|T_f\|^k.$$

If $\|T_f\|, \|T_c\| \le \rho$*, then*

$$\boxed{\|CT_f^n - T_c^n C\| \le n\rho^{n-1}\|\Delta\|.}$$

Corollary 11.4 (Polynomial defect)**.** *For*

$$p(z) = \sum_{n=0}^{N} a_n z^n,$$

we have

$$Cp(T_f) - p(T_c)C = \sum_{n=1}^{N} a_n \sum_{k=0}^{n-1} T_c^{n-1-k} \Delta T_f^k.$$

Proof. The constant terms cancel because $CI = IC$. Apply theorem 11.1 to every positive power and sum. □

Theorem 11.5 (Resolvent defect identity)**.** *If* $z - T_f$ *and* $z - T_c$ *are invertible, then*

$$C(z - T_f)^{-1} - (z - T_c)^{-1} C = (z - T_c)^{-1} \Delta (z - T_f)^{-1}.$$

Proof. Multiply the claimed identity on the left by $z - T_c$ and on the right by $z - T_f$. The left side becomes

$$(z - T_c)C - C(z - T_f) = CT_f - T_c C = \Delta.$$

Multiplying back by the two resolvents proves the formula. □

11.1 Inverse witnesses from one side and norm control

For continuous linear maps, the same identity gives the certified norm bound

$$\boxed{\|CR_f - R_c C\| \le \|R_c\| \, \|\Delta\| \, \|R_f\|,}$$

from a left inverse witness for the coarse scalar shift and a right inverse witness for the fine scalar shift. These witnesses suffice for the displayed identity. On their own, they need not define inverses on both sides and hence need not define resolvents. If $\Delta = 0$, Lean also proves, for every finite polynomial p, the exact compatibility

$$\boxed{C\, p(T_f) = p(T_c)\, C,}$$

in both the algebraic setting and the setting of continuous linear maps. Thus all powers and polynomial actions intertwine with coarse reconstruction. The stated inverse witnesses also intertwine. In particular, genuine resolvents do so when the scalar shifts are invertible. This is the linear prototype of the more general theory of refinement defects reserved for Paper III.

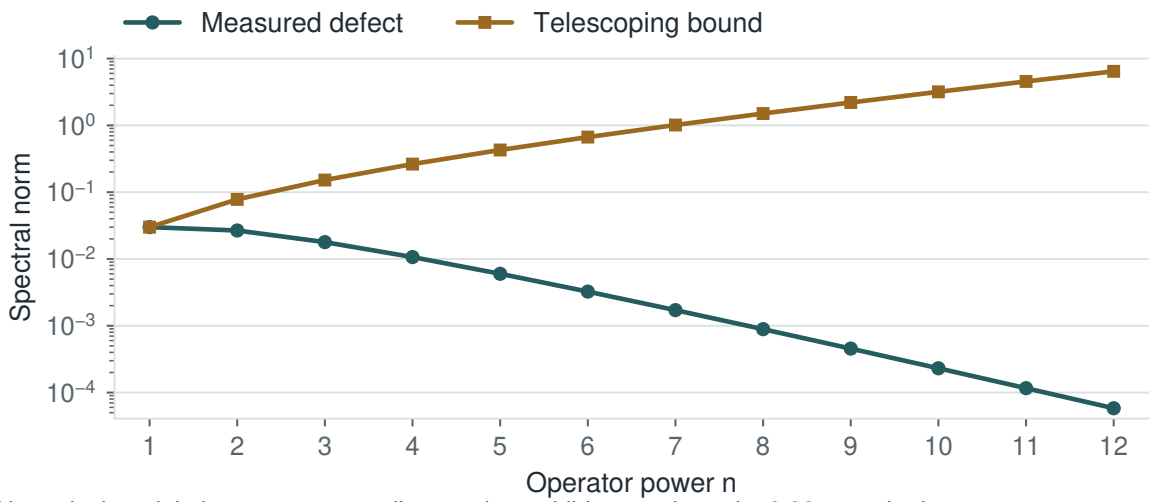


Figure 9. A controlled defect for one step and its telescoping norm bound under repeated operator application.

11.2 Analytic functional calculus

Theorem 11.6 (Holomorphic functional calculus defect)**.** *Let X_f and X_c be complex Banach spaces, and let $C : X_f \to X_c$, T_f, and T_c be bounded complex-linear maps. Let h be holomorphic on an open set U containing both spectra. Let Γ be an admissible finite rectifiable Cauchy cycle in U and in both resolvent sets, with winding number one on the union of the spectra and zero outside U. The length of a cycle counts multiplicities. Then*

$$Ch(T_f) - h(T_c)C = \frac{1}{2\pi i}\int_\Gamma h(z)(z - T_c)^{-1}\Delta(z - T_f)^{-1}\,dz.$$

Consequently,

$$\begin{aligned}\|Ch(T_f) - h(T_c)C\| \le \frac{\operatorname{length}(\Gamma)}{2\pi}\sup_{z\in\Gamma}|h(z)|\,\|\Delta\| \\ \times \sup_{z\in\Gamma}\|(z - T_c)^{-1}\|\sup_{z\in\Gamma}\|(z - T_f)^{-1}\|.\end{aligned}$$

Proof. The assertion is immediate if either space is zero. Otherwise, use the rectifiable-cycle form of the holomorphic functional calculus in [1, Chapter VII, §4, Theorem 4.1, formula (4.5), and Proposition 4.6]. For each operator T, compare Γ with a positive simple system around its spectrum. Their difference has zero winding outside $U \setminus \sigma(T)$, so Theorem 4.1 applied to the resolvent-valued integrand identifies the two integrals. Consequently,

$$h(T) = \frac{1}{2\pi i}\int_\Gamma h(z)(z - T)^{-1}\,dz.$$

Subtract the two formulas after composing with C, then use theorem 11.5 inside the integral. The norm estimate follows from the standard contour bound. □

Theorem 11.7 (Exponential defect identity). *Let $C : X_f \to X_c$, T_f, and T_c be bounded linear maps between Banach spaces over a common field. For real t, use real or complex spaces. For complex t, use complex spaces and complex-linear maps. Then*

$$Ce^{tT_f} - e^{tT_c}C = \int_0^t e^{(t-s)T_c}\Delta e^{sT_f}\,ds,$$

where the integral is taken along the straight segment from 0 to t when t is complex. For real $t \geq 0$,

$$\|Ce^{tT_f} - e^{tT_c}C\| \leq \|\Delta\| \int_0^t \|e^{(t-s)T_c}\|\|e^{sT_f}\|\,ds.$$

Proof. For real t, differentiate

$$F(s) = e^{(t-s)T_c}Ce^{sT_f}.$$

We obtain

$$F'(s) = e^{(t-s)T_c}\Delta e^{sT_f}.$$

Integration from 0 to t gives the identity. The complex statement follows by the same calculation along the straight segment. The norm bound follows from the integral formula. □

11.3 Several refinement levels

Let

$$C_\ell : X_{\ell+1} \longrightarrow X_\ell$$

be linear maps for $0 \leq \ell < L$, and let $T_\ell \in \mathrm{End}(X_\ell)$. Put

$$\Delta_\ell = C_\ell T_{\ell+1} - T_\ell C_\ell.$$

For $a < b$, write

$$C_{a,b} = C_a C_{a+1} \cdots C_{b-1},$$

with $C_{a,a} = I$.

Theorem 11.8 (Multilevel defect decomposition). *The total defect from level L to level zero is*

$$C_{0,L}T_L - T_0 C_{0,L} = \sum_{\ell=0}^{L-1} C_{0,\ell}\Delta_\ell C_{\ell+1,L}.$$

For bounded maps,

$$\|C_{0,L}T_L - T_0 C_{0,L}\| \leq \sum_{\ell=0}^{L-1} \|C_{0,\ell}\|\|\Delta_\ell\|\|C_{\ell+1,L}\|.$$

Proof. Insert the intermediate terms

$$C_{0,\ell}T_\ell C_{\ell,L}$$

for $0 \le \ell \le L$. Consecutive differences telescope. The difference between the terms indexed by $\ell + 1$ and ℓ is

$$C_{0,\ell}(C_\ell T_{\ell+1} - T_\ell C_\ell)C_{\ell+1,L}.$$

Summing gives the identity. The norm estimate follows from the triangle inequality. □

> **How to read the result**
>
> The multilevel formula keeps track of where each mismatch enters. A defect at one level is composed with the comparison maps needed to reach the input and output levels. The norm estimate therefore depends on both the local defects and the maps that carry them through the hierarchy. Controlling local mismatches alone does not control the total error if those maps amplify them.

Example 11.9 (A visible one step defect)**.** Let $X_f = \Bbbk^2$, let $X_c = \Bbbk$, and let

$$C(x_1, x_2) = x_1.$$

Take

$$T_f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad T_c = (a).$$

Then

$$\Delta = CT_f - T_c C = (0, b).$$

The coarse operator is exactly compatible with the fine operator precisely when the discarded coordinate does not feed into the retained coordinate. The entry c does not appear in the one step defect because it moves information in the opposite direction. It can still affect later powers through the terms in theorem 11.1.

Corollary 11.10 (Exact compatibility)**.** *If every one step defect Δ_ℓ vanishes, then*

$$C_{0,L}p(T_L) = p(T_0)C_{0,L}$$

for every polynomial p. The analogous resolvent, holomorphic, and exponential identities also hold whenever the relevant expressions are defined.

Proof. The total one step defect vanishes by theorem 11.8. Apply corollary 11.4 and theorems 11.5 to 11.7. □

12 Worked realizations in the common operator language

12.1 Signal polyphase coordinates

Let $x = (x_n)_{n\in\mathbb{Z}} \in c_{00}(\mathbb{Z}; R)$ and let the seed be δ_0 under the bilateral shifts $(S^r\delta_0)_n = \delta_{n,r}$. This makes the algebraic span and every displayed sum finite. The N packets are

$$x_j[m] = x[Nm + j].$$

Synthesis interleaves these sequences. A linear filter with a finite impulse response that is invariant under shifts becomes an $N \times N$ matrix of transfer blocks. Refining from N to M with $N \mid M$ splits each coarse phase into its descendants without changing the reconstructed signal. The same formulas extend

to standard sequence completions when the synthesis series converges. This is the usual polyphase viewpoint from multirate filter banks [10].

12.2 Sparsity of transfer blocks

A banded operator need not remain banded after residue reordering, but its nonzero couplings become a structured packet graph. With the convention $(S_r x)_n = x_{n-r}$, the only active transfer is

$$P_{N,j+r} S_r P_{N,j},$$

with indices taken modulo N. More general finite stencils produce a finite set of active phase differences.

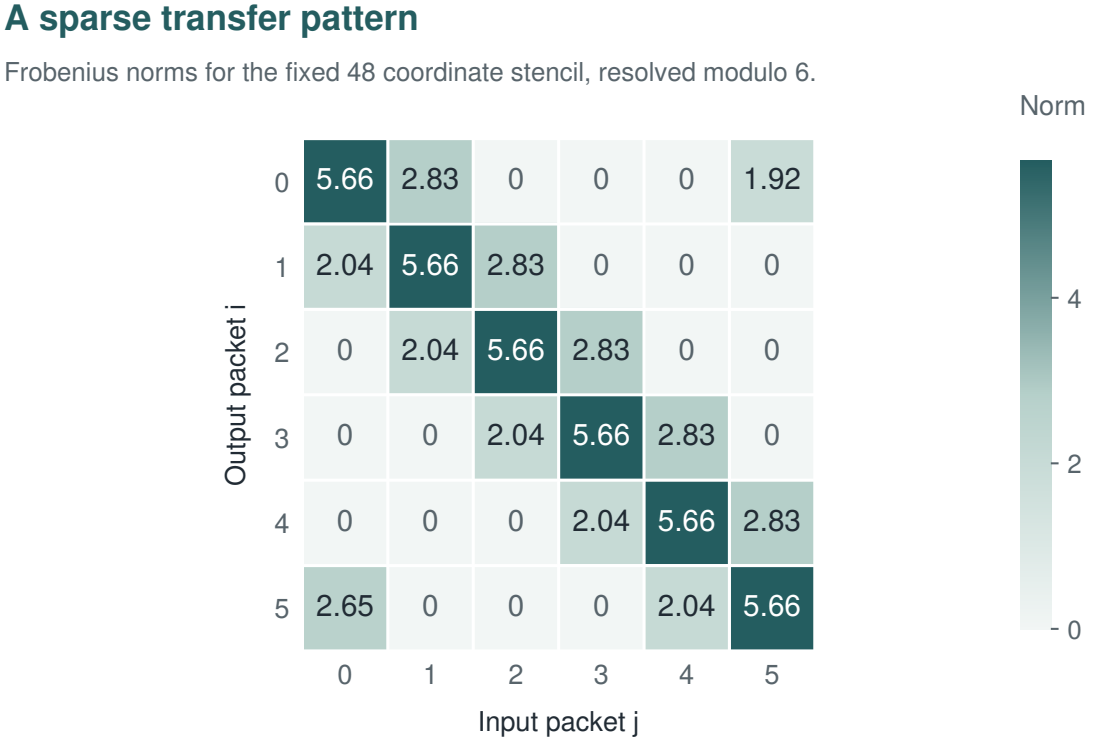


Figure 10. Norms of packet blocks for a finite stencil. The original operator is exactly reconstructed from a sparse transfer pattern.

12.3 A short realization in Smith chart coordinates

> **How to read the result**
>
> The Smith chart used here is a coordinate diagram for impedance and reflection. It is different from the Smith normal form used earlier to study integral congruences. The connection within this paper is the way a change of coordinates makes an action easier to describe. Here a rotation

in reflection coordinates becomes a fractional linear map in impedance coordinates. The sampled phase orbit is a geometric example and does not supply another linear projector decomposition.

Let Z and $Z_0 \neq 0$ be complex impedances, put $z = Z/Z_0$, and assume $z \neq -1$. Write

$$\Gamma = \frac{z-1}{z+1}.$$

On the open reflection disk $|\Gamma| < 1$, multiplication by any phase ω_N^k stays in the disk and has a finite inverse-chart image. The phase action is

$$\Gamma \mapsto \omega_N^k \Gamma.$$

Outside the open disk, the same chart formula requires $1 - \omega_N^k \Gamma \neq 0$. Pulling the phase action back on this domain gives

$$T_{N,k}(z) = \frac{(1+\omega_N^k)z + (1-\omega_N^k)}{(1-\omega_N^k)z + (1+\omega_N^k)}.$$

For $N = 2$ and $k = 1$,

$$T_{2,1}(z) = \frac{1}{z}, \qquad z \neq 0, -1.$$

Thus the nontrivial action at the first phase level exchanges normalized impedance and admittance. Calling these values "packets" is geometric terminology for the sampled orbit. It does not assert that the two values form linear idempotent ranges. Higher refinements at powers of 2 sample the same reflection circle at progressively finer electrical phases. Compare the standard Smith chart conventions in [7].

The symmetric weight

$$w_{\text{Smith}}(\Gamma) = \frac{1}{|1-\Gamma|^2} + \frac{1}{|1+\Gamma|^2}$$

indicates where small errors in reflection coordinates produce large impedance or admittance errors.

13 Finite computational certificates

The following results and tables are the recorded v1.2.1 finite implementation evidence. They do not certify the expanded lattice or analytic statements in Lean. Current reproduction outcomes are reported in the verification files accompanying this edition.

All numerical work in this paper is finite and reproducible. It tests the Python implementation of identities proved above or checked against certificates. It does not replace those arguments or the Lean kernel checks. Release 1.2.1 runs 48 tests, including contracts for invalid input, refinement transitivity, aggregation of transfer blocks, Fourier/DFT agreement, Smith fibre counts by exhaustive enumeration, and consistency of the stored JSON and CSV records.

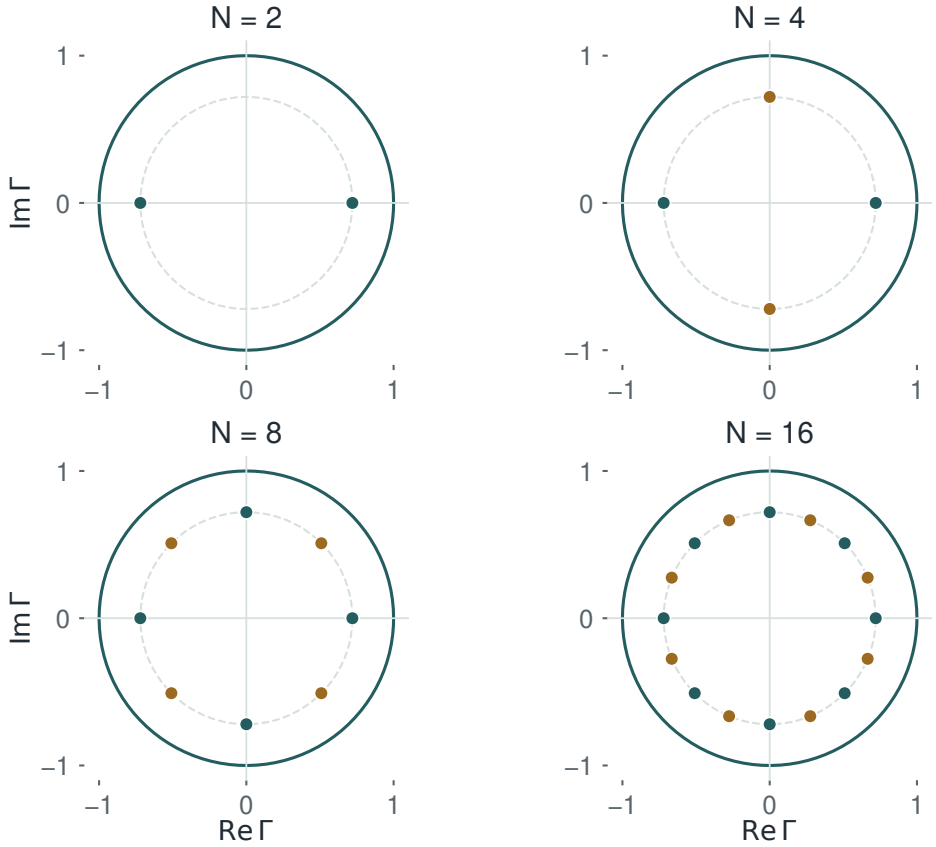
Numerical Certificate 13.1 (Weighted inverse)**.** For the tested diagonal-plus-transfer matrices, the filtered inverse agrees exactly with direct modular inversion at all tested precisions:

$$2^{10}, \qquad 3^8, \qquad 5^7.$$

The maximum modular residual is zero in every case.

Nested polyphase samples

Reflection coordinates retain their geometry at every dyadic refinement.

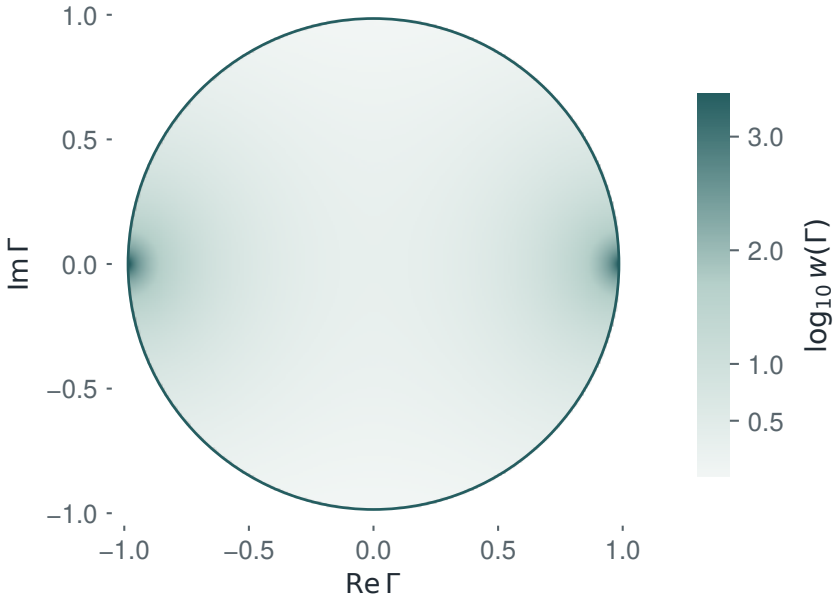


Every sample has |Γ| = 0.72. Gold marks points added at N = 4, 8, or 16.

Figure 11. Nested phase orbits at powers of 2 on the Smith reflection disk. The full treatment of the Smith chart is deferred to the signal paper.

Symmetric Smith refinement weight

The weight increases near the two boundary points Γ = −1 and Γ = 1.



$w(\Gamma) = |1-\Gamma|^{-2} + |1+\Gamma|^{-2}$

The 300 × 300 grid is shown only where $|\Gamma| < 0.985$.

Figure 12. A symmetric refinement weight concentrated near the boundary points for open and short circuits.

Table 1. Representative numerical and symbolic checks.

Check	Result
Packet synthesis residuals for moduli $2, 3, 4, 6, 8, 12$	0
Nested refinement residuals for $2 \to 4$, $2 \to 8$, $3 \to 6$, $3 \to 12$, $4 \to 8$, $4 \to 12$, $6 \to 12$	0
CRT residuals for dyadic and odd moduli for 4×3, 8×3, 4×5, 8×5	0
Maximum finite Fourier projector residual	6.60×10^{-15}
Weighted inverse residual modulo 2^{10}	0
Weighted inverse residual modulo 3^{8}	0
Weighted inverse residual modulo 5^{7}	0
Difference between filtered and direct modular inverses	0
Transfer zeta coefficients versus direct walk enumeration through weight 30	exact match
Dyadic sign projector resultant weight	$v_2(2) = 1$
Sign projector resultant weight at the odd prime 3	$v_3(2) = 0$
Power defect telescoping identity residual	at most 6.94×10^{-18}
Resolvent defect identity residual	1.09×10^{-17}
Smith $N = 2$ phase versus $z \mapsto 1/z$	residual below 1.4×10^{-16}

Numerical Certificate 13.2 (Transfer zeta)**.** The rational function

$$\frac{-t^3}{t^9 - t^7 - t^6 - t^5 + 2t^4 + t^3 + t^2 - 1}$$

was expanded through degree 30. Every coefficient agrees with direct weighted-walk enumeration.

Numerical Certificate 13.3 (Dyadic separation)**.** The code verifies exactly that the two rational sign projectors are idempotent, orthogonal, and complementary, that scaling by 2 makes both integral, and that the same projectors are already integral modulo every tested odd prime.

14 Scope and limits

The exact statements above have several deliberate boundaries.

First, refinement changes coordinates while preserving reconstruction. If a numerical method discards fine packets, the resulting error must be estimated separately.

Second, the weighted inverse is linear. A nonlinear map creates higher order interactions among packet coordinates. Those interactions are not contained in the path formula of theorem 5.2.

Third, a resultant is only a denominator bound. The exact weight depends on the lattice and is given by theorem 8.2. The two examples in examples 7.4 and 8.10 show that the same eigenvalue separation can give different integral packet weights.

Fourth, the dyadic sign example is a model rather than a classification of all phenomena at the prime two. Character averages, trace splittings, quadratic refinements, and deformation projectors must each be checked on their own lattices.

Finally, the defect formulas do not prove that a chosen coarse operator is a good approximation. They say exactly how a known one step defect propagates. A useful approximation theorem must also bound that defect in the problem under study.

A Notation table

Symbol	Meaning
$P_{\ell,\alpha}$	packet projector at refinement level ℓ
I_ℓ	packet address set at level ℓ
$\beta \succ \alpha$	fine packet β is a child of coarse packet α
$\mathcal{A}_\ell, \mathcal{S}_\ell$	packet analysis and synthesis maps
$J_{\ell+1,\ell}$	exact coordinate refinement map
$C_{\ell,\ell+1}$	exact coordinate regrouping from fine to coarse
$T^{(\ell)}_{\beta\alpha}$	transfer block $P_{\ell,\beta} T P_{\ell,\alpha}$
$P_{N,j}$	residue or phase selector for address $j \pmod N$
$w(e)$	positive filtration weight of a transfer edge
$W(\gamma)$	cumulative weight of a transfer word or graph walk
$A(t)$	polynomial weighted adjacency matrix
$Z_{s,t}(t)$	weighted transfer-zeta function from vertex s to vertex t

Symbol	Meaning
$s_{\pi,\Lambda}(E)$	separation weight of a rational projector on the chosen lattice, when a clearing exponent exists
Δ	linear refinement defect for one step $CT_f - T_cC$

B Algorithmic forms

B.1 Residue analysis and synthesis

```
def analyze_residues(x, N):
    require N > 0
    return [x[j::N] for j in range(N)]

def synthesize_residues(packets):
    N = len(packets)
    require packet lengths fit one N-way interleaving
    y = empty(sum(len(p) for p in packets))
    for j, packet in enumerate(packets):
        y[j::N] = packet
    return y
```

The verified contract is synthesize(analyze$(x, N)) = x$. The Lean counterpart uses packet coordinates restricted to projector ranges and proves both inverse laws. The finite Python routine checks the concrete array realization.

B.2 Weighted inverse at finite precision

```
Dinv = inverse(D)                       # certified on both sides
edge[e] = Dinv @ H_edge[e]
require Dinv @ H == sum(edge[e] for e in edges)
require edge[e] lies in filtration[weight[e]]
m = certified_lower_bound(weight)       # 1 <= m <= weight[e]
N = (L + m - 1) // m                    # so L <= m*N
S = 0
for n in range(N):
    for word in edge_tuples(edges, n):
        if sum(weight[e] for e in word) < L:
            S += (-1)**n * ordered_product(edge[e] for e in word)
return q_L(S @ Dinv)
```

Lean proves that the mapped pruned sum is exactly the mapped finite geometric inverse. The lower bound m gives $m|\gamma| \le W(\gamma)$, so $L \le mN$ derives $q_L(A^N) = 0$ from the filtration certificate and makes the returned value an inverse on both sides of $q_L(D + H)$. The Lean lemma $L \le m\lceil L/m \rceil$ justifies the displayed executable cutoff for $m > 0$. Taking $m = 1$ and $N = L$ gives the unconditional cutoff for positive weights. The Python implementation checks the DVR matrix specialization against direct modular inversion.

B.3 Walk construction for transfer zeta

```
def coefficient(G, start, target, n):
    total = 0
    for length in range(n + 1):
        for word in edge_tuples(G, length):
            if is_walk(word, start, target) and weight(word) == n:
                total += 1
    return total

A[i,j] = sum(t**weight(e) for distinct edges e from i to j)
T = I - A
Q = det(T)
H = adjugate(T)
P = H[start, target]
assert Q.coeff(0) == 1
Z = coefficient_series(G, start, target)
prove Q * Z == P
```

Positive weights make the coefficient loop finite because every contributing word has length at most n. Lean proves the enumeration correct, derives its recurrence from the first edge, proves $(I - A)Z = I$, constructs the determinant/adjugate inverse, and identifies the two matrices by inverse uniqueness. Thus the normalized $QZ = P$ identity and its induced recurrence follow from finite graph data. The Python companion repeats the exact symbolic construction as an executable check. The Lean proof does not use it as an oracle.

C Proof dependency map

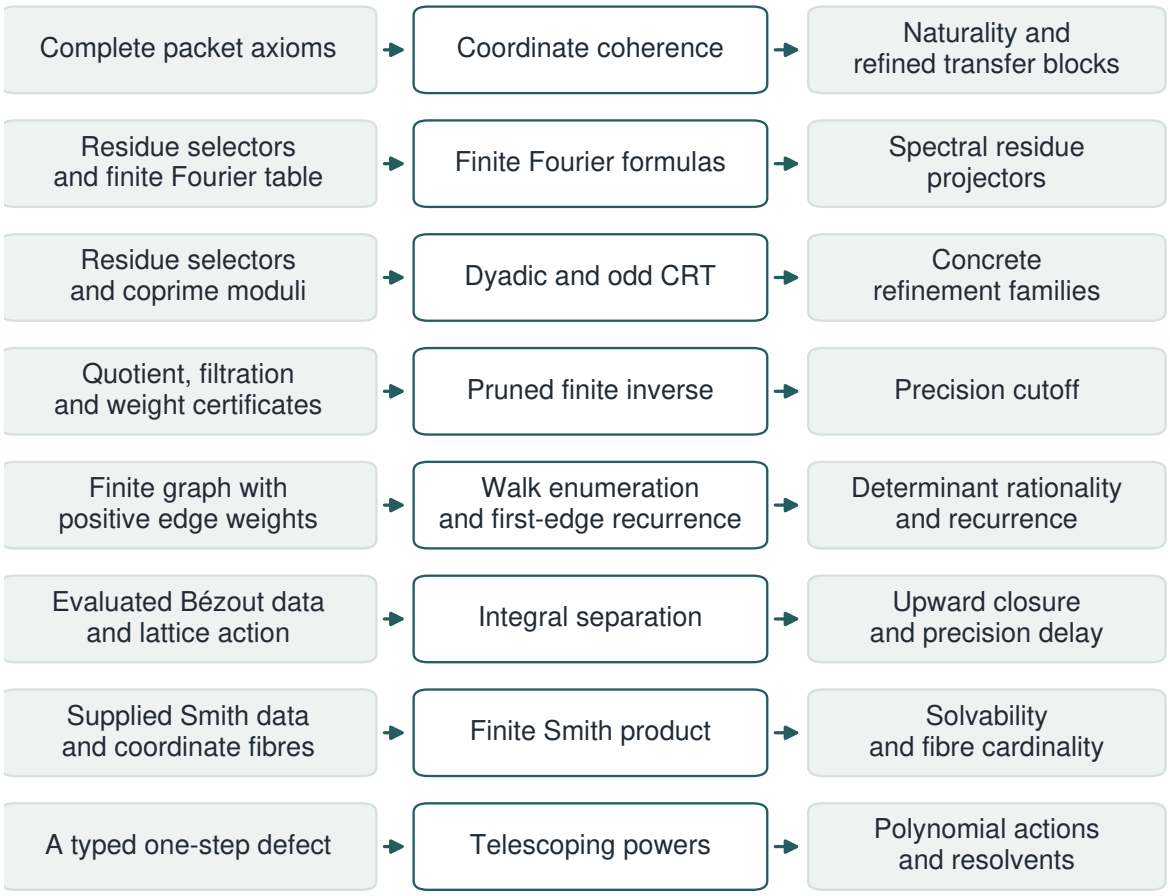
The proof order is:

complete packet axioms $\Rightarrow$ finite-packet uniqueness and coordinate coherence
$\Rightarrow$ coordinate naturality and refined transfer blocks,
residue selectors and a finite Fourier table $\Rightarrow$ Fourier formulas,
coprime moduli $\Rightarrow$ dyadic–odd formulas,
quotient, filtration, and edge-weight certificates $\Rightarrow$ explicit pruned inverse,
finite positive-weight graph $\Rightarrow$ semantic enumeration and first-edge recurrence
$\Rightarrow$ det / adj rationality and recurrence,
minimal evaluated Bézout data and inverse-unit lattice action
$\Rightarrow$ separation bound, upward closure, and precision delay,
concrete $\mathbb{Z}/p^L\mathbb{Z}$ coordinate fibres $\Rightarrow$ finite Smith product count,
one-step typed defect $\Rightarrow$ powers, polynomial actions, and witnessed resolvents.

The walk branch for transfer zeta is logically independent of operator inversion. Its rationality conclusion follows from finite graph data without a supplied $QZ = P$ premise. Likewise, CRT factorization follows directly from residue selectors and coprimality. It does not require the abstract refinement theorem, although it supplies a concrete refinement family. General Smith diagonalization and the construction of polynomial Bézout/resultant data remain inputs to the formalized branches that use those certificates.

Distinct proof branches

Each row identifies its own inputs and the conclusions they support.



The walk series is independent of operator inversion. General Smith and Bézout constructions remain upstream inputs to their certificate branches.

Figure 13. The dependency structure of Paper II. Coordinate coherence, residue/CRT, weighted inversion, automatic finite-graph transfer rationality, and integral separation are distinct branches with explicit inputs.

D Edition and proof scope

The traditional article is a single edition organized by topic. The expanded branch supplies the written proofs of common refinements, complete filtered inversion, lattice gluing, rectangular Smith congruences, and analytic and multilevel defects. The canonical v1.2.1 release supplies the inherited specializations, finite graph construction, worked realizations, and three algorithm specifications.

Current verification snapshot (19 October 2026). The current project is `formal/current/`. Its correspondence contains 81 source records, including 53 proof-level items. Current targets and historical records are kept separately, with exact source and proof hashes. The map and the Lean guide report each item's recorded status and release gates.

Current Lean scope. 53 of 53 proof-level source records have kernel checked, semantically accepted scope. No partial or pending proof records are listed in this correspondence. These are source-record counts under their stated hypotheses, not counts of independent discoveries.

The rectifiable analytic endpoints use a representation whose velocity length is bounded by the original geometric length. This inequality supplies the manuscript estimate. Equality of the two lengths is not required. The accepted correspondence identifies the exact canonical endpoints and their hypotheses.

The inherited Lean contract has 25 rows: 21 canonical numbered items, one Smith-phase identity, and three algorithm specifications. It remains a dated source contract. The current correspondence separately records the exact present statements, proof targets, source hashes, and execution evidence. A row may use several declarations, and several source labels may refer to one written theorem; these are distinct counting units.

The original branch sources and historical reports remain preserved. The current editorial log records the hypothesis, notation, and endpoint clarifications, including the verified rectifiable-cycle citation. Current navigation metadata records printed pages and PDF destinations. Historical proof-preservation reports retain their original scope and date; they do not describe later edited proof text.

The visual guide gives exact worked models. The Lean guide opens with the current proof scope and then explains the inherited declaration-level contract. Illustrations and finite checks support understanding and implementation. Universal claims depend on their written proofs and the exact formal statements where coverage is recorded.

This monograph includes the twenty-four worked plates after this account of the proof scope. Their statement references use the printed pages of this volume. The separate visual guide contains the same models for readers who want to follow the examples on their own.

E Visual atlas

Twenty-four worked plates follow the main constructions through explicit vectors, matrices, weighted walks, and congruence fibres. Their labels V1 to V24 stay fixed across the integrated atlas and the standalone guide. Each plate names the corresponding statement and its printed page in this monograph.

Three routes through the atlas

Exact coordinates. Read V1 to V8, then the finite filter in V20.
Weights and arithmetic. Read V9 to V19. The first group follows weighted operator words and graph walks. The second follows denominator clearing and Smith congruences.
Realizations and defects. Read V20 to V24. These plates separate the Smith impedance chart from Smith normal form, trace error propagation, and connect the examples to the formal contract.

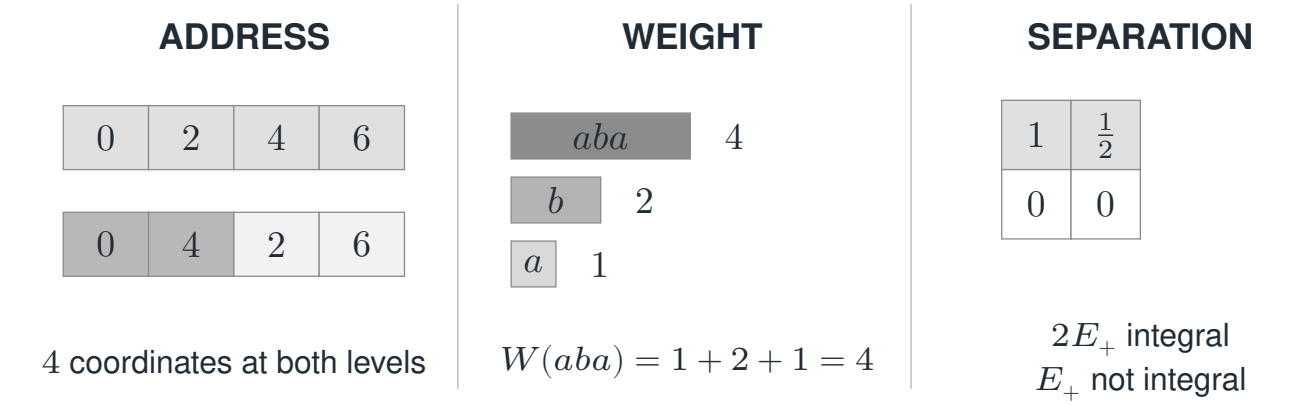
The matrices and finite ranges shown here are explicit models. The general proofs remain in the preceding sections. A picture or finite computation does not extend the hypotheses of a theorem or the coverage of a Lean declaration.

V1 / PAPER II / A WORKED VISUAL GUIDE

Three kinds of resolution

Traditional paper: Definition 3.4 (p. 4), Theorem 5.10 (p. 17), Theorem 7.7 (p. 26)

Paper II asks three different questions that can all sound like "increase the resolution." This guide separates them, then follows small objects far enough that the distinctions can be computed rather than merely named.



Three different operations. On the left, all coefficients survive a regrouping. In the middle, an integer weight controls which operator words can matter modulo a chosen power. On the right, a rational projector needs a denominator cleared before integral precision can be recovered.

Address resolution

A coefficient such as x_6 can sit in the even packet, then the residue-2 packet modulo 4, then the residue-6 packet modulo 8. Its value does not become more accurate. Its *address becomes more specific*, while the full object remains exactly reconstructible.

Arithmetic precision

An operator word with total filtration weight at least L vanishes in the specified depth-L quotient. Increasing L can make additional words relevant. This is a change in the precision of the ring in which the answer is required, not a lossless rearrangement of one fixed list.

Separation precision

For the integral involution used later, $E_+ = (I + A)/2$ is an exact rational projector but not an integral one at 2. One extra input digit suffices to recover the projected difference to the requested precision. The packet is present over the fraction field even when its integral separation is delayed.

> **Read this as:** Keep three quantities distinct: the packet address, the precision cutoff, and the denominator weight. Finer addresses do not automatically improve arithmetic precision or remove a spectral denominator.

V2 / ORIENTATION / SYMBOLS WITH JOBS

An atlas of the refinement layer

Traditional paper: Definition 3.2 (p. 3), Definition 3.4 (p. 4), Definition 7.6 (p. 26)

The same letter can have different roles in different realizations. The table fixes the roles used here. Every small matrix in the guide is explicitly supplied. No diagonalization or classification theorem is inferred from a picture.

Object	Role	Worked instance
$P_{N,j}$	Retain coefficients with address j modulo N.	Even/odd masks, then mod-4 and mod-8 masks.
$\mathcal{A}_N, \mathcal{S}_N$	Split into range-restricted packet vectors, then sum them.	A permutation/interleaving of eight coordinates.
J, C	Exact refinement and regrouping. In the defect chapter, C may instead be any supplied coarsening map.	A permutation in V5 and a shear in V22.
T_{ba}	Input packet a, output packet b: $P_b T P_a$.	Blocks of a finite cyclic filter.
D, H, A	$T = D + H = D(I + A)$, with $A = D^{-1}H$.	Explicit integral 2×2 matrices.
$w(e), W(\gamma)$	Positive certified edge gain and its additive word weight.	$w(a) = 1$, $w(b) = 2$.
L	Precision depth in a quotient such as $\mathbb{Z}/2^L\mathbb{Z}$.	$L = 5$ gives arithmetic modulo 32.
$A(t), Z(t)$	Polynomial graph adjacency and unsigned walk-count series.	Two parallel edges remain distinct.
E, Λ, s	A rational projector, a specified lattice, and a clearing exponent.	E_+ has local weight 1 at 2.
(s_i)	Directional weights of a supplied Smith diagonal.	$(1, 2)$ modulo 8 or $(0, 1, 3)$ across depths.
Δ	The failure of an independently chosen operator to intertwine: $CT_f - T_c C$.	One nonzero entry $1/16$.

Suggested reading routes

For exact coordinates, read V3 through V8, then the finite filter on V20. For weighted inversion and counting, read V9 through V13. For arithmetic separation and congruence fibers, read V14 through V19. The final pages distinguish nonlinear chart geometry, error propagation, and the formal contract.

> **Read this as:** Operator words carry signed matrix values. Graph walks carry unsigned counts. Smith normal form concerns linear congruences. The Smith impedance chart on V21 is a different, nonlinear construction.

V3 / MODEL A / LOSSLESS ADDRESS REFINEMENT

Follow one coefficient through the tower

Traditional paper: Theorem 3.3 (p. 3), Theorem 4.6 (p. 11)

Take a vector $x = (x_0, \dots, x_{15})$. A packet is not a single summary number: it retains every coefficient at its selected addresses. For example,

$$P_{2,0}x = (x_0, 0, x_2, 0, \dots, x_{14}, 0).$$

The rail below lists the addresses in each packet. Its display order groups descendants under their parents. The matrix example on V5 instead uses increasing residue order.

	$r=0$								$r=1$							
$N=2$	0	2	4	6	8	10	12	14	1	3	5	7	9	11	13	15	16
	$r=0$				$r=2$				$r=1$				$r=3$			
$N=4$	0	4	8	12	2	6	10	14	1	5	9	13	3	7	11	15	16
	$r=0$		$r=4$		$r=2$		$r=6$		$r=1$		$r=5$		$r=3$		$r=7$	
$N=8$	0	8	4	12	2	10	6	14	1	9	5	13	3	11	7	15	16

Each cell is an address n; its coefficient travels with it.

Every row contains all sixteen addresses exactly once. A heavier boundary separates packet ranges. The numbers are coefficient addresses, not new coefficient values. There is no interpolation, averaging, or omitted high-frequency tail.

The address 6 is even, is 2 modulo 4, and is 6 modulo 8. Its coefficient therefore lies in the successive ranges

$$P_{8,6}X \subseteq P_{4,2}X \subseteq P_{2,0}X.$$

The corresponding identities hold as operators:

$$P_{2,0} = P_{4,0} + P_{4,2}, \qquad P_{4,2} = P_{8,2} + P_{8,6}.$$

Applying either composite parent selector to $P_{8,6}x$ leaves it unchanged. Applying the odd selector kills it. This is absorption between a parent and a child, rather than an approximate resemblance between nearby scales.

Why the reconstruction is unique

If a vector is a sum of components in disjoint packet ranges, apply the projector for one range. All other components vanish and the selected component remains. Thus every child component is recovered uniquely from the sum.

> **Read this as:** Refinement is a sharper address book for the same coefficients. The total number of free coordinates does not change.

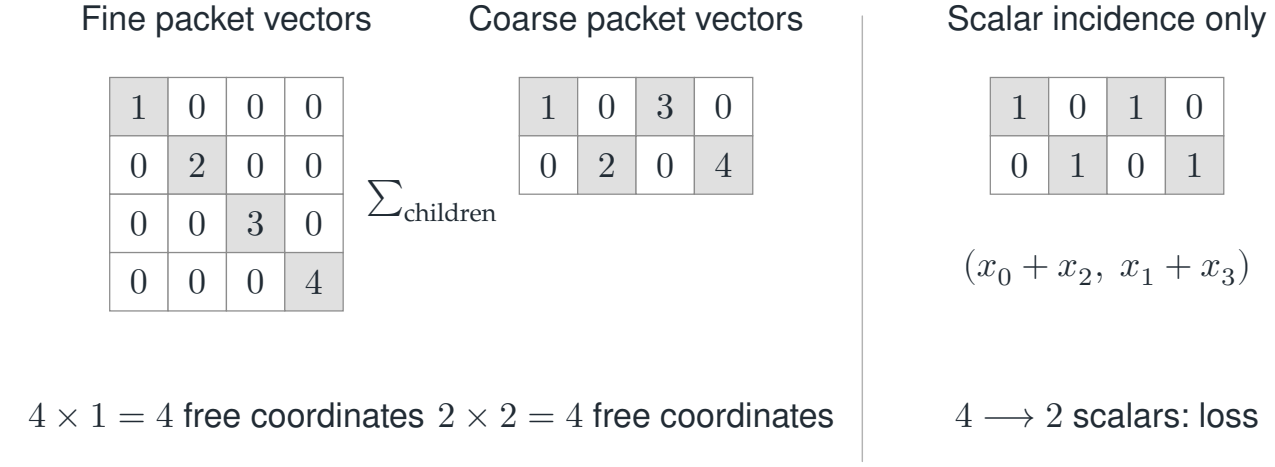
V4 / MODEL A / THE MEANING OF COARSENING

Regrouping is not averaging

Traditional paper: Definition 3.4 (p. 4), Theorem 3.5 (p. 4)

The word *coarsening* can suggest compression. In the exact coordinate theorem it means regrouping complete packet vectors. Replacing each packet by one scalar sum is a different operation and really does lose information.

For $x = (1, 2, 3, 4)$, the fine packets are the four coordinate-supported vectors. Regrouping gives $(1, 0, 3, 0)$ and $(0, 2, 0, 4)$. Every original coordinate remains available in one of these two vectors.



The left and middle panels store the same four independent coordinates. The right panel instead applies a 2×4 numerical incidence matrix to four scalar entries. That is not the range-restricted exact coordinate map.

Define

$$X_N^{\mathrm{pkt}} = \bigoplus_j P_{N,j} X, \qquad \mathcal{A}_N x = (P_{N,j} x)_j, \qquad \mathcal{S}_N (x_j)_j = \sum_j x_j.$$

Completeness and mutual annihilation give both inverse laws. Consequently,

$$J = \mathcal{A}_4 \mathcal{S}_2, \qquad C = \mathcal{A}_2 \mathcal{S}_4, \qquad CJ = I, \quad JC = I.$$

The apparently larger list of fine packets simply has smaller individual packet spaces.

A kernel witnesses genuine information loss

For the scalar incidence matrix B drawn on the right,

$$B(1, 0, -1, 0)^{\mathsf{T}} = 0.$$

The zero vector and $(1, 0, -1, 0)$ therefore have the same two scalar summaries. No inverse can recover the original four scalars from that output alone.

> **Read this as:** Ask what each packet stores. A list of full packet vectors can be exactly invertible. A list of packet means or totals generally is not.

V5 / MODEL A / COHERENCE AND NATURALITY

The operator changes coordinates too

Traditional paper: Theorem 3.5 (p. 4), Theorem 3.8 (p. 6)

Use a finite periodic carrier $X = \mathbb{Q}^8$ for this example. Let $Se_j = e_{j+1 \bmod 8}$ and let

$$T = I + 2S + S^2.$$

This is a concrete cyclic model, not an unannounced boundary convention for the finitely supported signal on V20. Write R_N for the permutation that groups coordinates in increasing residue order modulo N.

$\widehat{T}_2$

1	·	·	1	·	·	·	2
1	1	·	·	2	·	·	·
·	1	1	·	·	2	·	·
·	·	1	1	·	·	2	·
2	·	·	·	1	·	·	1
·	2	·	·	1	1	·	·
·	·	2	·	·	1	1	·
·	·	·	2	·	·	1	1

$J_{4,2}$

1	·	·	·	·	·	·	·
·	·	1	·	·	·	·	·
·	·	·	·	1	·	·	·
·	·	·	·	·	·	1	·
·	1	·	·	·	·	·	·
·	·	·	1	·	·	·	·
·	·	·	·	·	1	·	·
·	·	·	·	·	·	·	1

$\widehat{T}_4$

1	·	·	·	·	1	·	2
·	1	·	·	1	·	2	·
2	·	1	·	·	·	·	1
·	2	·	1	·	·	1	·
1	·	2	·	1	·	·	·
·	1	·	2	·	1	·	·
·	·	1	·	2	·	1	·
·	·	·	1	·	2	·	1

$$J_{4,2}\widehat{T}_2 = \widehat{T}_4 J_{4,2}$$

Actual 8×8 matrices. Dots are zero entries. The other entries are exact integers. Heavy lines mark the packet blocks: size four at level two and size two at level four.

The reordered operators and change of coordinates are

$$\widehat{T}_N = R_N T R_N^{-1}, \qquad J_{4,2} = R_4 R_2^{-1}.$$

Multiplication cancels the intermediate coordinate maps:

$$J_{4,2}\widehat{T}_2 = \widehat{T}_4 J_{4,2}, \qquad J_{8,4}J_{4,2} = J_{8,2}.$$

Thus it makes no difference whether the operator is applied before or after a consistent coordinate change. One must transport the operator, not merely reuse the same numerical matrix under a new ordering.

Refined blocks still reconstruct the same map

A coarse block expands as

$$P_b T P_a = \sum_{\beta \succ b,\, \alpha \succ a} P_\beta T P_\alpha.$$

The terms are maps embedded in the original source and target spaces. Adding their norms, or adding their matrix entries after forgetting the embeddings, is not this operator identity.

> **Read this as:** The fine matrix contains a different block organization of the same operator. An independently guessed coarse matrix is a separate approximation problem, measured later by a defect.

V6 / MODEL A / RESIDUE MASKS IN CHARACTERISTIC TWO

Selectors survive without division

Traditional paper: Proposition 4.2 (p. 9)

Work first in $\mathbb{F}_2$, where $1 + 1 = 0$. The eight coefficients below are just a finite display of a coefficient sequence. The even and odd projectors are diagonal masks with entries zero and one.

	0	1	2	3	4	5	6	7
x	1	1	0	1	1	0	1	1
$P_{2,0}x$	1	0	0	0	1	0	1	0
$P_{2,1}x$	0	1	0	1	0	0	0	1
$(P_{2,0} + P_{2,1})x$	1	1	0	1	1	0	1	1

Coefficient address n

No division by 2

All arithmetic in $\mathbb{F}_2$

Each nonzero input coefficient is retained by exactly one mask. Reconstruction is exact over $\mathbb{F}_2$. No average and no root of unity is used.

For a general coefficient ring and a positive integer N, the definition is

$$(P_{N,j}x)_n = \begin{cases} x_n, & n \equiv j \pmod{N}, \\ 0, & \text{otherwise.} \end{cases}$$

An address belongs to one residue class, so applying two masks either keeps that same coefficient or kills it. This proves

$$P_{N,i}P_{N,j} = \delta_{ij}P_{N,j}, \qquad \sum_j P_{N,j} = I.$$

The argument is coordinatewise and does not require the coefficients to form a field.

Why the usual even/odd average is a different formula

Over a field of characteristic not two, parity extraction can be represented using $F(z)$ and $F(-z)$ and division by 2. In characteristic two, $-z = z$ and $2 = 0$, so that formula cannot be used. The coefficient mask, however, still distinguishes even from odd *exponents*.

This distinction also explains the later involution example. Failure to reduce a rational spectral projector containing $1/2$ does not make every integral coefficient-address selector disappear.

> **Read this as:** The basic object is the address selector. A Fourier average is one possible realization of it and has additional algebraic hypotheses.

V7 / MODEL A / AN EXACT NINE-ENTRY TABLE

A Fourier certificate you can inspect

Traditional paper: Theorem 4.3 (p. 10)

Take $N = 3$ and work in $\mathbb{F}_7$. Set

$$\omega = 2, \qquad \omega^{-1} = 4, \qquad \eta = 3^{-1} = 5.$$

Here $2^3 = 1$ and neither 2 nor 2^2 is one. The unnormalized geometric sums are the entries in the first matrix.

$\sum_k 4^{jk}2^{rk} \pmod 7$ $\qquad$ $P_j(z^r) : \ 5 = 3^{-1} \pmod 7$

3	0	0
0	3	0
0	0	3

$\times 5$

1	0	0
0	1	0
0	0	1

$$\omega = 2, \quad \omega^3 = 1$$

$$\omega^{-1} = 4, \quad \eta = 5$$

$$j, r \in \{0, 1, 2\}$$

In $\mathbb{F}_3 \times \mathbb{F}_3$: order two does not imply orthogonality.

$$\omega = (-1, 1), \quad \eta = (2, 2), \quad \eta(1 + \omega) = (0, 1) \neq (0, 0)$$

All entries are exact residues modulo seven. Multiplying the diagonal value 3 by 5 produces 1. The lower calculation is a genuine failed Fourier certificate in a different coefficient ring.

On the monomial z^n, define $UF(z) = F(2z)$. The proposed projector acts by

$$5 \sum_{k=0}^{2} 4^{jk} 2^{nk}.$$

The identity $2^3 = 1$ makes this depend only on n mod 3. The nine-entry table therefore establishes the formula for every exponent once periodicity is used, rather than requiring a separate test at every coefficient.

Order alone is not sufficient over every ring

In $\mathbb{F}_3 \times \mathbb{F}_3$, the element $(-1, 1)$ has order two, but one component is a trivial character. Its nontrivial Fourier sum does not vanish in that component. Thus an element of order N in a ring with zero divisors is not, by itself, the full orthogonality certificate.

> **Read this as:** The finite table is a checkable sufficient input. Over a field, a primitive root supplies it. Over a more general ring, inspect the actual table instead of inferring orthogonality from order alone.

V8 / MODEL A / THE DYADIC AND ODD RECTANGLE

One address, two independent residues

Traditional paper: Corollary 4.9 (p. 11)

Because 4 and 3 are coprime, a residue modulo 12 is specified by a column $a = n \bmod 4$ and a row $b = n \bmod 3$. Each cell contains the unique combined address.

$a = n \bmod 4$

$b = n \bmod 3$	$a = 0$	$a = 1$	$a = 2$	$a = 3$
$b = 0$	0	9	6	3
$b = 1$	4	1	10	7
$b = 2$	8	5	2	11

$$(a, b) = (3, 1) \iff n = 7 \pmod{12}$$

$$7,\ 19,\ 31, \ldots$$

$$P_{12,7} = P_{4,3} P_{3,1}$$

The twelve cells are a permutation of $0, \ldots, 11$, not twelve extra degrees of freedom. The selected cell contains every integer congruent to 7 modulo 12.

An explicit reconstruction formula is

$$n \equiv 9a + 4b \pmod{12},$$

because 9 is one modulo four and zero modulo three, while 4 has the opposite pair of residues. For $(a, b) = (3, 1)$ this gives $31 \equiv 7$.

Applying the two masks in either order gives

$$P_{12,7} = P_{4,3} P_{3,1} = P_{3,1} P_{4,3}.$$

More generally, an address survives exactly when it satisfies both congruences. The coprime reconstruction identifies this intersection with one combined residue class.

The hypothesis can be seen failing

Replace 4 and 3 by 4 and 6. The requirements $n \equiv 0 \pmod 4$ and $n \equiv 1 \pmod 6$ are incompatible: the first makes n even and the second makes it odd. A full rectangular address product no longer exists. Compatible pairs must agree modulo the greatest common divisor.

> **Read this as:** CRT separates address coordinates. It does not claim that dyadic and odd spectral projectors have the same denominator weights or the same lifting behavior.

V9 / MODEL B / ORDERED TRANSFERS IN A FILTERED RING

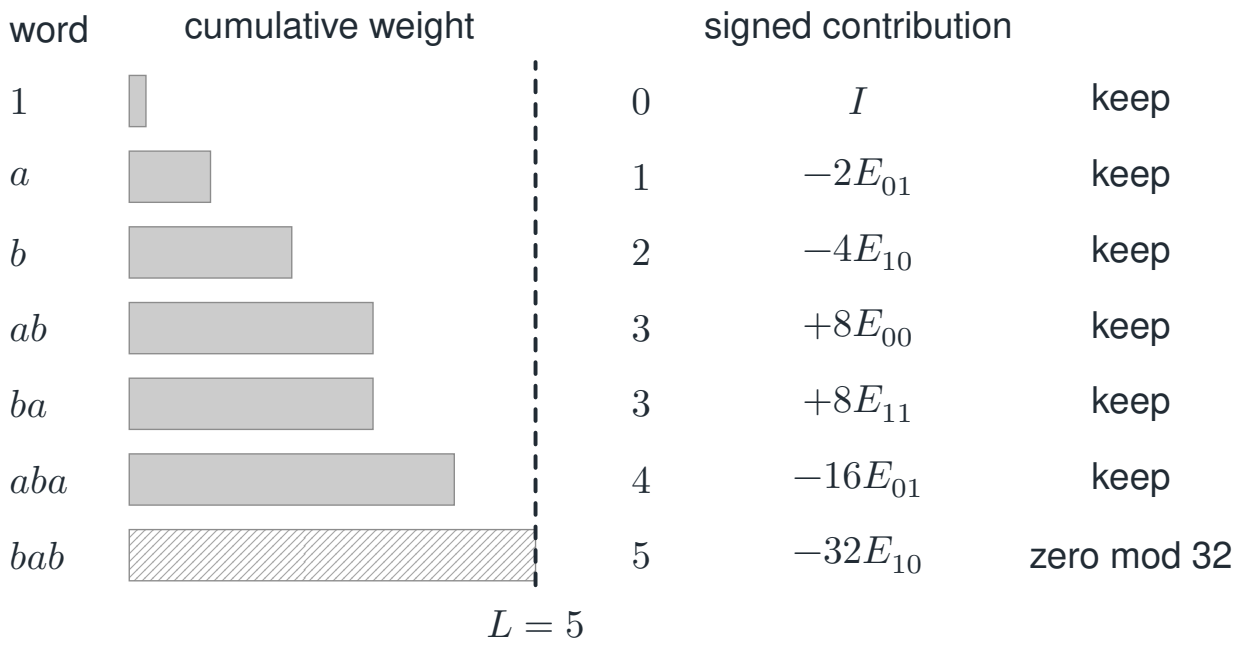
Which words can survive precision five?

Traditional paper: Theorem 5.10 (p. 17), Corollary 5.11 (p. 18)

Let E_{ij} send the jth basis vector to the ith. In integral 2×2 matrices set

$$a = 2E_{01}, \qquad b = 4E_{10}, \qquad A = a + b.$$

Their certified dyadic weights are 1 and 2. At precision $L = 5$, every word of weight at least five vanishes modulo 32.



All nonzero retained terms of the signed inverse sum, followed by the first discarded nonzero word. The empty word contributes I. Other words of weight below five can still be zero because $a^2 = b^2 = 0$.

The words are *written matrix products*: in ab, the map b acts first. Therefore

$$ab = 8E_{00}, \qquad ba = 8E_{11}, \qquad aba = 16E_{01}.$$

The two products ab and ba cannot be identified merely because they have equal weight. Both their order and their source/target information matter.

The pruning rule supplies a sufficient reason for a term to vanish. It is not an assertion that every word below the threshold contributes nontrivially. Packet incompatibility and algebraic relations can remove additional words.

> **Read this as:** Length controls the sign. Cumulative weight controls precision. The ordered matrix product controls the actual contribution. None of these three can replace the others.

V10 / MODEL B / A COMPLETE MODULO-32 CALCULATION

An inverse computed in both orders

Traditional paper: Theorem 5.10 (p. 17), Corollary 5.11 (p. 18)

Keep the matrices from V9 and choose

$$D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad D^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad T = D(I + A) = \begin{pmatrix} 5 & 3 \\ 4 & 1 \end{pmatrix}.$$

The six retained contributions sum to

$$S = I - a - b + ab + ba - aba = \begin{pmatrix} 9 & -18 \\ -4 & 9 \end{pmatrix}.$$

The required inverse is $K = SD^{-1}$ in the quotient, not $D^{-1}S$.

T

5	3
4	1

K

9	5
28	13

$TK \pmod{32}$

1	0
0	1

$KT \pmod{32}$

1	0
0	1

$$S = \begin{pmatrix} 9 & -18 \\ -4 & 9 \end{pmatrix}, \qquad K = SD^{-1} \pmod{32}$$

The displayed K uses canonical residues from 0 to 31. Both left and right products are the identity modulo 32. These are exact modular equalities, not floating-point residuals.

One can verify the claim without trusting the word expansion:

$$TK = \begin{pmatrix} 129 & 64 \\ 64 & 33 \end{pmatrix} \equiv I, \qquad KT = \begin{pmatrix} 65 & 32 \\ 192 & 97 \end{pmatrix} \equiv I \pmod{32}.$$

The cancellation of the exterior factor D explains the order. Since $T = D(I + A)$, multiply its inverse by D^{-1} on the *right* of the inverse of $I + A$.

What the finite theorem does not require

There is no claim here that the same integer series converges in the ordinary real matrix norm. The quotient kills all sufficiently deep products, so a finite sum suffices. The complete normed-ring theorem is a separate result with its own strict contraction hypothesis.

> **Read this as:** A correct noncommutative formula must work on both sides and preserve the order of its factors. Here the entire result can be checked by two integer matrix products.

V11 / MODEL B / A CERTIFIED CUTOFF, NOT A HEURISTIC

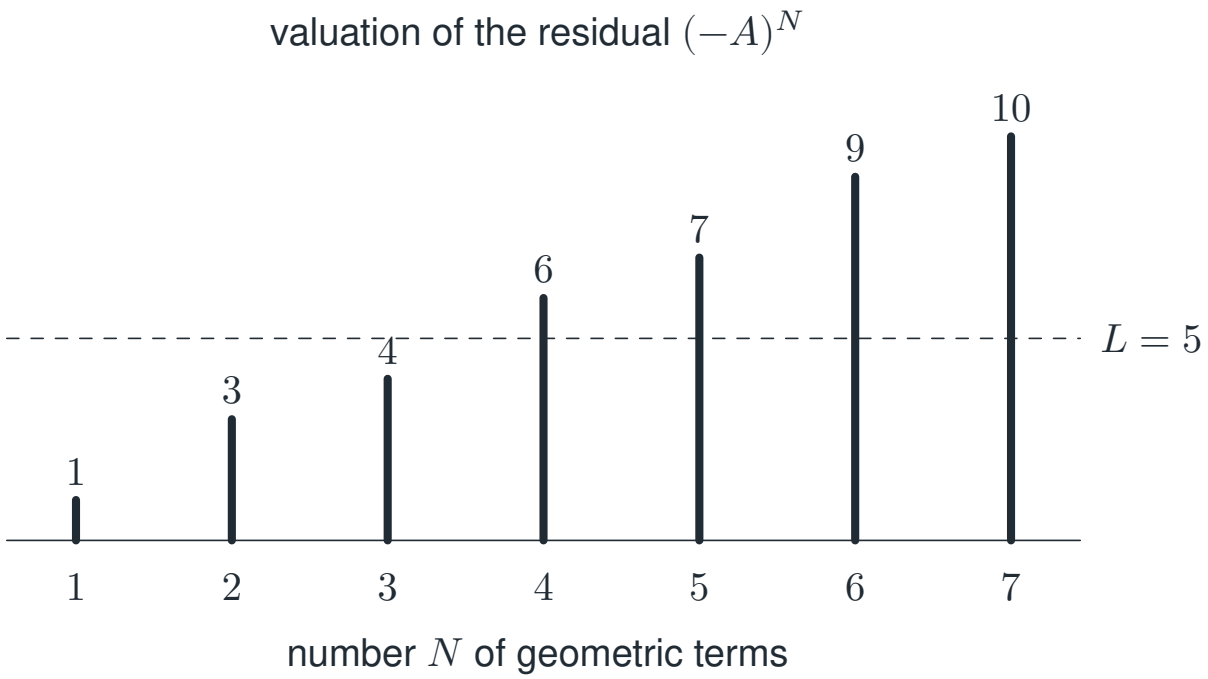
The residual tells you when to stop

Traditional paper: Theorem 5.10 (p. 17), Corollary 5.11 (p. 18), Section B.2 (p. 47)

For $S_N = \sum_{k=0}^{N-1} (-A)^k$, telescoping gives

$$(I + A)S_N = S_N(I + A) = I - (-A)^N.$$

Thus the exact residual is known. To certify precision L, it is enough to show that every residual entry is divisible by 2^L.



The plotted values are the minimum dyadic valuations of the nonzero entries of $(-A)^N$. The dashed line is the requested cutoff $L = 5$. This is an exact valuation profile, not a measured numerical convergence curve.

In this particular model, $A^2 = 8I$. Therefore

$$v_2(A^{2k}) = 3k, \qquad v_2(A^{2k+1}) = 3k + 1,$$

where the matrix valuation means the minimum entry valuation. Three geometric terms leave a residual of valuation four and do not suffice modulo 32. Four terms leave valuation six and do suffice.

The generic positive-weight estimate is more conservative. If every edge weight is at least $m > 0$, a word of length N has weight at least mN. Hence the cutoff

$$N = \left\lceil \frac{L}{m} \right\rceil$$

is sufficient. Here the certified minimum weight is $m = 1$, so the generic choice is five. The extra identity $A^2 = 8I$ justifies stopping at four in this example.

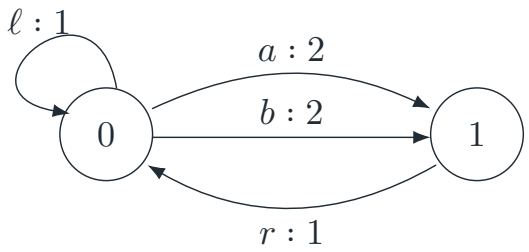
> **Read this as:** A structural identity may improve a cutoff. An observed small real norm or an arbitrarily larger claimed minimum edge weight does not supply that certificate.

V12 / MODEL C / A GRAPH WITH PARALLEL EDGES

Walks are not simple paths

Traditional paper: Theorem 6.6 (p. 21)

Consider the directed graph below. The loop ℓ has weight one, the two distinct edges a, b from 0 to 1 each have weight two, and the return edge r has weight one.



Edge labels carry positive integer weights.

All walks from 0 to 1 of weight 5

$\ell\ell\ell a$	$\ell\ell\ell b$
ara	arb
bra	brb

Every word on the right is read chronologically from left to right as an edge traversal. This is intentionally different from the right-to-left action of a written operator product in V9.

A walk may revisit a vertex and reuse an edge. At total weight five there are six walks from 0 to 1. Two use three loops followed by either a or b. Four travel to 1, return to 0, and travel to 1 again, with two independent choices of parallel edge.

Their multiplicities enter the polynomial adjacency matrix:

$$A(t) = \begin{pmatrix} t & 2t^2 \\ t & 0 \end{pmatrix}.$$

The coefficient 2 remembers the two distinct edges. Replacing it by 1 would count a different graph.

The first-edge decomposition

Let $a_n(s,t)$ count walks of total weight n from s to t. Removing the first edge gives

$$a_n(s,t) = \mathbf{1}_{s=t,\, n=0} + \sum_{\substack{e:\, \mathrm{source}(e)=s \\ w(e)\le n}} a_{n-w(e)}(\mathrm{target}(e), t).$$

The empty walk is allowed precisely on the diagonal at weight zero. Positive weights ensure that every contributing walk has length at most n, so enumeration at each fixed weight is finite.

> **Read this as:** The graph series counts unsigned walks, including repetitions and parallel-edge multiplicity. It does not count simple paths, and it does not know whether an associated operator product vanishes or cancels.

V13 / MODEL C / AUTOMATIC RATIONALITY AND RECURRENCE

A finite graph produces its denominator

Traditional paper: Theorem 6.6 (p. 21), Corollary 6.7 (p. 22)

For the graph on V12, compute the determinant and adjugate rather than guessing a recurrence:

$$I - A(t) = \begin{pmatrix} 1-t & -2t^2 \\ -t & 1 \end{pmatrix}, \quad Q(t) = 1 - t - 2t^3, \quad \operatorname{adj}(I-A)_{0,1} = 2t^2.$$

Since $Q(0) = 1$, its inverse exists as a formal power series with integer coefficients.

$$Z_{0,1}(t) = \frac{2t^2}{1 - t - 2t^3}$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12
a_n	0	0	2	2	2	6	10	14	26	46	74	126	218

$$a_8 = a_7 + 2a_5 = 14 + 2 \cdot 6 = 26$$

The coefficients are independently computed by the first-edge recurrence and checked against the rational series through weight 24. Literal edge-walk enumeration supplies a third check through weight 10.

From $QZ_{0,1} = 2t^2$ one obtains

$$a_0 = a_1 = 0, \quad a_2 = 2, \qquad a_n = a_{n-1} + 2a_{n-3} \quad (n \geq 3).$$

The numerator is responsible for the initial exception. Omitting it would incorrectly force the entire series to vanish.

Three boundaries of this calculation

This is a *formal* series identity. No analytic convergence at a chosen real or complex value of t is needed. The canonical determinant need not be the minimal denominator after cancellation. Finally, positive weights prevent infinitely many walks at a fixed weight: a single weight-zero loop would already give the infinitely many distinct walks $1, \ell, \ell^2, \ldots$ at weight zero.

> **Read this as:** Finite positive-weight graph data determine a rational series and an eventual recurrence. The proof does not begin by assuming a certificate $QZ = P$ that already contains that conclusion.

V14 / MODEL D / AN EVALUATED BÉZOUT CERTIFICATE

Separate before dividing

Traditional paper: Theorem 7.5 (p. 25), Definition 7.6 (p. 26)

Use the integral involution

$$A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \qquad A^2 = I.$$

With $F = A - I$ and $G = A + I$, the identities

$$-F + G = 2I, \qquad FG = 0$$

are an explicit evaluated Bézout certificate. Here all the required commutations follow because these operators are polynomials in the same matrix.

A

1	1
0	-1

$N = I + A$

2	1
0	0

$E_+ = N/2$

1	$\frac{1}{2}$
0	0

$$N^2 = 2N$$

$$E_+^2 = E_+$$

$$E_+(1,0) = (1,0), \qquad E_+(-1,2) = (0,0)$$

The denominator-free numerator is quasi-idempotent: $N^2 = 2N$. Division by the nonzero scalar 2 gives an idempotent over the fraction field. It does not yet make the projector integral on the selected lattice.

The numerator $N = G$ acts by 2 on $\ker(A - I)$ and by zero on $\ker(A + I)$. Thus

$$E_+ = \frac{1}{2}N = \begin{pmatrix} 1 & 1/2 \\ 0 & 0 \end{pmatrix}$$

is the required separator. The complementary projector is $E_- = I - E_+$.

A denominator bound need not be minimal

The same projector can be presented with the inflated certificate $-2F + 2G = 4I$. Its numerator is $2G$ and its normalization is still E_+. This supplies a two-digit clearing bound at 2, although one digit is enough. Thus the valuation of a supplied certificate gives an upper bound, not automatically the least possible weight.

For the original sign factors the concrete resultant is $\mathrm{Res}(x - 1, x + 1) = 2$ and the bound is sharp. That equality is special to this displayed calculation, not a universal minimality theorem for every resultant certificate.

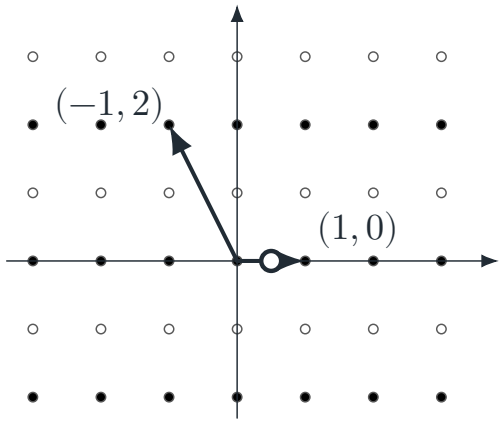
> **Read this as:** First check the evaluated identities and the numerator. Then normalize in the fraction field. Finally ask what denominator is needed on the particular lattice.

V15 / MODEL D / THE EIGENLATTICE HAS INDEX TWO

Where the dyadic digit is lost

Traditional paper: Proposition 9.3 (p. 32)

The eigenspaces are spanned by $(1,0)$ and $(-1,2)$. Their integral eigenvectors do not form an integral basis of $\mathbb{Z}^2$. The change-of-basis determinant is two.



$E_+(0,1) = (\frac{1}{2}, 0)$

$$B = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}, \quad \det B = 2$$

Filled: $B\mathbb{Z}^2 = \{(x,y) : y \text{ is even}\}$

Unfilled: the other lattice coset.

B^{-1} is integral at odd primes, not at 2.

An ordinary integer-lattice picture, not a drawing of the topology of $\mathbb{Q}_2$. Filled dots have even second coordinate. Unfilled dots lie in the other coset. The hollow point at $(1/2, 0)$ records the image of an integral vector under E_+.

Write

$$B = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}, \qquad AB = B\,\mathrm{diag}(1,-1), \qquad E_+ = B\,\mathrm{diag}(1,0)B^{-1}.$$

The index-two eigenlattice explains the denominator in B^{-1} and in the projector. Over the localized lattice $\Lambda_p = \mathbb{Z}^2_{(p)}$, the number 2 is a unit for every odd prime p. Hence the local denominator weight is zero there and one at $p = 2$.

This is a *local* assertion. Odd localization does not change the fact that the displayed eigenvectors span an index-two sublattice inside the ordinary integer lattice.

The special fiber does not retain the rational splitting

Modulo two, $x-1 = x+1$ and A becomes a nontrivial unipotent matrix. The entries $1/2$ in $E_\pm$ cannot be reduced modulo two. By contrast, the coefficient selectors on V6 use only zero and one and remain available.

> **Read this as:** The obstruction is in integral separation, not in the existence of the two rational eigenspaces. The lattice is part of the question.

V16 / MODEL D / A SHARP PRECISION-DELAY EXAMPLE

One extra digit is genuinely necessary

Traditional paper: Theorem 7.7 (p. 26), Proposition 9.3 (p. 32)

The projector acts by

$$E_+(x, y) = (x + y/2, 0).$$

Its factor $1/2$ can consume one dyadic digit. The claim concerns differences: if the inputs agree modulo $2^{m+1}\Lambda$, their projected differences lie in $2^m\Lambda$.

input pair	difference	same mod 8?	E_+ difference	same mod 8?
$0, (0, 8)$	$(0, 8)$	yes	$(4, 0)$	no
$0, (0, 16)$	$(0, 16)$	yes	$(8, 0)$	yes

$$m = 3: \quad 3 \text{ input digits fail; } 4 \text{ input digits suffice.}$$

For $m = 3$, inputs differing by $(0, 8)$ agree modulo eight, but their projected outputs differ by $(4, 0)$ and do not. Raising the input agreement to modulo sixteen makes the projected outputs agree modulo eight.

The example works at every positive depth:

$$E_+(0, 2^m) = (2^{m-1}, 0) \notin 2^m\Lambda, \qquad E_+(0, 2^{m+1}) = (2^m, 0) \in 2^m\Lambda.$$

Thus a zero-delay promise fails, while one extra input digit is sufficient. This witnesses sharpness for the specified projector and lattice.

The general calculation is just as short

If $\pi^s E\Lambda \subseteq \Lambda$ and $x - y = \pi^{m+s}v$, then

$$E(x - y) = \pi^m(\pi^s Ev) \in \pi^m\Lambda.$$

No rounding procedure is hiding in this identity. Nor does it authorize reducing E modulo π when its entries are nonintegral. Individual projected points may lie in a fractional lattice. The theorem controls their difference at the stated integral scale.

Accuracy and branching are different views

The later Smith example records how many inputs share one output at finite depth. Here we ask how accurately an input must be known to determine a projected output difference. Both involve directional divisibility, but they are different questions.

> **Read this as:** The phrase "one lost digit" has an exact meaning: both a sufficiency theorem and an explicit pair of inputs showing that a smaller delay fails.

V17 / MODEL E / TWO DIRECTIONAL WEIGHTS

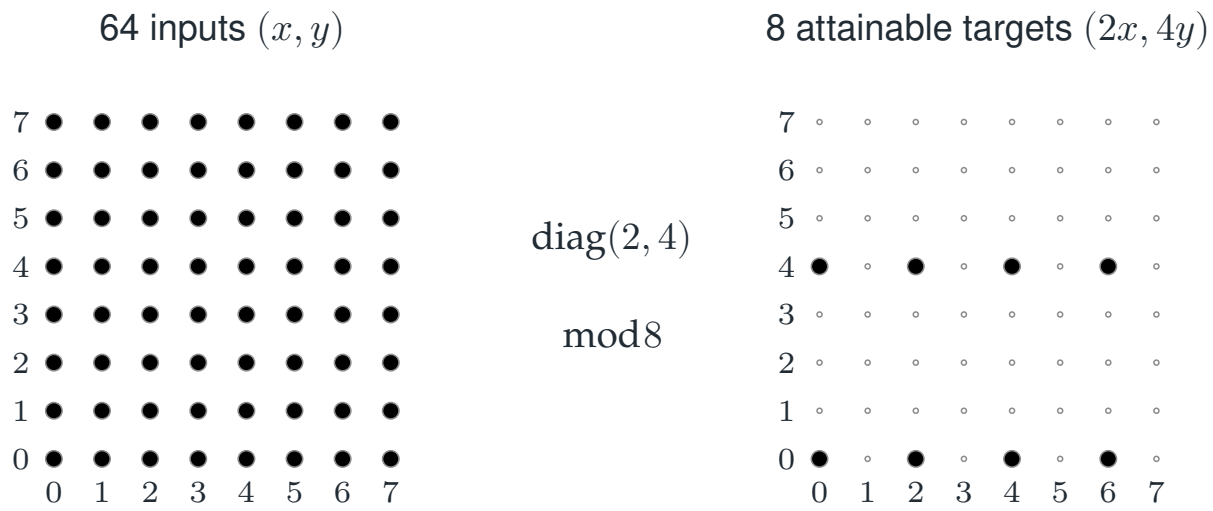
See every Smith fiber modulo eight

Traditional paper: Theorem 10.6 (p. 36)

Consider the actual map of finite sets

$$(\mathbb{Z}/8\mathbb{Z})^2 \longrightarrow (\mathbb{Z}/8\mathbb{Z})^2, \qquad (x, y) \longmapsto (2x, 4y).$$

The first coordinate of the output must be even. The second must be zero or four. Those are precisely the attainable targets in the right panel.



Every filled target has exactly $2 \cdot 4 = 8$ preimages.

All sixty-four input points and all sixty-four possible target addresses are shown. The eight filled target points are the image. Every attainable target has exactly eight preimages. The other fifty-six targets have none.

For a specific attainable target $(2, 4)$,

$$2x \equiv 2 \pmod 8 \iff x \in \{1, 5\}, \qquad 4y \equiv 4 \pmod 8 \iff y \in \{1, 3, 5, 7\}.$$

The coordinate choices are independent, giving $2 \cdot 4 = 8$ solutions. The entire carrier is partitioned consistently:

$$8 \text{ attainable targets } \times \ 8 \text{ inputs per target} = 64.$$

For an inadmissible target, such as $(1, 4)$ or $(2, 2)$, the fiber is empty. The same nonzero cardinality formula must not be applied to it.

The arithmetic behind the picture

Multiplication by p^s modulo p^L has kernel size $p^{\min(s,L)}$. Its image consists exactly of residues whose canonical representatives are divisible by that same power. The displayed model has $p = 2$, $L = 3$ and weights $(1, 2)$.

> **Read this as:** A directional weight has two consequences: a divisibility condition on the target and a conditional multiplicity of inputs. Solvability comes before counting.

V18 / MODEL E / EXPLICIT SMITH COORDINATES

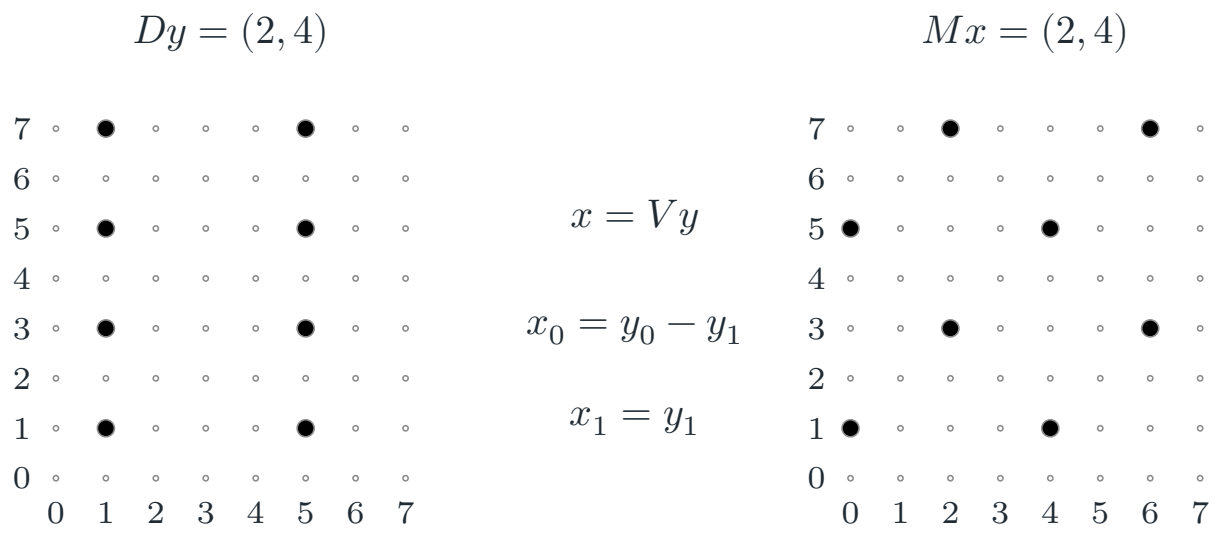
Transport the whole solution fiber

Traditional paper: Theorem 10.6 (p. 36)

The preceding diagonal system is connected to a nondiagonal one by the unimodular change of variables

$$M = \begin{pmatrix} 2 & 2 \\ 0 & 4 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad MV = \operatorname{diag}(2,4) = D.$$

Since $\det V = 1$, $x = Vy$ is bijective modulo eight. Thus $Mx = b$ is equivalent to $Dy = b$.



The eight solutions are permuted, not created or discarded.

The eight solutions for the target $(2,4)$, before and after the coordinate shear. Coordinates are residues in $\{0,\dots,7\}$. No interpolation on this modular grid is intended.

For example, the diagonal solution $y = (1,3)$ corresponds to

$$x = Vy = (-2,3) \equiv (6,3) \pmod 8,$$

and $M(6,3)^{\mathsf{T}} = (18,12)^{\mathsf{T}} \equiv (2,4)^{\mathsf{T}}$. Applying the inverse shear recovers the original solution.

Two boundary cases worth keeping visible

A unit diagonal coordinate has exactly one solution for every target. A literal zero coordinate has all eight possible inputs when its target is zero, and no input when the target is nonzero. A zero coordinate is not inverted or silently omitted.

The displayed diagonalization is verified by multiplication. The theorem takes diagonal data as input. A Smith normal form for an arbitrary matrix or general DVR requires a separate structure theorem.

> **Read this as:** An invertible change of coordinates transports existence, every solution, and the fiber count. The target must be transformed as prescribed when checking admissibility.

V19 / MODEL E / A DEPTH-BY-DEPTH FIBER BUDGET

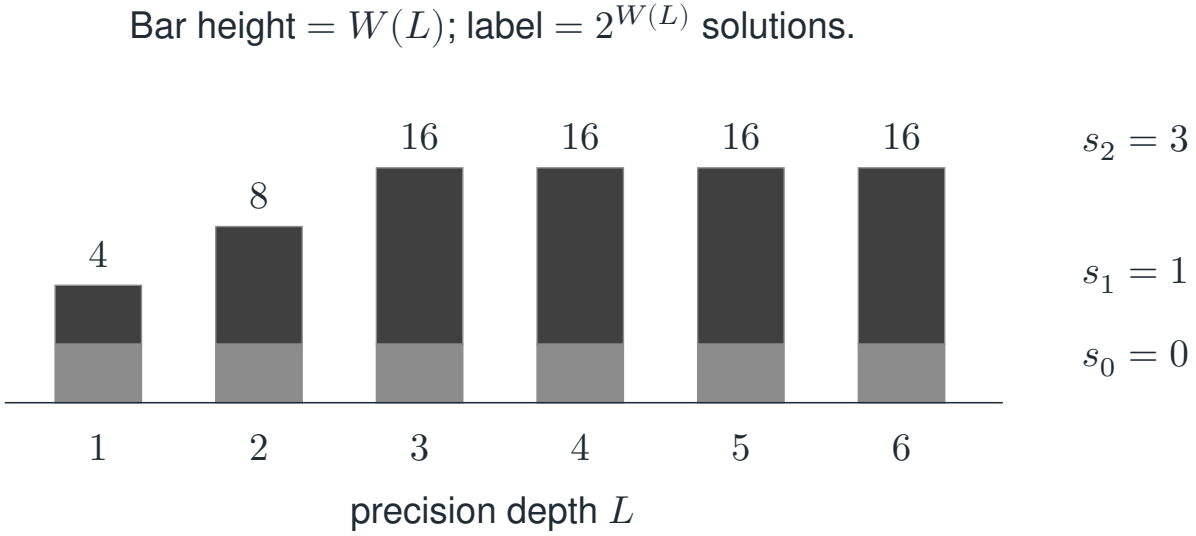
Precision exposes directional freedom

Traditional paper: Theorem 10.6 (p. 36)

Take $p = 2$ and the supplied diagonal weights $(s_0, s_1, s_2) = (0, 1, 3)$. At depth L, an admissible target has

$$2^{W(L)} \text{ solutions}, \qquad W(L) = \min(0, L) + \min(1, L) + \min(3, L).$$

Each direction contributes only the amount visible at the current depth.



The stacked height records the exponent $W(L)$. The label above each bar records the actual number of solutions. The zero-weight direction contributes no branching. This is a finite cardinality calculation, not an information-theoretic assumption about random inputs.

The values are

L	1	2	3	4	5	6
$W(L)$	2	3	4	4	4	4
# solutions	4	8	16	16	16	16.

Once $L \geq 3$, the nonzero weights have been fully exposed and the admissible fiber cardinality stabilizes at sixteen. The input space still grows as 2^{3L}. The number of attainable targets grows accordingly.

At depth six, $(1, 2, 8)$ is admissible and gives sixteen solutions. Replacing the last coordinate by four makes the target inadmissible and gives zero solutions, even though the weights and depth are unchanged.

What changes with a zero diagonal factor?

A literal zero factor contributes L, not a fixed eventual weight, and requires a zero target coordinate. Its fiber therefore keeps growing with the modulus. That case must not be mistaken for a finite nonzero Smith weight.

> **Read this as:** The scalar sum of weights gives the count, but the separate weights give the admissibility conditions. Keeping only a determinant valuation can hide which target coordinate fails.

V20 / MODEL F / A FINITE POLYPHASE REALIZATION

Filter first or split first: same output

Traditional paper: Section B.1 (p. 47)

Use the finitely supported signal $x = (1, 2, 3, 4, 5, 6)$ and the filter $h = (1, 2, 1)$. Outside the displayed support, the input is zero. Let $e_m = x_{2m}$ and $o_m = x_{2m+1}$.

x	1	2	3	4	5	6		
$h * x$	1	4	8	12	16	20	17	6
y_{even}	1	8	16	17				
y_{odd}	4	12	20	6				

$h = (1, 2, 1)$

zero extension

no periodic wrap

Exact integer convolution, including both output tail coefficients. The two bottom rows are the even and odd output phases. Interleaving them reconstructs the full eight-term output.

If w denotes one delay on a phase sequence, the filter becomes

$$\begin{pmatrix} y_{\text{even}} \\ y_{\text{odd}} \end{pmatrix} = \underbrace{\begin{pmatrix} 1+w & 2w \\ 2 & 1+w \end{pmatrix}}_{H_{\text{phase}}(w)} \begin{pmatrix} e \\ o \end{pmatrix}.$$

The cross-phase delay is not optional. A unit shift sends an odd sample to the next even position, while it sends an even sample to the odd position in the same phase block.

Here $e = 1 + 3w + 5w^2$ and $o = 2 + 4w + 6w^2$, so direct polynomial multiplication gives

$$y_{\text{even}} = 1 + 8w + 16w^2 + 17w^3, \qquad y_{\text{odd}} = 4 + 12w + 20w^2 + 6w^3.$$

This is the same result as ordinary convolution followed by residue extraction. No numerical transform or limiting argument is needed because every sequence is finitely supported.

> **Read this as:** A phase matrix is not just a sparsity pattern. Its delay factors carry the quotient-and-remainder information needed when a shift crosses a phase boundary.

V21 / MODEL G / A NONLINEAR PHASE ORBIT

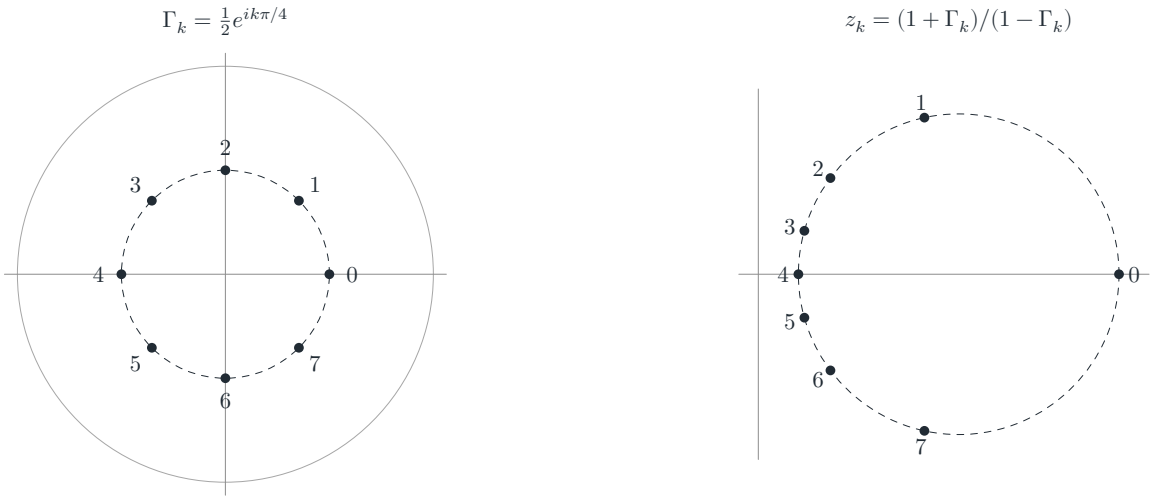
Two Smith constructions, different roles

Traditional paper: Section 12.3 (p. 42), Section 12.3, Smith-phase identity (p. 43)

Smith *normal form* concerns linear congruences. The Smith *impedance chart* uses the nonlinear coordinate change

$$\Gamma = \frac{z-1}{z+1}, \qquad z = \frac{1+\Gamma}{1-\Gamma}.$$

They share a name, not a mathematical definition. Sample the reflection circle at the exact algebraic points $\Gamma_k = \frac{1}{2}e^{ik\pi/4}$.



Matching labels identify the same point in the two coordinates. All eight points are computed algebraically. Only their placement on the page is rendered numerically. The outer left circle is $|\Gamma| = 1$, not part of the sampled radius-$1/2$ orbit.

The half-turn $\Gamma \mapsto -\Gamma$ gives

$$z \longmapsto \frac{1}{z}, \qquad z \neq 0, -1.$$

Thus the labels 0 and 4 are 3 and $1/3$. Label 2 is $(3+4i)/5$, and the opposite point is its reciprocal. The domain restrictions come from the forward and inverse coordinate denominators.

These sampled phase points are not linear idempotent ranges. Refining from two to four to eight samples inserts more points on the same orbit. This does not imply a linear packet decomposition of impedance values.

> **Read this as:** The chart is a geometric realization of phase action. Its nonlinear coordinate sensitivity near $\Gamma = \pm 1$ is not the Smith congruence-fiber count and not a new linear projector theorem.

V22 / MODEL H / EXACT TELESCOPING, EXPLICIT GROWTH

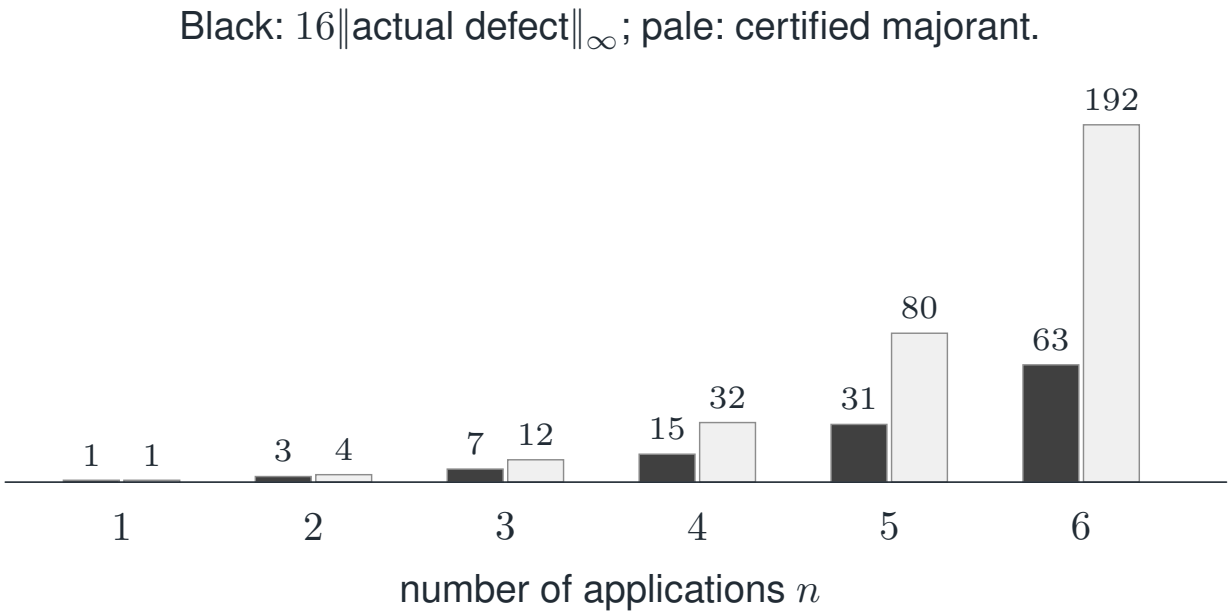
A small defect can grow under powers

Traditional paper: Theorem 11.1 (p. 37), Corollary 11.3 (p. 38)

Choose an invertible coordinate shear but independently keep the same numerical operator at both levels:

$$C = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}, \quad T_f = T_c = \operatorname{diag}(1, 2), \quad \varepsilon = \frac{1}{16}.$$

Then $\Delta = CT_f - T_c C = \varepsilon E_{01}$. Exact coordinate invertibility has not forced these independently selected operators to intertwine.



Actual defects and certified upper bounds in the induced infinity norm, both divided by ε. All heights are exact integers. The upper estimate is valid but need not be tight.

Direct multiplication gives

$$CT_f^n - T_c^n C = \varepsilon(2^n - 1)E_{01}.$$

The general telescoping formula explains every contribution:

$$CT_f^n - T_c^n C = \sum_{k=0}^{n-1} T_c^{n-1-k} \Delta T_f^k.$$

Since both operator norms are two, the source theorem bounds the error by $n2^{n-1}\varepsilon$. It does not promise a uniform-in-n small error merely because ε is small.

Had we chosen the transported operator $T_c' = CT_f C^{-1}$, the defect would be zero and every polynomial action would intertwine exactly. That is a different choice of coarse operator, not a stronger conclusion from the faulty one.

> **Read this as:** Separate coordinate coherence from operator consistency. A defect from one step is propagated by the dynamics on both sides. The telescoping identity tells you exactly how.

V23 / MODEL H / A DOMAIN-SENSITIVE COMPARISON

Resolvents amplify errors near poles

Traditional paper: Theorem 11.5 (p. 38)

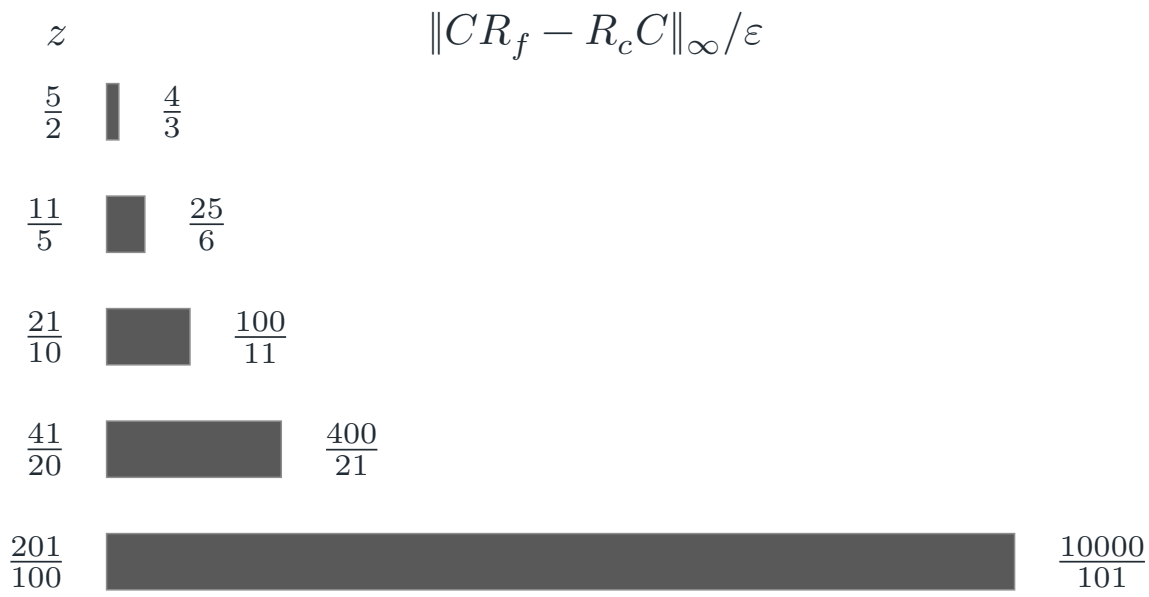
For the same model, take $z \notin \{1, 2\}$ and form the actual inverse matrices

$$R_f = (zI - T_f)^{-1}, \qquad R_c = (zI - T_c)^{-1}.$$

Their off-diagonal intertwining error is

$$CR_f - R_cC = R_c\Delta R_f = \frac{\varepsilon}{(z-1)(z-2)}E_{01}.$$

Both inverse hypotheses matter. Neither resolvent is asserted at a pole.



Exact rational samples; the point $z = 2$ itself is excluded.

The bars show exact amplification factors for five rational choices $z > 2$. Rows are separate samples, not equally spaced positions on the z axis. The coefficient $\varepsilon = 1/16$ has been divided out.

At $z = 3$ the amplification factor is only $1/2$. At $z = 201/100$, it is $10000/101$, so a small one-step defect can create a much larger resolvent defect. This follows from the exact formula rather than from unstable floating-point subtraction of two nearly singular inverses.

The certified general norm estimate is

$$\|CR_f - R_cC\| \le \|R_c\|\,\|\Delta\|\,\|R_f\|.$$

The paper also proves a one-sided-witness version: a left inverse for the coarse shift and a right inverse for the fine shift suffice for the algebraic identity. That does not turn arbitrary one-sided witnesses into a claim of two-sided resolvent existence.

> **Read this as:** A small local defect is meaningful only together with control of the surrounding operators or inverses. Near a pole, the domain and amplification factors are part of the theorem, not technical decoration.

V24 / PROOF BOUNDARY / HOW TO USE THIS GUIDE

Read the certificates with the pictures

Formal evidence: current correspondence and the inherited 25-row contract

The original release has a frozen twenty-five-item formal contract: twenty-one numbered main-text definitions and results, one Smith-phase identity, and three algorithm specifications. Its dated source and evidence remain preserved. Current proof additions have a separate correspondence and replay record. The table below describes the inherited interfaces; the current map identifies the broader constructions. The visual models illustrate the mathematics, and their finite checks have recorded ranges.

Current Lean scope. 53 of 53 proof-level source records have kernel checked, semantically accepted scope. No partial or pending proof records are listed in this correspondence. These are source-record counts under their stated hypotheses, not counts of independent discoveries.

Inherited interface	Inputs and checked scope	Boundary of that interface
Exact refinement	Complete projector families and a parent map when formulas relating parents to children are used.	A canonical refinement for every arbitrary pair of decompositions.
Fourier realization	A finite orthogonality table, or field/root hypotheses that provide it.	Orthogonality from multiplicative order alone over every ring.
Weighted inversion	A two-sided inverse of D, a multiplicative filtration, and positive certified gains.	Convergence in every possible complete filtered algebra.
Graph rationality	A finite multigraph with positive weights, together with its actual determinant and adjugate.	Simple-path counts, analytic convergence everywhere, or operator noncancellation.
Spectral separation	Evaluated Bézout identities and explicit lattice-preservation data.	Automatic construction or minimality of every resultant certificate.
Smith congruences	Concrete diagonal data modulo p^L, with $p > 1$.	An automatically constructed general-DVR Smith normal form.
Linear defects	Actual typed maps and the stated inverse or norm witnesses.	Small propagated errors independent of time or distance from a pole.

Reproduce the explanatory calculations

The preserved model source contains **118 named exact checks**. Each finite range is recorded in the file `EXACT_CHECKS.json`. They cover masks, CRT labels, modular inverse products, graph enumeration, rational recurrences, lattice denominators, complete finite fibers, phase identities, and defect formulas. Graphics are generated from those same exact model definitions. Numerical placement of geometric points is only a rendering step.

The historical monograph report records 48 passing Python tests, 118 exact visual checks, 22 byte-matching regenerated vector plates, 13 regenerated PDF/PNG figure pairs, and nine matching numerical summary rows. These finite computation records retain their tested ranges and dates. Current Lean replay and source coverage have separate records.

> **Read this as:** Use the guide to understand what a theorem says and why its hypotheses matter. Use the theorem map and preserved proof sources to identify the precise formal result. Neither a picture nor a finite test expands that result's scope.

References

[1] John B. Conway. *A Course in Functional Analysis*. Springer, second edition, 1990.

[2] Ingrid Daubechies. *Ten Lectures on Wavelets*, volume 61 of *CBMS NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics, Philadelphia, 1992.

[3] Israel Gohberg, Peter Lancaster, and Leiba Rodman. *Matrix Polynomials*, volume 58 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics, Philadelphia, 2009.

[4] Tosio Kato. *Perturbation Theory for Linear Operators*. Classics in Mathematics. Springer, Berlin, 1995.

[5] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer, New York, revised third edition, 2002.

[6] Morris Newman. *Integral Matrices*, volume 45 of *Pure and Applied Mathematics*. Academic Press, New York, 1972.

[7] David M. Pozar. *Microwave Engineering*. Wiley, 4 edition, 2012.

[8] Jean Pierre Serre. *Local Fields*, volume 67 of *Graduate Texts in Mathematics*. Springer, New York, 1979. Translated by Marvin Jay Greenberg.

[9] Richard P. Stanley. *Enumerative Combinatorics, Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012.

[10] P. P. Vaidyanathan. *Multirate Systems and Filter Banks*. Prentice Hall Signal Processing Series. Prentice Hall, Englewood Cliffs, New Jersey, 1993.