Lehmer’s conjecture

Ramanujan tau, signed packets, and the prime gap

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READER EDITION 1.0 / 9 OCTOBER 2026

Thirty visual studies of one fixed modular form. Exact coefficients, classical identities, project arguments and constructed local models are distinguished throughout. The program has not proved that every coefficient is nonzero.

WORKED EXAMPLES / EXACT COORDINATES / OPEN PROBLEMS

One product, infinitely many coefficients

DEFINITION / THE FIXED MODULAR FORM

$$\Delta(q) = q \prod_{m \geq 1} (1 - q^m)^{24}$$

$$q - 24q^2 + 252q^3 - 1472q^4 + \cdots$$

$$\tau(n) = [q^n]\Delta(q)$$

Fixed modular form. Positive integer index n.

READ THE FIGURE

The coefficient of q^n is $\tau(n)$. The initial q shifts the product coefficient by one: $\tau(n) = [q^{n-1}] \prod_{m \geq 1} (1 - q^m)^{24}$.

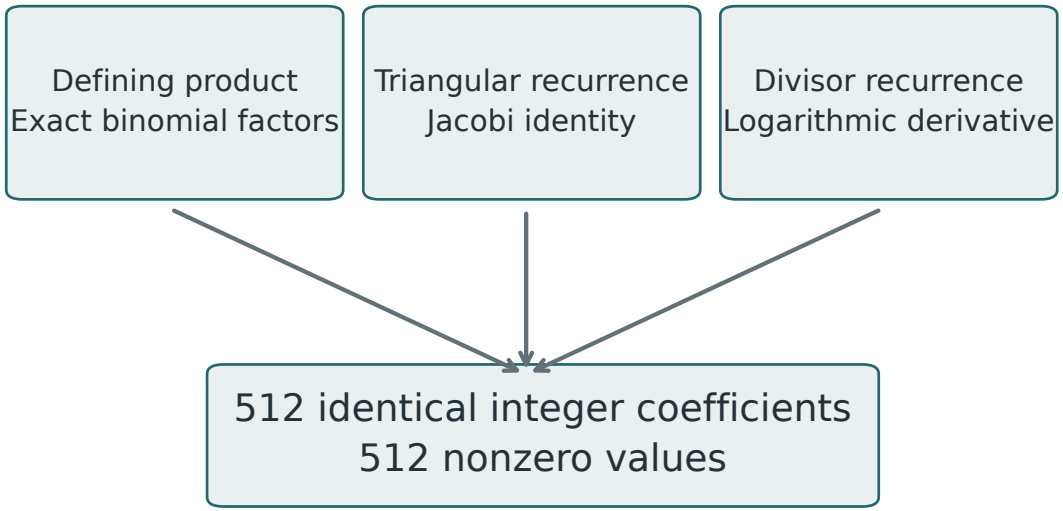
Lehmer's conjecture asks whether $\tau(n) \neq 0$ for every positive integer n. Large values and frequent sign changes do not answer that question.

The target is the actual normalized weight-12 cusp form Δ of level one.

Carry forward. A sequence constructed to obey some of the same rules is a separate mathematical object.

Three independent coefficient calculations

EXACT FINITE EVIDENCE / ALL INTEGERS THROUGH 512



READ THE FIGURE

The replay compares direct product expansion, a Jacobi triangular recurrence, and a divisor-sum recurrence from the logarithmic derivative.
If $F = \Delta/q = \sum a_m q^m$, the last method uses

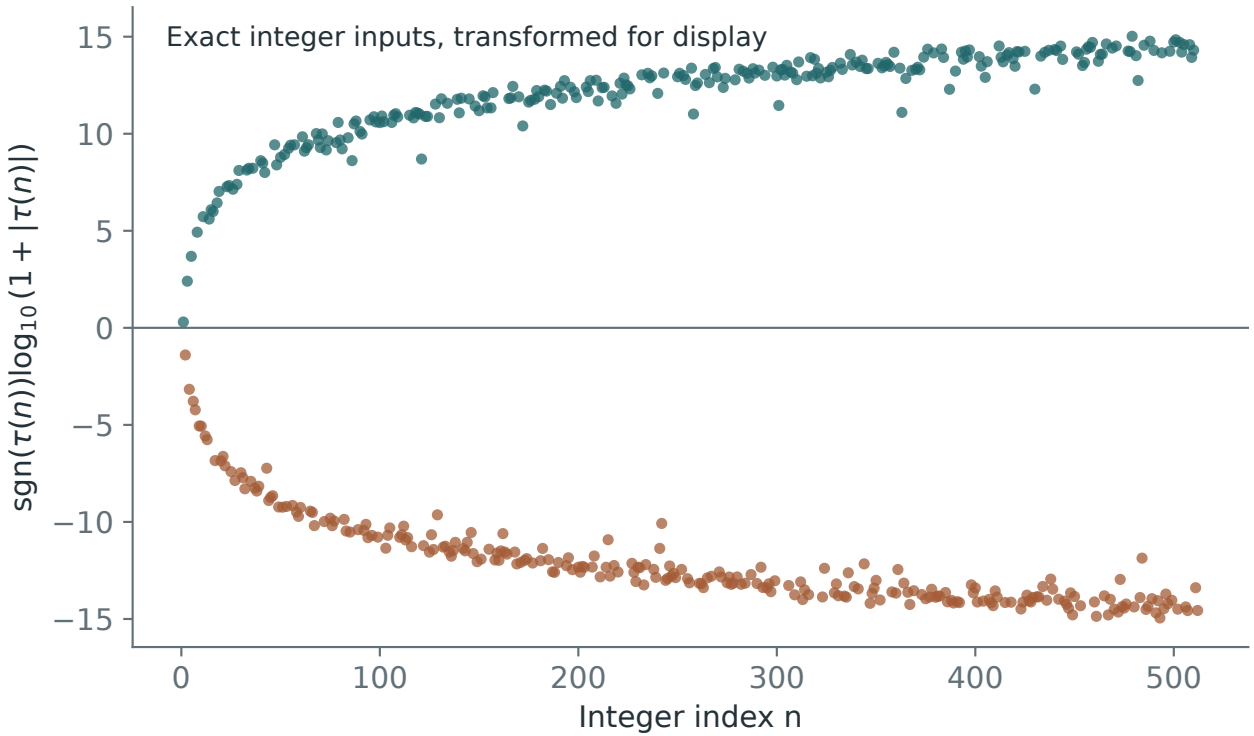
$$m a_m = -24 \sum_{j=1}^{m} \sigma_1(j) a_{m-j}.$$

All three integer methods agree on 512 coefficients. Every one is nonzero.

Carry forward. Agreement validates this bounded table. It is not a proof beyond the tested range.

The actual coefficients through 512

EXACT INPUTS / SIGNED LOGARITHMIC DISPLAY



READ THE FIGURE

The vertical coordinate compresses large integers while preserving their sign. Teal points are positive and copper points negative.

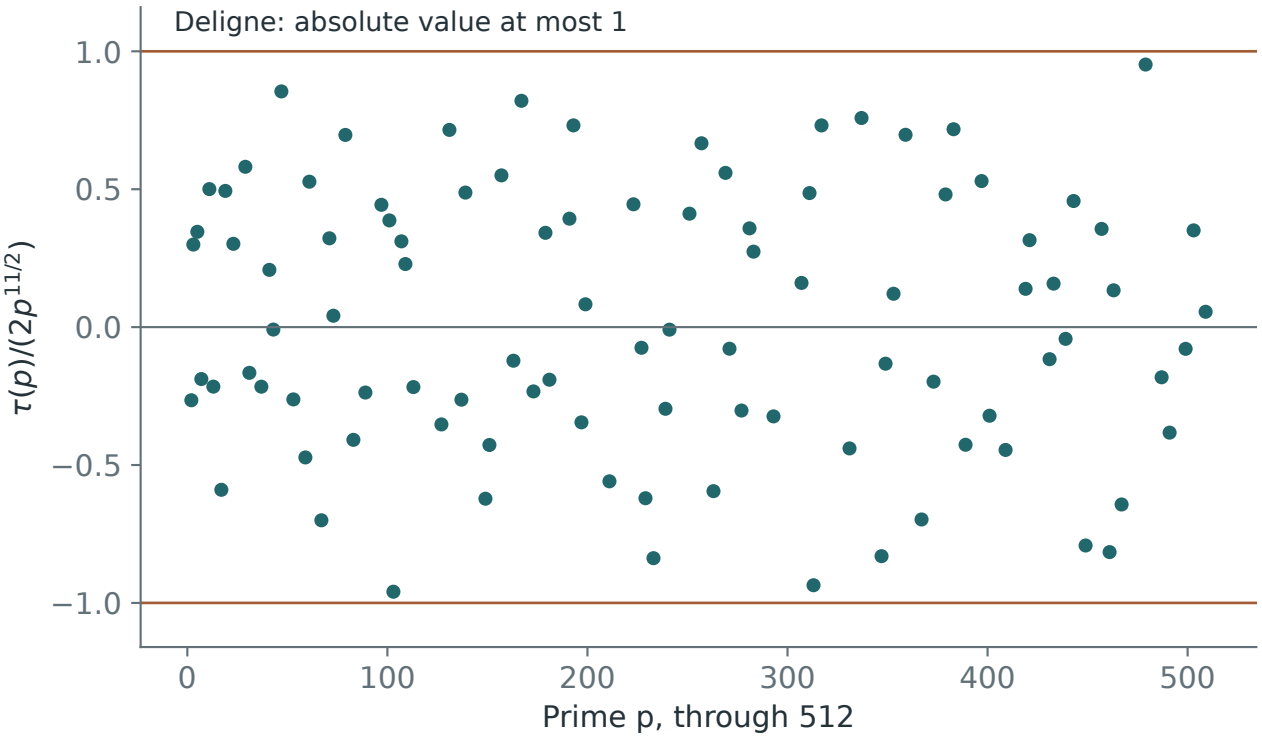
Every input is an exact integer. Floating-point logarithms display the magnitudes only.

This plot includes composite indices and primes. It is not a normalized prime distribution or a claimed limiting law.

Carry forward. A scatter plot supports exploration and checking. It cannot resolve the universal nonzero claim.

Deligne's bound controls size

CLASSICAL THEOREM / PRIME DATA THROUGH 512



READ THE FIGURE

For every prime p, Deligne gives

$$|\tau(p)| \leq 2p^{11/2}.$$

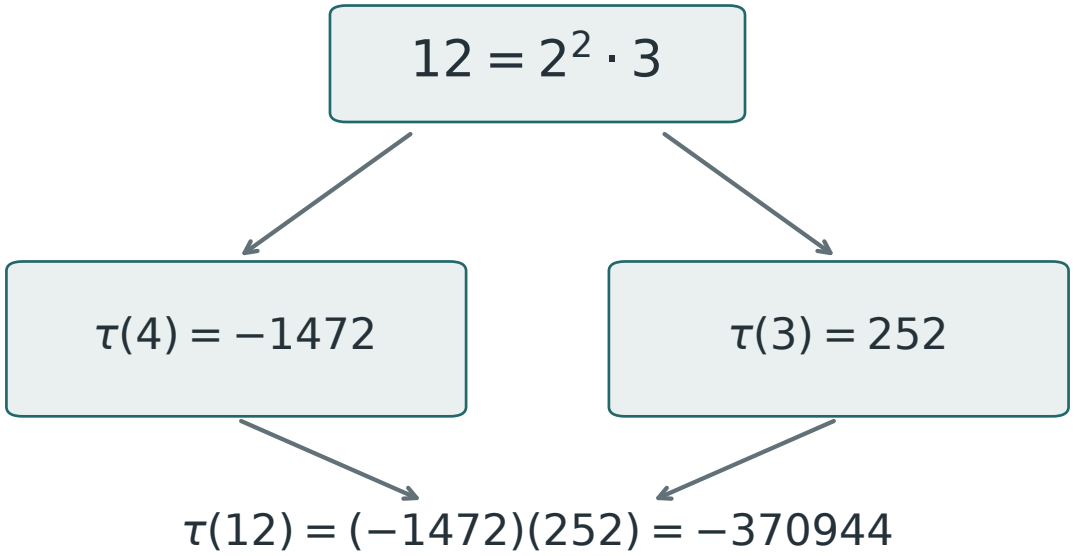
The plot divides each prime coefficient by this upper bound. All displayed values lie between the two horizontal lines.

Zero remains inside the allowed interval. The bound is fully compatible with a hypothetical zero coefficient.

Carry forward. The remaining step requires a nonzero conclusion. An upper bound supplies no positive lower bound.

Multiplicativity organizes the indices

CLASSICAL IDENTITY / AN EXACT WORKED EXAMPLE



READ THE FIGURE

If $\gcd(m, n) = 1$, then $\tau(mn) = \tau(m)\tau(n)$. Here 4 and 3 are coprime and their exact coefficients give $\tau(12) = -370944$.

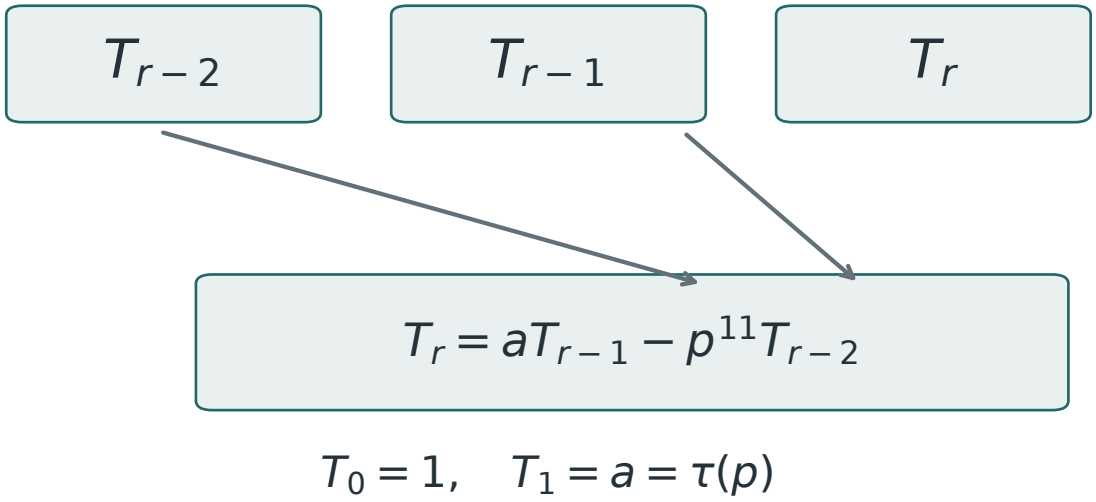
Unique prime factorization reduces all coefficients to prime-power blocks.

Coprimality matters: $\tau(4)$ is not $\tau(2)^2$. It uses the Hecke recurrence.

Carry forward. Multiplicativity identifies the correct local blocks. It does not establish their nonvanishing.

The recurrence lives at one prime

CLASSICAL HECKE IDENTITY / NOT INTERPOLATION IN p



Fixed prime p. Recurrence varies the exponent r, not the prime.

READ THE FIGURE

Fix a prime p. Write $T_r = \tau(p^r)$ and $a = \tau(p)$. The recurrence starts from $T_0 = 1$, $T_1 = a$ and applies for $r \geq 2$.

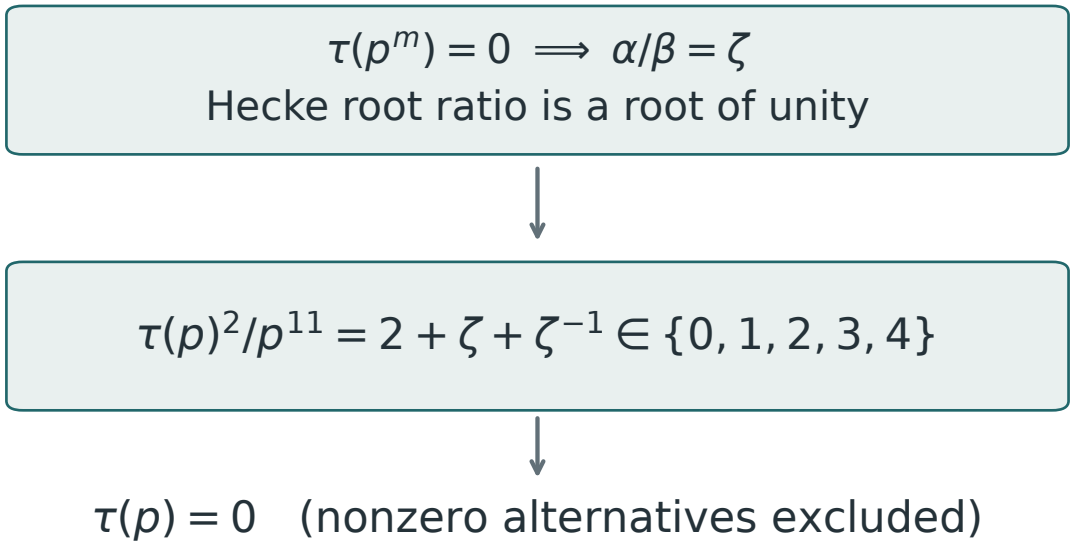
The roots of $X^2 - aX + p^{11}$ encode the same sequence.

This varies the exponent r. It supplies no single analytic function of the prime index p.

Carry forward. Interpolation across primes needs its own definition and proof. The exponent recurrence does not supply it.

Why settling primes is enough

CLASSICAL REDUCTION / THE ROOT-RATIO ARGUMENT



READ THE FIGURE

A prime-power zero forces the Hecke-root ratio to be a root of unity. The displayed rational algebraic integer can only be $0, 1, 2, 3, 4$.

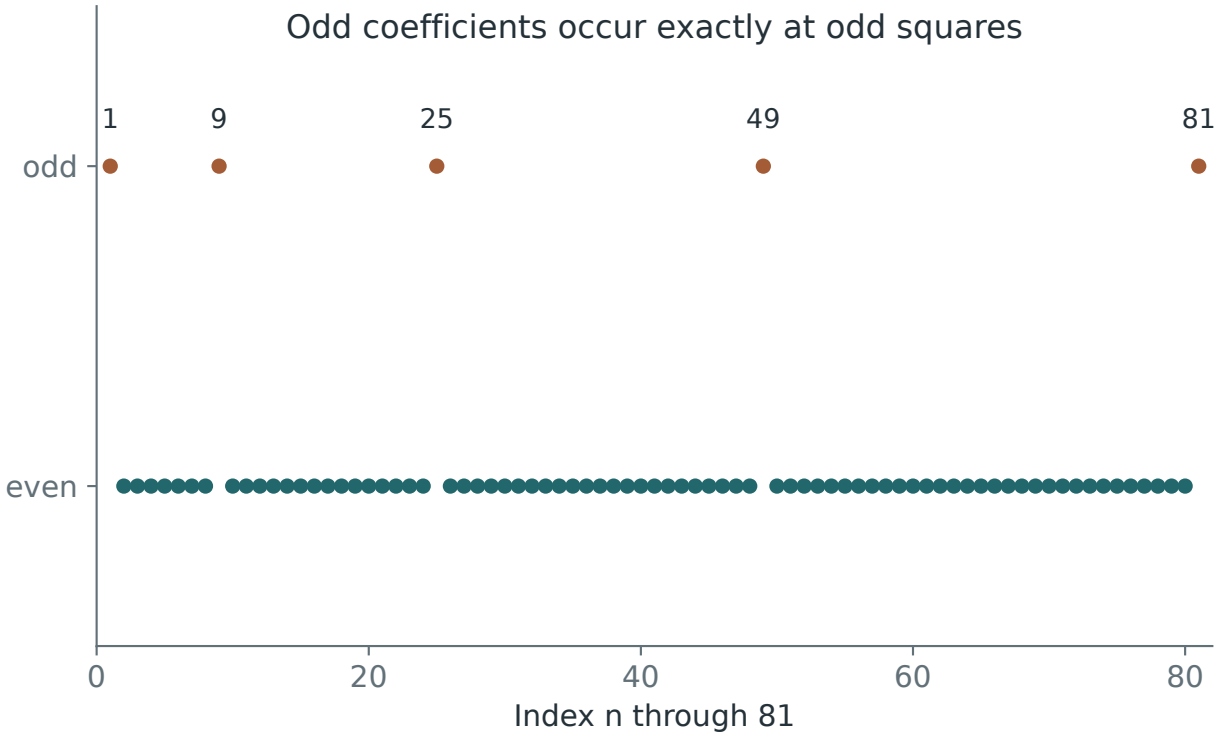
For $p \geq 5$, each nonzero alternative would make an integer square have odd p-adic valuation. The known values at 2 and 3 exclude their alternatives directly.

Multiplicativity completes the reduction.

Carry forward. The remaining target is $\tau(p) \neq 0$ at every prime, with no unexamined prime-power exception.

Parity has a precise square pattern

CLASSICAL IDENTITY / EXACT EXAMPLES THROUGH 81



READ THE FIGURE

Jacobi's identity yields

$$\tau(n) \text{ odd} \iff n \text{ an odd square.}$$

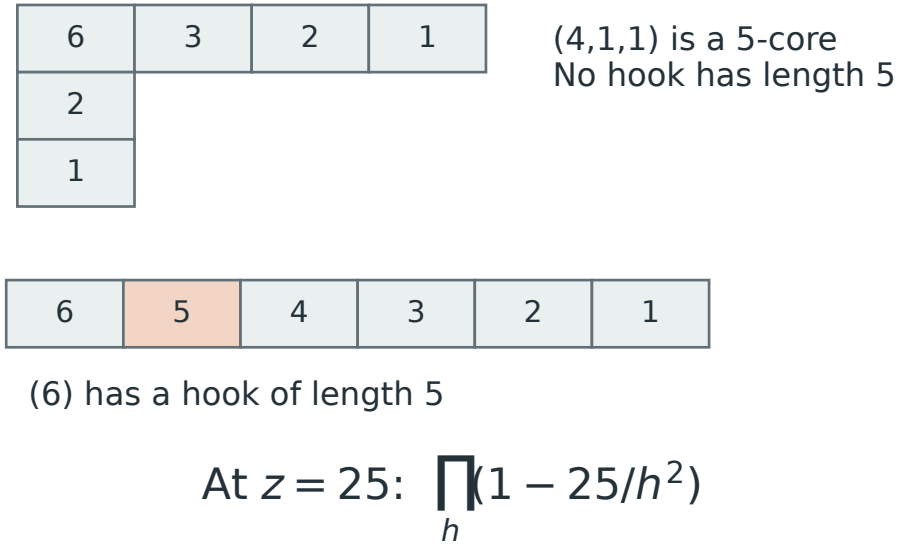
The marked indices are $1, 9, 25, 49, 81$. Every other displayed coefficient is even.

In particular an odd prime has an even coefficient. Evenness does not decide whether that integer is zero.

Carry forward. Parity rules out particular values. It leaves the prime zero value permitted.

Hook lengths select the five-cores

CLASSICAL HOOK IDENTITY / TWO PARTITIONS OF 6



READ THE FIGURE

Han's hook formula uses $\prod_h (1 - z/h^2)$. At $z = 25$, a hook of length 5 kills the contribution.

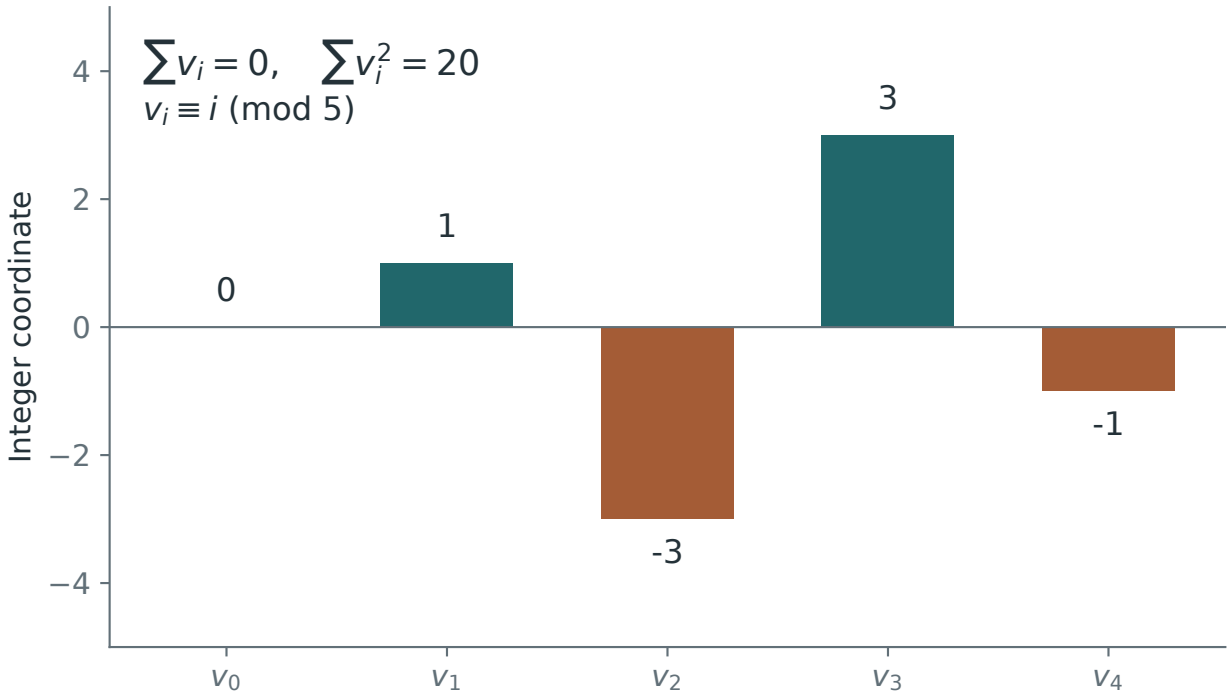
The surviving partitions are the 5-cores. The hooks of $(4,1,1)$ are $6,3,2,1,2,1$, so it survives. The row (6) contains a hook of length 5.

Partitions of size $n-1$ contribute to $\tau(n)$.

Carry forward. The formula exactly re-expresses the coefficient. Surviving weights still have mixed signs.

A packet is a five-coordinate lattice point

CLASSICAL FIVE-CORE CODING / THE UNIQUE PACKET AT n=2



READ THE FIGURE

The shell is

$$\mathcal{V}_n = \{v \in \mathbb{Z}^5 : \ v_i \equiv i \pmod 5, \quad \sum v_i = 0, \quad \sum v_i^2 = 10n\}.$$

Labels run from 0 to 4. Han's normalization is $|\lambda| = \|v\|^2/10 - 1$.

The unique packet at $n = 2$ has weight -24, exactly $\tau(2)$.

Carry forward. The shell is finite. A projection would not represent its full four-dimensional geometry.

Vandermonde weights are integers

EXACT ALGEBRA / EVERY PACKET WEIGHT IS NONZERO

$$W(v) = \frac{\prod_{i<j}(v_i - v_j)}{1!\ 2!\ 3!\ 4!}$$

$$W(v) = \det\left[\binom{v_i}{j}\right]_{i,j=0}^{4} \in \mathbb{Z}$$

Distinct residues modulo 5 make every factor nonzero

$$W(v) \equiv 1 \pmod 5$$

READ THE FIGURE

There are ten differences and denominator 288. Reversing all ten differences preserves the sign.

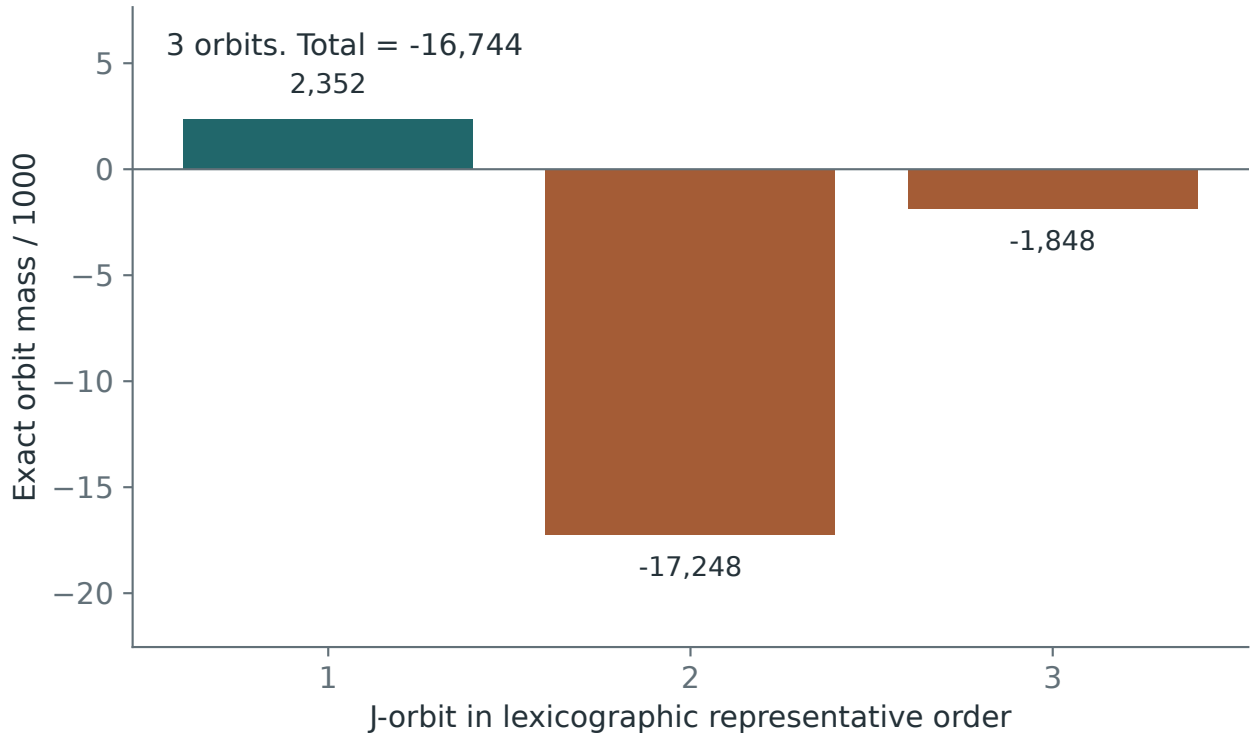
The binomial determinant proves integrality, including for negative coordinates. Distinct residues modulo 5 make all differences nonzero.

Every packet weight is 1 modulo 5. Thus each is a nonzero integer, even when their signed total might vanish.

Carry forward. Nonzero summands can cancel. The question concerns their total.

At 7, three orbit masses compete

EXACT SHELL ENUMERATION / ORIGINAL MULTIPLICITIES RETAINED



READ THE FIGURE

The involution

$$J(v)_i = -v_{(-i \bmod 5)}$$

preserves the shell and W. A bar sums one entire J-orbit.

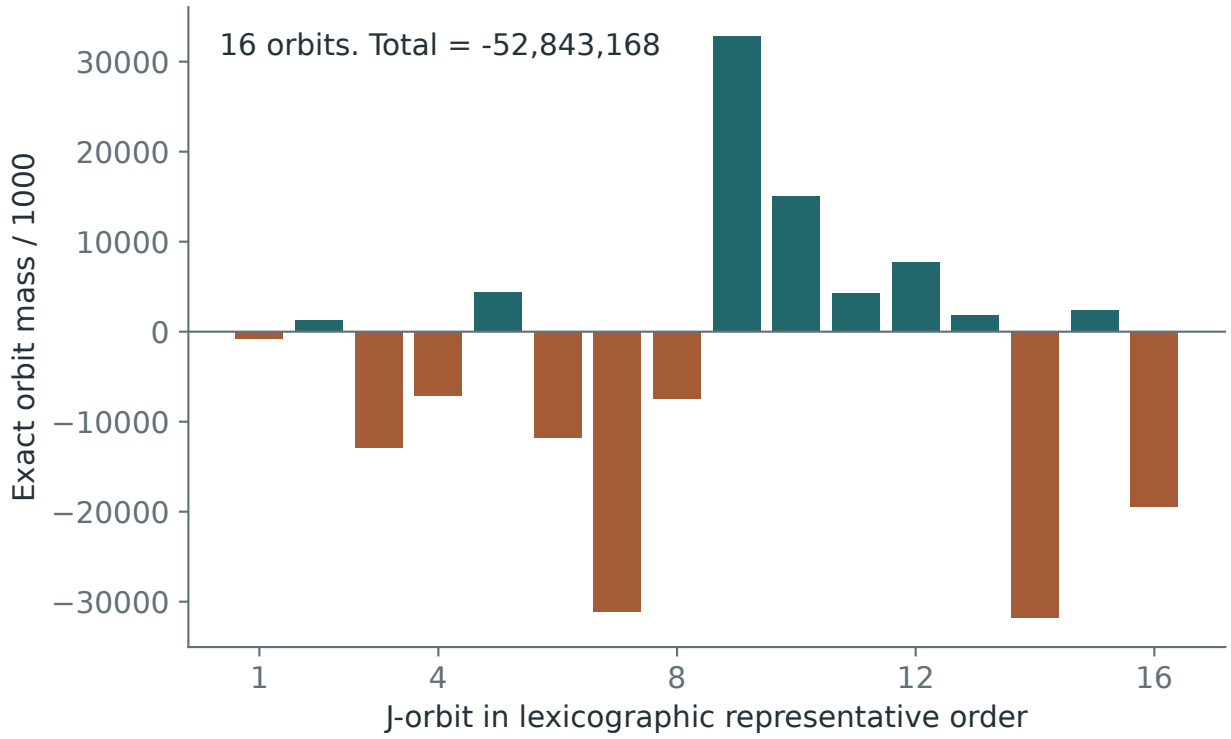
The six packets at 7 form three orbits with mixed signs. Their exact total is -16744.

Pairing by J simplifies bookkeeping. It does not match positive weights with negative weights.

Carry forward. An orbit weight includes one or two original packets. Retain its multiplicity in every total.

At 31, mixed signs remain visible

EXACT SHELL ENUMERATION / 16 J-ORBITS



READ THE FIGURE

The prime shell at 31 has 32 packets and 16 orbits. Eight orbit masses are positive and eight negative. Their total is

$$\tau(31) = -52843168.$$

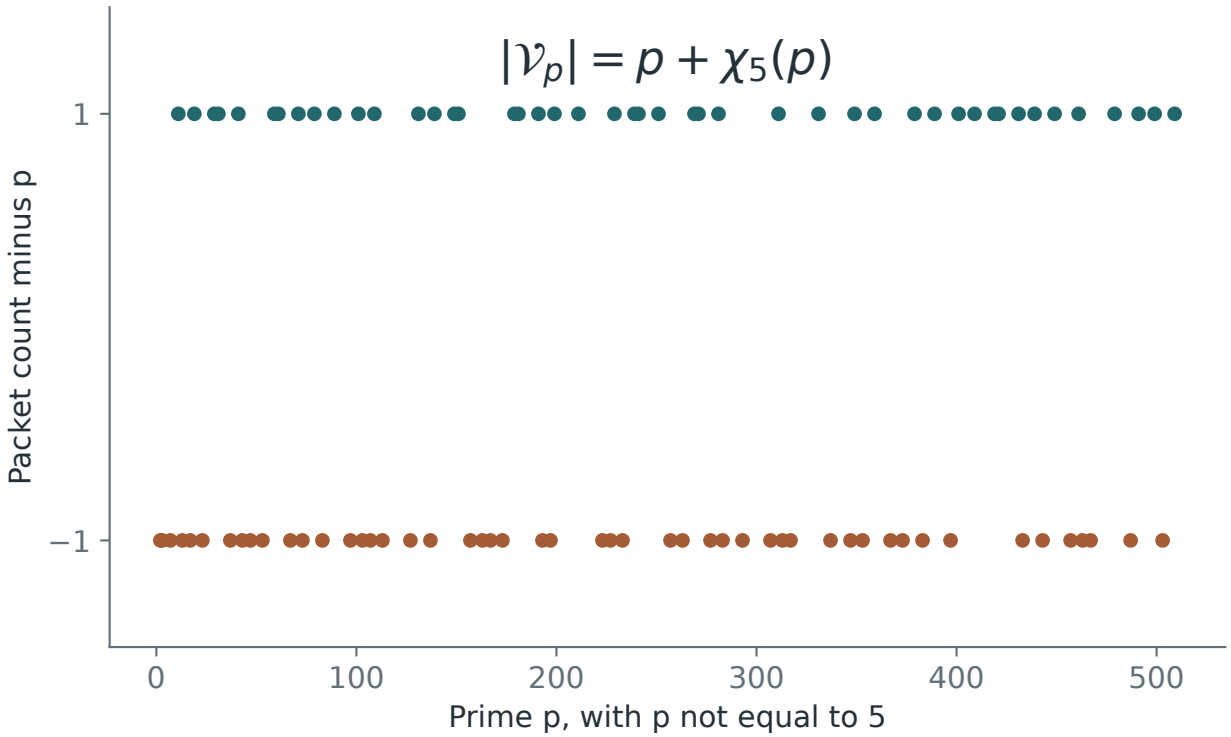
This is an exact finite coefficient calculation. Equal counts of positive and negative terms do not imply cancellation.

The bar height is mass divided by 1000.

Carry forward. Resolving all orbits exposes the arithmetic. It does not automatically choose a dominant sign or valuation.

Packet counts recover a congruence

CLASSICAL COUNT / EXACT PRIME FORMULA



READ THE FIGURE

The five-core count is

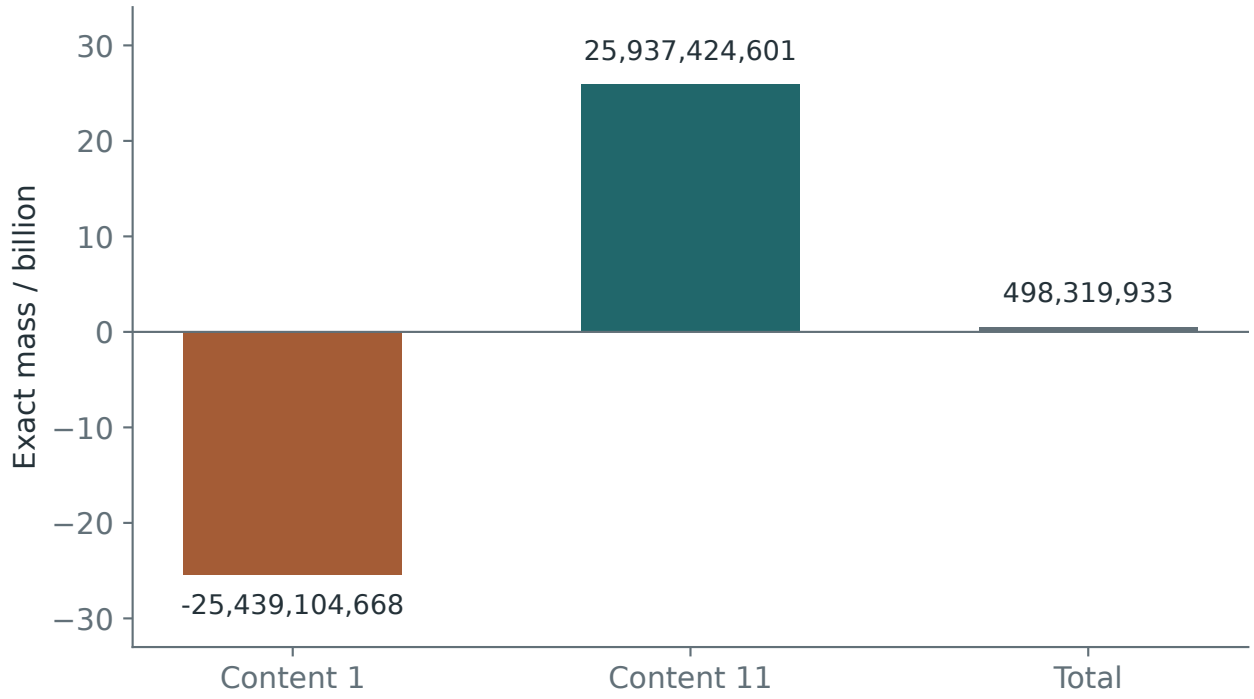
$$|\mathcal{V}_n| = \sum_{d|n} \chi_5(d) \frac{n}{d},$$

where $\chi_5(d) = (d/5)$, zero on multiples of 5.
At a prime it equals $p + \chi_5(p)$. Since every weight is 1 modulo 5, the weighted sum has the same residue.
For $p \neq 5$, a zero requires $p \equiv 4 \pmod 5$.

Carry forward. The congruence excludes three classes. An infinite residue class still needs additional arithmetic.

Content creates a conserved split

PROJECT ARGUMENT / EXACT MASSES AT n=121



READ THE FIGURE

Content is the positive gcd $g(v) = \gcd(v_0, \ldots, v_4)$. Its square divides n, and 5 does not divide it.

At 121, content-1 and content-11 masses have opposite signs. Their total is 498319933.

Every finite symmetric polynomial correction that is universally null conserves each content mass. It cannot make all corrected masses share one sign here.

Carry forward. This obstruction concerns multiple content classes. A prime shell has only content 1.

Primitive extraction has an exact inverse

PROJECT ALGEBRA / MOBIUS INVERSION ON SQUARE DIVISORS

$$\tau(n) = \sum_{g^2 \mid n} \chi_5(g) g^{10} R(n/g^2)$$

Möbius inversion on square divisors

$$R(n) = \sum_{d^2 \mid n} \mu(d) \chi_5(d) d^{10} \tau(n/d^2)$$

$R(p) = \tau(p)$ at every prime. Extraction preserves the prime target.

READ THE FIGURE

The primitive sum is

$$R(n) = \sum_{v \in \mathcal{V}_n,\ g(v)=1} W(v).$$

Rescaling an exact-content packet multiplies its weight by $\chi_5(g)g^{10}$. The displayed formulas are inverse convolutions.

The convention $\chi_5(g) = 0$ on multiples of 5 avoids nonexistent content sectors. At a prime, $R(p) = \tau(p)$.

Carry forward. Primitive extraction classifies radial contributions. It preserves the original prime obstruction.

The primitive local generating factor

PROJECT ALGEBRA / FORMAL POWER SERIES AT A FIXED PRIME

$$\sum_{r \geq 0} R(p^r)X^r = \frac{1 - \chi_5(p)p^{10}X^2}{1 - \tau(p)X + p^{11}X^2}$$

Numerator removes the square-content contribution

$$R(p^r) = \tau(p^r) - \chi_5(p)p^{10}\tau(p^{r-2})$$

for r at least 2

Formal series in X at one fixed prime p.

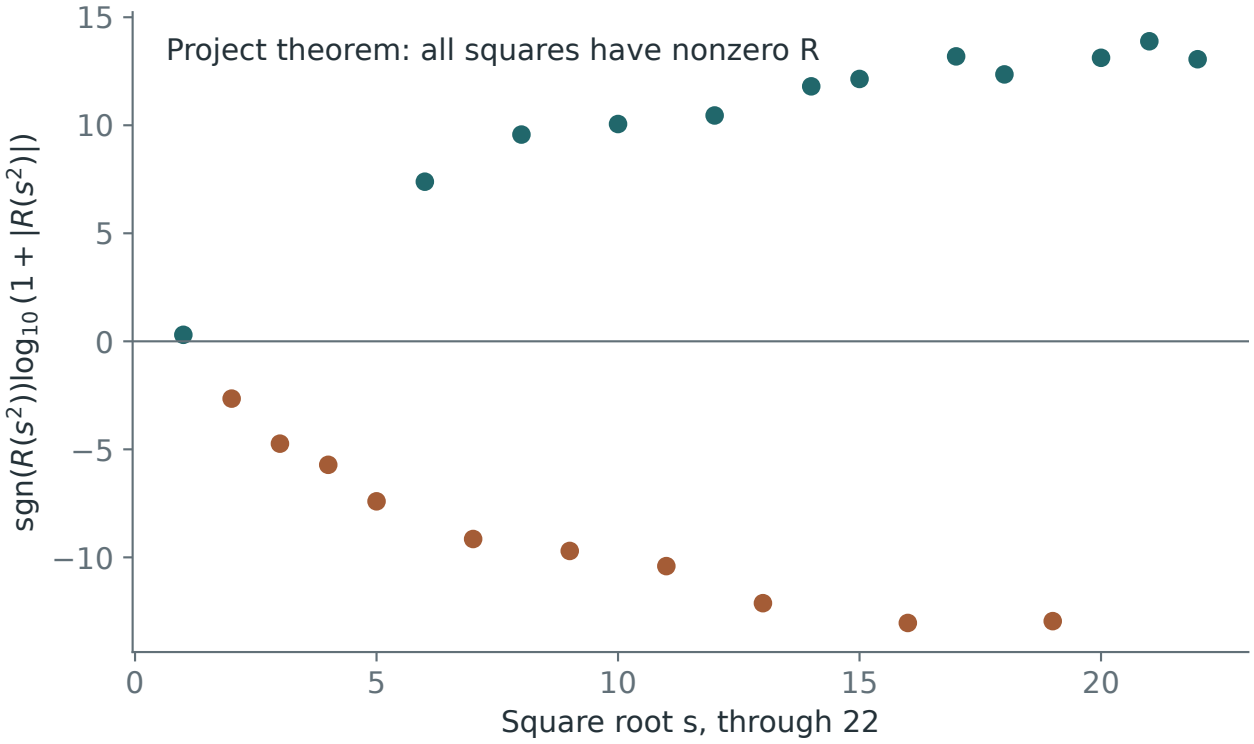
READ THE FIGURE

The denominator is the Hecke denominator. The numerator removes packets scaled by p.
The first coefficients are $R(1) = 1$ and $R(p) = \tau(p)$.
The subtraction formula starts at $r = 2$.
The denominator is invertible as a formal series because its constant coefficient is 1. No convergence assumption is needed.

Carry forward. Study the fixed-prime recurrence without assuming that its first nonconstant coefficient is nonzero.

Squares are settled for the primitive residual

PROJECT THEOREM / FINITE ILLUSTRATION AT s=1 THROUGH 22



READ THE FIGURE

The project proves $R(p^{2m}) \neq 0$ for every prime p and $m \geq 1$. Multiplicativity gives $R(s^2) \neq 0$ for every $s \geq 1$.

The proof includes $a = \tau(p) = 0$. For $p \equiv 4 \pmod 5$, it then gives

$$R(p^{2m}) = (-1)^m p^{11m-1}(p+1),$$

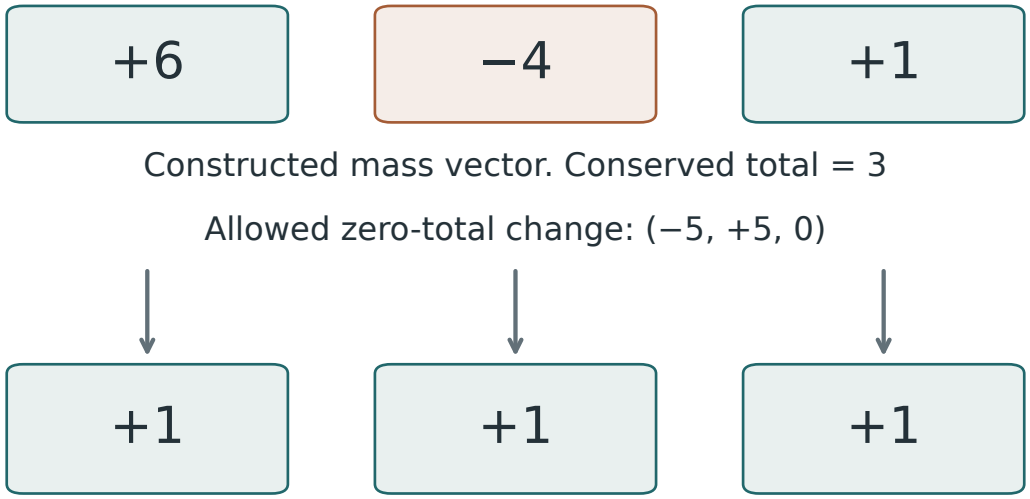
which is nonzero.

The plotted values illustrate the written proof.

Carry forward. The square theorem explicitly permits a zero prime parameter. It cannot force that parameter to be nonzero.

A null transfer preserves the signed total

CONSTRUCTED LINEAR-ALGEBRA EXAMPLE / NOT PACKET DATA



READ THE FIGURE

The displayed masses sum to 3. The change $(-5, +5, 0)$ preserves the total and leaves three positive masses.

For actual nonzero orbit masses w_i, a weighted-null correction satisfies $\sum w_i q_i = 0$. If $T = \sum w_i \neq 0$, set

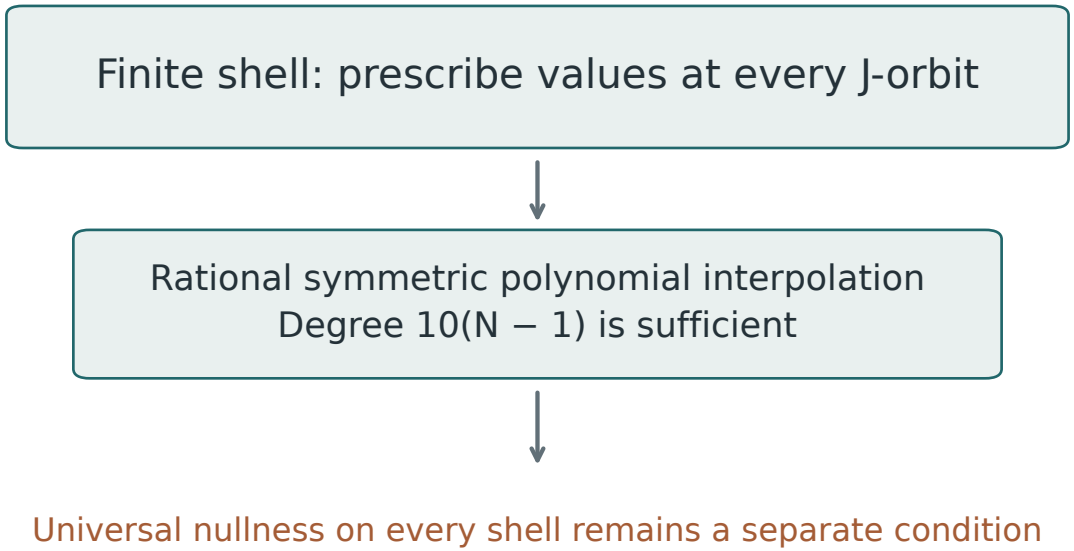
$$q_i = T/(Nw_i) - 1.$$

Every corrected mass becomes T/N. This choice already uses T.

Carry forward. Even a complete space of zero-total transfers leaves the need to prove that the total is nonzero.

Interpolation is not universal nullness

PROJECT ARGUMENT / THE FINITE-SHELL BOUNDARY



READ THE FIGURE

Even symmetric coordinates identify the J-orbits on one shell. A rational homogeneous symmetric polynomial of degree $10(N-1)$ can interpolate arbitrary values on N orbits.

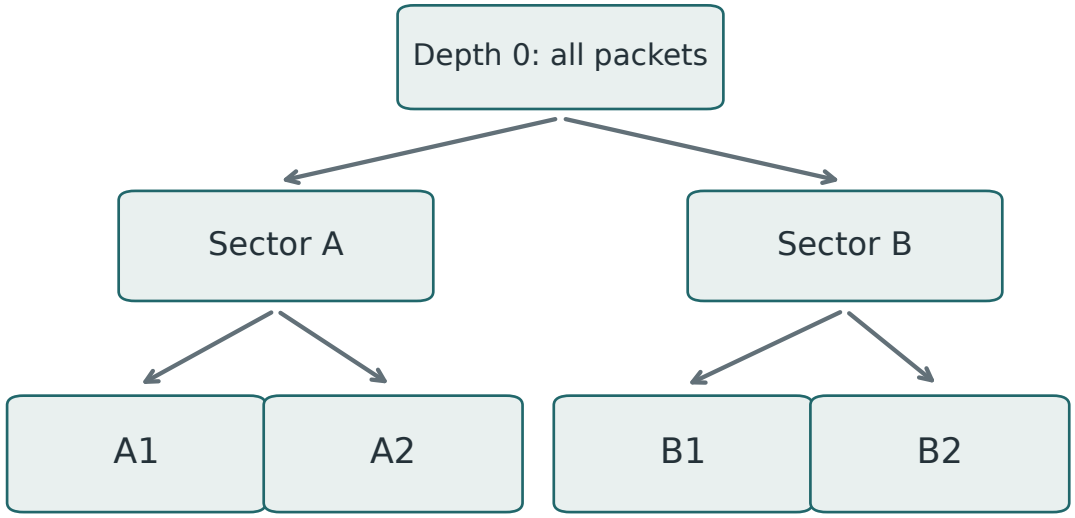
That prescribes only one shell. The natural interpolated null at 7 fails already at shell 1.

Top-degree harmonic projection also loses values: it annihilates $s_2 D$, where D is the Vandermonde.

Carry forward. Interpolation supplies neither universal nullness nor the missing nonzero arithmetic margin.

Refinement separates more residue data

SCHEMATIC TREE / c MODULO POWERS OF TWO



Schematic refinement. Actual child counts depend on the shell.

READ THE FIGURE

Write $c_i = (v_i - i)/5$. A depth-k sector fixes c modulo 2^k, allowing J.

Each child belongs to one parent. The aggregation matrix A gives

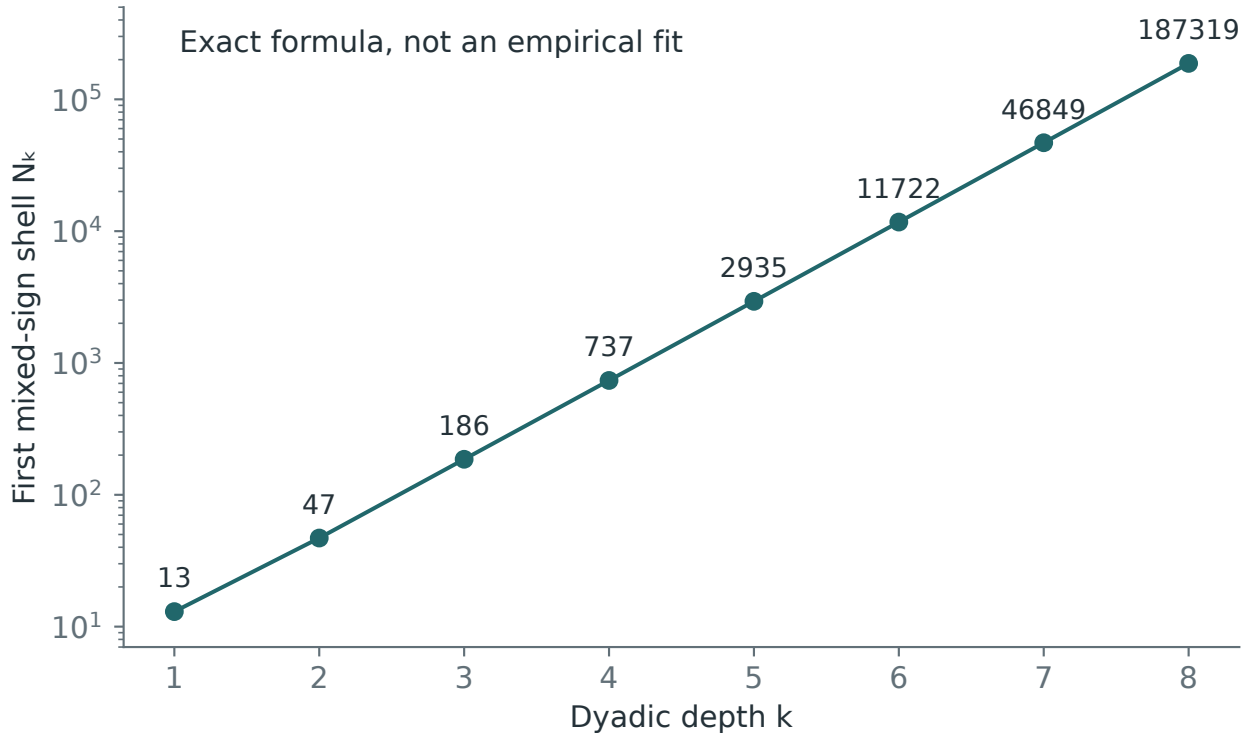
$$t_k = At_{k+1}, \qquad M_k = AM_{k+1}.$$

This binary picture is schematic. Actual shells have varying occupied-child counts.

Carry forward. Refinement adds constraints to sector-preserving corrections. It does not automatically add needed transfers.

The first mixed-sign boundary is exact

PROJECT THEOREM / ALL DYADIC DEPTHS k AT LEAST 1



READ THE FIGURE

For $M = 2^k$, the first mixed shell is

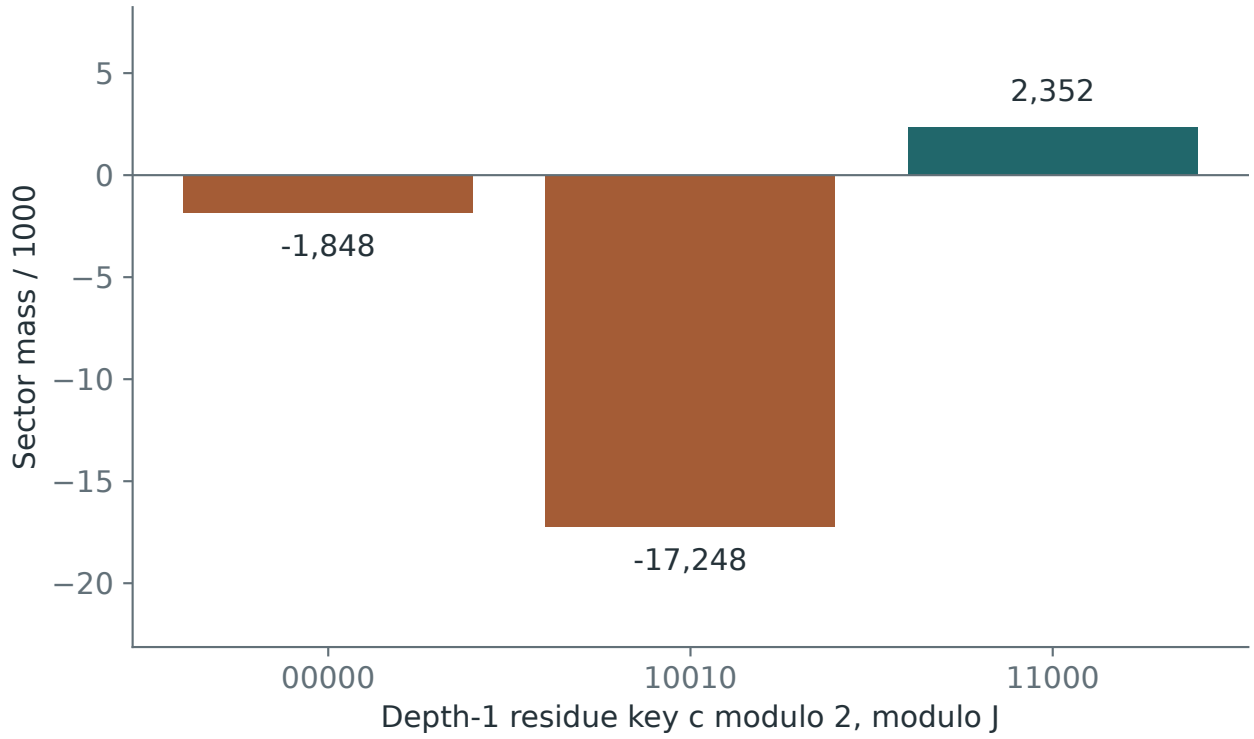
$$N_k = (20M^2 + 2M + \gamma_k)/7,$$

where $\gamma_k = 7, 1, 6$ for $k \equiv 1, 2, 0 \pmod 3$.
Every shell below it has sign-pure depth-k sectors.
At the boundary, an explicit opposite-sign pair occupies one sector.
The logarithmic plot shows the first eight formula values.

Carry forward. Sign-purity differs from orbit uniqueness. Neither statement compares global positive and negative totals.

A global null can fail inside each sector

EXACT COUNTEREXAMPLE / DEPTH 1 AT p=7



READ THE FIGURE

These occupied parity-sector masses are $-1848, -17248, 2352$, totaling -16744.

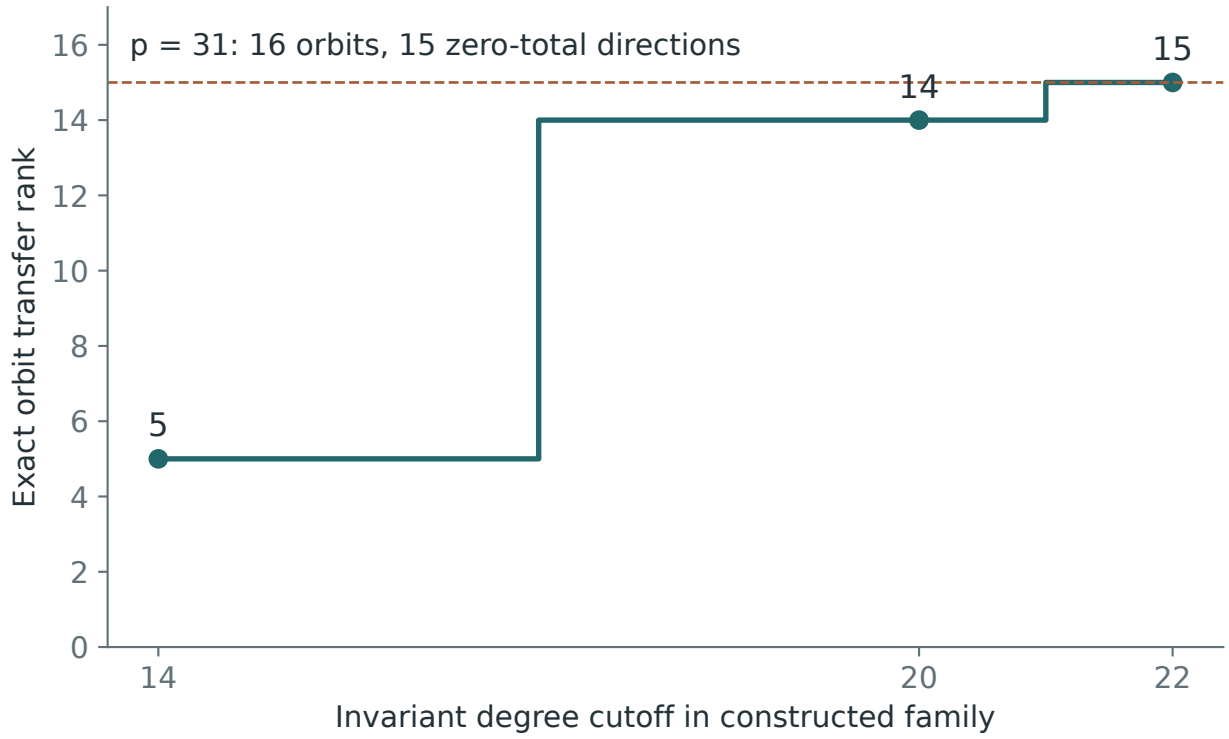
The polynomial Q_8 is globally null. Its three sector sums are nonzero and add to zero. Multiplication by a sector indicator therefore destroys the null identity.

A correction preserving each displayed sector cannot remove their opposing signs.

Carry forward. A valid localized construction must solve its parent cancellation conditions explicitly.

A finite rank defect can disappear

EXACT MATRIX RANKS / THE CONSTRUCTED FAMILY AT p=31



READ THE FIGURE

There are 16 orbit sectors and 15 zero-total directions. The orbit-transfer ranks are 5, 14, 15 at cutoffs 14, 20, 22.

Degree-22 additions remove the remaining one-dimensional defect. The recorded rational certificate preserves coarse masses during its fine solve. These are cutoffs in the constructed family. No optimality claim is made.

Carry forward. Full rank at 31 establishes this example. It proves neither uniform rank nor a uniformly nonzero survivor.

Auxiliary roots use a different variable

EXACT POLYNOMIAL EXAMPLES / n=499 AND 16249

$$P_n(5) = 288\tau(n), \quad G_n(U) = 5^{-\mu_n} P_n(5U)$$

$p = 499$
$\overline{G}(U) = 2 + 3U^2$
Simple root at 1

$p = 16249$
$\overline{G}(U) = (U-1)^2(U-2)$
Double root at 1

Auxiliary variable U. The target coefficient is evaluated at U = 1.

READ THE FIGURE

The shell polynomial has degree at most ten and $P_n(5) = 288\tau(n)$. The bar in $\overline{G}$ denotes reduction modulo 5.
Its integral normalization satisfies

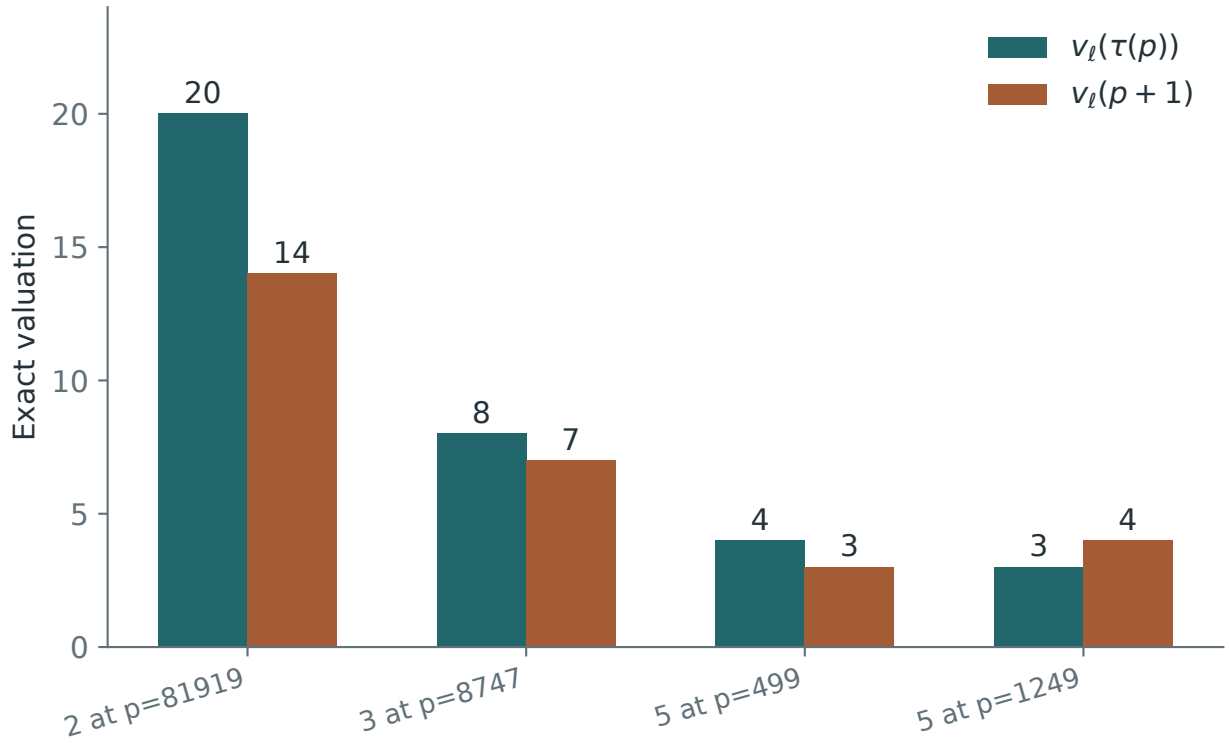
$$v_5(\tau(n)) = \mu_n + v_5(G_n(1)).$$

At 499, the simple root near 1 lifts to 16 modulo 25. At 16249, every $U \equiv 1 \pmod 5$ gives $G(U) \equiv 15 \pmod{25}$, so the double residual root does not lift.

Carry forward. Root analysis varies U and evaluates at 1. It is not interpolation in the prime index.

Three proposed valuation shortcuts fail

EXACT COUNTEREXAMPLES / AUXILIARY EQUALITIES, NOT LEHMER



READ THE FIGURE

The proposed identity was

$$v_\ell(\tau(p)) = v_\ell(p+1).$$

Among primes $p \neq \ell$ with $\ell \mid (p+1)$, the first failures in the exhaustive search through 150000 occur at 81919, 8747, 499 for $\ell = 2, 3, 5$.

The 1249 example gives the 5-adic discrepancy in the opposite direction. Decisive coefficients were independently checked with an Eisenstein convolution.

Carry forward. These nonzero values refute proposed valuation identities. They are not counterexamples to Lehmer.

Local filtered data allow trace zero

CONSTRUCTED LOCAL MODEL / EVERY PRIME p

$$\varphi(e_1) = e_2, \qquad \varphi(e_2) = -p^{11} e_1$$

$$\operatorname{tr}\varphi = 0, \quad \det\varphi = p^{11}$$

Hodge jumps	Newton slopes
0 and 11	11/2 and 11/2

Constructed local filtered module. No global identification with Δ.

READ THE FIGURE

The displayed Frobenius has polynomial $X^2 + p^{11}$, determinant p^{11} and trace zero.

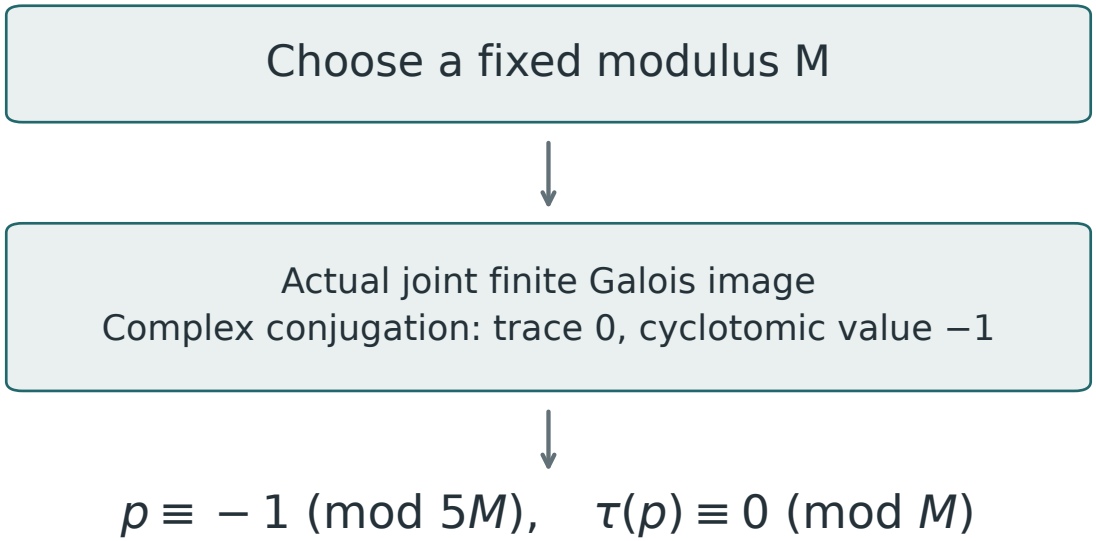
Choose a line L. Set $\mathrm{Fil}^i D = D$ for $i \leq 0$, L for $1 \leq i \leq 11$, and 0 afterward. Then $t_H = t_N = 11$.

A rational stable line would require $\lambda^2 = -p^{11}$ in $\mathbb{Q}_p$, impossible by odd valuation. The module is weakly admissible.

Carry forward. The local object permits zero trace. It is not identified with the restriction of the global form Δ.

Every fixed modulus has actual survivors

CLASSICAL-THEOREM CONSEQUENCE / ACTUAL TAU COEFFICIENTS



READ THE FIGURE

Use the actual joint finite image of Deligne's integral reductions and the cyclotomic character modulo $5M$.

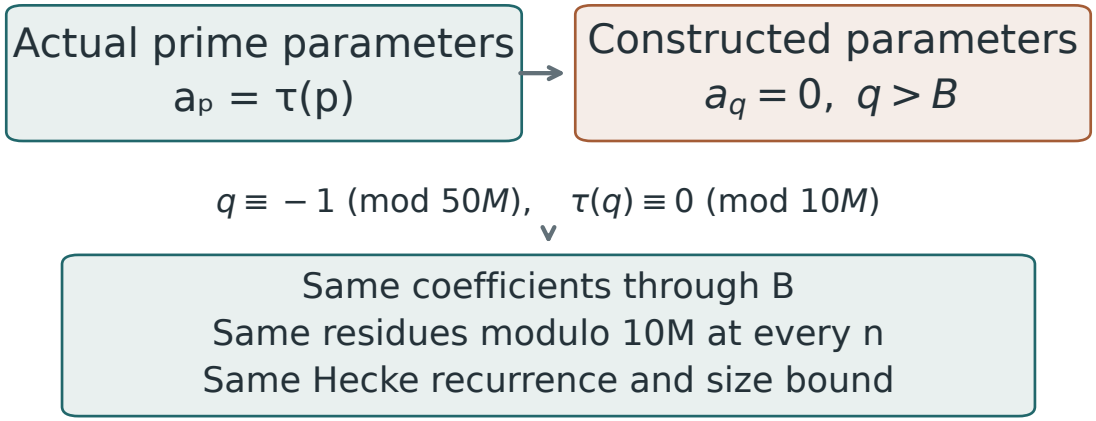
Complex conjugation has exact trace zero before reduction, including at auxiliary prime 2. Chebotarev gives positive-density prime classes with the displayed residues.

The primitive factor reduces to $(1 - X^2)/(1 - X^2) = 1$. Thus every $R(p^r)$ for $r \geq 1$ is zero modulo M.

Carry forward. No independence or image-surjectivity is assumed. A fixed list of residue tests cannot exclude all these actual primes.

A constructed zero can pass weaker tests

PROJECT COUNTERMODEL / NO ACTUAL TAU ZERO ESTABLISHED



The weaker retained premises permit a prime zero

READ THE FIGURE

For any bound B and modulus M, choose a prime $q > B$ with $q \equiv -1 \pmod{50M}$ and $\tau(q) \equiv 0 \pmod{10M}$. Change its prime parameter to zero and retain all other prime parameters.

Extend by Hecke recurrence and multiplicativity. The sequence f matches τ through B and modulo $10M$ at every index. It retains the size bound, parity, modulo-5 congruence and primitive-square theorem.

Yet $f(q) = 0$.

Carry forward. This shows insufficiency of those premises. It does not claim unconditionally that f differs from τ or is nonmodular.

The global gap is a quantifier gap

LOGICAL DISTINCTION / THE UNRESOLVED PRIME OBLIGATION

$$\forall M\ \exists p: \quad M \mid \tau(p)$$

Different moduli may use different primes

does not imply

$$\exists p\ \forall M: \quad M \mid \tau(p)$$

This would force the same integer τ(p) to be zero

READ THE FIGURE

The proven statement chooses a prime after a modulus. That prime can change when the modulus changes.

Lehmer concerns each actual coefficient at its own prime. Divisibility by every integer at the same prime would force that coefficient to be zero.

Adaptive moduli and additional global consequences of Δ remain possible. The current routes do not supply the required uniform nonzero conclusion.

Carry forward. The next useful step is a specific property of actual Δ excluding a zero prime parameter without presupposing Lehmer.

Sources and the evidence boundary

READ WITH THE COMPANION MONOGRAPH AND ITS PINNED REPORTS

Classical source leaves.

Han, *The Nekrasov–Okounkov hook length formula*, arXiv:0805.1398v1, Theorems 1.1 and 1.2. Five-core coding, normalization and signed packets.

Balakrishnan–Craig–Ono, *Variations of Lehmer's Conjecture for Ramanujan's tau-function*, arXiv:2005.10345v1: equation (1.2), p.1, for modulo 5; Theorem 2.2, p.4, for Hecke laws and bound; §3, p.7, for parity.

Deligne, *Formes modulaires et représentations ℓ-adiques*, Bourbaki exposé 355, 1968/69, published 1971, pp.139–172. Representation construction and Theorem 5.6.

Deligne, *La conjecture de Weil I*, IHÉS 43 (1974), Theorem 1.6. With the earlier implication, this yields the bound.

Serre, *Congruences et formes modulaires*, Bourbaki exposé 416, 1971/72, published 1973, §3.1: compatible representations, full-level unramified scope.

Garvan–Kim–Stanton, *Cranks and t-cores*, Invent. Math. 101 (1990), author manuscript, Theorem 4 and equation (5.3), p.8: five-core count.

Stevenhagen–Lenstra, *Chebotarëv and his density theorem*, 1996, pp.31,34: natural density. Colmez–Fontaine, *Construction des représentations p-adiques semi-stables*, 2000, author manuscript §3.4: weak admissibility.

Ibukiyama–Williams, arXiv:2505.23497v1, §3.4, pp.18–19: harmonic theta modularity. Stein, *Modular Forms of Level 1*, Theorem 2.11: valence. The vector-to-scalar descent is a project argument.

Exact URLs, versions and access limits are in the monograph bibliography and checked source contracts.

Project arguments and exact data.

The 26 September packet, tree, content-transport and final-push archives are historical inputs. Their argument, computation, literature and definition reviews are distinct. Project status implies neither priority nor formal proof-assistant certification.

The coefficient table is the current independent integer replay through 512. Shell figures are regenerated by integer enumeration. Logarithms and normalized heights convert data for display.

Threshold, primitive-square, content-conservation, interpolation and counter-model spreads summarize written arguments. Finite illustrations do not prove universal statements.

The trace-zero spread constructs a local filtered module. The fixed-modulus spread uses actual finite Galois images and Chebotarev. These two kinds of local information differ.

Release status. This program has not proved prime nonvanishing and has found no actual prime zero. The companion reader preserves the exact remaining implication.

Carry forward. Exact examples, scoped theorems and route obstructions sharpen the problem while preserving its universal prime quantifier.