Lehmer Conjecture

The prime that must survive

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A mathematical reader on Ramanujan's tau function, exact signed packets, primitive content, local arithmetic, and the remaining global nonvanishing problem.

EXACT IDENTITIES / FINITE EVIDENCE / OPEN CONJECTURE

A note to the reader

Ramanujan's tau function is easy to define and remarkably hard to keep away from zero. Its coefficients arise from one fixed infinite product,

$$\Delta(q) = q \prod_{m \geq 1} (1 - q^m)^{24} = \sum_{n \geq 1} \tau(n) q^n.$$

The question in this volume is whether every one of those integer coefficients survives. Lehmer's conjecture says that it does. A great deal of structure is known, including strong upper bounds, exact recurrences, congruences and large verified ranges. None of these facts, by itself, provides a lower bound that excludes an exact zero at every prime.

This reader develops the Operation Komorebi packet program as a mathematical argument. We begin with the classical modular form, move through its convolution and five-core descriptions, construct permitted changes to its signed packet sums, and study what those changes preserve. The strongest conclusions concern the arithmetic of primitive packets and the exact limits of several proposed finishing mechanisms. They sharpen the problem rather than solve it.

Claim boundary

This edition does not prove Lehmer's conjecture and does not present an integer counterexample. It proves and explains auxiliary statements, records exact finite certificates and identifies the remaining obligation for the actual modular form Δ. Written proofs, source-dependent theorems and bounded computations retain their distinct meanings throughout.

The primitive residual $R(n)$ plays a central role. It removes packets having a common coordinate factor. We will prove that $R(s^2)$ never vanishes and that $R(n) = 0$ exactly when $\tau(n) = 0$. These are useful structural theorems. They do not dispose of a prime zero, because $R(p) = \tau(p)$. At a prime the radial extraction leaves the original coefficient untouched.

The visual guide accompanies this book as a separate reading path. Its diagrams show cancellation, refinement, conserved content and the quantifier distinction behind the congruence argument. The source and evidence bundle contains the retained research packages, the edition's replay tools and the inputs required to inspect the finite claims. Its files are part of the mathematical record, but the reader need not follow an execution log to follow the proofs here.

This is Series 07, authored by Komorebi P. for Operation Komorebi. The reader edition is dated 9 October 2026. The underlying Lehmer research checkpoints were recorded on 26 September 2026 and have been preserved with their original scope. No claim of historical priority follows from their inclusion.

How to read the argument

The first five chapters establish the object and its exact residual descriptions. Chapters 6 and 7 turn the coefficient into a finite geometric sum and a genuine auxiliary polynomial. Chapters 8–10 explain why some polynomial corrections preserve every coefficient and show what a real sign certificate looks like. Chapters 11–14 distinguish geometric refinement from arithmetic conservation. Chapters 15 and 16 prove the primitive nonvanishing and zero-equivalence theorems. Chapters 17 and 18 extend the conservation law and separate interpolation from universal nullness. Chapters 19–21 examine actual congruence survivors, a compatible local trace-zero object and an arithmetic countermodel. The final chapter isolates the all-prime statement still needed.

Some prerequisites are used as contracts rather than reproved from the foundations. These include the modularity and Hecke properties of Δ, harmonic theta modularity, the level-one valence formula, Deligne's representations and bound, Chebotarev's theorem, and the filtered-module classification. Their precise applications and normalizations are stated where used. The bibliography and source appendix identify the relevant primary or author-hosted passages. A reader may take those contracts as established and verify the project-specific algebra in the main text.

Three meanings of evidence

A universal theorem has quantified hypotheses and a proof. A finite certificate proves a claim about its stated shell, matrix or polynomial by exact arithmetic. A bounded search describes its tested range and can locate examples or counterexamples within that range. A large count of passing tests does not change a bounded search into a universal theorem.

The text uses $v_\ell(x)$ for the exponent of the prime ℓ in a nonzero integer x, with $v_\ell(0) = \infty$. The quadratic character χ_5 is zero on multiples of 5, is 1 on residues $1, 4$, and is -1 on residues $2, 3$. A shell is a set of integral vectors with a fixed norm and prescribed coordinate residues. A packet is one vector in that shell. Its weight can be positive or negative, but is never zero.

The recurring difficulty is cancellation. A representation as a sum of nonzero integers does not imply a nonzero sum. A method that changes individual contributions must state exactly why the total is unchanged and exactly why the changed contributions have a common nonzero sign. A method that supplies divisibility conditions must track whether its modulus is fixed or depends on the prime. These distinctions will do much of the conceptual work.

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CHAPTER 1

A fixed modular form and an exact question

1.1 The coefficient that must not disappear

For $|q| < 1$ the product defining Δ converges. As a formal power series it is even simpler: the coefficient of q^N depends only on the first N product factors. Thus each $\tau(n)$ is an unambiguously defined integer. The opening terms are

$$\begin{aligned}\Delta(q) = q &- 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 \\ &- 16744q^7 + 84480q^8 - 113643q^9 - 115920q^{10} + \cdots .\end{aligned} \tag{1.1}$$

The signs already change repeatedly. Nonvanishing cannot be inferred from positivity of the defining product, since positivity of a function at a real argument does not require positivity, or nonvanishing, of its Taylor coefficients.

Conjecture 1.1 (Lehmer)**.** *For every integer $n \geq 1$, one has $\tau(n) \neq 0$.*

The conjecture concerns Ramanujan's tau function. It is distinct from the conjecture bearing Lehmer's name about Mahler measure and from questions about Euler's totient. Throughout this book there is one fixed weight-12 cusp form, one fixed coefficient sequence and one fixed universal target. Auxiliary sequences introduced later will be explicitly named and will not silently replace τ.

Lehmer's 1947 paper records the multiplicative and prime-power structure that makes the prime version of the problem decisive [1]. Modern verified ranges are extraordinarily large. For example, Derickx, van Hoeij and Zeng proved nonvanishing for

$$0 < n < 816212624008487344127999$$

in their 2014 source [3]. This is a source-certified historical range. It is not a range recomputed for this edition, and it is not asserted here to be the latest record. Its logical role is still finite: no finite endpoint reaches every positive integer.

1.2 Upper bounds and lower bounds

Deligne's theorem gives

$$|\tau(p)| \leq 2p^{11/2} \tag{1.2}$$

at a prime p [5]. The exponent $11/2$ comes from weight 12: the associated Frobenius eigenvalues have weight 11. An upper bound confines a possible coefficient to an interval. It does not remove zero from that interval.

Integrality makes a residue argument useful. If $x \in \mathbb{Z}$, $|x| < L$ and $L \mid x$, then $x = 0$. If instead a nonzero residue modulo L is established, then $x \neq 0$ without any size estimate. At

primes $p \geq 5$, (1.2) yields

$$2p^{11/2} < p^6.$$

Therefore

$$\tau(p) = 0 \quad \Longleftrightarrow \quad p^6 \mid \tau(p), \qquad p \geq 5. \tag{1.3}$$

This is an exact criterion. It replaces equality to zero by a divisibility condition with a modulus depending on p. It does not show that the divisibility condition fails.

For a nonzero actual coefficient, the same comparison gives $v_p(\tau(p)) \leq 5$. A hypothetical assertion $v_p(\tau(p)) \leq 3$ would be stronger than needed, and hence sufficient, but requires its own proof. One must not obtain such a valuation bound merely by rewriting the coefficient as another exact sum.

1.3 What a successful certificate must contain

There are several legitimate ways to prove a particular coefficient survives. An exact expansion can evaluate it. A congruence can show it is nonzero modulo a specified integer. A sign certificate can rewrite it as a sum of one weak sign with at least one nonzero term. A valuation certificate can identify a uniquely least-valued summand in a nonarchimedean field. Each method has two parts: an identity connecting the certificate to the actual $\tau(p)$, and an independently proved property that forces the identity's value away from zero.

The second part is the delicate one. If a correction is chosen using the already known total, equalizing the summands can be circular as a uniform argument. If a modular form identity introduces a new cusp coefficient, its nonvanishing is a new obligation. If an enlarged correction space can express every total-preserving change, that completeness does not determine whether the preserved total is zero.

The exact prime obligation

Prove $\tau(p) \neq 0$ for every prime p, using information about the actual fixed form Δ that does not already assume this conclusion. A verified actual prime with $\tau(p) = 0$ would instead disprove the conjecture.

The developments below are organized around this obligation. Some routes yield stronger identities or cleaner decompositions. Some proposed final steps fail by explicit examples. Both outcomes are useful when they identify precisely what has and has not been supplied.

CHAPTER 2

Exact coefficient engines

2.1 The product and a logarithmic recurrence

Let $\sigma_j(n) = \sum_{d|n} d^j$ and write $\sigma = \sigma_1$. Logarithmic differentiation of the product gives

$$\mathcal{D}\Delta = \left(1 - 24\sum_{m\geq 1}\sigma(m)q^m\right)\Delta, \qquad \mathcal{D} = q\frac{d}{dq}. \tag{2.1}$$

Indeed, the contribution from the factor $(1-q^r)^{24}$ is

$$-24\frac{rq^r}{1-q^r} = -24\sum_{k\geq 1} rq^{rk}.$$

Collecting the terms with $rk = m$ gives $-24\sigma(m)$. The leading factor q contributes 1.

Comparing coefficients of q^n in (2.1), for $n > 1$, yields

$$(n-1)\tau(n) = -24\sum_{m=1}^{n-1}\sigma(m)\tau(n-m), \qquad \tau(1) = 1. \tag{2.2}$$

This recurrence uses only smaller indices. Division by $n-1$ is exact because the recurrence came from the integral product, although an implementation should still check divisibility rather than silently round a quotient. It is a convenient independent coefficient engine at a moderate range.

For example,

$$\tau(2) = -24\sigma(1)\tau(1) = -24.$$

At $n = 3$, the right side is $-24(-24+3) = 504$, and division by 2 gives 252. These tiny checks fix the normalization, especially the leading q and the exponent 24.

2.2 Jacobi's triangular series

Jacobi's identity is

$$\prod_{m\geq 1}(1-q^m)^3 = \sum_{j\geq 0}(-1)^j(2j+1)q^{j(j+1)/2}. \tag{2.3}$$

Raising this series to the eighth power and multiplying by q gives Δ. The sparse triangular exponents make it useful for coefficient calculations. The source check in [2] also uses this identity to establish the parity criterion discussed below. It is an exact formal series identity, not a numerical approximation.

Two independent routes to the same coefficients matter because a shared normalization error can pass many internal tests. An implementation of a triangular recurrence can be checked against a direct truncated product. A logarithmic recurrence uses a different algebraic derivation. A modular identity involving Eisenstein series gives another convolution.

Agreement between distinct formulations authenticates the finite result more effectively than running the same recurrence twice.

The edition's independent replay compares a defining-product calculation and a logarithmic recurrence through $n = 512$. The historical arithmetic branch used the triangular engine through 10000 and a separate product check through 128. The historical packet-tree branch carried its own product comparison through 512. These are different bounded contracts, and their endpoints must not be merged into an invented universal verification.

2.3 Eisenstein coefficients

With the standard constant-one normalization,

$$E_4(q) = 1 + 240 \sum_{n\geq 1} \sigma_3(n) q^n, \tag{2.4}$$

$$E_6(q) = 1 - 504 \sum_{n\geq 1} \sigma_5(n) q^n, \tag{2.5}$$

and

$$\Delta = \frac{E_4^3 - E_6^2}{1728}. \tag{2.6}$$

Since $E_8 = E_4^2$, one may equivalently write $(E_4 E_8 - E_6^2)/1728$. This last form supplies an independent divisor-sum convolution for selected large coefficients in the valuation branch. The denominator 1728 is exact, and an integer implementation can test the required divisibility before extracting a coefficient.

The first coefficient of E_4^3 is 720, whereas that of E_6^2 is -1008. Their difference is 1728, so the leading coefficient of the quotient is 1. A constant-one Eisenstein series is essential here. Some modular-form texts use a scaled Eisenstein basis. The normalization must be translated before coefficient identities are used.

2.4 Parity from the actual product

Modulo 2, the Frobenius identity gives

$$\prod_{m\geq 1} (1 - q^m)^{24} \equiv \prod_{m\geq 1} (1 - q^{8m})^3.$$

Using (2.3), whose coefficients $2j + 1$ are odd, gives

$$\Delta(q) \equiv \sum_{j\geq 0} q^{1+4j(j+1)} = \sum_{j\geq 0} q^{(2j+1)^2} \pmod 2.$$

Distinct nonnegative j give distinct exponents. Therefore

$$\tau(n) \text{ is odd} \iff n \text{ is an odd square.} \tag{2.7}$$

In particular, $\tau(p)$ is even at every odd prime. This elementary global property will later make the primitive odd-block multiplier odd. It is useful arithmetic, but is consistent with a zero at an odd prime because zero is even.

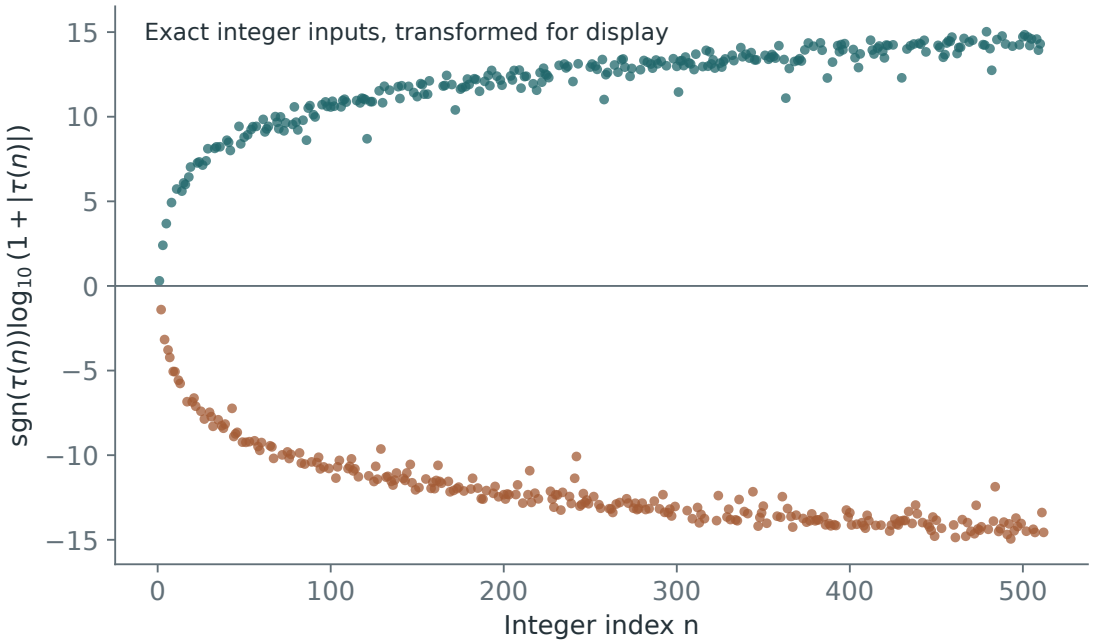


Figure 2.1: The exact coefficients through 512 on a signed logarithmic scale. The plotted ordinate is $\operatorname{sgn}(\tau(n)) \log_{10}(1 + |\tau(n)|)$. Sign changes and large values are visible, while every plotted coefficient is nonzero. This finite plot does not establish the conjecture.

2.5 A replay is an assertion with a domain

A finite coefficient engine should report its formula, exact range, input hashes, output hashes and runtime. The value of a large coefficient is a mathematical assertion. The statement that a checker passed is an execution assertion. Both require provenance, but they are not interchangeable. A recorded result without replay may still be a retained historical certificate. A fresh replay provides new execution evidence for the same finite statement. Neither establishes an unbounded theorem by accumulation.

The source bundle retains these distinctions. It includes the original packages unchanged, release replay records and the independent coefficient comparison. This makes numerical examples in the text inspectable without replacing their algebraic proofs by a count of successful assertions.

CHAPTER 3

Why the problem lives at primes

3.1 The Hecke contracts

The normalized form Δ is a Hecke eigenform. Its coefficients satisfy

$$\tau(mn) = \tau(m)\tau(n), \qquad \gcd(m,n) = 1, \tag{3.1}$$

$$\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1}), \qquad r \geq 1. \tag{3.2}$$

The exponent 11 again records weight 12. The constant term for the local recurrence is $\tau(1) = 1$, and the first parameter is $a = \tau(p)$. These classical statements occur in Lehmer's original paper and in the modern source [1, 2].

For a fixed prime put $T_r = \tau(p^r)$ and $b = p^{11}$. The generating series is

$$\sum_{r\geq 0} T_r X^r = \frac{1}{1 - aX + bX^2}. \tag{3.3}$$

Multiplication by the denominator gives $T_0 = 1$, $T_1 = a$ and the recurrence. Thus this is a formal identity over $\mathbb{Z}[[X]]$. No analytic continuation in the prime index is involved.

The first local blocks are

$$T_2 = a^2 - b, \qquad T_3 = a^3 - 2ab, \qquad T_4 = a^4 - 3a^2b + b^2.$$

Even if $a \neq 0$, it is not formally obvious from these expressions that none vanishes. The reduction to primes needs a short arithmetic argument.

3.2 The two roots and a root of unity

Let α, β be the complex roots of

$$X^2 - aX + p^{11} = 0.$$

They are nonzero because their product is p^{11}. They are distinct: equality would require $a^2 = 4p^{11}$, but the p-adic valuation of the right side is odd for odd p, and is 13 for $p = 2$.

Solving the recurrence gives

$$T_r = \frac{\alpha^{r+1} - \beta^{r+1}}{\alpha - \beta} = \sum_{j=0}^{r} \alpha^{r-j}\beta^j. \tag{3.4}$$

If $T_r = 0$ for some $r \geq 1$, then $\zeta = \alpha/\beta$ is a root of unity with $\zeta^{r+1} = 1$. Consequently

$$\frac{a^2}{p^{11}} = \frac{(\alpha+\beta)^2}{\alpha\beta} = 2 + \zeta + \zeta^{-1}. \tag{3.5}$$

The left side is rational. The right side is an algebraic integer, because roots of unity are algebraic integers. A rational algebraic integer is an integer. It is real and lies between 0 and 4. Thus

$$\frac{a^2}{p^{11}} \in \{0, 1, 2, 3, 4\}.$$

This argument uses the root-of-unity consequence of a zero, rather than assuming that an arbitrary ratio of Hecke roots is a root of unity.

3.3 Excluding the nonzero integer cases

For $p \geq 5$, every $c \in \{1, 2, 3, 4\}$ is prime to p. The equality $a^2 = cp^{11}$ would give an odd p-adic valuation on a square. It is impossible. Hence the only possible value in (3.5) is 0, and $a = 0$.

At $p = 2, 3$, the actual parameters are

$$\tau(2) = -24, \qquad \tau(3) = 252.$$

Their quotients a^2 / p^{11} are not integers. Thus no prime power of 2 or 3 vanishes. We obtain the exact local criterion:

$$\tau(p^r) = 0 \quad \Longleftrightarrow \quad \tau(p) = 0 \text{ and } r \text{ is odd.} \tag{3.6}$$

The forward implication is what we just proved. For the reverse implication and the even case, set $a = 0$ in the recurrence:

$$T_{2m} = (-p^{11})^m, \qquad T_{2m+1} = 0.$$

Theorem 3.1 (Prime reduction)**.** *For every positive integer n,*

$$\tau(n) = 0 \quad \Longleftrightarrow \quad \begin{array}{l} \textit{some prime } p \textit{ occurs to an odd exponent in } n \\ \textit{and } \tau(p) = 0. \end{array}$$

In particular, Lehmer's conjecture is equivalent to nonvanishing at every prime.

Proof. Factor $n = \prod_p p^{r_p}$. Multiplicativity expresses $\tau(n)$ as a finite product of local blocks. An integral product is zero exactly when one factor is zero. Apply (3.6) to each factor. □

One immediate corollary is that $\tau(s^2) \neq 0$ for every s. This square theorem does not rely on the truth of Lehmer at primes. A possible prime zero would propagate only through odd exponents. Later the primitive residual will have the same zero criterion but a different proof of its square nonvanishing.

3.4 A local parameter is not the global form

The recurrence makes sense for every integer a. Setting $a = 0$ gives a consistent sequence at one prime. Thus the recurrence alone cannot disallow a zero parameter. The actual modular form chooses its parameters through global geometry and modularity. A proof of Lehmer must use some part of that actual structure to forbid the parameter $a = 0$ at every prime.

This distinction will recur in the local filtered-module construction and in the global arithmetic countermodel. They test whether the properties retained by a proposed argument already distinguish the actual Δ. When they admit zero parameters, they expose a missing premise rather than produce a counterexample to the conjecture.

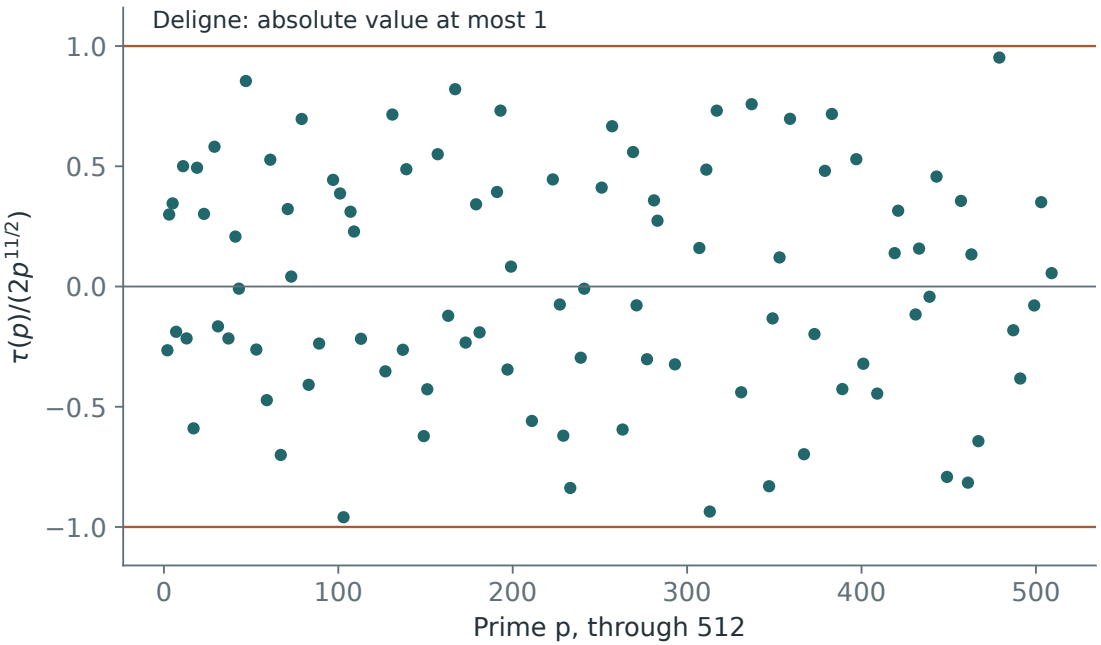


Figure 3.1: Actual prime coefficients through 512, normalized by $2p^{11/2}$. Deligne's bound confines the values to $[-1, 1]$. The bound is universal, whereas these plotted coefficients form a finite set. The permitted interval includes zero.

CHAPTER 4

The differential residual that survives extraction

4.1 A divisor convolution with the original cusp term

Define the fourth divisor moment

$$S_4(n) = \sum_{k=1}^{n-1} k^4 \sigma(k)\sigma(n-k).$$

It is a positive integer when $n > 1$. The weights k^4 and the two divisor sums suggest a route through elementary arithmetic. The exact identity, however, retains the cusp coefficient:

$$840S_4(n) = n^4\big[50\sigma_3(n) + (35 - 84n)\sigma(n)\big] - \tau(n). \tag{4.1}$$

Thus the difference between two explicit divisor expressions is exactly $\tau(n)$. Positivity of $S_4(n)$ does not imply that this difference is nonzero.

At a prime p, $\sigma(p) = 1 + p$ and $\sigma_3(p) = 1 + p^3$. Put

$$C(p) = 85 - 49p - 84p^2 + 50p^3.$$

Equation (4.1) becomes

$$\tau(p) = p^4 C(p) - 840S_4(p). \tag{4.2}$$

For $p > 7$, the coefficient 840 is a p-adic unit. Reducing modulo p^4 gives

$$p^4 \mid S_4(p) \quad \Longleftrightarrow \quad p^4 \mid \tau(p). \tag{4.3}$$

The proposed assertion that p^4 never divides $S_4(p)$ would therefore assert a uniform bound $v_p(\tau(p)) \le 3$. It is a sufficient strengthening of Lehmer, not a consequence of the displayed convolution.

4.2 Deriving the identity in the differential ring

Set

$$A(q) = \sum_{n\ge1} \sigma(n)q^n, \qquad B(q) = \sum_{n\ge1} \sigma_3(n)q^n.$$

Then $E_2 = 1 - 24A$ and $E_4 = 1 + 240B$. The Ramanujan differential relations are

$$\mathcal{D}E_2 = (E_2^2 - E_4)/12, \tag{4.4}$$
$$\mathcal{D}E_4 = (E_2E_4 - E_6)/3, \tag{4.5}$$
$$\mathcal{D}E_6 = (E_2E_6 - E_4^2)/2. \tag{4.6}$$

These normalized relations are recorded in [12], and the constant-one formula for Δ is checked in [11]. They define a derivation on the polynomial ring $\mathbb{Q}[E_2, E_4, E_6]$. Repeated differentiation and collection of terms yield

$$E_2\mathcal{D}^2E_2 = 4\mathcal{D}^3E_2 + \frac{3}{10}\mathcal{D}^2E_4, \tag{4.7}$$

$$E_2\mathcal{D}^4E_2 = \frac{12}{5}\mathcal{D}^5E_2 + \frac{1}{7}\mathcal{D}^4E_4 - \frac{24}{35}\Delta. \tag{4.8}$$

These are polynomial identities in the differential ring. They can be verified by expanding each derivative using the three relations and checking a zero rational remainder. The cusp term in (4.8) is essential.

Substituting $E_2 = 1 - 24A$, $E_4 = 1 + 240B$ into (4.8) gives

$$-24\mathcal{D}^4A + 576A\mathcal{D}^4A = -\frac{288}{5}\mathcal{D}^5A + \frac{240}{7}\mathcal{D}^4B - \frac{24}{35}\Delta.$$

Multiplication by $-35/24$ and rearrangement gives

$$\Delta = 50\mathcal{D}^4B + 35\mathcal{D}^4A - 84\mathcal{D}^5A - 840A\mathcal{D}^4A. \tag{4.9}$$

The coefficient of $A\mathcal{D}^4A$ at q^n is precisely $S_4(n)$ after the summation variable is reversed if necessary. Coefficient extraction proves (4.1).

For completeness, $\Delta = (E_4^3 - E_6^2)/1728$ has the expected product normalization. The differential relations imply $\mathcal{D}\Delta = E_2\Delta$. The product in Chapter 2 satisfies the same equation by logarithmic differentiation. Both begin with q. Coefficient recursion therefore identifies them as formal power series. This avoids treating a matching first coefficient as sufficient on its own.

4.3 The exact forbidden residue

Combine (1.3) and (4.2). For $p > 7$,

$$\tau(p) = 0 \quad \Longleftrightarrow \quad 840S_4(p) \equiv p^4C(p) \pmod{p^6}. \tag{4.10}$$

If $p^4 \mid S_4(p)$, division by p^4 is legitimate as an integer operation, and the same condition is

$$840\frac{S_4(p)}{p^4} \equiv 85 - 49p \pmod{p^2}.$$

The terms $-84p^2$ and $50p^3$ have disappeared modulo p^2. This is a compact description of the remaining residue problem. It is not a contradiction. A successful proof through this formula must establish that the displayed residue is never attained by the actual convolution.

At $p = 2411$, retained exact values are

$$\tau(p) = 4542041100095889012, \qquad S_4(p) = 28168797627693083506845.$$

Both have p-adic valuation 1. The normalized residues are

$$\frac{\tau(p)}{p} \equiv 150 \pmod{p}, \qquad \frac{S_4(p)}{p} \equiv 947 \pmod{p}.$$

They satisfy $-840 \cdot 947 \equiv 150 \pmod{p}$. This example shows that substantial cancellation can coexist with a clean exact identity. It neither refutes the stronger p^4 bound nor supplies it.

4.4 Why a projector does not finish the proof

A differential projector can remove Eisenstein components and isolate a cusp component. That is a structural accomplishment: it explains where the coefficient sits in the algebra. But a projector selecting Δ also selects its unknown prime coefficient. The coefficient $-1/840$ in (4.1) is a p-adic unit for $p > 7$. There is no automatic extra factor of p left behind.

The retained symbolic verification checks the differential identities exactly and checks their Fourier consequences at finite indices. The universal identity is justified by the differential-ring calculation. The finite checks authenticate the implementation and normalization. The all-prime residue exclusion remains a separate mathematical task.

CHAPTER 5

Trace moments and a unit triangular inversion

5.1 The Hurwitz-moment description

Another exact route uses Hurwitz class numbers. Let $H(D)$ denote the conventional weighted Hurwitz class number for a positive discriminant parameter D, with the standard exceptional automorphism weights. At a prime p define

$$s_{2j}(p) = \frac{1}{2} \sum_{|t|<2\sqrt{p}} t^{2j} H(4p - t^2). \tag{5.1}$$

Here the symbol $s_{2j}(p)$ is a trace moment and is unrelated to the coordinate power sums $s_j(v)$ introduced later. The argument of H is positive in the displayed range. The normalization, including the factor $1/2$, must be kept fixed when a trace formula is imported.

The Eichler–Selberg trace identities give, for $r \geq 1$,

$$\sum_{j=0}^{r} (-1)^{r-j} \binom{r+j}{r-j} p^{r-j} s_{2j}(p) = -1 - \operatorname{Tr} T_{2r+2}(p), \tag{5.2}$$

together with $s_0(p) = p$. Here $T_k(p)$ is the Hecke operator at p on $S_k(SL_2(\mathbb{Z}))$. It is distinct from the scalar recurrence notation $T_r = \tau(p^r)$ at a fixed prime. The trace normalization and Catalan inversion are checked against [15], Sections 4–6. The cusp spaces in weights $4, 6, 8, 10$ are zero. The weight-12 cusp space is spanned by Δ, so its Hecke trace is $\tau(p)$.

This leads to a lower triangular system for the moments. Its diagonal entries are all 1. The inversion can recover the moments exactly, but the unit last diagonal also means that the last cusp trace survives with a unit coefficient.

5.2 The Catalan inverse

The inverse matrix has coefficients

$$\left[\binom{2r}{r-j} - \binom{2r}{r-j-1} \right] p^{r-j}, \qquad 0 \leq j \leq r, \tag{5.3}$$

where a binomial coefficient outside its usual range is zero. The bracket is a ballot-number form of the Catalan triangle. It is an integer, and at $j = r$ it is 1.

The resulting first moments are

$$s_2(p) = p^2 - 1, \tag{5.4}$$
$$s_4(p) = 2p^3 - 3p - 1, \tag{5.5}$$
$$s_6(p) = 5p^4 - 9p^2 - 5p - 1, \tag{5.6}$$
$$s_8(p) = 14p^5 - 28p^3 - 20p^2 - 7p - 1, \tag{5.7}$$
$$s_{10}(p) = 42p^6 - 90p^4 - 75p^3 - 35p^2 - 9p - 1 - \tau(p). \tag{5.8}$$

The integer 42 is the fifth Catalan number. It records the contribution from $s_0(p) = p$ after inversion. The lower-degree terms come from the constant -1 in the trace formula. The coefficient of $\tau(p)$ is -1 because the last diagonal entry is 1.

One can check the inversion without computing class numbers. Multiply the lower triangular matrices symbolically after factoring the appropriate powers of p. The binomial identities give the identity matrix. The retained exact test checks both products through size 32, which is an implementation test for the universal binomial inversion rather than its proof.

5.3 A worked first row

At $r = 1$, (5.2) reads

$$-ps_0 + s_2 = -1.$$

Since $s_0 = p$, this gives $s_2 = p^2 - 1$. At $r = 2$ it reads

$$p^2 s_0 - 3ps_2 + s_4 = -1.$$

Substitution gives $s_4 = 2p^3 - 3p - 1$. The same triangular procedure produces the next three rows. At the fifth row, the right side is $-1 - \tau(p)$ rather than -1. The inversion has not supplied a reason for that extra term to be nonzero.

This small calculation reveals the general mechanism. The early vanishing cusp spaces make the early moments polynomial in p. The first nonzero cusp space introduces the actual modular coefficient. That distinction is geometric, not a numerical defect of the inversion.

5.4 An exact reformulation can retain the hard part

Equation (5.8) can be read as a formula for $\tau(p)$ in terms of a weighted class-number moment. Setting its last term formally equal to zero is consistent with the triangular linear system. To exclude a zero one needs an additional arithmetic restriction on the actual Hurwitz sum, beyond the identities used to obtain the system.

For example, positivity of $H(D)$ makes $s_{10}(p)$ nonnegative. The polynomial on the right of (5.8) with $\tau(p)$ omitted is also compatible with nonnegativity for large p. A positivity statement of this general kind does not separate the two possible values. A useful inequality would have to be quantitatively strict in a way that excludes the exact polynomial value for every prime.

The historical check independently enumerated reduced positive definite binary quadratic forms for all primes through 499 and for $p = 2411$. There are 96 prime rows in that scope. The resulting moments agree with the tau coefficients from the separate triangular engine. This gives exact evidence for the normalization and finite rows. It is not an exhaustive class-number theorem at all primes.

5.5 Two residuals, one obligation

The differential formula and the trace formula look different. One uses divisor convolutions and quasimodular differentiation. The other uses class numbers and Hecke traces. Both isolate the cusp coefficient with a unit coefficient. The original nonvanishing problem has therefore survived both extractions.

CHAPTER 6

Five-core packets and signed integral geometry

6.1 From hooks to an integral shell

A partition is a finite weakly decreasing list of positive integers. Its Young diagram has one box for each unit in the sum of its parts. The hook of a box consists of that box, the boxes to its right and the boxes below it. A 5-core is a partition with no hook length divisible by 5. Equivalently, it has no hook of length 5, the criterion relevant to the specialization below.

The Nekrasov–Okounkov identity in Han's formulation is

$$\prod_{m\geq 1}(1-q^m)^{z-1}=\sum_{\lambda}q^{|\lambda|}\prod_{h\in H(\lambda)}\left(1-\frac{z}{h^2}\right). \tag{6.1}$$

At $z=25$, a partition having a hook of length 5 contributes zero. The surviving terms are the 5-cores. Multiplication by q shows that the coefficient $\tau(n)$ is the sum of the hook products over 5-cores of size $n-1$.

Han's V-coding identifies these cores with

$$\mathcal{V}_n=\left\{v\in\mathbb{Z}^5 : v_i\equiv i \pmod 5,\quad \sum_{i=0}^{4}v_i=0,\quad \sum_{i=0}^{4}v_i^2=10n\right\}. \tag{6.2}$$

The shift by 1 in the partition size is important: the coding formula is $|\lambda|=\|v\|^2/10-1$. Define

$$D(v)=\prod_{0\leq i<j\leq 4}(v_i-v_j),\qquad W(v)=\frac{D(v)}{288}. \tag{6.3}$$

Here $288=1!2!3!4!$. The hook product becomes $W(v)$, with positive prefactor for $t=5$. Therefore

$$\boxed{\tau(n)=\sum_{v\in\mathcal{V}_n}W(v).} \tag{6.4}$$

This is a classical identity with translated conventions, authenticated against Han's Theorems 1.1 and 1.2 [8]. The finite shell is a new arena for the same coefficient, not a new coefficient definition.

6.2 Every packet is nonzero and integral

Distinct coordinate residues guarantee $v_i\neq v_j$ for $i\neq j$. Hence $D(v)$ and $W(v)$ are nonzero. Their signs can differ from packet to packet.

Integrality follows from an integer-valued basis. For every integer x, including negative integers, the generalized binomial coefficient $\binom{x}{j}$ is an integer. Its leading coefficient as a

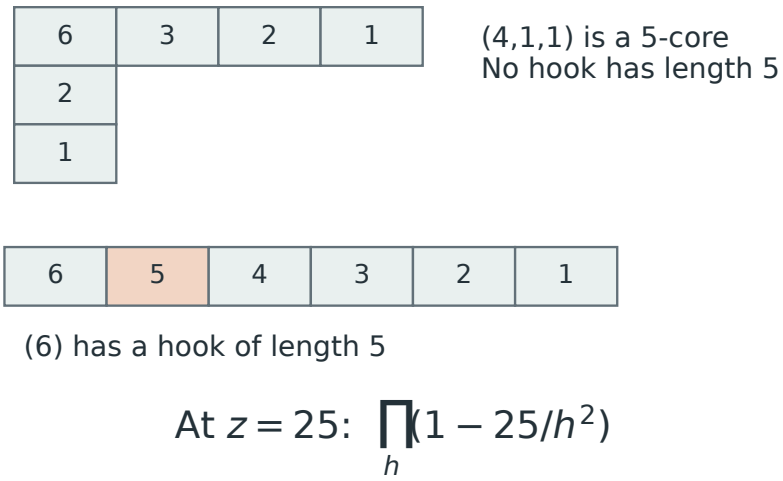


Figure 6.1: The hook specialization retains 5-cores and annihilates partitions with a hook of length 5. The displayed size-6 examples illustrate the exact mechanism behind the coefficient shift $|\lambda| = n - 1$.

polynomial in x is $1/j!$. Thus

$$\det\left[\binom{v_i}{j}\right]_{0\le i,j\le 4} = \frac{\prod_{i<j}(v_j - v_i)}{\prod_{j=0}^{4} j!}. \tag{6.5}$$

There are ten pairs, so reversing every difference changes the sign by $(-1)^{10} = 1$. The determinant is $W(v)$ and is integral. This proof is independent of the fact that the total in (6.4) is an integer.

Modulo 5, the denominator 288 is a unit and every difference has the fixed residue $i - j$. Hence

$$W(v) \equiv \frac{\prod_{i<j}(i - j)}{288} = 1 \pmod 5. \tag{6.6}$$

Every individual contribution is therefore a 5-adic unit. A total divisible by 5 arises from cancellation of those units, including their multiplicities.

6.3 Counting packets and restricting a possible zero

Let χ_5 be the quadratic character modulo 5, extended by zero at multiples of 5. The classical 5-core count [9], Theorem 4 and equation (5.3), gives

$$|\mathcal{V}_n| = \sum_{d\mid n} \chi_5(d)\frac{n}{d}. \tag{6.7}$$

At a prime,

$$|\mathcal{V}_p| = p + \chi_5(p).$$

Together with (6.6), this gives

$$\tau(p) \equiv p + \chi_5(p) \pmod 5. \tag{6.8}$$

For $p \neq 5$, $\chi_5(p) \equiv p^2 \pmod 5$, so this is also the familiar congruence $\tau(p) \equiv p(1+p) \pmod 5$ [2]. It excludes zeros at primes congruent to $1, 2, 3$ modulo 5. Since $\tau(5) = 4830 \neq 0$, the unexcluded prime family is

$$p \equiv 4 \pmod 5.$$

The restriction is substantial but still leaves infinitely many primes.

The formula for the unweighted count must not be confused with the weighted sum. At $p = 1249$ there are 1250 packets, whose count has 5-adic valuation 4. The weighted coefficient has valuation 3. At $p = 499$, the count has valuation 3 while the weighted coefficient has valuation 4. Weights that are all 1 modulo 5 recover the first residue, but not all higher divisibility.

6.4 A complete shell enumerator

The shell is finite because every coordinate has absolute value at most $\sqrt{10n}$. Enumeration can reduce its dimension without discarding any solution. Choose v_0, v_1, v_2 in their residue classes within the bound. The remaining coordinates $x = v_3$, $y = v_4$ must satisfy

$$x + y = S = -v_0 - v_1 - v_2, \qquad x^2 + y^2 = T = 10n - v_0^2 - v_1^2 - v_2^2.$$

Then

$$(x - y)^2 = 2T - S^2.$$

The right side must be a nonnegative square. For each permitted sign of its square root, the formulas $x = (S + (x-y))/2$ and $y = (S - (x-y))/2$ give the candidates. Their integrality and residues modulo 5 are checked directly. This is a complete quadratic solution, rather than a sampling procedure.

A second enumerator can loop through four coordinates and solve the fifth from the sum. Agreement between these independent organizations is a useful bounded control. The historical packet-tree verifier compared them through $n = 32$. Shell sums were then compared with an independent tau engine at a larger finite scope.

6.5 The index-prime unit obstruction

If $p > 20$ is prime and $v \in \mathcal{V}_p$, then for each pair

$$0 < (v_i - v_j)^2 \leq 2(v_i^2 + v_j^2) \leq 20p < p^2.$$

Thus p divides none of the differences and does not divide 288. We obtain

Proposition 6.1. *For every prime $p > 20$ and every $v \in \mathcal{V}_p$, one has $v_p(W(v)) = 0$.*

All packets are units at the index prime. There is no increasing termwise p-adic depth automatically built into this representation. At $p = 2411$, for example, all 2412 packet weights are 2411-adic units while their sum has valuation 1. The divisibility belongs to the cancellation, not to an individually deeper term.

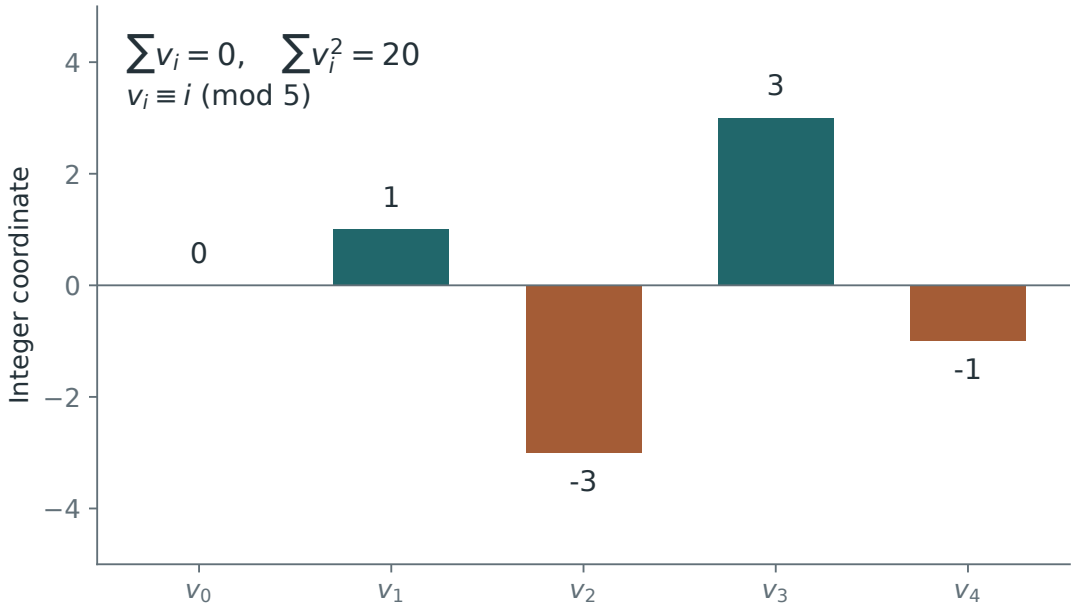


Figure 6.2: The shell $\mathcal{V}_2$ is a concrete intersection of a zero-sum hyperplane, an integral residue condition and a sphere. Coordinates are labeled by their residues modulo 5. A shell point is a packet, with its separate nonzero integral Vandermonde weight.

This rules out importing a termwise filtration without constructing it. Groupings may still have useful divisibility, and new transformations may still reveal structure. The identity itself does not provide such a filtration.

CHAPTER 7

A genuine auxiliary polynomial and its local roots

7.1 A polynomial at each fixed shell

Write $v_i = i + 5c_i$ for $v \in \mathcal{V}_n$. Define

$$P_n(T) = \sum_{v \in \mathcal{V}_n} \prod_{i<j} \left[(i-j) + T(c_i - c_j) \right] = \sum_{j=0}^{10} B_j(n) T^j. \tag{7.1}$$

For a fixed n, this is an integral polynomial of degree at most ten. It is obtained by summing products of ten linear factors. Evaluation at $T = 5$ returns the Vandermonde numerators:

$$P_n(5) = 288\tau(n), \qquad B_0(n) = 288|\mathcal{V}_n|. \tag{7.2}$$

The variable T is auxiliary. Nothing in the construction says that $\tau(p)$ is one analytic function of the prime index p in a p-adic or real variable.

Normalize the coefficient valuations by

$$\mu_n = \min_{B_j(n) \neq 0} \big(j + v_5(B_j(n)) \big), \qquad G_n(U) = 5^{-\mu_n} P_n(5U).$$

Every coefficient of G_n is integral, and at least one is a unit. Because 288 is a 5-adic unit,

$$v_5(\tau(n)) = \mu_n + v_5(G_n(1)). \tag{7.3}$$

This formula remains meaningful if $\tau(n) = 0$, with valuation ∞. The normalization does not assume nonvanishing: its constant coefficient comes from a positive packet count on a nonempty shell.

The bounded factorization experiment found all 300 aggregate polynomials at primes $7 \leq p \leq 2000$ irreducible over $\mathbb{Q}$. This observation is unused in the universal arguments. Excluding $U = 1$ throughout the prime family still requires further arithmetic restrictions on the moments $B_j(p)$.

7.2 A simple root at $p = 499$

The exact coefficients $B_0, \ldots, B_{10}$ at 499 are

$$\begin{aligned}(&144000, -360000, 1063040, 28418240, -148872808,\\ &209474504, 2422451240, 16445000, 8107402848,\\ &-11893316304, -1160632800).\end{aligned}$$

Their weighted valuations are

$$(3, 5, 3, 4, 4, 5, 7, 11, 8, 9, 12).$$

Hence $\mu_{499} = 3$ and reduction gives

$$\overline{G}_{499}(U) = 2 + 3U^2 = 3(U-1)(U+1) \quad \text{in } \mathbf{F}_5[U].$$

At the relevant value $U = 1$,

$$G_{499}(1) \equiv 10 \pmod{25}, \qquad G'_{499}(1) \equiv 1 \pmod{5}.$$

Substitute $U = 1 + 5d$. The integral Taylor identity gives

$$G_{499}(1 + 5d) \equiv 10 + 5d \pmod{25}.$$

The root in this residue class is therefore $\rho \equiv 16 \pmod{25}$. Hensel's lemma lifts it uniquely to $\mathbb{Z}_5$. The divided difference $(G(U) - G(\rho))/(U - \rho)$ is a unit on $U \equiv 1 \pmod{5}$, since its reduction is $G'(1) \neq 0$. Thus

$$v_5(G_{499}(U)) = v_5(U - \rho)$$

on that residue class. At $U = 1$ the difference from ρ has valuation 1, giving

$$v_5(\tau(499)) = 3 + 1 = 4.$$

The root has been located in the auxiliary variable. This proves a finite coefficient valuation, without suggesting that 499 is near a root in the prime-index variable.

7.3 A double residual root with no lift

At $p = 16249$, the shell has 16250 packets. The weighted coefficient valuations of $P_p(5U)$ are

$$(4, 5, 4, 4, 6, 5, 7, 8, 9, 9, 11),$$

so $\mu_p = 4$. The reduction is

$$\overline{G}_p(U) = 3 + U^2 + U^3 = (U-1)^2(U-2) \pmod{5}.$$

The residual root at 1 is double. Exact coefficient reduction gives

$$G_p(1) \equiv 15 \pmod{25}, \qquad G'_p(1) \equiv 0 \pmod{5}.$$

For every $y \in \mathbb{Z}_5$, polynomial Taylor expansion has an integral remainder:

$$G_p(1 + 5y) = G_p(1) + 5yG'_p(1) + 25y^2H(y), \qquad H \in \mathbb{Z}_5[y].$$

Consequently

$$G_p(1 + 5y) \equiv 15 \pmod{25} \quad \text{for every } y \in \mathbb{Z}_5. \tag{7.4}$$

No point in this entire residue class is a root. Its valuation is exactly 1, and

$$v_5(\tau(16249)) = 4 + 1 = 5.$$

The exact coefficient is 133116215274958169206250. The universal statement in (7.4) follows from the Taylor identity and the two residues. A finite test at 625 choices of y is only a secondary code check.

$$P_n(5) = 288\tau(n), \quad G_n(U) = 5^{-\mu_n} P_n(5U)$$

$p = 499$	$p = 16249$
$\overline{G}(U) = 2 + 3U^2$	$\overline{G}(U) = (U-1)^2(U-2)$
Simple root at 1	Double root at 1

Auxiliary variable U. The target coefficient is evaluated at U = 1.

Figure 7.1: Two exact auxiliary root certificates. At 499, the simple residual root near 1 lifts and sits one 5-adic digit away from the evaluation point. At 16249, the double residual root has no lift in the class 1 mod 5, by an integral Taylor congruence. These are roots in U, not roots in the prime index.

7.4 What a degree bound does not control

The comparison cases are

p	Packets	μ_p	$v_5(G_p(1))$	$v_5(\tau(p))$
499	500	3	1	4
1249	1250	3	0	3
2999	3000	3	1	4
4999	5000	3	0	3
16249	16250	4	1	5
18749	18750	3	0	3

The degree-10 bound controls the number of roots over an algebraic closure, counted with multiplicity. It does not give a uniform bound for a value's valuation. Even the degree-one polynomial $U - (1 + 5^k)$ has valuation k at $U = 1$. A root can lie arbitrarily close to the evaluation point as the polynomial changes.

Individual packet polynomials have rational finite roots in $[-4, 4]$, since each root is $-(i-j)/(c_i - c_j)$ with a nonzero integer denominator. The aggregate polynomial need not retain that root region. Exact values at $p = 31$ are

$$P_{31}(10) = -56767541184 < 0, \qquad P_{31}(11) = 4645170845280 > 0.$$

It has a real root between 10 and 11. Summing signed real-rooted factors does not preserve their root bounds.

CHAPTER 8

Symmetry and the harmonic theta mechanism

8.1 An involution that preserves the weight

The coordinate labels are residue labels, so negation alone does not preserve the standard shell. The correct involution is

$$J(v)_i = -v_{(-i \bmod 5)}. \tag{8.1}$$

It preserves the residues, sum and norm. Its coordinate permutation is $(1\,4)(2\,3)$, which is even. Negation contributes $(-1)^{10} = 1$ to the degree-10 Vandermonde. Hence

$$W(Jv) = W(v).$$

Write $s_m(v) = \sum_i v_i^m$. Then $s_m(Jv) = (-1)^m s_m(v)$. In particular, every odd homogeneous symmetric polynomial Q satisfies $Q(Jv) = -Q(v)$. Pairing by J gives

$$\sum_{v \in \mathcal{V}_n} W(v)Q(v) = 0. \tag{8.2}$$

At a fixed point of J, such a Q is zero, so the argument covers both orbit sizes. Odd corrections alone do not change the sum of an entire J-orbit. Even corrections will be the useful source of transfers between orbits.

8.2 Why the Vandermonde is harmonic

Work on the four-dimensional Euclidean hyperplane

$$H = \{v \in \mathbb{R}^5 : \sum_i v_i = 0\}.$$

The Laplacian on H commutes with coordinate permutations. Since D is alternating, its Laplacian is alternating and has degree 8. Every alternating polynomial on H vanishes when $v_i = v_j$. The ten distinct linear factors $v_i - v_j$ therefore divide it. A degree-8 polynomial cannot be divisible by their degree-10 product unless it is zero. Thus

$$\Delta_H D = 0.$$

This degree argument is useful because it does not require expanding the ten-factor product.

More generally, if Q is symmetric and DQ is harmonic homogeneous of degree $10 + d$, its weighted shell theta series is

$$\Theta_Q(q) = \sum_{n \geq 1} \left(\sum_{v \in \mathcal{V}_n} W(v)Q(v) \right) q^n. \tag{8.3}$$

The harmonic theta theorem initially supplies a vector-valued modular form. Obtaining a scalar full-level cusp form requires the following lattice specialization, rather than an unexplained assertion that every shell theta series has level one.

8.3 The regular discriminant cosets

Let $A_4 = H \cap \mathbb{Z}^5$ and $L = \sqrt{5}A_4$. Its discriminant module is

$$L^*/L \cong A_4^*/5A_4, \qquad |L^*/L| = 5^5 = 3125.$$

A representative of L^* can be written

$$x = \frac{1}{\sqrt{5}}(a_i + t/5)_{i=0}^4, \qquad 0 \le t < 5, \qquad \sum_i a_i = -t.$$

If two a_i have the same residue modulo 5, the transposition of those coordinates fixes the discriminant coset. An alternating theta component on that coset must equal its own negative and is zero.

If all five residues are distinct, they are $0, 1, 2, 3, 4$ in some order. Their sum is zero modulo 5, so $\sum_i a_i = -t$ forces $t = 0$. These regular cosets form one free S_5-orbit of size 120. The sign representation occurs exactly once in the corresponding permutation module. It does not occur in an irregular orbit, since an odd stabilizer kills the alternating part. Thus the alternating theta vector lies on a one-dimensional sign line.

For a regular representative, $\sum_i a_i^2$ is divisible by 5, since the squared residues sum to 30. It is even, since $a_i^2 \equiv a_i \pmod 2$ and $\sum_i a_i = 0$. The quadratic exponent $\sum_i a_i^2/10$ is integral. The Weil operator T therefore acts as 1 on the sign line. The Weil action commutes with S_5, so this line is invariant. The rank is even, and the representation descends to $SL_2(\mathbb{Z})$. On the line the relation $(ST)^3 = S^2$ reduces to $S^3 = S^2$, giving $S = 1$.

The full-level scalar conclusion has now been justified by the discriminant calculation. The ordinary residue coset has $x = v/\sqrt{5}$, and its exponent $\|x\|^2/2$ is exactly $\|v\|^2/10 = n$. Scaling the homogeneous polynomial changes an overall scalar, not the modular weight or the zero conclusion.

8.4 The modularity contract and the cusp condition

The authenticated harmonic theta theorem states that a homogeneous harmonic polynomial of degree ν on a rank-r positive definite even lattice gives a vector-valued modular form of weight $\nu + r/2$ for the Weil representation [10]. Here $r = 4$ and $\nu = 10 + d$. The project-specific sign-line argument above therefore gives

$$\Theta_Q \in S_{12+d}(SL_2(\mathbb{Z})) \tag{8.4}$$

for the even-degree harmonic construction. The zero vector contributes no constant term. The full modular group has one cusp, so the form is cuspidal. This is the exact external-to-internal dependency: the general theorem supplies vector modularity, and the explicit coset argument supplies the scalar full-level specialization.

The convergence needed for the analytic theta function is harmless here. The lattice is positive definite, the polynomial has finite degree and Gaussian decay dominates polynomial growth on each compact subset of the upper half-plane. Termwise differentiation is therefore valid. The corresponding formal Fourier coefficients are finite shell sums.

CHAPTER 9

Universal null polynomials

9.1 The correction that does not change any coefficient

A universally null polynomial is a symmetric polynomial Q with

$$\sum_{v \in \mathcal{V}_n} W(v)Q(v) = 0 \quad \text{for every } n \geq 1. \tag{9.1}$$

This is much stronger than being null on one selected shell. If a finite family satisfies (9.1), any rational linear combination of it does too. Thus

$$F(v) = 1 + \sum_j a_j Q_j(v)$$

has the exact property

$$\sum_{v \in \mathcal{V}_n} W(v)F(v) = \tau(n) \tag{9.2}$$

at every index, after the coefficients a_j have been fixed. The choice of coefficients may depend on a target shell. The null identity does not thereby become merely a shell-specific identity.

The retained construction obtains (9.1) by combining harmonicity with a finite vanishing certificate and the valence formula. This is one place where finite checks support a universal implication for a stated mathematical reason.

9.2 A low-degree limit

Every alternating polynomial of total degree at most 13 is D times a symmetric quotient of degree at most 3. On $s_1 = 0$, such a quotient is a linear combination of $1, s_2, s_3$. On a shell, $s_2 = 10n$, while the s_3 contribution vanishes under J. Its weighted sum is thus $(a + 10bn)\tau(n)$ for constants a, b.

This low-degree space cannot extract a new independent arithmetic residual. It is a scoped limitation on alternating polynomial refinements through total degree 13. At quotient degree 4, s_4 appears, and harmonic theta forms of higher weights introduce new moment identities. These will still require care: new cusp coefficients do not automatically give a nonzero lower bound for the original one.

9.3 An exact operator on symmetric power sums

Define $\mathcal{L}$ by

$$\Delta_H(DQ) = D\mathcal{L}Q.$$

The Vandermonde is harmonic. Its logarithmic derivative and the product rule give, with $s_0 = 5$ and $s_1 = 0$,

$$\mathcal{L}s_m = m \sum_{a=0}^{m-2} s_a s_{m-2-a} - \frac{m(m-1)}{5} s_{m-2}, \qquad m \geq 2. \tag{9.3}$$

The chain-rule metric is

$$\nabla_H s_m \cdot \nabla_H s_n = mn \left(s_{m+n-2} - \frac{s_{m-1} s_{n-1}}{5} \right). \tag{9.4}$$

For a polynomial in s_2, s_3, s_4, s_5, these two formulas determine $\mathcal{L}Q$ by the ordinary second-order product rule. Higher power sums are reduced through Newton's identities for five variables.

The apparent denominators from differentiating D cancel pairwise. The quotient

$$\frac{v_i^{m-1} - v_j^{m-1}}{v_i - v_j}$$

is a polynomial. Thus the operator really has rational polynomial coefficients on the symmetric ring. For example, (9.3) gives $\mathcal{L}s_2 = 48$. This agrees with the direct radial identity $\Delta_H(s_2 D) = 48D$.

At each even invariant degree d, one lists monomials $s_2^a s_3^b s_4^c s_5^e$ with $2a + 3b + 4c + 5e = d$. Writing an unknown rational linear combination, the equation $\mathcal{L}Q = 0$ is a finite linear system. The initial theta coefficients supply further rational linear equations. No floating-point approximation is needed to define this kernel.

9.4 Why a finite initial segment suffices

The level-one valence formula implies that a nonzero holomorphic modular form of weight k has order at the cusp at most $k/12$ [11]. A cusp form whose coefficients of $q^1, \dots, q^{\lfloor k/12 \rfloor}$ all vanish has order at least $\lfloor k/12 \rfloor + 1 > k/12$. It must be zero.

For a harmonic DQ of invariant degree d, the theta weight is $k = 12 + d$. Thus the constructor imposes

$$[q^m]\Theta_Q = 0, \qquad 1 \le m \le \left\lfloor \frac{12 + d}{12} \right\rfloor.$$

Harmonicity supplies membership in the specified modular-form space, and valence supplies the implication from this finite certificate to all coefficients. Without harmonicity and the modularity theorem, testing the same initial segment would have no universal force.

The distinction between the three degree conventions is essential. The symmetric correction has invariant degree d. The alternating product DQ has total degree $10 + d$. Its theta form has modular weight $12 + d$. A claim about corrections through degree 20 means the first of these unless stated otherwise.

9.5 The first explicit corrections

The construction gives

$$\begin{aligned} Q_8 = {} & 6273 s_2^4 - 33750 s_2^2 s_4 - 56800 s_2 s_3^2 \\ & + 69600 s_3 s_5 + 45000 s_4^2, \end{aligned} \tag{9.5}$$

and

$$\begin{aligned} Q_{10} = {} & 2091 s_2^5 - 11250 s_2^3 s_4 - 145800 s_2^2 s_3^2 \\ & + 409600 s_2 s_3 s_5 + 15000 s_2 s_4^2 \\ & - 30000 s_3^2 s_4 - 264000 s_5^2. \end{aligned} \tag{9.6}$$

Both DQ_8 and DQ_{10} are harmonic. The first shell has the vector $(0, 1, 2, -2, -1)$, with

$$s_2 = 10, \quad s_3 = 0, \quad s_4 = 34, \quad s_5 = 0.$$

Both polynomials vanish there. Their theta weights are 20 and 22. The cusp spaces are respectively spanned by ΔE_4^2 and $\Delta E_4 E_6$, each with nonzero first coefficient. The zero first coefficient therefore kills the whole theta form. This is also the simplest valence argument for these two degrees.

The first retained family has fourteen corrections, with counts

$$8 : 1, \quad 10 : 1, \quad 12 : 1, \quad 14 : 2, \quad 16 : 2, \quad 18 : 3, \quad 20 : 4.$$

The transport continuation adds 22 more: four at degree 22, five at 24, six at 26 and seven at 28. The total is 36. These are explicit permitted corrections, not a proof that every universally null polynomial lies in their span.

9.6 Moment identities that introduce other forms

The harmonic polynomial $D(130s_4 - 47s_2^2)$ has weight-16 theta form. Its first coefficient is -280. Write $E_4\Delta = \sum a_{16}(n)q^n$. Then

$$13 \sum_{v \in \mathcal{V}_n} W(v) s_4(v) = 470 n^2 \tau(n) - 28 a_{16}(n). \tag{9.7}$$

Likewise $D(90s_2s_4 - 27s_2^3 - 320s_3^2)$ gives

$$900n \sum W(v) s_4(v) - 27000 n^3 \tau(n) - 320 \sum W(v) s_3(v)^2 = 3600 a_{18}(n), \tag{9.8}$$

where $E_6\Delta = \sum a_{18}(n)q^n$ and the packet sums are on $\mathcal{V}_n$. These are exact identities. Their additional cusp coefficients must be controlled if the identities are used to infer nonvanishing. A new modular moment is not automatically a sign or residue certificate.

CHAPTER 10

What an exact sign certificate proves

10.1 The sufficient sign condition

Take a finite rational combination of universally null corrections and set $F = 1 + \sum a_j Q_j$. If all real numbers $W(v)F(v)$ on a shell have one weak sign and at least one is nonzero, their sum is nonzero. By (9.2), that sum is $\tau(n)$.

This is a valid certificate. Its verification requires the universal null identities, the exact rational coefficients, every packet in the shell and every claimed sign. A numerical optimizer may propose coefficients, but the certificate is established by exact rational checks. A failed optimizer does not establish infeasibility without a separate exact obstruction.

10.2 Six packets reduced to two at $n = 7$

Choose

$$a = \frac{383}{1411344000000}, \qquad b = \frac{139}{61158240000000}, \qquad F = 1 + aQ_8 + bQ_{10}.$$

There are three J-orbits, each of size two. Their representatives and weights are

Representative v	$W(v)$	$F(v)$
$(-5,-4,2,3,4)$	1176	0
$(-5,6,2,-2,-1)$	-8624	0
$(0,-4,7,-2,-1)$	-924	$299/33$

For the three rows, the (Q_8, Q_{10}) values are

$$(712800000, -525096000000),$$
$$(-3790800000, 12636000000),$$
$$(36288000000, -786240000000).$$

Substitution verifies the displayed F values exactly. Four packets disappear, and the remaining two are negative. Therefore

$$\tau(7) = 2(-924)\frac{299}{33} = -16744. \tag{10.1}$$

The solve can be performed by requiring only the first two rows to vanish. It need not use a preassigned value of $\tau(7)$. The surviving value then supplies the coefficient, and the independent product engine checks it.

An integral form of the same certificate multiplies F by 183474720000000:

$$183474720000000 + 49790Q_8 + 417Q_{10}.$$

Its surviving value is 1662392160000000. Clearing denominators is useful for a short exact checker, but does not change the sign argument.

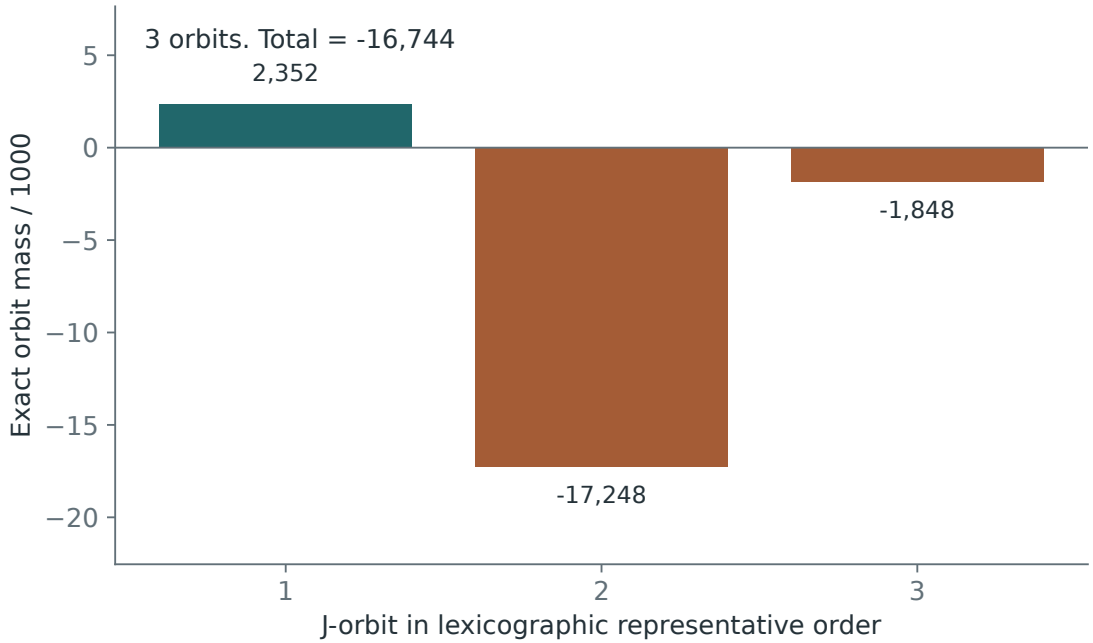


Figure 10.1: The three original J-orbit masses at $n = 7$, each containing two packets. The exact certificate in the text removes the positive orbit and one negative orbit, then rescales the remaining negative one. The unchanged total is certified nonzero by the displayed rational coefficients.

10.3 An exact obstruction in a specified smaller space

At $n = 31$, the seven retained corrections through degree 16 do not admit a nontrivial common-sign certificate. The shell has sixteen J-orbits. Let $\epsilon_i = \operatorname{sgn} W(v_i)$ for their representatives. The retained exact dual has strictly positive rational entries y_i satisfying

$$\sum_i y_i = 1, \qquad \sum_i y_i \epsilon_i = 0, \qquad \sum_i y_i \epsilon_i Q_j(v_i) = 0$$

for all seven correction columns. Consequently, for every coefficient vector,

$$\sum_i y_i \epsilon_i \left(1 + \sum_j a_j Q_j(v_i)\right) = 0.$$

If every corrected packet had one weak sign and one were nonzero, every $\epsilon_i F(v_i)$ would have that sign and one would be strict. A strictly positive weighted average could not be zero. This contradiction proves infeasibility in the specified seven-column span, for either sign.

The scope matters. The dual is not a theorem about every polynomial of degree at most 16, unless completeness of the retained list in that larger class is separately proved. It is not a theorem about all universally null polynomials. It does not show that $\tau(31)$ vanishes. It describes one exact finite correction space.

10.4 The obstruction can disappear when the space changes

Adding the retained corrections at degrees 18 and 20 gives an exact strictly negative certificate at 31. The previous dual does not annihilate the larger family. This is an instructive example

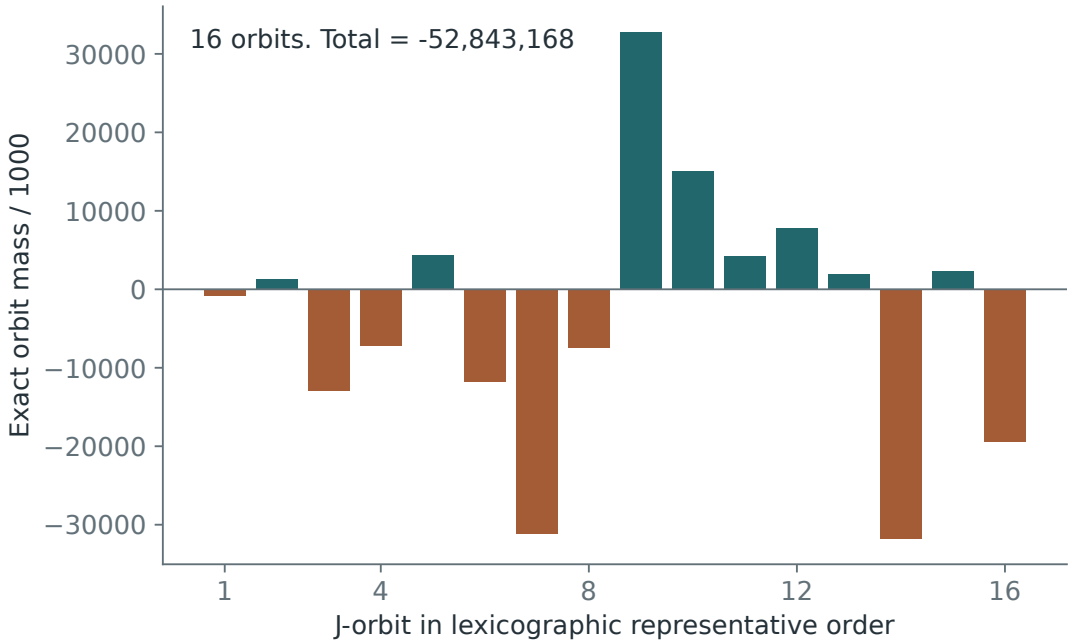


Figure 10.2: The sixteen original J-orbit masses at 31. Eight are positive and eight negative. The coefficients sum to the nonzero total -52843168, but individual nonzero masses and a balance of sign counts do not force that total.

of a removable obstruction. It cannot support the assertion that every obstruction at every prime will be removable.

The retained strict rational certificates cover the following primes:

p	2	3	5	7	11	13
Degree	8	8	8	10	8	12
Sign	$-$	$+$	$+$	$-$	$+$	$-$
p	17	19	23	29	31	37
Degree	8	14	14	16	20	20
Sign	$-$	$+$	$+$	$+$	$-$	$-$
p	47	59				
Degree	14	20				
Sign	$+$	$-$				

The degrees used are not claimed optimal. In particular, no exact degree-18 impossibility at 31 is asserted. The fresh release replay verifies all fourteen stored rational certificates and all 300 packet margins in their combined finite scope.

10.5 Completeness retains the total

Suppose a shell has nonzero orbit masses $w_1, \dots, w_N$ and total T. Even allow every rational correction vector q with $\sum_i w_i q_i = 0$. Then

$$\exists q : \ w_i(1+q_i) \text{ have one weak sign and are not all zero} \quad \Longleftrightarrow \quad T \neq 0. \tag{10.2}$$

Necessity follows from conservation. For sufficiency, if $T \neq 0$, take

$$q_i = \frac{T}{N w_i} - 1.$$

Every corrected mass is T/N and the correction has zero weighted total. This construction proves an equivalence. If used as a proof of $T \neq 0$ without independent information about T, it would be circular.

Thus a theorem that all zero-total correction vectors are available would explain the existence of a sign certificate once the total is known to survive. It would not independently establish that survival. A concrete sign certificate can still be decisive, as at 7. A general completeness theorem must be combined with an independently established nonzero sign or arithmetic condition to settle every prime.

CHAPTER 11

The exact dyadic mixed-sign boundary

11.1 Sectors and two different notions of separation

For a packet put $c_i(v) = (v_i - i)/5$. At depth $k \geq 1$, write $M = 2^k$. A sector is the residue of $c(v)$ modulo M, identifying the residues obtained from v and Jv. One may represent it by the lexicographically smaller of $c(v)$ mod M and $c(Jv)$ mod M. All sector sums retain the original packet multiplicities.

A sector is mixed if it contains both positive and negative weights. Sign-purity is weaker than resolving a sector into one J-orbit. Several different orbits can have the same sign. A theorem about the first mixed sector therefore answers a different question from a theorem about the first collision of distinct orbits.

The sign has an elementary coding description. For $i < j$, the difference

$$v_i - v_j = (i - j) + 5(c_i - c_j)$$

is positive exactly when $c_i > c_j$. There are ten pairs, so

$$\operatorname{sgn} W(v) = (-1)^{\#\{i<j : c_i > c_j\}}. \tag{11.1}$$

Equal c_i, c_j give a negative difference because $i - j < 0$. Thus ties in the coding do not make a zero Vandermonde.

11.2 The sharp threshold

Theorem 11.1. *Let $k \geq 1$, $M = 2^k$, and set*

$$N_k = \left\lceil \frac{100M^2 + 10M + 2}{35} \right\rceil = \frac{20M^2 + 2M + \gamma_k}{7}, \tag{11.2}$$

where

$$\gamma_k = \begin{cases} 7 & k \equiv 1 \pmod 3, \\ 1 & k \equiv 2 \pmod 3, \\ 6 & k \equiv 0 \pmod 3. \end{cases}$$

Every depth-k sector in every shell $n < N_k$ is sign-pure. The shell $n = N_k$ has a mixed sector.

This threshold ranges over all positive integer shell indices. It does not assert that N_k is prime. For example, $N_3 = 186$ is composite.

11.3 The geometric lower bound

Take opposite-sign packets v, w in one sector. Replace w by Jw if needed so that

$$w - v = 5Md, \qquad d \in A_4 \setminus \{0\}.$$

Set $a = 5M/2$ and $m = (v + w)/2$. Since M is even, both are integral. Equal norms give

$$m \cdot d = 0, \qquad 10n = \|m\|^2 + a^2\|d\|^2. \tag{11.3}$$

The squared norm of a nonzero integral zero-sum vector is a positive even integer. It cannot be 2 here. Such a vector is $\pm(e_i - e_j)$, and $m \cdot d = 0$ would force $v_i - v_j = \pm 5M$. This is divisible by 5, contrary to the distinct coordinate residues.

If $\|d\|^2 \geq 6$, (11.3) gives

$$n \geq \frac{15}{4}M^2 > \frac{100M^2 + 10M + 2}{35}, \qquad M \geq 2.$$

The remaining case is $\|d\|^2 = 4$. Then d has two coordinates 1, two -1 and one 0. Of the ten coordinate pairs, two have equal d entries and eight have unequal entries. For a pair with $\delta = d_i - d_j$,

$$(v_i - v_j)(w_i - w_j) = (m_i - m_j)^2 - a^2\delta^2.$$

The two equal-entry pairs have positive products. If all eight unequal-entry pairs had negative products, their product would also be positive, contradicting the opposite signs of $D(v)$ and $D(w)$. None of these pair products is zero, since both packets have distinct coordinates. Hence at least one unequal pair has a positive product, giving the integral inequality

$$|m_i - m_j| \geq a|\delta| + 1.$$

Project $e_i - e_j$ orthogonally to $d^\perp$. Its squared norm is

$$2 - \frac{\delta^2}{4}.$$

Because $m \perp d$, Cauchy's inequality gives

$$\|m\|^2 \geq \frac{(a|\delta| + 1)^2}{2 - \delta^2/4}.$$

For $|\delta| = 1$ this is $4(a+1)^2/7$. For $|\delta| = 2$ it is $(2a+1)^2$, a larger quantity. Therefore

$$10n \geq 4a^2 + \frac{4}{7}(a+1)^2,$$

which simplifies to the lower bound inside the ceiling in (11.2). Taking the ceiling proves that no earlier shell can be mixed. Since powers of 2 modulo 7 cycle as $2, 4, 1$, direct division gives the displayed γ_k values.

11.4 Witnesses attaining the bound

At $k = 1$, take

$$v = (-5, -4, 2, -2, 9), \qquad w = (5, 6, -8, -2, -1).$$

They belong to $\mathcal{V}_{13}$, have opposite weights and differ coordinatewise by multiples of 10. Thus they lie in one depth-one sector.

For $k \geq 2$, explicit affine families attain the lower bound. The following table lists $7v$ and $7w$, so that the displayed entries remain small integer affine expressions.

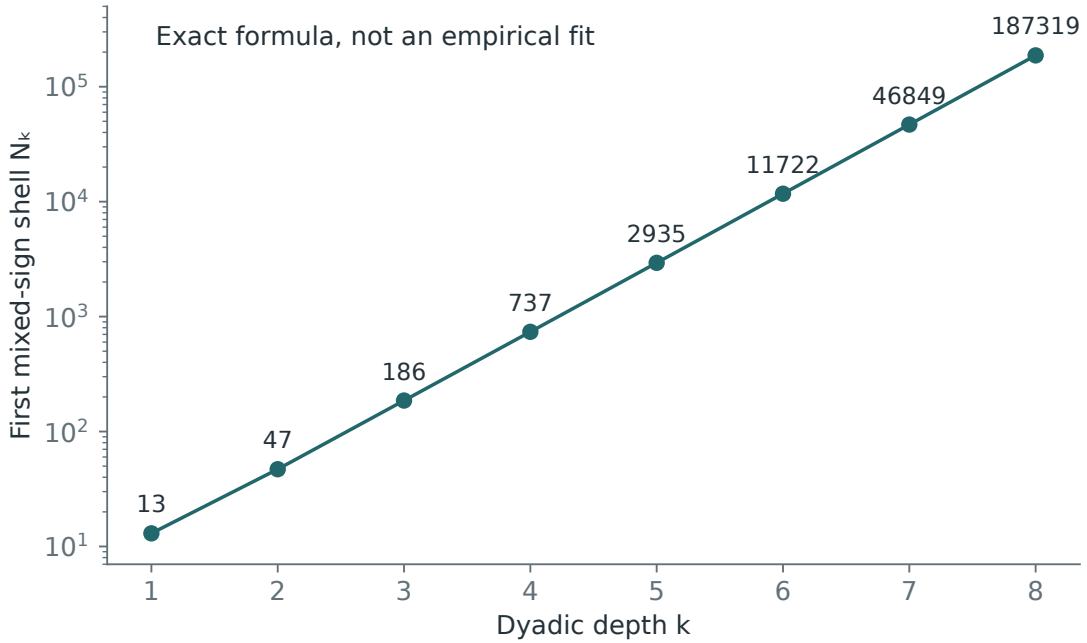


Figure 11.1: The first mixed shell N_k at depths 1 through 8. These exact values belong to an all-depth geometric theorem. The index range is all positive integer shells, and the thresholds need not be prime.

k mod 3	$7v$	$7w$
$1, k \geq 4$	$(-20M+5, -10M-8, -10M-1, 20M+16, 20M-12)$	$(15M+5, -10M-8, 25M-1, -15M+16, -15M-12)$
$2, k \geq 2$	$(-25M-5, 10M+2, -15M+4, 15M-4, 15M+3)$	$(10M-5, 10M+2, 20M+4, -20M-4, -20M+3)$
$0, k \geq 3$	$(-20M-15, 10M-3, 10M+4, 20M+1, -20M+13)$	$(15M-15, -25M-3, 10M+4, -15M+1, 15M+13)$

The appropriate powers of 2 make each entry divisible by 7. After division, each vector has the standard residues, sum zero and norm squared $10N_k$. The norm identities before division are

$$\sum_i (7v_i)^2 = \sum_i (7w_i)^2 = 1400M^2 + 140M + 70\gamma_k = 490N_k.$$

In the first two rows, $(w-v)/(5M) = (1, 0, 1, -1, -1)$. In the third it is $(1, -1, 0, -1, 1)$. Thus the packets share a sector.

Every coordinate difference is affine in M. Its sign is fixed on the indicated range. Checking the ten linear differences in each row proves that the two Vandermonde signs are opposite throughout that range. This is an all-depth argument using polynomial identities and linear inequalities, not an inference from a list of sample powers.

The first thresholds are

$$13, 47, 186, 737, 2935, 11722, 46849, 187319.$$

The verifier checks witnesses through $k = 64$ as an implementation control. The affine identities and lower-bound proof establish the theorem for every depth.

11.5 A geometric theorem without a nonzero total

Refining sectors eventually separates distinct J-orbits on a fixed finite shell. The sharp threshold describes how quickly opposite signs can share a sector. Neither assertion compares the total positive mass with the total negative mass. Geometric separation has not supplied an arithmetic inequality between them.

This is why the proposed sector-preserving finishing argument must be examined separately. Resolving signs can make cancellation visible. It does not remove cancellation if each resolved mass is then conserved.

CHAPTER 12

Why strict localization and automatic valuation fail

12.1 Conserved sectors can preserve opposite signs

Partition a shell into sectors S_r, with masses $t_r = \sum_{v \in S_r} W(v)$. Suppose every permitted correction is null inside every sector:

$$\sum_{v \in S_r} W(v) Q_j(v) = 0.$$

Then $F = 1 + \sum a_j Q_j$ leaves each corrected sector mass equal to t_r. If two sector masses have opposite nonzero signs, all corrected packets cannot have one weak sign. Summing within those two sectors proves the obstruction immediately.

At $p = 7$, depth-one sector data are

Sector key	Packets	Mass	$\sum W Q_8$
$(0,0,0,0,0)$	2	-1848	-67060224000000
$(1,0,0,1,0)$	2	-17248	65383718400000
$(1,1,0,0,0)$	2	2352	1676505600000

The three correction sums total zero, but no individual one is zero. Thus the known universal null Q_8 is not sector-null when restricted by a parity indicator. The sign certificate at 7 works by transferring mass between sectors. A completion that preserves every one of these sectors would forbid the needed transfer.

This is a failure of the proposed localization principle, not a failure of the universal null identity. Multiplying Q by an arbitrary sector indicator changes its theta series and may destroy universal nullness. A correct localized operation must be constructed by solving for combinations whose parent effects cancel, as in Chapter 14.

12.2 Aggregate parity identities have a larger scope

The number of odd coordinates of a packet is even. Since squares modulo 4 are 0 or 1,

$$\#\{i : v_i \text{ odd}\} \equiv 10n \equiv 2n \pmod 4.$$

For odd n there are two odd coordinates. For even n there are zero or four. Let A, B, C be the weighted generating series for these three aggregate classes, respectively.

The all-even class scales by 2. The bijection $u_j = v_{2j \bmod 5}/2$ has a coordinate permutation of sign -1, so

$$A(q) = -2^{10} \Delta(q^4).$$

Using the Hecke recurrence at 2 gives

$$B(q) = -24\Delta(q^2) - 2^{10}\Delta(q^4), \tag{12.1}$$
$$C(q) = \Delta(q) + 24\Delta(q^2) + 2^{11}\Delta(q^4). \tag{12.2}$$

For a homogeneous universally null correction, scaling kills its aggregate A series, and odd and even support then kill the corresponding aggregate B and C series. This does not imply nullity inside each of the six finer parity sectors making up class C. The explicit $p = 7$ table is a counterexample to that stronger inference.

At sufficient depth, a sector is one J-orbit. For an even symmetric Q, its weighted correction sum on the orbit is its nonzero mass times $Q(v)$. Sector nullity then forces $Q(v) = 0$ there. Imposing sector-nullity at every depth eventually removes every effective even correction. Refinement adds constraints rather than creating new available transfers.

12.3 A unique least valuation is a sufficient criterion

If one summand in an integer sum has strictly smaller ℓ-adic valuation than all others, the sum has that least valuation and is nonzero. This follows by dividing by the least power of ℓ and observing one unit plus terms divisible by ℓ.

Terminal packet resolution does not guarantee this hypothesis. Every packet is 1 modulo 5, and each J-orbit has size 1 or 2. Every orbit mass is therefore a 5-adic unit. At an index prime $p > 20$, Proposition 6.1 makes every orbit mass a p-adic unit as well. When several orbits occur, there is no unique least valuation at either of these primes.

At $p = 31$, depth 2 already resolves all sixteen J-orbits. Eight masses are positive and eight negative. Four have the minimum 2-adic valuation 3. At depth 1, two sector masses tie at that minimum. The unique-minimum test therefore fails at every positive depth in this example, although

$$\tau(31) = -52843168, \qquad v_2(\tau(31)) = 5.$$

Further residue information about tied sums may be useful. The counterexample excludes the proposed automatic criterion, rather than every possible valuation argument.

12.4 A prime-index interpolation that cannot be integral

An earlier proposed shortcut treated

$$F(u) = \frac{\tau(2^{14}u - 1)}{2^{14}}$$

at prime arguments as the restriction of a single integral 2-adic power series. Any convergent integral restricted power series is 1-Lipschitz: subtracting its values at x, y factors by $x - y$ with an integral convergent quotient.

Take the prime indices

$$(u_1, p_1) = (15, 245759), \qquad (u_2, p_2) = (47, 770047).$$

The exact values are

$$\tau(245759) = 77900595298037436891536424960,$$
$$\tau(770047) = -37987261739315675365604729538 1504.$$

Their residues modulo 2^{19} are 393216 and 147456. Dividing by 2^{14} gives

$$F(15) \equiv 24 \pmod{32}, \qquad F(47) \equiv 9 \pmod{32}.$$

Thus $v_2(F(47) - F(15)) = 0$, while $v_2(47 - 15) = 5$. This violates the necessary Lipschitz condition and refutes the stated integral interpolation model.

Different valuations alone would not disprove analyticity. The obstruction is this divided-difference failure. The example does not exclude models with denominators, finer residue classes or additional variables. It also does not concern the legitimate auxiliary polynomial $G_n(U)$ at a fixed shell. The two constructions use different variables and different hypotheses.

12.5 Exact lifting equalities also need proof

Another proposed shortcut was

$$v_\ell(\tau(p)) = v_\ell(p+1)$$

for candidate primes $p \neq \ell$ with $p \equiv -1 \pmod{\ell}$, at $\ell = 2, 3, 5$. All three equalities fail. Exhaustive search through 150000 gives the first failures within these respective candidate domains:

ℓ	p	$v_\ell(\tau(p))$	$v_\ell(p+1)$
2	81919	20	14
3	8747	8	7
5	499	4	3

At $p = 1249$, the 5-adic discrepancy goes the other way: the valuations are 3 and 4. The decisive integers were independently checked through the Eisenstein convolution. These are counterexamples to the auxiliary valuation equalities, never to Lehmer. They show that higher cancellation cannot be read off from $p + 1$ by the stated formula.

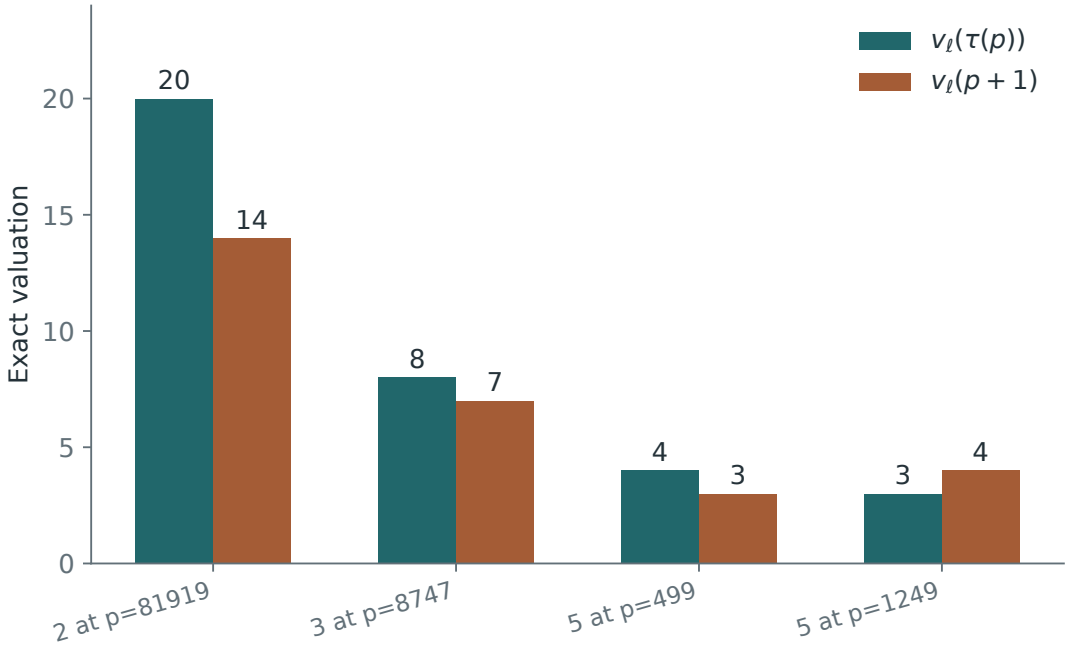


Figure 12.1: The first failures of the three proposed valuation equalities in their stated candidate domains, found by exhaustive prime search through 150000. All decisive values were independently checked. The equalities fail, while Lehmer is not refuted.

CHAPTER 13

Primitive content and conserved arithmetic sectors

13.1 The coordinate gcd is a real invariant

For $v \in \mathcal{V}_n$, let $g(v)$ be the positive gcd of its five coordinates. Since the coordinates occupy all residues modulo 5, one has $5 \nmid g(v)$. Call $g(v)$ the content of the packet. A primitive packet has content 1.

The content is compatible with the shell in a precise way. If $g = g(v)$, define

$$u_j = \frac{v_{gj \bmod 5}}{g}, \qquad j = 0, \dots, 4. \tag{13.1}$$

The coordinate permutation restores the standard residues after division. The vector u has sum zero and gcd 1. Its norm squared is $10n/g^2$. For every integral standard-residue vector of sum zero, the norm is divisible by 5, because its squared residues sum to 30, and by 2, because squares and coordinates agree modulo 2. Thus $10n/g^2$ is 10 times an integer, and

$$g^2 \mid n.$$

The map is a bijection from exact-content-g packets on shell n to primitive packets on shell n/g^2.

The permutation $j \mapsto gj \bmod 5$ has sign $\chi_5(g)$. For the four units modulo 5, multiplication by 1 is the identity, multiplication by 2 or 3 is a four-cycle on the nonzero residues, and multiplication by 4 is two transpositions. Scaling the degree-10 Vandermonde therefore gives

$$W(v) = \chi_5(g) g^{10} W(u). \tag{13.2}$$

The character is not an arbitrary normalization. It records the sign of the residue-restoring permutation.

13.2 The primitive residual and its inverse

Define

$$R(n) = \sum_{\substack{v \in \mathcal{V}_n \\ g(v)=1}} W(v). \tag{13.3}$$

Partitioning the packets by content and using (13.2) yields

$$\tau(n) = \sum_{g^2 \mid n} \chi_5(g) g^{10} R(n/g^2). \tag{13.4}$$

Terms with $5 \mid g$ have character zero and correspond to no actual content sector. They may be included in the formula without adding contributions.

The square-supported arithmetic kernel in (13.4) has inverse

$$d^2 \longmapsto \mu(d)\chi_5(d)d^{10}.$$

Indeed, at a square index m^2, the convolution of the two kernels is

$$\chi_5(m)m^{10}\sum_{d\mid m}\mu(d),$$

which is 1 at $m = 1$ and 0 otherwise. At nonsquare indices both kernels have no support. Consequently

$$\boxed{R(n) = \sum_{d^2\mid n}\mu(d)\chi_5(d)d^{10}\tau(n/d^2).} \tag{13.5}$$

Both formulas are exact for every positive integer. They are not fitted from the first few shells.

At $n = 1$, $R(1) = 1$. At a prime, only $d = 1$ occurs in (13.5), giving

$$R(p) = \tau(p). \tag{13.6}$$

The primitive extraction has no radial contribution to remove at a prime. This is the simplest reason it cannot, by itself, settle the prime nonvanishing obligation.

13.3 Homogeneous universal nulls preserve each content

Theorem 13.1. *Let Q be a symmetric homogeneous polynomial of degree d that is universally null. Then its weighted sum is zero separately in every exact-content sector on every shell.*

Proof. Let

$$R_Q(n) = \sum_{\substack{v\in\mathcal{V}_n\\ g(v)=1}} W(v)Q(v).$$

Symmetry removes the residue permutation from Q, and homogeneity gives $Q(v) = g^dQ(u)$. The scaling bijection yields

$$\sum_{v\in\mathcal{V}_n} W(v)Q(v) = \sum_{g^2\mid n}\chi_5(g)g^{10+d}R_Q(n/g^2). \tag{13.7}$$

The left side is zero. At $n = 1$, only $g = 1$ contributes, so $R_Q(1) = 0$. Inductively, the $g = 1$ term is $R_Q(n)$ and every nonzero term with $g > 1$ has smaller argument. Thus $R_Q(n) = 0$ for every n. The contribution of exact content g is its scaling factor times $R_Q(n/g^2)$, so it is zero as well. □

Finite linear combinations of homogeneous universal nulls inherit this conservation law even when their degrees differ. Chapter 17 will prove the stronger result for every finite symmetric polynomial that is universally null, without initially requiring a decomposition into known homogeneous nulls.

Parity sectors failed to conserve general null transfers. Content sectors do conserve them. This is the distinction between a geometric partition chosen for convenience and an arithmetic decomposition forced by a scaling identity.

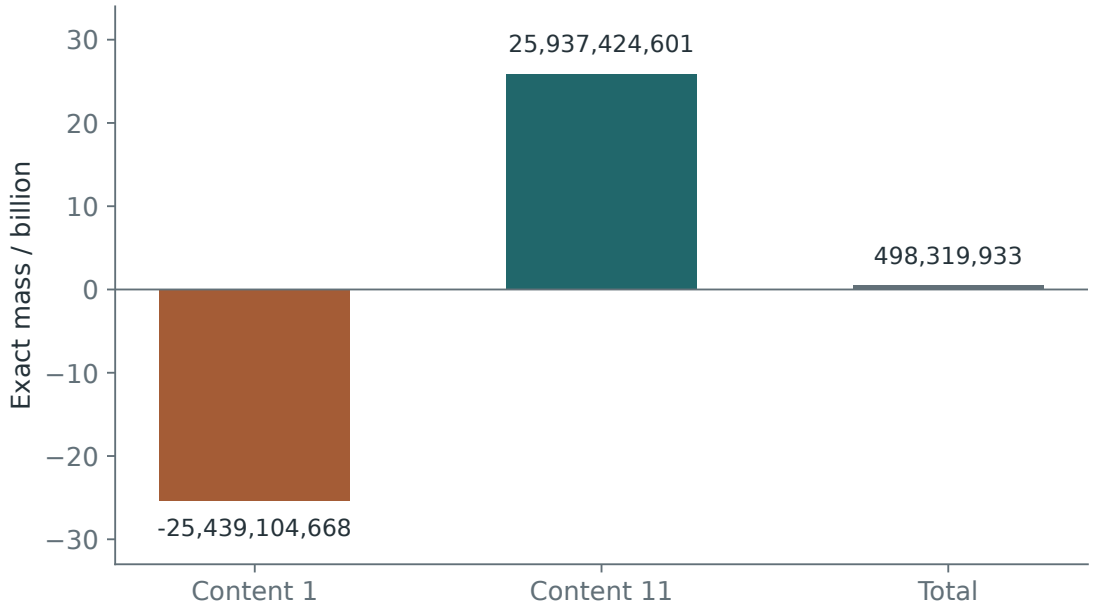


Figure 13.1: The two exact-content masses at 121 have opposite signs. Their sum survives, but every permitted finite symmetric polynomial universal null conserves the two masses separately. A common-sign correction at this composite shell is therefore impossible in that class.

13.4 A composite obstruction with opposite conserved masses

At $n = 121$, the contents are 1 and 11. The exact masses are

$$R(121) = -25439104668, \qquad 11^{10}R(1) = 25937424601.$$

Their sum is

$$\tau(121) = 498319933.$$

One can verify these values independently from $\tau(11) = 534612$:

$$\tau(121) = \tau(11)^2 - 11^{11}, \qquad R(121) = \tau(121) - 11^{10}.$$

The two content masses have opposite nonzero signs. Every retained homogeneous universal-null correction preserves them separately. Hence no correction in that entire permitted span can make all real corrected packet weights have one common sign at 121. Increasing the degree cannot remove this obstruction.

The obstruction is composite-specific. A prime shell has content 1 only. There the conservation law adds no invariant beyond the total. It must not be promoted into an impossibility theorem for prime-specific sign constructions.

13.5 A rank bound from content

Suppose a shell has N J-orbits occupying c nonempty content classes. A transfer vector produced by a homogeneous universal null has zero sum in each class. That subspace has dimension $N - c$, so every such transfer matrix has rank at most $N - c$.

The constructed family through degree 28 attains this bound at selected composite shells:

n	J-orbits	Content classes	Rank
9	4	$1, 3$	2
49	22	$1, 7$	20
81	31	$1, 3, 9$	28

These are exact finite ranks. They illustrate the conserved split and the effectiveness of the constructed transfers. They do not prove saturation of the content-preserving space at every shell.

CHAPTER 14

Refinement and exact transfer spaces

14.1 Mass vectors and aggregation

Fix a shell, a finite rational family $Q_1, \dots, Q_h$ of universally null corrections, and the occupied sectors at depth k. Let their number be s_k. Define

$$t_{k,r} = \sum_{v \in S_{k,r}} W(v), \qquad (M_k)_{rj} = \sum_{v \in S_{k,r}} W(v) Q_j(v).$$

The vector t_k contains sector masses. The columns of M_k are the effects of the permitted corrections. Coefficients $a \in \mathbb{Q}^h$ change the mass vector to $t_k + M_k a$.

Every depth-$k+1$ sector belongs to exactly one parent at depth k. Let A be the corresponding 0–1 aggregation matrix. Summing child contributions gives the exact identities

$$t_k = A t_{k+1}, \qquad M_k = A M_{k+1}. \tag{14.1}$$

Every occupied parent has a child. Each correction column has global total zero.

Put

$$H_k = \ker(\mathbf{1}^T : \mathbb{Q}^{s_k} \to \mathbb{Q}), \qquad T_k = \operatorname{im} M_k, \qquad E_k = H_k / T_k.$$

The quotient E_k is the vector space of total-preserving changes that are unavailable from the chosen family. It is not a count of failed sign certificates. A sign certificate may exist even when E_k is nonzero.

14.2 The exact refinement sequence

Theorem 14.1. *There is a short exact sequence*

$$0 \longrightarrow \frac{\ker A}{M_{k+1}(\ker M_k)} \longrightarrow E_{k+1} \longrightarrow E_k \longrightarrow 0. \tag{14.2}$$

Consequently, if $r_k = \operatorname{rank} M_k$,

$$\dim E_{k+1} - \dim E_k = (s_{k+1} - s_k) - (r_{k+1} - r_k). \tag{14.3}$$

Proof. The aggregation map sends H_{k+1} onto H_k. Given a zero-total parent vector, place each parent entry in one selected child and zero in its other children. The resulting child vector has zero total and maps to the required vector.

Equation (14.1) gives $A(T_{k+1}) = T_k$, so aggregation induces a surjective map between the quotients. Its kernel is the quotient of $\ker A$ by $T_{k+1} \cap \ker A$. A transfer $M_{k+1}a$ lies in $\ker A$ exactly when

$$A M_{k+1} a = M_k a = 0.$$

Thus $T_{k+1} \cap \ker A = M_{k+1}(\ker M_k)$, which proves (14.2). Finally $\dim E_k = s_k - 1 - r_k$, giving (14.3). □

This is ordinary exact linear algebra applied to the packet data. Refinement introduces $s_{k+1} - s_k$ new detail directions. Effective new column rank resolves $r_{k+1} - r_k$ of them. The formula says exactly how much of the new geometric detail remains unavailable.

14.3 Localization by cancellation, rather than indicators

Let the columns of a rational matrix N form a basis of $\ker M_k$. Then

$$B_k = M_{k+1}N, \qquad AB_k = 0. \tag{14.4}$$

The columns of B_k transfer mass only within the parents: their effects on every parent cancel. They are still combinations of universal null polynomials. This is the valid repair of the failed sector-indicator proposal.

There is a cost. The finer effects must be computed and solved for, and their span may be smaller than all child changes preserving parent masses. Equation (14.2) identifies that remaining detail defect. One does not obtain local freedom simply by declaring that a global null polynomial is available in each sector separately.

14.4 The exact calculation at 31

The shell at 31 has six coarse occupied sectors and sixteen terminal J-orbit sectors. The constructed ranks are

Degree cutoff	Coarse rank	Orbit rank	Orbit defect	Detail rank
14	5	5	10	0
20	5	14	1	9
22	5	15	0	10

The zero-total coarse space has dimension 5, and the zero-total orbit space has dimension 15. At degree 22, the available columns fill both. The detail space has dimension $15 - 5 = 10$.

The stored two-stage certificate first eliminates five coarse masses, using only five elimination equations. It then uses coefficient vectors in $\ker M_1$ to eliminate all but one fine mass while preserving every coarse total. The survivor is

$$-52843168.$$

The product engine independently identifies this with $\tau(31)$. The solve does not require a preassigned tau value in its elimination equations. The final survivor remains an exact finite calculation, not a formula that independently forces a nonzero survivor at every prime.

The degree-20 family has an orbit-transfer defect of dimension 1, but it already supplied a strictly negative sign certificate at 31. Filling every zero-total direction is stronger than needed for that finite sign task. Conversely, filling them at degree 22 does not by itself prove a nonzero total. These two facts prevent rank from being misinterpreted as a nonvanishing test.

14.5 Other finite saturation profiles

Full zero-total orbit rank is obtained in the retained family at

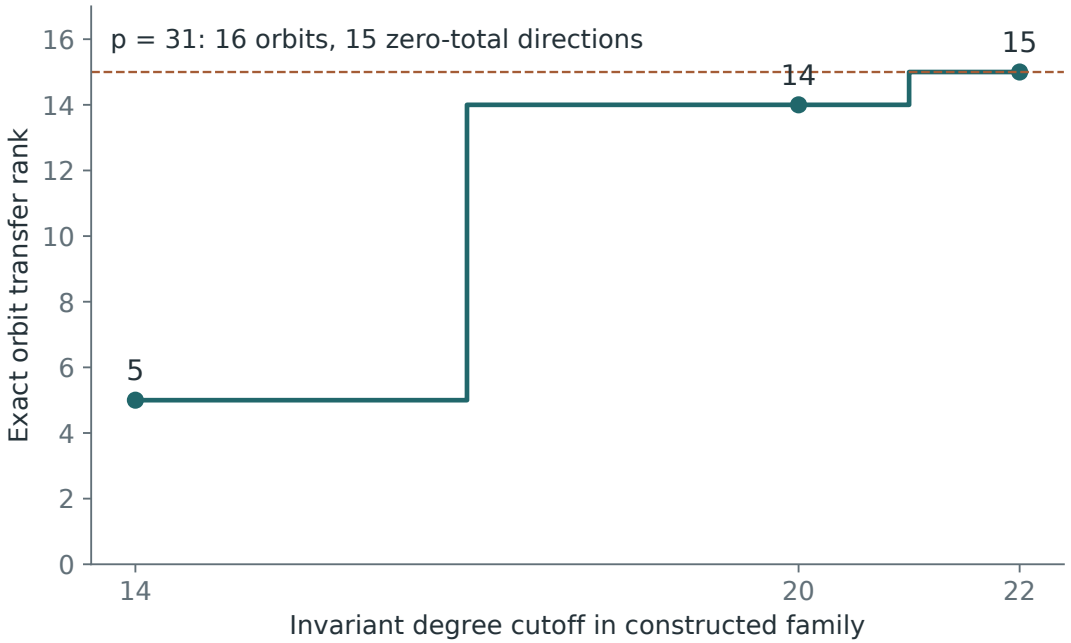


Figure 14.1: The exact ranks at 31. The coarse rank is already full at degree 14, while the terminal orbit rank reaches 15 at degree 22. A degree-20 sign certificate exists despite its one-dimensional transfer defect. Rank and nonvanishing are different statements.

Prime	J-orbits	Degree cutoff
31	16	22
37	19	22
41	22	24
59	30	26
71	36	28

These cutoffs are not asserted optimal among all polynomials. The first five corrections through degree 14 also had full coarse zero-total rank at all 94 odd primes below 500. Among the 250 odd indices below 500, the only coarse failures were 9 and 81. The content split explains genuine conserved directions in those composite examples.

The fresh transport replay verifies 144 refinement ranks and the complete rational certificate at 31. Its results authenticate these finite matrices. A uniform theorem about every prime, every degree requirement or every content-preserving transfer is not obtained by these rank profiles.

14.6 What further completeness would mean

A plausible geometric question is whether all-degree universal nulls span every content-preserving transfer at each shell. Chapter 18 explains why ordinary polynomial interpolation does not settle it. Even if such completeness were proved, the conserved total at a prime would remain $\tau(p)$.

The useful implication of completeness would be that every compatible redistribution is expressible by legitimate corrections. The missing arithmetic implication would still be that the compatible total is nonzero. Treating these as separate statements prevents a successful linear-algebra calculation from silently doing the work of the unsolved conjecture.

CHAPTER 15

Every primitive square survives

15.1 The local numerator introduced by content

The transform in (13.5) is a Dirichlet convolution of multiplicative functions, so R is multiplicative. At a prime p, write

$$a = \tau(p), \qquad b = p^{11}, \qquad c = \chi_5(p)p^{10}.$$

The local generating factor is

$$\boxed{\sum_{r\geq 0} R(p^r)X^r = \frac{1 - cX^2}{1 - aX + bX^2}.} \tag{15.1}$$

It follows either by multiplying the Hecke factor by the square-supported Möbius factor, or by reading the two possible square divisors with nonzero Möbius value at a prime power. In particular,

$$R(1) = 1, \qquad R(p) = a, \qquad R(p^r) = T_r - cT_{r-2} \quad (r \geq 2).$$

The subtraction begins at $r = 2$. No undefined T_{-1} or T_{-2} is needed.

The even primitive blocks are not the same as the even tau blocks. For example,

$$R(p^2) = a^2 - p^{10}(p + \chi_5(p)).$$

An argument that $\tau(p^{2m}) \neq 0$ does not alone imply the corresponding difference is nonzero. We now prove the primitive statement directly.

Theorem 15.1. *For every prime p and every integer $m \geq 1$,*

$$R(p^{2m}) \neq 0.$$

Consequently $R(s^2) \neq 0$ for every positive integer s.

The proof is stronger than the actual-tau specialization in a useful way. For $p \neq 5$, it works for every integer recurrence parameter a with $a \equiv p + \chi_5(p) \pmod 5$. At $p = 5$, it works for every integer a. It therefore includes a hypothetical parameter $a = 0$ whenever that parameter is compatible with the congruence. The square theorem has not secretly assumed prime nonvanishing.

15.2 A residue eliminates three prime classes

Suppose $p \neq 5$ and put $\chi = \chi_5(p)$. Modulo 5, $\chi \equiv p^2$, so the two recurrence roots may be taken as p and χ:

$$p + \chi \equiv a, \qquad p\chi \equiv p^{11} \pmod 5.$$

The complete homogeneous recurrence coefficient is

$$T_r \equiv C_r = \sum_{i=0}^{r} p^{r-i}\chi^i \pmod 5.$$

Also $\chi p^{10} \equiv 1 \pmod 5$. Since $\chi^2 = 1$, for $r \geq 2$ the sum telescopes:

$$R(p^r) \equiv C_r - C_{r-2} = p^{r-1}(p+\chi) \pmod 5. \tag{15.2}$$

The same expression holds at $r = 1$. It is nonzero for $p \equiv 1, 2, 3 \pmod 5$. It proves nonvanishing for all positive exponents in those prime classes. Only $p = 5$ and $p \equiv 4 \pmod 5$ remain.

15.3 The recurrence polynomial and untied valuations

For $p \equiv 4 \pmod 5$, $\chi = 1$ and $a \equiv 0 \pmod 5$. The exact recurrence expansion is

$$T_r = \sum_{j=0}^{\lfloor r/2 \rfloor} (-1)^j \binom{r-j}{j} b^j a^{r-2j}. \tag{15.3}$$

It follows by induction: multiplying the previous row by a and subtracting b times the row before it combines neighboring binomial coefficients. Equivalently, expand the rational generating factor formally. The extreme coefficients are 1, so a strictly least valuation at either end cannot be raised by a coefficient divisible by p.

First suppose $a \neq 0$, and write $t = v_p(a)$. In T_{2m}, the term indexed by j has valuation at least

$$2mt + j(11 - 2t).$$

If $t \leq 4$, the $j = 0$ term a^{2m} has strictly least valuation $2mt$. Likewise T_{2m-2} has valuation $(2m-2)t$. The correction $p^{10}T_{2m-2}$ has valuation

$$2mt + (10 - 2t) > 2mt.$$

Thus

$$v_p(R(p^{2m})) = 2mt, \qquad t \leq 4. \tag{15.4}$$

If $t \geq 6$, the constant-in-a term $(-b)^m$ is uniquely least-valued in T_{2m}, with valuation $11m$. The analogous term gives $v_p(T_{2m-2}) = 11(m-1)$. The correction has valuation $11m - 1$, now smaller than the first block. Hence

$$v_p(R(p^{2m})) = 11m - 1, \qquad t \geq 6. \tag{15.5}$$

Actual nonzero tau parameters at $p \geq 5$ cannot have $t \geq 6$ by Deligne's bound. The wider argument is nonetheless a useful exact control for the recurrence class.

If $a = 0$, do not assign it a finite t. Direct substitution gives

$$R(p^{2m}) = (-1)^m p^{11m-1}(p+1), \tag{15.6}$$

which is nonzero. This branch is indispensable: a theorem purporting to survive a hypothetical prime zero must actually evaluate that case.

15.4 The tied valuation $t = 5$

The preceding endpoint arguments leave $t = 5$. Write $a = p^5 u$ with $p \nmid u$, and set

$$H_r(u) = p^{-5r} T_r.$$

Equation (15.3) shows that H_r is an integral polynomial. Its recurrence is

$$H_0 = 1, \qquad H_1 = u, \qquad H_r = uH_{r-1} - pH_{r-2}.$$

The primitive block becomes

$$R(p^{2m}) = p^{10m}\big(H_{2m}(u) - H_{2m-2}(u)\big). \tag{15.7}$$

The expression in parentheses is a monic polynomial $F_m(x)$ in $x = u^2$. Its constant term and reduction modulo p are

$$F_m(0) = (-1)^m p^{m-1}(p+1), \qquad F_m(x) \equiv x^{m-1}(x-1) \pmod{p}. \tag{15.8}$$

The leading term comes from u^{2m}. The constant term comes from $H_{2m}(0) = (-p)^m$ and $H_{2m-2}(0) = (-p)^{m-1}$.

Suppose $R(p^{2m}) = 0$. Then $x = u^2$ is a positive integer root of the monic integer polynomial F_m. The integer-root theorem gives $x \mid p^{m-1}(p+1)$. Because $p \nmid u$, one obtains $x \mid p+1$. Reduction in (15.8) gives $x \equiv 1 \pmod{p}$. Since $1 \le x \le p+1$, the only possibilities are

$$u^2 = 1 \quad \text{or} \quad u^2 = p+1.$$

The first is impossible: a is divisible by 5 and p is not, so u is divisible by 5. The second gives

$$p = (|u|-1)(|u|+1).$$

For a prime this forces $|u| = 2$ and $p = 3$, outside the family $p \equiv 4 \pmod{5}$. Both possibilities are excluded. This closes the tied case for every exponent m, rather than for a tested list of exponents.

For $m = 1$, the polynomial is simply $F_1(x) = x - (p+1)$. For larger m, its degree grows, but the same constant-term and residue argument controls any integral square root. That is why no bound on m is needed.

15.5 The prime 5 and multiplication over squares

At $p = 5$, $\chi_5(5) = 0$, so $R(5^r) = T_r$. If $a = 0$, its even blocks are $(-5^{11})^m$. If $a \neq 0$, then $2v_5(a)$ is even and cannot equal 11. In the expansion (15.3), either the first term or the constant-in-a term has a unique least valuation at an even exponent. Thus no even block vanishes.

All prime cases have now been covered. If $s = \prod p^{m_p}$, multiplicativity gives

$$R(s^2) = \prod_p R(p^{2m_p}).$$

Every factor is nonzero, and $R(1) = 1$. This proves Theorem 15.1.

The theorem gives survival, not a common sign. For example,

$$R(4) = -448, \qquad R(9) = -54594, \qquad R(25) = -25499225,$$

while other square values can have different signs. Nor does it furnish a fixed auxiliary-prime unit: a nonzero integer can be divisible by arbitrarily high preassigned moduli at suitable varying primes. Chapter 19 makes that last observation into an actual-prime theorem.

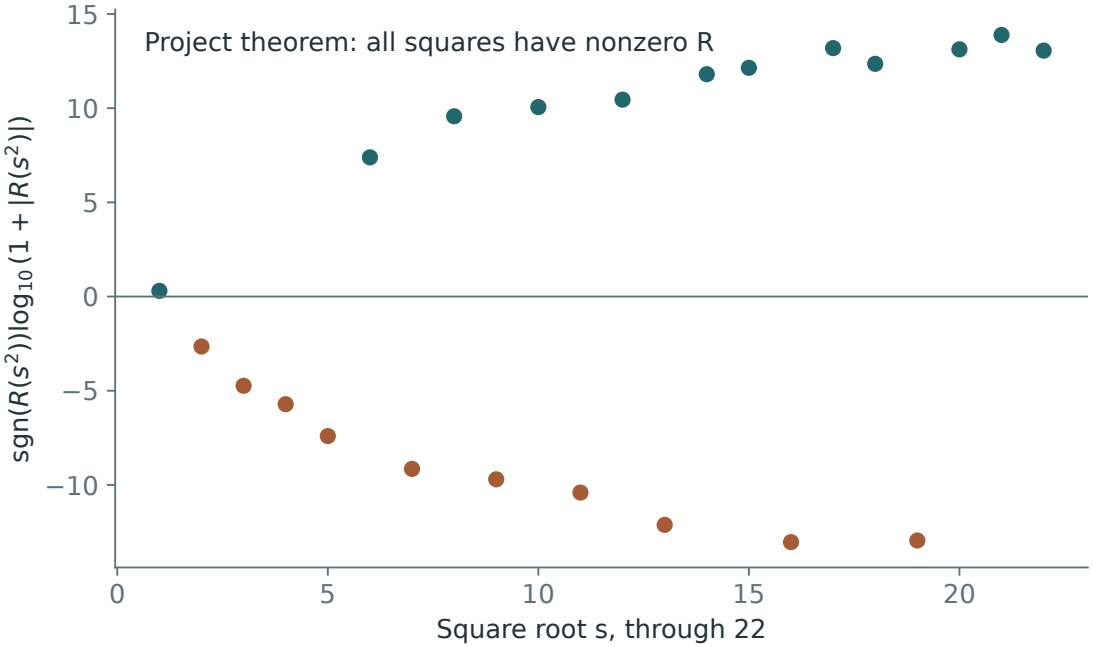


Figure 15.1: Exact primitive residuals at square indices s^2 for $1 \leq s \leq 22$, displayed on a signed logarithmic scale. Their survival is a consequence of the universal theorem, whose proof also allows a compatible zero prime parameter. The graph illustrates the theorem rather than proves it.

CHAPTER 16

Odd primitive blocks retain exactly the prime obstruction

16.1 Polynomial division before evaluation

The odd Hecke polynomial T_{2m+1} is divisible by a. For $m \geq 1$, so is

$$T_{2m+1} - cT_{2m-1}.$$

Define its quotient as an integer polynomial in a^2 before substituting the actual value $a = \tau(p)$. This matters at a hypothetical zero: the quotient exists algebraically even when numerical division by a would be undefined.

Theorem 16.1. *For every odd prime $p \neq 5$ and every $m \geq 0$, an odd integer $U_{p,m}$ satisfies*

$$R(p^{2m+1}) = \tau(p)U_{p,m}.$$

The multiplier is defined by the recurrence polynomial, including when $\tau(p) = 0$. If $\tau(p) \neq 0$, then

$$v_2(R(p^{2m+1})) = v_2(\tau(p)).$$

Proof. For $m = 0$, take $U_{p,0} = 1$. For $m \geq 1$, use the quotient

$$U(a^2) = \frac{T_{2m+1} - cT_{2m-1}}{a}$$

as polynomial division. The constant term in a^2 comes from the linear-in-a terms in the two blocks. By (15.3), it is

$$(-1)^m b^{m-1}\big((m+1)b + mc\big). \tag{16.1}$$

For an odd prime $p \neq 5$, both $b = p^{11}$ and $c = \chi_5(p)p^{10}$ are odd. The bracket in (16.1) is congruent to $(m+1) + m = 2m + 1$ modulo 2. Thus the constant term is odd.

The actual $a = \tau(p)$ is even by (2.7). Every nonconstant term in the polynomial in a^2 is even after evaluation. Hence the evaluated quotient is odd. It cannot be zero. Multiplication by a gives the theorem, and multiplying a nonzero integer by an odd integer preserves its 2-adic valuation. □

This proof uses parity rather than a delicate size estimate for the quotient. It also exhibits the zero parameter honestly: if $a = 0$, every odd primitive block is zero and every even primitive block remains nonzero by Chapter 15.

16.2 The primes 2 and 5

At $p = 2$, the residue formula (15.2) gives

$$R(2^r) \equiv 2^{r-1}(2-1) \not\equiv 0 \pmod 5$$

for every $r \geq 1$. Thus no local block at 2 vanishes.

At $p = 5$, the character is zero and $R(5^r) = \tau(5^r)$. The actual parameter is $\tau(5) = 4830$, with $v_5(a) = 1$. In (15.3), the $j = 0$ term has valuation r and every other term has valuation at least $r + 9j$. Hence

$$v_5(R(5^r)) = r$$

for every positive exponent, including odd ones.

16.3 The exact zero-equivalence theorem

Theorem 16.2. *For every positive integer n,*

$$R(n) = 0 \quad \Longleftrightarrow \quad \tau(n) = 0.$$

More precisely, either side is zero exactly when some prime occurring to an odd exponent in n has $\tau(p) = 0$.

Proof. The primitive transform is multiplicative. Its even prime-power blocks are nonzero by Theorem 15.1. Its odd blocks at odd primes other than 5 are the prime parameter times an odd, hence nonzero, integer by Theorem 16.1. The exceptional local blocks at 2 and 5 are nonzero by the preceding calculations. Thus its product over the prime factorization of n has precisely the claimed zero criterion. Theorem 3.1 gives the same criterion for $\tau(n)$. □

Primitive extraction introduces no new zero. It also removes no zero of the original coefficient sequence. This is stronger and more informative than agreement at a finite list of shells. It describes exactly how the two arithmetic functions encode the same conjecture.

16.4 The theorem's gain and its limit

The primitive square theorem and the odd multiplier theorem classify the new local factors. They show that the extraction does not add an accidental vanishing caused by the numerator $1 - \chi_5(p)p^{10}X^2$. They also give a precise dyadic valuation law at every odd nonzero block.

The remaining obstruction is concentrated in the first prime parameter. For $p \equiv 4 \pmod 5$, the hypotheses used in the even theorem permit $a = 0$. The resulting blocks are

$$\begin{aligned} R(p^{2m}) &= (-1)^m p^{11m-1}(p+1), && m \geq 1, \\ R(p^{2m+1}) &= 0, && m \geq 0. \end{aligned}$$

The constant block is $R(1) = 1$. This is consistent with the congruence, parity and recurrence inputs. Therefore those inputs and the new square theorem cannot jointly establish $a \neq 0$. They have isolated its role.

At a prime shell there is no imprimitive part, so the universal target is still

$$\sum_{\substack{v \in \mathcal{V}_p \\ g(v)=1}} W(v) = R(p) = \tau(p) \neq 0.$$

The content geometry has clarified the radial arithmetic. The prime total still needs a global or uniform arithmetic mechanism excluding cancellation to zero.

CHAPTER 17

All finite symmetric polynomial nulls conserve content

17.1 Removing the homogeneity restriction

The earlier content proof used $Q(gv) = g^d Q(v)$. A finite nonhomogeneous polynomial does not have one scaling degree. In principle, cancellation between its homogeneous pieces might make it universally null while allowing individual pieces to have nonzero sums. The following theorem excludes that escape.

Theorem 17.1. *Let Q be any finite symmetric polynomial over $\mathbb{C}$, restricted to $\sum_i v_i = 0$. Suppose*

$$\sum_{v \in \mathcal{V}_n} W(v) Q(v) = 0$$

for every positive integer n. Write $Q = \sum_d Q_d$ in homogeneous degrees. Then every Q_d is separately universally null. Moreover, the weighted Q-sum is zero in each exact-content sector of every shell.

The finite-polynomial hypothesis is substantive. The argument below separates finitely many modular weights. It says nothing about an arbitrary infinite series, arbitrary functions on the packets or nonsymmetric corrections. For a sign consequence, corrected weights must be real, as they are for the retained rational corrections. Complex coefficients have no ordered sign.

17.2 Harmonic decomposition of an alternating polynomial

For a homogeneous Q_d, put $P = DQ_d$ and $M = 10 + d$. The standard decomposition into harmonic components on the rank-four Euclidean hyperplane is

$$P = \sum_{j \geq 0} s_2^j h_{M-2j}, \tag{17.1}$$

where h_ℓ is harmonic homogeneous of degree ℓ. It can be constructed recursively from the Laplacian. For a harmonic h_ℓ,

$$\Delta_H(s_2^j h_\ell) = 4j(\ell + j + 1) s_2^{j-1} h_\ell. \tag{17.2}$$

The nonzero coefficients in this formula give uniqueness of the decomposition. The operator commutes with coordinate permutations, so every h_ℓ is alternating when P is alternating. Hence D divides every nonzero h_ℓ, and $\ell \geq 10$. Thus only $j \leq d/2$ can occur.

When d is odd, the shell sum vanishes directly under J as in (8.2), separately inside each content class. We may therefore concentrate on even d, when all harmonic degrees in (17.1) are even and the scalar harmonic theta argument applies.

17.3 Radial factors become derivatives

For an alternating harmonic h_ℓ, write

$$\theta_h(\tau) = \sum_{\substack{v_i \equiv i \ (5) \\ \sum v_i = 0}} h(v) \exp\left(2\pi i \tau \frac{\|v\|^2}{10}\right).$$

Chapter 8 establishes that it is a scalar cusp form of weight $\ell + 2$. The normalization includes only a fixed scalar if one divides by 288.

Each application of $\mathcal{D} = q\,d/dq$ multiplies the shell coefficient by $\|v\|^2/10$. Therefore

$$\theta_{s_2^j h} = 10^j \mathcal{D}^j \theta_h. \tag{17.3}$$

Differentiating a modular transformation law shows that $\mathcal{D}^j f$, for modular weight w, is quasimodular of weight $w + 2j$ and depth at most j. Combining (17.1) and (17.3), the homogeneous shell series

$$F_d(\tau) = \sum_{n \geq 1} \left(\sum_{v \in \mathcal{V}_n} W(v) Q_d(v) \right) q^n$$

is quasimodular of weight

$$(M - 2j + 2) + 2j = 12 + d,$$

independent of the radial index j. Its depth is at most $d/2$. Polynomial growth and Gaussian decay justify the termwise differentiations.

The common weight for all terms within one homogeneous degree is the key. Different homogeneous degrees produce different weights. A nonhomogeneous null identity would have to make those different weights cancel, and the next step shows that they cannot.

17.4 Separating the quasimodular weights

We need only an elementary weight-separation statement. Suppose finitely many quasimodular functions F_k of distinct nonnegative integer weights satisfy

$$F_k(\gamma z) = \sum_{r=0}^{p_k} f_{k,r}(z) c^r (cz + d)^{k-r}, \qquad f_{k,0} = F_k, \qquad p_k \leq k,$$

for $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in SL_2(\mathbb{Z})$. Constants from the usual quasimodular convention can be absorbed into the coefficient functions. Assume $\sum_k F_k = 0$ identically.

Fix z in the upper half-plane. The polynomial

$$P_z(c, d) = \sum_k \sum_r f_{k,r}(z) c^r (cz + d)^{k-r} \tag{17.4}$$

vanishes at every relatively prime integer pair (c, d), because every such pair is the bottom row of a modular matrix. These pairs are Zariski dense. To see this without a geometric theorem, fix any positive prime c. Infinitely many integers d are coprime to it, so the polynomial in d vanishes identically. Each of its coefficient polynomials in c then vanishes at infinitely many primes, so is identically zero. Hence P_z is the zero polynomial.

Every summand belonging to weight k is homogeneous of total degree k in (c, d). Different homogeneous components of a zero polynomial vanish separately. Evaluate the degree-k

component at $(c, d) = (0, 1)$. Only $r = 0$ survives, giving $F_k(z) = 0$. Since z was arbitrary, every F_k is zero.

Apply this to the finite even homogeneous degrees with weights $12 + d$. Universal nullness of Q says their sum is zero, after the odd pieces have already vanished. Thus every F_d is zero. Equality of Fourier coefficients proves that every homogeneous Q_d is universally null.

17.5 Finishing the content proof

Each homogeneous piece now satisfies Theorem 13.1. Its exact-content contribution is zero on every shell. Summing the finitely many pieces proves the full nonhomogeneous claim.

The argument has two independent mechanisms. Modular transformation laws prevent different homogeneous degrees from canceling globally. The arithmetic scaling identity prevents a homogeneous null from moving mass between exact contents. Both are needed. Merely applying the homogeneous scaling formula to a nonhomogeneous Q would skip the first mechanism.

17.6 The strengthened composite boundary

At $n = 121$, the opposed content masses remain conserved under every finite symmetric polynomial universal-null correction. This is now true without a degree cutoff, a prescribed finite family or an initial homogeneity assumption. Any real corrected weights of one common sign would force both content totals to have that sign, contradicting the exact values.

The theorem closes one possible escape from the content invariant. It does not constrain all nonpolynomial operations, and it does not produce a new prime invariant. At a prime there is still one content class, and its total is still $\tau(p)$.

CHAPTER 18

Interpolation is not universal nullness

18.1 Coordinates that identify exactly the orbits

Fix a nonempty shell and put $C = 10n$. Consider the normalized even symmetric invariants

$$I(v) = \left(\frac{s_4}{C^2}, \frac{s_3^2}{C^3}, \frac{s_3 s_5}{C^4}, \frac{s_5^2}{C^5} \right). \tag{18.1}$$

They are unchanged by J.

Proposition 18.1. *On a fixed shell, $I(v) = I(w)$ if and only if $w = v$ or $w = Jv$.*

Proof. Equality gives equal s_4 and equal rank-one outer products of the real vectors $(s_3(v), s_5(v))$ and $(s_3(w), s_5(w))$. Those vectors agree up to a common sign, including the zero case. Thus the first five power sums of v and either w or $-w$ agree: $s_1 = 0$, $s_2 = C$ and the remaining three by the preceding comparison.

Newton's identities recover the elementary symmetric polynomials from those power sums. The coordinate multisets therefore agree. Each multiset has exactly one coordinate in each residue class modulo 5, so equal multisets imply $v = w$. Negated multisets imply $w_i = -v_{-i \bmod 5} = J(v)_i$. The converse follows from the even invariance of all four coordinates in (18.1). □

This removes a potential geometric explanation for an evaluation defect. If an all-degree universal-null family has an additional annihilator on a shell, it cannot arise from two distinct J-orbits secretly having the same even symmetric coordinates. It must reflect a further arithmetic or modular restriction.

18.2 An explicit homogeneous interpolation basis

Let $v_1, \dots, v_N$ represent the distinct J-orbits. For an integer $t \geq 0$, define the degree-10 symmetric polynomial

$$E_t = s_4 s_2^3 + t\, s_3^2 s_2^2 + t^2 s_3 s_5 s_2 + t^3 s_5^2. \tag{18.2}$$

For two distinct orbits, $E_t(v_i) - E_t(v_j)$ is a nonzero polynomial in t of degree at most 3, by Proposition 18.1. It has at most three roots. Among all pairs there are at most $3N(N-1)/2$ forbidden values. Hence some integer

$$0 \leq t \leq \frac{3N(N-1)}{2}$$

separates all $e_i = E_t(v_i)$.

For that t, set

$$L_i(v) = \prod_{j \neq i} \frac{E_t(v) - e_j (s_2(v)/C)^5}{e_i - e_j}. \tag{18.3}$$

Each factor is homogeneous of degree 10, and the denominators are nonzero rational numbers. The resulting polynomial is symmetric homogeneous of degree $10(N-1)$. On the shell, $s_2/C = 1$, so

$$L_i(v_j) = \delta_{ij}.$$

Consequently, any prescribed rational orbit values $a_1, \dots, a_N$ are interpolated by $Q = \sum_i a_i L_i$. If $N = 1$, an ordinary constant suffices.

This is a constructive interpolation theorem, with a degree bound and explicit basis. It imposes neither harmonicity nor universal nullness.

18.3 A shell-null interpolant can fail on the first shell

Let m_i be the nonzero mass of the ith orbit. Interpolation makes the map $Q \mapsto (m_i Q(v_i))$ onto $\mathbb{Q}^N$. Thus any zero-total transfer on this one shell can be produced by a homogeneous polynomial.

The condition that its weighted sum vanish on this shell is weaker than requiring it to vanish on every shell. The natural interpolation that kills all but the first orbit at $n = 7$ has degree 20 and zero correction total at that target. Its corrected survivor is -16744. Yet its weighted correction sum on shell 1 is

$$-\frac{449}{42371287350} \neq 0.$$

The analogous degree-150 construction at 31 also fails already at shell 1. These exact failures occur at small shells where different, universal nulls are known to supply sign certificates.

Thus ordinary interpolation cannot be substituted for the modular kernel construction. Both use polynomials, but they solve different systems of conditions.

18.4 Harmonic projection does not preserve values

One might try to interpolate first and then take a harmonic projection. There is no general principle preserving the target-shell values under this operation. Since D is harmonic of degree 10 on a rank-four space,

$$\Delta_H(s_2 D) = 48D.$$

The degree-12 top harmonic component of $s_2 D$ is zero: the entire polynomial is a radial factor times a lower-degree harmonic polynomial. Yet $s_2 D$ is nonzero at every packet on a nonempty shell.

More generally, the top harmonic projection of $s_2^k D$ at degree $10 + 2k$ is zero for every $k > 0$. Radial information can disappear. Replacing an interpolant by its top harmonic piece may therefore destroy exactly the values it was designed to have.

Dimension growth does not repair this missing implication. Even if harmonic-polynomial dimensions grow quadratically while the number of modular coefficient constraints grows linearly, one must prove independence between the evaluation rows and the modular rows. A large kernel need not evaluate surjectively onto a prescribed finite subspace. The finite saturation profiles in Chapter 14 are useful examples, not that uniform independence theorem.

18.5 The strongest conceivable content fullness

Let the orbits be grouped by exact content and let T_g be the mass of each group. Suppose, as an unproved hypothesis, that all-degree universal null transfers span every vector with zero sum in each content class. Then a strictly positive corrected mass on every orbit exists exactly when every nonempty T_g is positive. Necessity is conservation. For sufficiency, distribute T_g equally among its orbits, then use fullness to realize the difference from the original mass vector.

The negative version is identical. For a weakly nonnegative certificate with at least one strict mass, the criterion is $T_g \geq 0$ for every class and $\sum_g T_g > 0$.

At a prime there is one class. Under even this maximal fullness hypothesis, the existence of a common-sign certificate is therefore equivalent to $\tau(p) \neq 0$. A concrete independently verified certificate can prove nonvanishing. The mere availability of every conserved redistribution cannot decide whether the conserved total vanishes.

The all-degree universal-null fullness theorem itself remains unproved in this program. Its proof would answer an interesting geometric and modular question. It would still leave an arithmetic sign or residue input to complete Lehmer.

CHAPTER 19

Every fixed congruence depth has actual prime survivors

19.1 A theorem about actual primes

Congruences can disprove a zero at a particular prime, but no fixed finite congruence depth excludes every prime. The following statement concerns the actual coefficients of Δ, not a synthetic recurrence.

Theorem 19.1. *For every integer $M \geq 1$, a set of rational primes of positive natural density satisfies*

$$p \equiv -1 \pmod{5M}, \qquad \tau(p) \equiv 0 \pmod{M}. \tag{19.1}$$

At every such prime, simultaneously for every $r \geq 1$,

$$R(p^r) \equiv 0 \pmod{M}. \tag{19.2}$$

Natural density here is relative to the rational primes. The proof exhibits a positive-density Chebotarev subset of the displayed set. The full set is itself a union of conjugacy classes in the same finite quotient, so also has positive density. No numerical density for the full set is calculated.

The theorem is a consequence of the classical Galois representations and Chebotarev, followed by a translation to the primitive local factor. It is not a new general density theorem. Its usefulness here is to delimit one proposed nonvanishing strategy exactly.

19.2 Integral representations and complex conjugation

For each prime ℓ, the representation attached to Δ has the arithmetic-Frobenius normalization

$$\rho_\ell : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{Q}_\ell),$$

is continuous and unramified outside ℓ, and satisfies

$$\det(X - \rho_\ell(\mathrm{Frob}_p)) = X^2 - \tau(p)X + p^{11}, \qquad p \neq \ell. \tag{19.3}$$

Its determinant is the eleventh power of the ℓ-adic cyclotomic character. The original cohomological formulation uses a dual convention for Frobenius. The trace and determinant contract in this normalization is explicitly checked in the primary sources [4, 6]. Both conventions agree at complex conjugation for the argument below.

A continuous representation of the compact group $G_{\mathbb{Q}}$ has a stable $\mathbb{Z}_\ell$ lattice. To see this directly, take an initial lattice L_0. Compactness bounds all its translates inside $\ell^{-a}L_0$ for some a. The module

$$L = \sum_{g \in G_{\mathbb{Q}}} \rho_\ell(g) L_0$$

is contained in $\ell^{-a}L_0$ and contains L_0. It is finitely generated, since a submodule of a finite free $\mathbb{Z}_\ell$ module is finitely generated, and it is invariant by construction. Reductions modulo ℓ^e are therefore defined. No image-surjectivity theorem is required.

Let c be complex conjugation. Then $c^2 = 1$ and the cyclotomic character sends c to -1. Consequently $\det \rho_\ell(c) = -1$. In characteristic zero an involution is diagonalizable, with eigenvalues among $1, -1$. The determinant forces one of each, so

$$\operatorname{Tr} \rho_\ell(c) = 0 \tag{19.4}$$

as an exact $\mathbb{Z}_\ell$ identity. This reasoning also covers $\ell = 2$: the trace is computed before reduction. Diagonalizing an involution in characteristic 2 would not justify the same step.

19.3 The actual joint image

Factor $M = \prod_{\ell^e \| M} \ell^e$ and form the single joint finite map

$$\Psi : G_{\mathbb{Q}} \longrightarrow \left(\prod_{\ell^e \| M} GL_2(\mathbb{Z}/\ell^e\mathbb{Z}) \right) \times (\mathbb{Z}/5M\mathbb{Z})^\times. \tag{19.5}$$

Its components are the integral representation reductions and the cyclotomic character modulo $5M$. Let G be its actual image. The kernel defines a finite Galois extension of $\mathbb{Q}$. The image need not be the whole displayed product. Relations between different representations and determinants are allowed.

Let C be the conjugacy class of $\Psi(c)$ in G. Chebotarev gives unramified primes whose Frobenius lies in C, with density $|C|/|G| > 0$ [17]. The trace in each representation component is zero by (19.4), and the cyclotomic component is -1. Thus every ℓ^e dividing M divides $\tau(p)$, while $p \equiv -1 \pmod{5M}$. This proves (19.1).

This use of the actual image is decisive. There is no assumption that one may prescribe unrelated Frobenius data independently in a product of groups. The required element is already present: it is the image of one actual complex conjugation. Chebotarev supplies prime Frobenius classes matching that element in the finite quotient.

For $M = 1$, the representation product is empty. The cyclotomic quotient modulo 5 suffices. Finitely many ramified primes are omitted in every case, which does not affect positive density.

19.4 All primitive exponents vanish modulo the fixed modulus

At a prime in (19.1), one has $\chi_5(p) = 1$, $p^{10} \equiv 1 \pmod{M}$, $p^{11} \equiv -1 \pmod{M}$ and $\tau(p) \equiv 0 \pmod{M}$. In the formal power series ring $(\mathbb{Z}/M\mathbb{Z})[[X]]$, the primitive factor becomes

$$\sum_{r \geq 0} R(p^r)X^r = \frac{1 - p^{10}X^2}{1 - \tau(p)X + p^{11}X^2} \equiv \frac{1 - X^2}{1 - X^2} = 1.$$

The denominator has constant coefficient 1, so it is invertible even when M is composite or even. No field hypothesis is needed.

Coefficient extraction gives (19.2). The constant coefficient remains $R(1) = 1$. Equivalently, the reduced Hecke recurrence has $T_0 = 1$, $T_1 = 0$ and $T_r = T_{r-2}$. Even T_r are 1, odd T_r are 0, and the primitive subtraction kills every positive exponent.

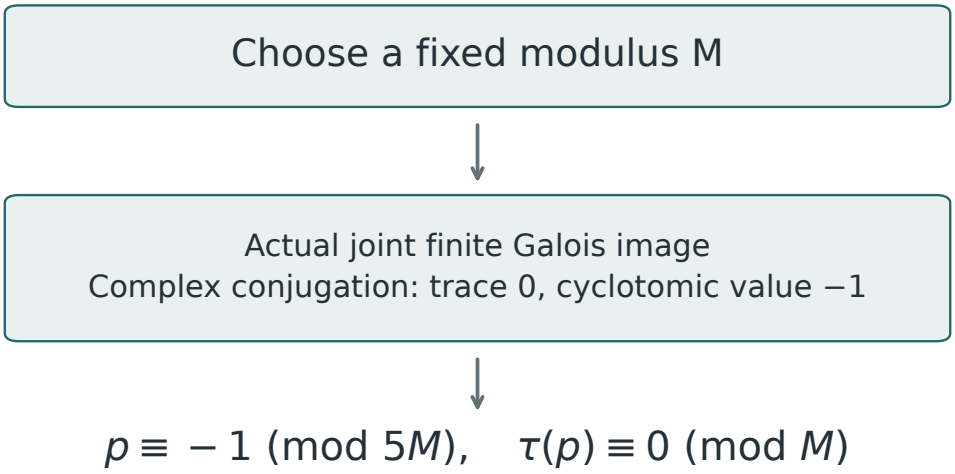


Figure 19.1: The simultaneous congruence proof uses the actual image of a joint representation. Complex conjugation supplies one existing trace-zero element with cyclotomic value -1, and Chebotarev supplies matching prime classes. No independence of the representation factors is assumed.

This includes nonzero even primitive blocks. Theorem 15.1 says they are nonzero integers. The present theorem says that at suitable varying primes they are divisible by any preassigned M. The two statements are compatible.

19.5 The quantifiers cannot be reversed

Given a fixed finite list of moduli $m_1, \ldots, m_s$, apply Theorem 19.1 to their least common multiple. Infinitely many actual primes defeat every test in that list whose decisive step is finding a nonzero residue of $\tau(p)$ or any $R(p^r)$. This remains true even if the test is allowed all positive local exponents.

The assertion is

$$\forall M \; \exists \text{ a positive-density set of suitable } p.$$

It is not

$$\exists p \; \forall M : \; M \mid \tau(p).$$

The surviving primes may change with M. No passage to an exact integer zero is available by reversing the quantifiers.

Nor does the theorem exclude a modulus chosen adaptively from p, such as the p^6 criterion. It excludes a fixed finite collection of nonzero-residue tests of the stated type. Congruences used for other purposes and global arguments remain possible.

The fresh arithmetic replay searched twenty-four selected moduli through 10000. Twenty-two had a simultaneous witness, producing 704 checked primitive congruences through exponent 32. The moduli 11 and 19 had no witness in that bounded range. Their absence is consistent with a positive-density theorem and does not refute it. Synthetic recurrence tests

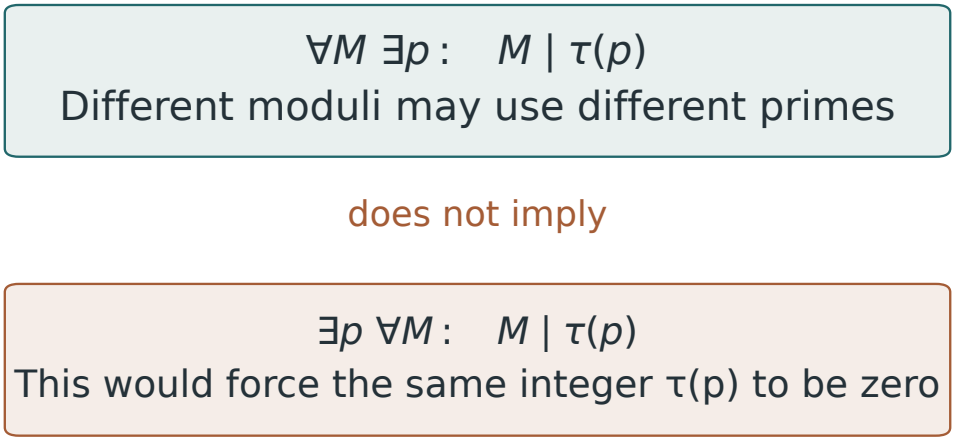


Figure 19.2: For each fixed modulus there are actual prime survivors. The primes may vary with the modulus. The quantifiers therefore do not assert one prime whose integer coefficient is divisible by every modulus.

add 24576 exact checks, but the universal prime-existence proof is the Galois argument above, not those counts.

CHAPTER 20

A locally compatible trace-zero object

20.1 The local conditions do not contradict zero trace

A zero coefficient at an actual prime would mean zero Frobenius trace. It is tempting to expect the weight, slopes or filtration to make that impossible. The following explicit local construction shows that the standard conditions admit such a trace.

Theorem 20.1. *For every prime p, there is a weakly admissible two-dimensional filtered φ-module over $\mathbb{Q}_p$, with monodromy $N = 0$, filtration jumps $0, 11$ and Frobenius characteristic polynomial $X^2 + p^{11}$.*

This is a local existence theorem. It neither identifies the object with the restriction of Δ's global representation nor asserts the existence of a level-one modular form with a prescribed zero coefficient.

20.2 The matrix and filtration

Take $D = \mathbb{Q}_p e_1 + \mathbb{Q}_p e_2$ and define

$$\varphi(e_1) = e_2, \qquad \varphi(e_2) = -p^{11} e_1, \qquad N = 0.$$

In the displayed basis,

$$\varphi = \begin{pmatrix} 0 & -p^{11} \\ 1 & 0 \end{pmatrix}, \qquad \operatorname{Tr} \varphi = 0, \qquad \det \varphi = p^{11}, \qquad \varphi^2 = -p^{11} I.$$

Choose any $\mathbb{Q}_p$-line $L \subset D$ and put

$$\operatorname{Fil}^i D = \begin{cases} D & i \le 0, \\ L & 1 \le i \le 11, \\ 0 & i \ge 12. \end{cases}$$

The two nonzero associated graded pieces are $\operatorname{gr}^0 D = D/L$ and $\operatorname{gr}^{11} D = L$, each of dimension 1.

20.3 Checking weak admissibility

For the chosen cohomological convention, define

$$t_H(D) = \sum_i i \dim \operatorname{gr}^i D, \qquad t_N(D) = v_p(\det \varphi).$$

Weak admissibility requires $t_H(D) = t_N(D)$ and $t_H(D') \le t_N(D')$ for every φ-stable subobject D', with the induced filtration. The total equality is

$$t_H(D) = 11 = t_N(D).$$

There is no proper nonzero φ-stable $\mathbb{Q}_p$-subspace. In dimension two such a subspace would be a line, and φ on it would have an eigenvalue $\lambda \in \mathbb{Q}_p$ satisfying

$$\lambda^2 = -p^{11}.$$

This would require $2v_p(\lambda) = 11$, impossible for an integer-valued valuation on $\mathbb{Q}_p^\times$. Thus the proper-subobject inequalities are vacuous. The zero object imposes equality $0 = 0$, and the whole object satisfies the total equality. The filtered module is weakly admissible.

The proof works at every p, including 2. It does not rely on a sign argument over $\mathbb{Q}_p$ or on quadratic-residue classifications. The odd determinant valuation is sufficient.

20.4 The realization theorem and conventions

Colmez–Fontaine's theorem makes a weakly admissible filtered module admissible [18]. In the present setting $K = K_0 = \mathbb{Q}_p$ and the base Frobenius is the identity. The theorem supplies a two-dimensional semistable $\mathbb{Q}_p$-representation realizing the filtered module. Since $N = 0$, the comparison is crystalline, using $B_{\mathrm{cris}} = B_{\mathrm{st}}^{N=0}$.

The filtration jumps $0, 11$ and determinant p^{11} are stated in the cohomological convention. Dualizing changes the Galois and Hodge sign conventions. The arithmetic-Frobenius representation in Chapter 19 and this local filtered object should not be identified merely because a polynomial has been written in a familiar form. The construction's role is compatibility, not that missing identification.

20.5 Purity and slopes

The Newton slopes are $11/2, 11/2$. The complex roots of $X^2 + p^{11}$ are

$$\pm i\, p^{11/2}.$$

They are algebraic integers, since they solve a monic integral polynomial, and each has the expected weight-11 absolute value. Thus zero trace is compatible simultaneously with integrality, purity, Deligne's size bound and weak admissibility.

The Newton polygon lies above the Hodge polygon with jumps $0, 11$, and the total endpoints agree. There is no contradiction in those polygons. A local condition that is satisfied by this object cannot, by itself, forbid zero trace.

For the actual nonzero $\tau(p)$ at $p \geq 5$, Deligne's estimate forces $v_p(\tau(p)) \leq 5$. A value with valuation at least 6 would be zero by the size bound. This does not exclude the zero case. It is precisely the case the conjecture asks us to eliminate.

20.6 What is missing globally

The actual coefficients belong to one global modular form, with compatible Galois representations and the packet realization already described. The local module constructed here is not asserted to belong to that system. A successful local-to-global argument must use a further property that distinguishes the actual restriction from this compatible object.

Finite matrix checks can authenticate the formula $\varphi^2 = -p^{11} I$. The release replay checks it at all 1229 primes through 10000. The absence of a rational eigenline at every prime

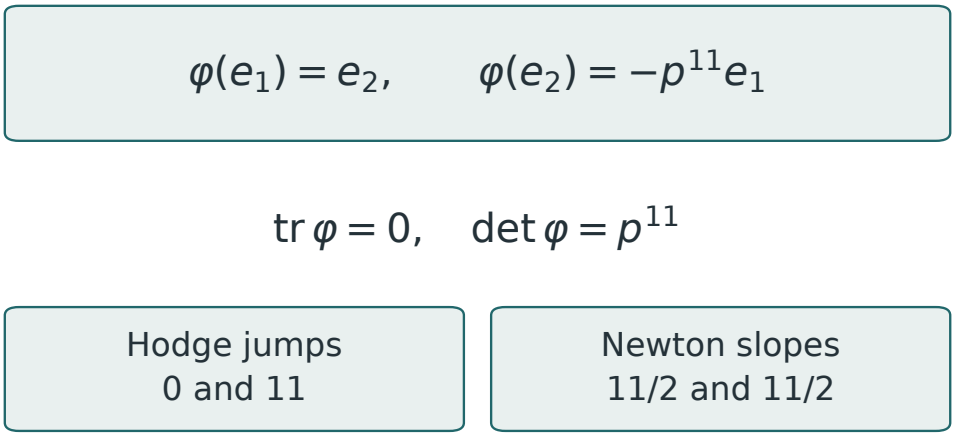


Figure 20.1: The constructed rank-two filtered Frobenius module has trace zero, determinant p^{11} and filtration jumps $0, 11$. An odd valuation excludes a rational eigenline, making the proper-subobject inequalities vacuous. This local compatibility is distinct from an actual global identification with Δ.

follows from the valuation proof, not from that finite matrix range. Conversely, neither the finite checks nor the realization theorem provides the additional global exclusion needed for Lehmer.

CHAPTER 21

A global arithmetic countermodel to the retained premises

21.1 An exact test of what the premises imply

The recurrence, parity, congruences, size bounds and primitive square theorem are all strong facts. It is natural to ask whether their combination already forces nonvanishing. The next construction gives an arithmetic sequence satisfying all of them and having a zero.

Theorem 21.1. *For every real $B \geq 1$ and integer $M \geq 1$, there exist a prime $q > B$ and a multiplicative integer sequence f with the following properties:*

1. $f(1) = 1$ *and* $f(n) = \tau(n)$ *for every* $n \leq B$.
2. $f(n) \equiv \tau(n) \pmod{10M}$ *for every positive integer* n.
3. *Every weight-*12 *prime-power recurrence holds:*
$$f(p^r) = f(p)f(p^{r-1}) - p^{11}f(p^{r-2}), \qquad r \geq 2.$$
4. $|f(n)| \leq d(n)n^{11/2}$, *where* $d(n)$ *counts the positive divisors of* n.
5. $f(n)$ *is odd exactly when* n *is an odd square, and* $f(n) \equiv n\sigma(n) \pmod{5}$.
6. *The arithmetic primitive transform* R_f *is nonzero at every square.*
7. *Nevertheless* $f(q) = R_f(q) = 0$.

This is a countermodel to an implication from the listed premises alone. It is not a counterexample to Lehmer for the fixed actual form. The sequence is not unconditionally described as nonmodular, because if the chosen actual $\tau(q)$ were already zero, the construction could coincide with τ.

21.2 Choosing an actual prime

Apply Theorem 19.1 with $L = 10M$. Its positive-density set contains a prime $q > B$ with
$$q \equiv -1 \pmod{50M}, \qquad \tau(q) \equiv 0 \pmod{10M}.$$
Enlarging q avoids the finitely many prime divisors of $10M$. In particular q is odd, $q \neq 5$ and $\chi_5(q) = 1$.

Set prime parameters
$$a_p = \tau(p) \quad (p \neq q), \qquad a_q = 0.$$
For every prime define integral local blocks
$$F_{p,0} = 1, \qquad F_{p,1} = a_p, \qquad F_{p,r} = a_p F_{p,r-1} - p^{11}F_{p,r-2}.$$
For $n = \prod p^{r_p}$, define $f(n) = \prod_p F_{p,r_p}$. Unique factorization makes this a well-defined multiplicative integer sequence with the stated recurrence.

21.3 Agreement at every fixed modulus and finite prefix

Only the parameter at q has changed. Every $n \le B$ is coprime to q, so $f(n) = \tau(n)$ there. At q, the old and new first parameters agree modulo $10M$. Induction in the integral recurrence gives

$$F_{q,r} \equiv \tau(q^r) \pmod{10M}$$

for every exponent. All other local blocks agree exactly. Multiplying local congruences gives $f(n) \equiv \tau(n) \pmod{10M}$ for every n, not only through a finite range.

The parity criterion follows from agreement modulo 2, and the modulo-5 congruence follows from agreement modulo 5. Thus even globally quantified versions of these retained properties do not remove the zero parameter.

One can incorporate any finite list of fixed congruence moduli by taking their common multiple in M. One can incorporate any finite coefficient check by increasing B. The resulting sequence and prime may change with B and M. No statement that one altered sequence agrees with τ modulo every modulus simultaneously is made.

21.4 The full size bound survives

At primes other than q, the Hecke roots have absolute value $p^{11/2}$ by Deligne. Formula (3.4) therefore gives

$$|\tau(p^r)| \le (r+1)p^{11r/2}.$$

At the modified prime,

$$F_{q,2m} = (-q^{11})^m, \qquad F_{q,2m+1} = 0.$$

These obey the same local bound at every exponent, including zero. Multiplying the local estimates gives

$$|f(n)| \le \prod_{p^{r_p} \| n} (r_p+1)p^{11r_p/2} = d(n)n^{11/2}.$$

A zero coefficient is fully compatible with an upper bound. The global estimate has not detected the modified parameter.

21.5 All primitive squares still survive

Define the arithmetic transform

$$R_f(n) = \sum_{d^2 \mid n} \mu(d)\chi_5(d)d^{10}f(n/d^2).$$

It is multiplicative. Its prime blocks are $R_f(p) = a_p$, and for $r \ge 2$,

$$R_f(p^r) = F_{p,r} - \chi_5(p)p^{10}F_{p,r-2}.$$

For $p \ne q$, these are the actual R blocks and their even terms are nonzero. At q, direct substitution and $\chi_5(q) = 1$ give

$$R_f(q^{2m}) = (-1)^m q^{11m-1}(q+1) \ne 0, \qquad m \ge 1.$$

Thus every positive-exponent factor of $R_f(s^2)$ is nonzero, and $R_f(1) = 1$, so the transform is nonzero at every square. Meanwhile $R_f(q) = a_q = 0$.

The square property is universal in this construction. It is not merely checked through a chosen exponent. This makes the countermodel stronger than a finite imitation of the actual sequence.

21.6 The logical conclusion and the remaining information

The theorem refutes

$$\text{the listed arithmetic properties} \quad \Longrightarrow \quad \text{every coefficient is nonzero.}$$

The construction does not impose a modular-form realization, the packet-shell identity or the compatible global Galois representations of Δ. Those are potential sources of additional information, rather than properties already contained in the weaker premises.

There is no need to know whether $\tau(q)$ is nonzero. If it is nonzero, f differs from τ. If it is zero, $f = \tau$ and Lehmer is already false. In either case, an arithmetic sequence with the listed properties and a zero exists. Calling f nonmodular without the extra nonzero assumption would be logically stronger than what has been proved.

Any successful resumption must therefore identify a particular additional consequence of the actual global modular form or its actual packet geometry, and show how it excludes a zero prime parameter. More fixed congruences, a larger finite prefix, the same recurrence, the same upper bound and the same all-square theorem do not supply that implication, even together.

CHAPTER 22

The exact obligation that remains

22.1 The target after every valid reduction

The constructions in this reader return to one fixed arithmetic question. They have not replaced Δ by a more convenient sequence, and they have not changed nonvanishing into a statement about typical primes. Their target is still every actual prime coefficient.

Proposition 22.1 (The remaining family)**.** *Lehmer's conjecture is equivalent to the assertion*

$$R(p) = \tau(p) \neq 0 \qquad \text{for every prime } p \equiv 4 \pmod 5. \tag{22.1}$$

Proof. Theorem 3.1 reduces the conjecture to prime indices. The congruence (6.8) excludes a zero at primes congruent to $1, 2, 3$ modulo 5, and the explicit value $\tau(5) = 4830$ handles 5. The primitive transform equals τ at a prime by (13.6). Thus exactly the displayed unexcluded family remains. □

The word *remaining* describes the family after these universal reductions. Many of its individual coefficients have also been certified, both in the finite examples here and in the much larger historical computation cited in Chapter 1. Accepting the latter as an external input lets one restrict attention to primes at or above its stated endpoint. It does not remove the universal prime quantifier.

On each such prime shell, all packets have content one. The residual is therefore the full signed sum

$$\tau(p) = \sum_{v \in \mathcal{V}_p} W(v). \tag{22.2}$$

The summands are nonzero integers, and for $p > 20$ they are units at the index prime. Neither property forbids their sum from being zero. The unresolved statement concerns the exact balance of their signed masses.

22.2 What the program has proved

The side theorems are useful precisely because their quantifiers and dependencies are now explicit. Primitive extraction has a complete local description. It gives a nonzero residual at every square and introduces no new zeros anywhere. The polynomial construction supplies universal null corrections, with harmonicity and valence as the justification. Finite rational sign certificates prove their individual coefficients survive. The tree theorem identifies the first mixed-sign shell at every dyadic depth. Content conservation persists for every finite symmetric polynomial universal null, without a degree cutoff or a homogeneity assumption.

Established statement	Exact implication for the target
$R(s^2) \neq 0$ for all s	Every even primitive local block survives, including a hypothetical zero prime parameter.
$R(n) = 0 \iff \tau(n) = 0$	Primitive extraction encodes the same zero problem, with $R(p) = \tau(p)$.
Universal polynomial nulls	Permitted corrections preserve every coefficient and every exact-content mass.
Sharp dyadic boundary	Geometric separation is controlled at all depths, without comparing the global positive and negative totals.
Exact finite certificates	The stated shells, matrix ranks and polynomial residues are settled in their declared ranges.

These statements are written proofs and exact finite certificates. They are not a compiled proof-assistant solution of Lehmer. Their inclusion claims no historical priority. In particular, a translated classical identity and a project-specific consequence of that identity have different provenance even when both are correct.

The arithmetic survivor theorem also has an all-modulus quantifier, but it is a limitation theorem for a proposed method. For every fixed positive modulus M, a positive-density set of actual primes satisfies

$$p \equiv -1 \pmod{5M}, \qquad R(p^r) \equiv 0 \pmod{M} \quad (r \geq 1).$$

It applies to the actual joint finite image, with no independence assumption. This is stronger than observing survivors in a bounded search. It still says nothing about an integer zero at one fixed prime, because that prime may change with M.

22.3 The missing arrows cannot be supplied by repetition

Several attractive final steps have failed in precisely stated forms. The integral prime-index interpolation violates a necessary divided-difference condition. The aggregate auxiliary polynomial does not inherit its individual factors' root interval. The proposed lifting equalities at $2, 3, 5$ have exact prime counterexamples. Strict preservation of sectors with opposite masses cannot yield a common-sign sum. Interpolation on one shell does not give universal nullness, and top-degree harmonic projection does not preserve the interpolated values.

Other statements are unproved rather than refuted. The all-degree universal-null space has not been shown to fill every content-preserving transfer space at every shell. Its finite rank profiles are examples, not a uniform theorem. Even a proof of that fullness would retain the question of whether the conserved prime total is zero.

The local filtered Frobenius construction and the arithmetic sequence f test different kinds of retained information. The former realizes compatible local zero-trace data. The latter satisfies a specified collection of arithmetic rules while having a prime zero. Neither is identified unconditionally with an actual zero coefficient of Δ. Their role is to show why those weaker premises alone cannot force nonvanishing.

A new attempt should therefore state which missing arrow it supplies. Rechecking the recurrence, increasing a coefficient endpoint or adding finitely many fixed congruence tests does not create that arrow. This is a logical conclusion about the retained premises. It does not declare every possible use of recurrences, computation or congruences exhausted.

22.4 A sufficient adaptive residue input

One precise route would be a uniform construction of residue certificates. Suppose there are explicitly defined functions on prime indices, with integer values $m(p) \geq 2$ and $b(p)$, for which one proves

$$\tau(p) \equiv b(p) \pmod{m(p)}, \qquad b(p) \not\equiv 0 \pmod{m(p)} \tag{22.3}$$

for every prime in (22.1). These two assertions would settle that family and hence Lehmer. The modulus may depend on p, so the fixed-modulus survivor theorem does not refute this proposed form of input.

The issue is the proof of the second assertion for the actual coefficient, together with the identity linking the construction to τ. Defining $b(p)$ to be the unknown residue and then assuming it is nonzero would only restate the target. A useful construction must give an independently justified nonzero residue, with all exceptional primes and all precision requirements covered.

The index-dependent criterion already available from Deligne's bound is

$$\tau(p) = 0 \iff p^6 \mid \tau(p), \qquad p \geq 5.$$

Combined with the divisor convolution, it becomes

$$840S_4(p) \not\equiv p^4 C(p) \pmod{p^6}, \qquad p > 7, \tag{22.4}$$

as the exact desired exclusion. This gives a correctly typed target for a convolution route. No uniform arithmetic property excluding that residue has been proved here. A stronger valuation bound may suffice, but its proof must permit the zero case as a possibility until the bound itself excludes it.

22.5 A sufficient sign input

A second precise route would construct permitted corrections with an independently proved sign. At each remaining prime, choose finitely many rational universal null polynomials $Q_{p,j}$ and rational coefficients $a_{p,j}$, and put

$$F_p(v) = 1 + \sum_j a_{p,j} Q_{p,j}(v).$$

The choice may depend on p. Universal nullness must hold after the choice, so its weighted sum is still the actual $\tau(p)$.

Suppose the construction proves a sign $\varepsilon_p \in \{-1, 1\}$ for which

$$\varepsilon_p W(v) F_p(v) \geq 0 \quad \text{for every } v \in \mathcal{V}_p, \tag{22.5}$$

and proves that at least one of these finitely many quantities is strictly positive. Then summing (22.5) gives $\varepsilon_p \tau(p) > 0$. This would be a valid all-prime certificate theorem.

The strict clause cannot be discarded. If the total were zero, a correction making every term zero would satisfy the weak inequalities. Nor can the strict clause be obtained merely from full transfer rank. Equalizing corrected masses to $\tau(p)/N$ already uses the total whose nonvanishing is at issue.

The finite certificates at 7 and 31 show that this form of argument can work at concrete shells. A uniform construction needs a proof covering all remaining primes, with coefficients and inequalities derived from actual packet structure. The current family and its finite successes do not provide such a theorem.

22.6 Where a global input would have to enter

The actual form has more structure than an arbitrary multiplicative recurrence sequence. A new global theorem could impose a relation or constraint that is absent from the countermodel and that excludes a zero prime parameter. The constraint must be stated exactly and derived for Δ. Merely naming modularity or compatibility does not prove a particular trace is nonzero.

Local compatibility also has a direction. A weakly admissible trace-zero object establishes that the selected local conditions permit zero. It does not establish that this object occurs as the restriction of the actual global representation. Proving such an identification at a prime, together with a proved comparison identifying its normalized Frobenius trace with $\tau(p)$, would give an actual zero and would disprove Lehmer. Proving a global obstruction to that identification could instead contribute to nonvanishing. Neither direction has been supplied by the local construction alone.

Similarly, showing that the synthetic series $\sum_{n\geq 1} f(n)q^n$ is a holomorphic cusp form of level one and weight 12 would be new information. Its normalization $f(1) = 1$ and the one-dimensional cusp space would then identify it with Δ. The present construction does not establish that premise. Calling the sequence nonmodular without an independent argument would also go beyond what was proved, since it could coincide with τ if the chosen prime were already an actual zero.

An effective continuation needs a concrete additional input, its exact hypotheses and a proof that it changes the zero test. That input may concern a prime-shell identity, an adaptive arithmetic restriction or a global relation. This reader establishes none of those prospective finishing theorems.

22.7 The closeout statement

The publication preserves both positive results and failed implications. A failed shortcut is identified by its exact domain and counterexample. An open span or global realization statement remains open. A finite run retains its finite range. A universal side theorem retains the proof that licenses its universal quantifier.

One actual coefficient at every remaining prime

For every prime $p \equiv 4 \pmod 5$, prove that the actual signed prime-shell total in (22.2) is nonzero. Supply an independent nonzero residue, a legitimate sign certificate with a strict term, or another fully justified mechanism excluding exact cancellation. A verified actual prime zero would instead disprove the conjecture.

The program has clarified the objects, their conserved quantities and several boundaries on their use. Its final mathematical status is that the prime total is still unresolved. The source and evidence make this unresolved status inspectable.

CHAPTER A

Definitions and conventions

The following conventions fix the object under discussion and the translations between its arithmetic, geometric and local descriptions. They are part of the claim. Changing a coefficient shift, the Vandermonde orientation or a Frobenius convention changes the formulas that a computation must verify.

A.1 The coefficient and its local parameters

The normalized form is

$$\Delta(q) = q\prod_{m\ge 1}(1-q^m)^{24} = \sum_{n\ge 1}\tau(n)q^n, \qquad \tau(1)=1.$$

All coefficient indices in this book are positive integers. A stored value at index zero is a programming sentinel unless a displayed formula explicitly introduces the coefficients of Δ/q. For the latter coefficients, $f_m = \tau(m+1)$ and $f_0 = 1$.

At a prime p, the default local parameters are $a = \tau(p)$ and $b = p^{11}$. The scalar recurrence has $T_0 = 1$, $T_1 = a$ and $T_r = aT_{r-1} - bT_{r-2}$ for $r \ge 2$. Thus $T_r = \tau(p^r)$. A construction that permits an arbitrary integer a states that change explicitly. Such a recurrence need not be the local sequence of Δ.

The symbol $T_k(p)$ in the trace-formula chapter denotes a Hecke operator on the weight-k cusp-form space. It is an operator, whereas T_r in a scalar recurrence is an integer or an integer polynomial. The weight k and the exponent r serve different roles.

The valuation $v_\ell(x)$ is the exponent of ℓ in a nonzero rational number x, with $v_\ell(0) = \infty$. The quadratic character used in the radial extraction is

$$\chi_5(d) = \begin{cases} 0 & 5 \mid d, \\ 1 & d \equiv 1,4 \pmod 5, \\ -1 & d \equiv 2,3 \pmod 5. \end{cases}$$

The character value and the tenth power in the scaling formula are both essential.

A.2 Packets, content and the coefficient shift

Coordinates have indices $0,1,2,3,4$. A shell is

$$\mathcal{V}_n = \left\{ v \in \mathbb{Z}^5 : v_i \equiv i \pmod 5, \quad \sum_i v_i = 0, \quad \sum_i v_i^2 = 10n \right\}.$$

The standard orientation is

$$D(v) = \prod_{0\le i<j\le 4}(v_i - v_j), \qquad W(v) = D(v)/288.$$

The identity is $\tau(n) = \sum_{v \in \mathcal{V}_n} W(v)$. No absolute values occur in that sum. Distinct residue classes ensure that no Vandermonde factor is zero. Integrality of W is proved separately from the core identity.

The corresponding five-core partition has size $n-1$. In particular, the coefficient at $n = 1$ corresponds to the empty partition. The involution is $J(v)_i = -v_{-i \bmod 5}$. A J-orbit may contain one or two packets, and its mass is the sum of their weights.

The content of a packet is the positive gcd of its five coordinates. Content-one packets are primitive. Multiplication followed by the prescribed residue reindexing gives

$$\tau(n) = \sum_{g^2 \mid n} \chi_5(g) g^{10} R(n/g^2), \qquad R(n) = \sum_{d^2 \mid n} \mu(d) \chi_5(d) d^{10} \tau(n/d^2).$$

The boundary values are $R(1) = 1$ and $R(p) = \tau(p)$. At a prime shell the only possible content is one, so the extraction has not created a new prime coefficient.

A.3 Three polynomial operations

The ordinary hyperplane Laplacian is Δ_H. The conjugated operator $\mathcal{L}$ is characterized by $\Delta_H(DQ) = D\mathcal{L}Q$. Harmonicity of DQ means $\mathcal{L}Q = 0$. It does not mean that Q itself is harmonic.

A universal null is a polynomial Q for which

$$\sum_{v \in \mathcal{V}_n} W(v) Q(v) = 0$$

for every positive n. A shell-null polynomial satisfies that equation only at a specified shell. A polynomial interpolating prescribed values on a shell need satisfy neither condition. The difference between these three operations is responsible for several of the obstruction results.

For a universal null, replacing $W(v)$ by $W(v)(1 + Q(v))$ preserves the shell total. A sign conclusion requires real corrected weights and at least one strict nonzero contribution. There is no order relation on complex weights. The all-polynomial content theorem assumes a finite symmetric polynomial and makes no claim for arbitrary infinite series or nonsymmetric functions.

A.4 Local and global conventions

For $\ell \neq p$, the arithmetic-Frobenius convention in the integral ℓ-adic representation gives the characteristic polynomial

$$X^2 - \tau(p) X + p^{11}.$$

The cohomological conventions in the filtered-module discussion are stated there separately. An abstract trace-zero filtered module and a representation realizing it are local objects. A comparison identifying its normalized Frobenius trace with the actual coefficient $\tau(p)$ is additional mathematical data.

The fixed-modulus survivor theorem concerns the actual joint finite image, including the cyclotomic factor. It does not assume that its constituent images are independent. Its quantifiers are

$$\text{for every fixed } M, \quad \text{there exist suitable actual primes } p.$$

The prime may change with M. The theorem does not supply one prime for every modulus.

CHAPTER B

The symmetric Laplacian in coordinates

The universal null construction depends on an exact differential operator. This appendix derives that operator from the Euclidean metric, rather than treating its power-sum formulas as a black box. It also explains the Newton reductions used by the implementation and gives two low-degree harmonic checks. All polynomial identities are on the hyperplane $H = \{v \in \mathbb{R}^5 : \sum_i v_i = 0\}$, with its induced Euclidean metric. The modularity and valence contracts remain the ones stated in Chapters 8 and 9.

Write $s_m = \sum_{i=0}^{4} v_i^m$, with $s_0 = 5$ and $s_1 = 0$, and let $D = \prod_{i<j}(v_i - v_j)$. We distinguish the ordinary Laplacian Δ_H from the conjugated operator $\mathcal{L}$ defined by $\Delta_H(DQ) = D\mathcal{L}Q$. The factor D changes radial coefficients substantially. Confusing the two operators would invalidate the harmonicity test.

B.1 Two descriptions of the same metric

The orthogonal projection from $\mathbb{R}^5$ to H is

$$P = I_5 - \frac{1}{5}\mathbf{1}\mathbf{1}^T, \tag{B.1}$$

where $\mathbf{1} = (1,1,1,1,1)^T$. Indeed, subtracting the coordinate mean leaves a zero coordinate sum. The matrix is symmetric, satisfies $P^2 = P$, and has image H. For an ambient polynomial F, its gradient along H is $P\nabla F$. Its Laplacian along H is

$$\Delta_H(F|_H) = \left(\sum_{i=0}^{4} \partial_i^2 F - \frac{1}{5}\left(\sum_{i=0}^{4} \partial_i\right)^2 F\right)\Big|_H. \tag{B.2}$$

Only tangential derivatives enter. Thus this expression is independent of the polynomial extension away from H.

For direct symbolic computation, use four coordinates $x_0, \ldots, x_3$ and substitute

$$v = (x_0, x_1, x_2, x_3, -x_0 - x_1 - x_2 - x_3).$$

The coordinate tangent vectors are $b_a = e_a - e_4$. Their Gram matrix and its inverse are

$$G = I_4 + \mathbf{1}\mathbf{1}^T, \qquad G^{-1} = I_4 - \frac{1}{5}\mathbf{1}\mathbf{1}^T, \tag{B.3}$$

now with a four-entry vector $\mathbf{1}$. Matrix multiplication verifies the inverse. The factor $1/5$ remains because the missing fifth coordinate is determined by the four chosen coordinates.

The chart Laplacian is consequently

$$\Delta_H = \sum_{a,b=0}^{3} \left(\delta_{ab} - \frac{1}{5}\right) \frac{\partial^2}{\partial x_a \partial x_b}. \tag{B.4}$$

The coordinates x_a are not orthonormal. Using only the four diagonal second derivatives would use a different metric. As a normalization check, $s_2 = x^T G x$, so its Hessian is $2G$ and

$$\Delta_H s_2 = \operatorname{tr}(G^{-1} 2G) = 8.$$

The rank of the Euclidean space is four, giving twice that rank. Later we obtain $\mathcal{L} s_2 = 48$, an intentionally different number.

B.2 Conjugating by the Vandermonde

The Vandermonde is alternating under coordinate permutations. Since Δ_H commutes with those permutations, $\Delta_H D$ is alternating of degree eight. Every alternating polynomial vanishes on each hyperplane $v_i = v_j$. In the polynomial ring of H, the ten corresponding linear factors are distinct and divide it. Their product has degree ten. Therefore $\Delta_H D = 0$.

On the open set where $D \neq 0$, the product rule gives

$$\frac{\Delta_H(DQ)}{D} = \Delta_H Q + 2\nabla_H \log D \cdot \nabla_H Q. \tag{B.5}$$

Here $\nabla \log D$ denotes the rational logarithmic derivative $\nabla D / D$. It does not require choosing a complex logarithm or assuming that D is positive. For symmetric Q, the apparent poles cancel and the right side extends to a polynomial on all of H.

There is a second explanation of the extension. The polynomial $\Delta_H(DQ)$ is alternating, so D divides it. Its quotient is symmetric. Thus the operator $\mathcal{L}$ is well defined on symmetric polynomials even at coordinate collisions. The packet shell itself has no collisions because its residues are distinct. The polynomial extension is nevertheless needed when constructing and comparing symbolic identities.

Translation invariance of D gives

$$\sum_i \frac{\partial_i D}{D} = 0, \qquad \frac{\partial_i D}{D} = \sum_{j \neq i} \frac{1}{v_i - v_j}.$$

Its ambient logarithmic gradient is already tangential. This removes a potential extra projection term in the derivation below.

B.3 Deriving the power-sum operator

For $m \geq 2$, differentiation and projection give

$$(\nabla_H s_m)_i = m \left(v_i^{m-1} - \frac{1}{5} s_{m-1} \right), \tag{B.6}$$

$$\Delta_H s_m = \frac{4}{5} m(m-1) s_{m-2}. \tag{B.7}$$

In the logarithmic term, the coordinate-mean subtraction vanishes because the logarithmic gradient has zero sum. Pair the ordered indices i, j and j, i to obtain

$$2\nabla_H \log D \cdot \nabla_H s_m = 2m \sum_{i<j} \frac{v_i^{m-1} - v_j^{m-1}}{v_i - v_j}.$$

The divided difference is the finite polynomial

$$\frac{v_i^{m-1} - v_j^{m-1}}{v_i - v_j} = \sum_{a=0}^{m-2} v_i^a v_j^{m-2-a}.$$

The sum over a is symmetric in the two coordinates. Hence twice the sum over $i < j$ equals the sum over $i \neq j$. For each a, that ordered sum is $s_a s_{m-2-a} - s_{m-2}$. There are $m-1$ values of a. We obtain

$$2\nabla_H \log D \cdot \nabla_H s_m = m \sum_{a=0}^{m-2} s_a s_{m-2-a} - m(m-1)s_{m-2}.$$

Adding (B.7) proves

$$\boxed{\mathcal{L}s_m = m \sum_{a=0}^{m-2} s_a s_{m-2-a} - \frac{m(m-1)}{5} s_{m-2}.} \tag{B.8}$$

This formula is asserted for $m \geq 2$. The boundary values are separately $\mathcal{L}s_0 = 0$ and $\mathcal{L}s_1 = 0$, since s_0 is constant and s_1 vanishes on H. No negative-index power sum is introduced.

The first values are

$$\mathcal{L}s_2 = 48, \qquad \mathcal{L}s_3 = 0, \qquad \mathcal{L}s_4 = \frac{188}{5} s_2, \qquad \mathcal{L}s_5 = 46 s_3. \tag{B.9}$$

For example, at $m = 2$ the first term is $2s_0^2 = 50$, while the subtraction is $2s_0/5 = 2$. The additional 40 compared with $\Delta_H s_2 = 8$ comes from differentiating the degree-ten factor D.

B.4 The chain-rule metric and polynomial assembly

The same projected gradients give

$$\Gamma_{mn} := \nabla_H s_m \cdot \nabla_H s_n = mn \left(s_{m+n-2} - \frac{1}{5} s_{m-1} s_{n-1} \right). \tag{B.10}$$

Expanding the product in (B.6) gives the first power sum once and the coordinate-mean product with coefficient $-1/5$. This identity includes $m, n \geq 1$, with the evident zero tangential gradient for s_1.

If $Q = F(s_2, s_3, s_4, s_5)$, the ordinary second-order chain rule and (B.5) yield

$$\mathcal{L}Q = \sum_{m=2}^{5} \frac{\partial F}{\partial s_m} \mathcal{L}s_m + \sum_{m,n=2}^{5} \frac{\partial^2 F}{\partial s_m \partial s_n} \Gamma_{mn}. \tag{B.11}$$

The second sum runs over ordered pairs. Thus a mixed derivative contributes twice when $m \neq n$. If one instead sums over $m \leq n$, the off-diagonal terms require a factor two. These are two equivalent conventions, but their factors must not be mixed.

Useful instances are

$$\mathcal{L}(s_2^2) = 104 s_2, \tag{B.12}$$

$$\mathcal{L}(s_2^3) = 168 s_2^2, \tag{B.13}$$

$$\mathcal{L}(s_2 s_4) = 64 s_4 + \frac{188}{5} s_2^2, \tag{B.14}$$

$$\mathcal{L}(s_3^2) = 18 s_4 - \frac{18}{5} s_2^2. \tag{B.15}$$

For the first identity, $2s_2\mathcal{L}s_2 + 2\Gamma_{22} = 96s_2 + 8s_2$. For the mixed product, $s_4\mathcal{L}s_2 + s_2\mathcal{L}s_4 + 2\Gamma_{24}$ gives the displayed coefficients. Each calculation is a check of the ordered-pair convention.

The metric formula may produce s_6, s_7, s_8. They are not new independent variables. The invariant ring on H is generated by s_2, s_3, s_4, s_5, and Newton's identities reduce higher indices exactly.

B.5 Newton reductions with zero first power sum

Let e_j be the elementary symmetric polynomials in the five coordinates. Their characteristic polynomial is

$$\prod_i (T - v_i) = T^5 + e_2T^3 - e_3T^2 + e_4T - e_5,$$

because $e_1 = s_1 = 0$. The first Newton identities give

$$e_2 = -\frac{1}{2}s_2, \qquad e_3 = \frac{1}{3}s_3, \tag{B.16}$$

$$e_4 = \frac{1}{8}s_2^2 - \frac{1}{4}s_4, \qquad e_5 = \frac{1}{5}s_5 - \frac{1}{6}s_2s_3. \tag{B.17}$$

For instance, the degree-five identity is $s_5 + e_2s_3 - e_3s_2 - 5e_5 = 0$. The two middle terms combine to $-5s_2s_3/6$, fixing the sign in e_5.

Every coordinate satisfies its characteristic polynomial. Multiplying that equality by v_i^{k-5} and summing, for $k \geq 6$, gives

$$s_k = -e_2s_{k-2} + e_3s_{k-3} - e_4s_{k-4} + e_5s_{k-5}. \tag{B.18}$$

This recurrence uses only lower indices. Substitution at $k = 6, 7$ yields

$$s_6 = \frac{3}{4}s_2s_4 + \frac{1}{3}s_3^2 - \frac{1}{8}s_2^3, \tag{B.19}$$

$$s_7 = \frac{7}{10}s_2s_5 + \frac{7}{12}s_3s_4 - \frac{7}{24}s_2^2s_3. \tag{B.20}$$

The same rule gives

$$s_8 = \frac{1}{4}s_4^2 + \frac{1}{4}s_2^2s_4 - \frac{1}{16}s_2^4 + \frac{8}{15}s_3s_5. \tag{B.21}$$

Each term has the required invariant degree. This supplies a useful local check before substitution into a larger operator expression.

At the first packet $v = (0, 1, 2, -2, -1)$, the first power sums are $s_2 = 10$, $s_3 = 0$, $s_4 = 34$, $s_5 = 0$. Formula (B.19) gives $255 - 125 = 130$, matching the direct sum of sixth powers. Formula (B.20) gives zero, matching cancellation of the opposite coordinates. Formula (B.21) gives 514. These examples fix normalizations. The polynomial recurrences, rather than these examples, establish the identities on every vector in H.

B.6 Radial factors and harmonic projection

Let h_ℓ be homogeneous harmonic of degree ℓ on the rank-four space H. Euler's identity is $v \cdot \nabla_H h_\ell = \ell h_\ell$. The product rule for the radial factor s_2^j gives

$$\Delta_H(s_2^j h_\ell) = 4j(\ell + j + 1)s_2^{j-1}h_\ell, \qquad j \geq 1. \tag{B.22}$$

At $j = 0$ the left side is zero. At $j = 1, \ell = 0$ the coefficient is 8, agreeing with $\Delta_H s_2$. At $j = 1, \ell = 10$ it is 48, agreeing with $\Delta_H(s_2 D) = 48D$.

For a homogeneous polynomial P of degree M, a concrete top-degree harmonic projection in dimension four is

$$\mathcal{H}_M(P) = \sum_{j=0}^{\lfloor M/2 \rfloor} \frac{(-1)^j (M-j)!}{4^j j! M!} s_2^j \Delta_H^j P. \tag{B.23}$$

For $M = 0$ or 1 the sum contains only $j = 0$. In general, if a_j denotes its coefficient, the product rule gives the cancellation condition

$$a_j + 4(j+1)(M-j)a_{j+1} = 0.$$

The factorial coefficients satisfy this condition. The last possible iterated Laplacian has no further nonzero derivative, so the resulting polynomial is harmonic. Its difference from P is divisible by s_2.

Repeating the construction on that lower-degree quotient produces the harmonic decomposition

$$P = \sum_{j=0}^{\lfloor M/2 \rfloor} s_2^j h_{M-2j}. \tag{B.24}$$

It is unique. If such a sum were zero, apply the Laplacian as many times as the largest occupied radial exponent. All smaller-exponent terms vanish. Formula (B.22) leaves a nonzero scalar times the last harmonic polynomial, so that polynomial must be zero. Descending removes every term.

When $P = DQ$ is alternating, each projected harmonic component is alternating because the Laplacian and multiplication by s_2 commute with permutations. Each nonzero component is divisible by D and has degree at least ten. Thus for a correction of homogeneous degree d, only $j \le d/2$ can occur. This is the degree bound used in the quasimodular argument for nonhomogeneous content conservation.

Harmonic projection is not preservation of shell values. The elementary counterexample is $P = s_2 D$, of degree 12. Since $\Delta_H P = 48D$ and the next Laplacian is zero,

$$\mathcal{H}_{12}(s_2 D) = s_2 D - \frac{1}{48} s_2 (48D) = 0.$$

Yet on $\mathcal{V}_n$ the original polynomial equals $10nD$, which is nonzero at every packet. The operation removes radial information. It cannot silently turn a shell interpolant into a universally null polynomial with the same values.

B.7 Two explicit harmonic moment checks

The degree-four correction

$$K_4 = 130 s_4 - 47 s_2^2$$

has

$$\mathcal{L} K_4 = 130 \frac{188}{5} s_2 - 47(104 s_2) = 0.$$

Thus DK_4 is harmonic. At the first packet, $W = 1$ and $K_4 = 130 \cdot 34 - 47 \cdot 100 = -280$. Its theta form has weight 16, whose cusp space is spanned by ΔE_4 with first coefficient

one. Consequently the theta form is $-280\Delta E_4$, giving the moment identity already stated in Chapter 9.

Likewise, set

$$K_6 = 90s_2s_4 - 27s_2^3 - 320s_3^2.$$

Using (B.13)–(B.15), its operator image is

$$\begin{aligned}\mathcal{L}K_6 &= (5760 - 5760)s_4 \\ &\quad + (3384 - 4536 + 1152)s_2^2 = 0.\end{aligned}$$

At the first packet its value is $90 \cdot 10 \cdot 34 - 27 \cdot 1000 = 3600$. The weight-18 cusp space is spanned by ΔE_6, again normalized with first coefficient one. Its theta form is therefore $3600\Delta E_6$.

These checks have different roles from the null certificates. Both polynomials are harmonic, but neither is universally null: their first coefficients are nonzero. Harmonicity supplies a modular-form space. The initial coefficients then identify the form or, when enough initial coefficients vanish, force it to be zero by valence. Skipping that second step would misidentify a nonzero moment as a permitted null correction.

The retained programs organize these same operations using sparse rational polynomials. A direct four-coordinate implementation can instead substitute $v_4 = -\sum_{a=0}^{3} x_a$ into DQ and apply (B.4). Agreement between these organizations checks the symbolic construction through different algebraic representations. The universal justification remains the metric calculation, the exact polynomial identities and the stated harmonic theta and valence contracts. None of those statements alone excludes the unresolved zero prime parameter of Δ.

CHAPTER C

An exact computation workbook

This appendix gives calculations that a reader can reproduce without trusting a status flag. The objects are the normalized coefficients of Δ, the auxiliary shell polynomials, and one rational transfer certificate at 31. Each calculation ends at its actual mathematical boundary. Agreement of finite outputs does not supply the missing all-prime nonvanishing argument.

The complete integer inputs and outputs are in the companion evidence archive. Paths below are relative to its extracted project root. A JSON integer remains an integer even when it has many digits. An encoded rational pair $[a, b]$ means a/b, with a nonzero denominator. Converting these objects to floating point before checking an equality changes the contract.

C.1 Three engines and one coefficient shift

Write

$$F(q) = \frac{\Delta(q)}{q} = \prod_{k\geq 1}(1-q^k)^{24} = \sum_{m\geq 0} f_m q^m.$$

Then $f_0 = 1$ and $\tau(n) = f_{n-1}$. This shift is essential. A program that stores $\tau(0) = 0$ as a sentinel must not confuse that entry with the constant coefficient f_0.

For direct product expansion, terms of degree at most m depend only on factors with $k \leq m$. The exact binomial expansion

$$(1-q^k)^{24} = \sum_{j=0}^{24}(-1)^j \binom{24}{j} q^{kj}$$

therefore supplies a finite algorithm for each desired coefficient. Three short calculations are

$$\begin{aligned} f_1 &= -24, \\ f_2 &= \binom{24}{2} - 24 = 276 - 24 = 252, \\ f_3 &= -\binom{24}{3} + (-24)(-24) - 24 \\ &= -2024 + 576 - 24 = -1472. \end{aligned}$$

The last line comes respectively from the q^3 term in the first factor, the product of its q term with the second factor's q^2 term, and the third factor's q^3 term. Thus it computes $\tau(4)$, not $\tau(3)$.

The logarithmic derivative provides a different exact engine. With $\sigma_1(j) = \sum_{d|j} d$, comparison of coefficients in

$$q\frac{F'(q)}{F(q)} = -24\sum_{j\geq 1}\sigma_1(j)q^j$$

gives

$$m f_m = -24\sum_{j=1}^{m}\sigma_1(j) f_{m-j} \qquad (m \geq 1). \tag{C.1}$$

For $m = 3$, the divisor sums are $1, 3, 4$. Hence

$$f_3 = -8\big(252 + 3(-24) + 4\big) = -8 \cdot 184 = -1472.$$

An implementation should check that the numerator in (C.1) is divisible by m, rather than rounding the quotient.

Jacobi gives a third engine. Put $T_j = j(j+1)/2$ and $a_j = (-1)^j(2j+1)$. The series

$$A(q) = \prod_{k\geq 1}(1-q^k)^3 = 1 + \sum_{j\geq 1} a_j q^{T_j}$$

satisfies $F = A^8$. The identity $AqF' = 8FqA'$ yields

$$m f_m = \sum_{\substack{j\geq 1 \\ T_j \leq m}} a_j(9T_j - m) f_{m-T_j}. \tag{C.2}$$

At $m = 3$, only $T_1 = 1$, $T_2 = 3$ contribute. Its numerator is

$$(-3)(9-3)(252) + 5(27-3)(1) = -4536 + 120 = -4416.$$

Division by 3 again gives -1472. The direct product, divisor recurrence and triangular recurrence reach the same integer through different algebraic operations.

m	$f_m = \tau(m+1)$	m	$f_m = \tau(m+1)$
0	1	5	-6048
1	-24	6	-16744
2	252	7	84480
3	-1472	8	-113643
4	4830	9	-115920

The current independent comparison covers every $\tau(n)$ for $1 \leq n \leq 512$. The historical final-arithmetic run's direct-product comparison stopped at 128, while the packet-tree run compared through 512. These are different documented ranges. The new three-engine table supplies its own bound and does not infer one from an older summary.

C.2 Reconstructing the auxiliary polynomial

For a packet v, set $c_i = (v_i - i)/5$ and form

$$P_v(T) = \prod_{0\leq i<j\leq 4} \big((i-j) + T(c_i - c_j)\big).$$

Its physical value is $P_v(5) = 288W(v)$. Summing the full shell gives

$$P_n(T) = \sum_{v\in\mathcal{V}_n} P_v(T) = \sum_{j=0}^{10} B_j T^j, \qquad P_n(5) = 288\tau(n).$$

The polynomial has a different variable from the prime index. A root in T or in a rescaled variable does not represent a nearby prime coefficient.

For a nonzero P_n, use the convention $v_5(0) = +\infty$ and define

$$\mu_n = \min_j \bigl(j + v_5(B_j)\bigr), \qquad G_n(U) = 5^{-\mu_n} P_n(5U) = \sum_{j=0}^{10} C_j U^j.$$

Every $C_j = B_j 5^{j-\mu_n}$ is an integer. The power of 5 should be removed only after verifying divisibility for each coefficient. Since 288 is a 5-adic unit,

$$v_5(\tau(n)) = \mu_n + v_5(G_n(1)). \tag{C.3}$$

All coefficients in the next two tables are exact integers from the freshly reproduced shell data. They allow a reader to check the normalization and every displayed residue without re-enumerating a large shell.

C.3 The simple branch at 499

At $n = 499$ the shell has 500 packets. Of their weights, 262 are positive and 238 negative. Their sum is

$$\tau(499) = -108877719272500.$$

The minimum layer valuation is $\mu_{499} = 3$.

j	B_j in P_{499}	C_j in G_{499}	$j + v_5(B_j)$
0	144000	1152	3
1	−360000	−14400	5
2	1063040	212608	3
3	28418240	28418240	4
4	−148872808	−744364040	4
5	209474504	5236862600	5
6	2422451240	302806405000	7
7	16445000	10278125000	11
8	8107402848	25335633900000	8
9	−11893316304	−185833067250000	9
10	−1160632800	−90674437500000	12

Summing the second coefficient column and using the normalization checks gives

$$P_{499}(5) = -31356783150480000 = 288\tau(499),$$
$$G_{499}(1) = -250854265203840.$$

Reducing the normalized coefficients gives

$$G_{499}(U) \equiv 2 + 3U^2 \pmod 5,$$
$$G_{499}(U) \equiv 2 + 8U^2 + 15U^3 + 10U^4 \pmod{25}.$$

The first polynomial has the simple roots 1 and 4 modulo 5. At $U = 1$, the exact sum is 10 modulo 25 and its derivative is 1 modulo 5. Consequently

$$G_{499}(1 + 5h) \equiv 10 + 5h \pmod{25}.$$

The five possibilities are $10, 15, 20, 0, 5$ as h runs from 0 to 4. The root in the branch $U \equiv 1 \pmod 5$ therefore lifts to $U = 16 \pmod{25}$.

This lifted root is not the physical target. The target remains $U = 1$, where $v_5(G_{499}(1)) = 1$. Formula (C.3) gives $v_5(\tau(499)) = 3 + 1 = 4$.

The following exact root history records both simple branches. Each entry lies between 0 and $5^k - 1$ and evaluates to zero modulo 5^k. Consecutive entries agree modulo the preceding power. These conditions are stronger than displaying an approximate root.

k	5^k	Root reducing to 1	Root reducing to 4
1	5	1	4
2	25	16	4
3	125	41	54
4	625	541	554
5	3125	3041	2429
6	15625	9291	14929
7	78125	9291	77429
8	390625	165541	233679
9	1953125	1337416	1014929
10	9765625	7196791	1014929
11	48828125	7196791	20546179
12	244140625	153681166	215858679

The fixed degree 10 does not impose a universal ceiling on the valuation of polynomial values at varying inputs. A simple 5-adic root supplies arbitrarily accurate integer approximations. That observation concerns the auxiliary variable. It neither changes the value at 1 nor supplies interpolation of the prime-indexed family.

C.4 The double branch at 16249

At 16249 the shell contains 16250 packets, with 8284 positive and 7966 negative weights. The exact coefficient is

$$\tau(16249) = 13311621527495816920625 0.$$

Here $\mu_{16249} = 4$. The full coefficient table is as follows.

j	B_j in P_{16249}	C_j in G_{16249}	$j + v_5(B_j)$
0	4680000	7488	4
1	−13320000	−106560	5
2	−199541600	−7981664	4
3	−26375464320	−5275092864	4
4	737398074300	737398074300	6
5	−39838758304992	−199193791524960	5
6	−746041848478320	−18651046211958000	7
7	−6990990388478280	−873873798559785000	8
8	21389931029028020	13368706893142512500	9
9	−64003552771660728	−200011102411439775000	9
10	3937759175521630080	61527487117525470000000	11

The exact sums are

$$P_{16249}(5) = 38337469999187952731400000,$$
$$G_{16249}(1) = 61339951998700724370240.$$

The first equals $288\tau(16249)$, and the second equals the first divided by 625. Modulo 5, the normalized polynomial is

$$3 + U^2 + U^3 = (U-1)^2(U-2).$$

Modulo 25, the table gives

$$G_{16249}(U) \equiv 13 + 15U + 11U^2 + 11U^3 + 15U^5.$$

Thus $G_{16249}(1) \equiv 15 \pmod{25}$ while $G'_{16249}(1) \equiv 0 \pmod{5}$. For every integer or 5-adic integer h, polynomial Taylor expansion gives

$$G_{16249}(1+5h) \equiv G_{16249}(1) + 5hG'_{16249}(1) \equiv 15 \pmod{25}.$$

Terms of order at least two have a factor 25. This is a uniform obstruction throughout that residue branch, not merely the observation that five trial inputs fail. Every input congruent to 1 modulo 5 has normalized valuation exactly 1.

The other residual root should not be discarded. At $U = 2$, the normalized value is 5 modulo 25 and its derivative is 1 modulo 5. That simple branch lifts to 22 modulo 25. The conclusion is that the double branch at 1 does not lift, rather than that the polynomial has no 5-adic roots anywhere.

At the physical input 1, the coefficient valuation is $4 + 1 = 5$. The two examples have different root geometry but both settle their actual target value by an exact modulo-25 calculation.

C.5 A first failure needs a domain

The tested auxiliary equality was

$$v_\ell(\tau(p)) = v_\ell(p+1), \qquad p \neq \ell, \quad \ell \mid (p+1), \quad p \text{ prime}.$$

The complete search through 150000 contains 13848 primes. The following counts refer to the indicated candidate sets, not to every prime without a residue condition.

ℓ	First p	$v_\ell(\tau(p))$	$v_\ell(p+1)$	Candidates	Discrepancies
2	81919	20	14	13847	1
3	8747	8	7	6941	5
5	499	4	3	3451	50

The exact decisive values, including the discrepancy in the opposite 5-adic direction, are

$$\begin{aligned}\tau(499) &= -108877719272500,\\ \tau(1249) &= 27757672778617250,\\ \tau(8747) &= 348766170858826454196,\\ \tau(81919) &= -904107824789465028682055680.\end{aligned}$$

For example, $1250 = 2 \cdot 5^4$, while the second integer has 5-adic valuation 3. Hence a proposed one-sided lift $v_5(\tau(p)) \geq v_5(p+1)$ also fails at this actual prime.

These four coefficients were checked independently by the Eisenstein convolution. If $\sigma_r(n) = \sum_{d|n} d^r$, the coefficient of $(E_4E_8 - E_6^2)/1728$ is obtained from

$$\begin{aligned}1728\tau(n) = {} & 240\sigma_3(n) + 480\sigma_7(n) + 1008\sigma_5(n)\\ & + 115200 \sum_{k=1}^{n-1} \sigma_3(k)\sigma_7(n-k)\\ & - 254016 \sum_{k=1}^{n-1} \sigma_5(k)\sigma_5(n-k).\end{aligned}$$

This computes the selected integers without the triangular recurrence. The numerator is divided exactly by 1728. The replay checks this independent formula for $1 \leq n \leq 128$ and at the four displayed primes, giving 132 checks.

The phrase “first failure” has two limits. It concerns the ordered search in its declared candidate set, and its exhaustive range ends at 150000. A larger prime may supply another failure. The displayed nonzero integers refute an auxiliary valuation identity, not Lehmer’s conjecture.

C.6 Reading the rational transfer at 31

The degree-at-most-22 certificate has six coarse sectors and sixteen fine sectors. Both correction matrices have eighteen columns. Let

$$A \in \mathbb{Z}^{6\times 16}, \quad M_c \in \mathbb{Z}^{6\times 18}, \quad M_f \in \mathbb{Z}^{16\times 18}.$$

The fine-to-coarse parent map completely specifies A: its column j has a 1 in the parent row and zero elsewhere. The original fine masses are given below. Let s_f denote their sixteen-entry column vector. Indices start at 0.

j	Parent	Fine mass	j	Parent	Fine mass
0	0	2354352	8	0	−11844448
1	2	−19467448	9	1	−7162848
2	4	−832048	10	0	−31156048
3	4	15074752	11	2	4356352
4	4	1881152	12	2	32821152
5	5	4248552	13	2	1262352
6	5	7759752	14	0	−7455448
7	4	−31808448	15	3	−12874848

For instance, row 0 aggregates fine indices $0, 8, 10, 14$:

$$2354352 - 11844448 - 31156048 - 7455448 = -48101592.$$

The six coarse masses are therefore

$$s_c = (-48101592, -7162848, 18972408, -12874848, -15684592, 12008304)^t.$$

Their sum, and the sum of the fine masses, is -52843168. These original masses differ in order from the lexicographic orbit bars because the certificate orders the residue-sector keys.

The full machine certificate is

`evidence/computation/replays/transport/p31_transport_certificate.json`.

It supplies an 18×13 rational matrix N whose columns form a basis of $\ker M_c$, and the detail matrix $B = M_f N$ of size 16×13. The basic exact checks are

$$AM_f = M_c, \quad As_f = s_c, \quad M_c N = 0, \quad B = M_f N, \quad AB = 0.$$

The last equation shows that a detail correction changes no coarse mass.

The rational row-reduction results are

$$\operatorname{rank} M_c = 5, \qquad \operatorname{rank} M_f = 15, \qquad \operatorname{rank} B = 10.$$

An independent check obtains the same lower bounds over both prime fields $\mathbb{F}_{65537}$ and $\mathbb{F}_{1000003}$. Every denominator used in a reduction is invertible there. A nonzero minor in either field proves a nonzero rational minor after clearing denominators.

The corresponding upper bounds have direct structural explanations. Each correction matrix has column sum zero, giving bounds 5 and 15. Every row of A has a nonempty, disjoint support, so $\operatorname{rank} A = 6$ and $\dim \ker A = 10$. Since $AB = 0$, the detail rank is at most 10. The upper and lower bounds establish the displayed ranks exactly.

The coarse coefficient vector $a \in \mathbb{Q}^{18}$ makes

$$x = s_f + M_f a, \qquad Ax = (0, 0, 0, 0, 0, -52843168)^t.$$

The certificate then chooses a coefficient vector $d \in \ker M_c$. Its final fine vector $y = x + M_f d$ is zero except at index 6, where it equals -52843168.

One parent gives a concrete view of this second stage. Parent 5 has fine children $5, 6$. Just after the coarse stage their values are

$$x_5 = -\frac{161598180629860124657534386744}{23300823161076013875625},$$
$$x_6 = -\frac{106969113220917073742448593256}{23300823161076013875625}.$$

Their exact sum is -52843168. The detail stage adds $-x_5$ to child 5 and x_5 to child 6, giving $(0, -52843168)$ in that parent. All other parents retain their zero totals. This is a rational conservation calculation, not a numerical near-zero test.

To verify the whole certificate, parse the rational pairs, check the matrix dimensions, verify all five identities, and compare both stored stages entry by entry. Then check $M_c d = 0$ and the single surviving final entry. The script `code/check_transport_certificate.py` performs the independent equality and prime-field checks.

This example does not assume a uniform rank theorem. Nor does full rank prove a uniformly nonzero survivor. At a shell whose original total were zero, complete zero-total transfer could still leave every final mass zero. At 31 the separately computed exact total is nonzero, so the displayed survivor is authenticated for this one shell.

C.7 Reproducing a run and interpreting its record

From the extracted source-set root, a new independent coefficient run is requested with

```
python3 code/run_replays.py tau_independent \
  --output-root evidence/computation/reader_coefficients
```

The destination must be fresh. The launcher refuses to replace an existing run directory. Omitting the suite name requests the complete declared sequence of suites. For that complete reproduction, use a fresh destination such as `evidence/computation/reproduced`. The large modular comparison consumes the newly generated dyadic result in that same destination, so these dependent suites belong in their declared order.

A version 2 manifest records the actual argument vector, exact arithmetic domain, semantic bounds, software versions and executed limits. Its `input_artifacts` contain input checksums before and after execution. Its `declared_results` name the scientific files the child must create. Output checksums cover those files and the execution logs.

Three questions remain separate. Did execution finish within its limits with unchanged inputs and all required outputs? Does the code implement the intended finite assertion? Does a proof show that the finite assertion decides the universal claim? An exit code of zero answers only part of the first question.

The count 704 means 22 selected modulus examples times 32 exponents, not 704 distinct primes. Of 24 selected moduli, 11 and 19 had no observed matching prime through 10000. That finite absence does not contradict positive density. The 24576 synthetic cases mean 256 moduli times three trace parameters times 32 exponents. Their parameter $p = 5M - 1$ need not be prime, and their trace is not asserted to equal $\tau(p)$.

A checksum pins bytes. Run provenance records execution. A timeout records an execution limit. A reproduced obstruction can refute its stated auxiliary implication while leaving Lehmer unchanged. Attach each exact calculation to a claim of the correct scope.

CHAPTER D

External theorem contracts

This edition checks the specific theorem statements used in the argument. The check is targeted, not an exhaustive review of the literature or a certification of novelty. The sixteen source contracts below are recorded with versions, locators, access outcomes and limitations in the companion evidence. The bibliography supplies the corresponding primary or author-hosted references.

The source-dependent input and the project argument should be read together. For example, a vector-valued harmonic theta theorem does not by itself prove the scalar sign-line descent used here. That descent is developed in the main text. Similarly, a trace formula computes an exact difference, but its existence does not exclude a zero difference.

D.1 Classical coefficient structure

S01: Lehmer's original reduction. Lehmer's 1947 paper is the historical reference for the least-zero prime reduction and the multiplicative structure [1]. The original scan was not text-readable through the checked access route, so this edition does not claim to have freshly reconstructed its original proof. The necessary coefficient identities are separately checked in readable modern primary sources, and the prime-power zero criterion is proved in the main text.

S06: Deligne's bound and normalization. The modular representation and purity inputs give $|\tau(p)| \leq 2p^{11/2}$ [4, 5]. The modular paper's representation construction and the Weil theorem supply different parts of this implication. Their Frobenius conventions require translation. This upper bound includes zero and is never used as a positive lower bound.

S07: classical congruences. Serre's 1967/68 exposition supplies the checked modulo-691 identity and the modulo-23 prime classification [7]. The latter excludes $p = 23$. Representation existence is still conjectural in the relevant passage of that early exposition. The existence input in this book therefore uses the later established representation statement.

S10: a cited historical computation. Derickx, van Hoeij and Zeng establish the strict historical range

$$n < 816212624008487344127999$$

in their checked 2014 version, Corollary 1.2 [3]. The endpoint is not asserted to be a zero. This source-certified range was not rerun for the present edition and is not presented as the latest record.

S11: scoped contemporary context. The checked 2026 version supplies a contemporary formulation and readable classical recurrence and parity statements [19]. Only those classical introductory inputs are used. The 2025 submission titled as a proof was withdrawn [20]. Its

title is not accepted as evidence of a theorem. No exhaustive current-status or priority claim follows from this targeted check.

S16: readable classical leaves. Balakrishnan, Craig and Ono provide the modulo-5 congruence at equation (1.2), the multiplicativity and recurrence statements at Theorem 2.2, and the direct odd-square parity argument in Section 3 of the checked 2020 version [2]. These are the exact leaves used here. This edition has not audited the paper's separate integral-point or omitted-value arguments.

D.2 Convolutions and trace moments

S12: the Hurwitz moment normalization. Lygeros and Rozier's 2010 primary paper gives the trace moments and Catalan inversion used in Chapter 5 [15]. The class-number function includes imprimitive forms with its specified weights. The $t = 0$ term has a half-weight in the zeroth moment. The integral normalization is $6H$. Removing any of these qualifications would change the rows of the inverse system.

S14: Niebur's convolution. The universal formula is checked in the primary paper of Moriya and Smyth, Section 7, rather than inferred from finite coefficient agreement [14]. The original 1975 Niebur proof was inaccessible through the checked route [13]. The attribution is retained with that access limitation. The size and divisibility alternatives remain compatible with a prime zero.

S15: differential equations. Fonseca records the normalized Ramanujan differential relations used in the polynomial computation [12]. The project-specific S_4 cancellation is verified algebraically in the main text and by an exact symbolic remainder. Modularity and the level-one dimension input for the cusp identity are separate contracts [11]. A finite output check is not substituted for the universal algebraic identity.

D.3 Cores, harmonic theta series and valence

S02: Han's packet identity. Han's Definition 1.1 and Theorems 1.1–1.2 in the checked first arXiv version provide the five-core specialization and Vandermonde product [8]. The specialization $t = 5$, $z = 25$ gives the shell norm $10n$ after the coefficient shift. With the orientation used here, the normalization is $D/288$. The identity permits cancellation and does not imply that its terms have one sign.

S13: packet cardinality. Garvan, Kim and Stanton's Theorem 4 and equation (5.3) give the five-core count [9]. In the present conventions this reads

$$a_5(n-1) = \sum_{d \mid n} \chi_5(d) \frac{n}{d}.$$

For a prime different from 5, the shell cardinality is $p + \chi_5(p)$. This is the number of packets, not their weighted sum.

S03: harmonic theta modularity. Ibukiyama and Williams, Section 3.4 of the checked version, give the vector-valued harmonic theta theorem with weight $\nu + r/2$ [10]. Here the rank is four and the harmonic polynomial is alternating of degree $10 + d$. The scalar full-level transformation follows from the separate sign-line descent in Chapter 8. It is not silently imported as part of the general theorem.

S04: the valence step. Stein's author exposition supplies the level-one valence bound and the relevant cusp-form dimensions [11]. The vanishing test is applied only after modularity, holomorphy and the weight are established. Vanishing of sufficiently many initial Fourier coefficients then proves that specific modular form is zero. It does not establish an arbitrary polynomial's universal nullness without the modular prerequisites.

D.4 Global congruences and local realization

S05: the actual compatible representations. Serre's 1973 exposition gives the established integral rank-two representations and the arithmetic-Frobenius polynomial [6]. The fixed-modulus proof uses their actual joint finite quotient. It does not require an image classification, surjectivity onto an ambient matrix group, or independence across primes ℓ.

S08: Chebotarev's theorem. The checked Stevenhagen–Lenstra exposition states the nonempty conjugacy-class theorem and distinguishes natural density [17]. In a finite Galois quotient the density is the class size divided by the group size. Complex conjugation supplies the class in the actual image. The theorem is applied separately for each fixed modulus. Nothing in the density formula interchanges the quantified modulus and prime.

S09: weak admissibility and admissibility. The Colmez–Fontaine author manuscript supplies the implication from weak admissibility to admissibility and the $N = 0$ crystalline realization in the specified setting [18]. The constructed rank-two module has equal Hodge and Newton totals and no proper Frobenius-stable rational line. Those checks concern a local object. They do not identify it with the global representation of Δ.

D.5 Access and attribution limits

The companion source records retain the exact checked versions and successful alternative author-hosted access routes. The two original-proof access limits above are explicit. The book does not label a cached excerpt, a source title, a search result or a historical status field as a completed proof audit.

The internal packet transport, nonhomogeneous content argument, interpolation analysis and arithmetic model constructions are reviewed as written arguments. Their correctness is distinct from whether any formulation has appeared elsewhere. This edition makes no historical-priority claim for them and no exhaustive novelty claim.

CHAPTER E

Reproduction and evidence boundaries

The companion archive is a reproducible record of this edition's finite claims. It preserves six recovered research ZIPs and all their original extracted file members. New executions write to separate directories. The present prose and audits determine what this reader claims. Historical labels remain available for comparison without being treated as current validation.

E.1 What was rerun

Sixteen fresh suites completed. Their version-two execution manifests validate current inputs, scientific results and logs. Thirteen historical scientific outputs agree exactly with the fresh results after removing only elapsed-time metadata fields named `runtime`, `runtime_seconds` and `seconds`. Three additional suites provide independent coefficient, transfer and large-residue checks.

The most direct coefficient check covers every $1 \leq n \leq 512$. The triangular recurrence, the defining binomial product and an independent logarithmic-derivative divisor-sum engine agree exactly. All three use integers. This range is distinct from the historical final-arithmetic program's product check through 128, and from the independent product check in the packet-tree program through 512.

The arithmetic survivor search uses τ through 10000 and twenty-four selected moduli. Twenty-two moduli have a matching prime in that bounded search, supplying $22 \times 32 = 704$ local recurrence checks. Moduli 11 and 19 have no observed witness in that search. This is a search outcome, not a contradiction of the density theorem. Modulus one is included as a trivial case.

The synthetic recurrence suite has 256 moduli, three parameter choices and thirty-two positive exponents, for 24576 checks. Its value $p = 5M - 1$ need not be prime. Separately, the 1229 prime-indexed companion matrices through 10000 are constructed local models. They are not sampled global Frobenius matrices.

E.2 Packet and polynomial checks

The five-core suite checks 405 shell sums and 326585 packet polynomials, with 308 prime shell counts. Twenty-five coefficients also agree with an independent hook-partition calculation. The tree suite enumerates all shells through 512 and the two additional shells 737 and 2411, giving 514 shell sums and 95937 packets. It checks 3084 sector partitions and explicit sharp-boundary witnesses at sixty-four depths.

The shell at 737 is a sparse additional witness. The replay does not enumerate every shell from 513 through 736. The universal threshold formula rests on the separate lower-bound proof and the symbolic witness families.

The retained universal-null suite checks fourteen exact rational sign certificates. Its degree-at-most-sixteen dual at 31 concerns the seven stored corrections, not all possible corrections.

The primitive transport suite checks thirty-six symbolic harmonic certificates, 128 Mobius identities, 144 profiles and the stored rational transfer. The independent 31 calculation obtains coarse, fine and detail ranks 5, 15 and 10 over the rationals and the two prime fields 65537 and 1000003, with structural upper bounds.

The content obstruction is the exact shell 121. Its masses are

$$-25439104668 + 25937424601 = 498319933.$$

The computation establishes these integers. The theorem that every finite symmetric universal null conserves each content is a separate proof. Likewise, the interpolation suite's finite orbit checks and two explicit failures do not prove an all-degree fullness theorem.

E.3 Large modular coefficients

The valuation suite regenerates coefficients at all 13848 primes through 150000. Its proposed valuation equalities are tested only on their stated candidate residue classes. Independent Eisenstein convolutions verify the selected counterexample coefficients. The dyadic obstruction uses twelve selected primes through 1032191, with nineteen fresh modular comparisons.

The CRT/NTT computation returns residues modulo 2^{19}. It is not a reconstruction of those large integer tau coefficients. Within each convolution, however, the integer coefficient is reconstructed before reduction. With

$$L = 1032191, \qquad M = 524288,$$

the nonnegative representative bound is

$$L(M-1)^2 = 283725419316756479 < 998244353 \cdot 1004535809 = 1002772198720536577.$$

The common transform length 2^{21} divides both finite-field multiplicative periods. This bound, primitive-root checks and integer-type limits are part of the computation contract. Modular agreement alone would not justify CRT reconstruction without them.

E.4 How to reproduce the record

Paths in this section are relative to the extracted project root. The executable entry point is

```
python3 code/run_replays.py --output-root evidence/computation/reproduced
```

The output directory must be fresh. Individual suite names may precede the output option. The tested runtime and dependency versions are recorded in `evidence/BUILD_ENVIRONMENT.json`. Each execution records its actual arguments, inputs, result hashes and resource outcomes.

The full replay uses a 600-second wall-clock bound per suite, a 570-second per-process CPU cap, inherited one-core affinity and a two-million-byte combined log limit. Numerical-library thread caps are cooperative. No address-space cap is claimed. Exact arithmetic programs can consume substantial time or memory outside the recorded ranges.

The two readers rebuild with

```
python3 build.py --figures
python3 quality_check.py
python3 validate_fonts.py
```

Rendering creates immutable snapshots for inspection. Automated checks locate unresolved references, clipped text and font problems. Every final page in this edition also has an individual visual review record tied to the final PDF hash. A fresh rebuild may change PDF metadata, so its layout approval must be tied to its own output rather than copied from this edition.

E.5 What the evidence does not claim

No scientific replay failed. One inventory collector initially read the wrong schema field, then was corrected. Its failure and correction are recorded rather than hidden as a successful mathematical execution.

The recovered project contains written mathematics and exact programs, with no recovered Lehmer proof-assistant source project or compiled certificate. The formal inventory is an inventory and definition audit. It is not a Lean verification of the theorems in this book.

Each exact certificate proves its stated finite equality, rank or sign claim. Each range check ends at its stated bound. The universal deductions use the written proofs and the external contracts in Appendix D. The all-prime nonvanishing statement remains open in this work.

Bibliography

[1] D. H. Lehmer, *The vanishing of Ramanujan's function $\tau(n)$*, Duke Mathematical Journal **14** (1947), 429–433. DOI: 10.1215/S0012-7094-47-01436-1. Original scan. The scan was authenticated, but its original proof was not readable through the source-check route. The prime reduction is a classical attributed input and is proved separately in this reader.

[2] Jennifer S. Balakrishnan, William Craig, and Ken Ono, *Variations of Lehmer's Conjecture for Ramanujan's tau-function*, arXiv:2005.10345v1, 20 May 2020. Checked classical leaves: equation (1.2), p. 1, for the congruences; Theorem 2.2, p. 4, for the Hecke relations and bound; the beginning of §3, p. 7, for the Jacobi proof of the odd-square parity criterion. The omitted-value and integral-point proofs are outside this edition's source check.

[3] Maarten Derickx, Mark van Hoeij, and Jinxiang Zeng, *Computing Galois representations and equations for modular curves $X_H(\ell)$*, arXiv:1312.6819v2, 18 March 2014. Corollary 1.2, p. 2; its computational proof in §4, p. 11. The cited nonvanishing range has the strict endpoint $n < 816212624008487344127999$. This is a historical source result, not an edition replay or a verified latest record.

[4] Pierre Deligne, *Formes modulaires et représentations ℓ-adiques*, Séminaire Bourbaki 1968/69, exposé 355, published 1971, 139–172. NUMDAM original. Theorem 5.6 supplies the implication from the Weil conjectures to the Ramanujan bound. The author's English translation, Proposition 4.8 and equation (5.7), was also checked. Its cohomological F is inverse arithmetic Frobenius; the reader uses the rank-two arithmetic normalization specified in [6].

[5] Pierre Deligne, *La conjecture de Weil: I*, Publications Mathématiques de l'IHÉS **43** (1974), 273–307. DOI: 10.1007/BF02684373. Theorem 1.6, p. 276. For the modular specialization giving $|\tau(p)| \leq 2p^{11/2}$, this purity theorem is used with the implication in [4].

[6] Jean-Pierre Serre, *Congruences et formes modulaires (d'après H. P. F. Swinnerton-Dyer)*, Séminaire Bourbaki 1971/72, exposé 416, Lecture Notes in Mathematics **317** (1973), 319–338. NUMDAM PDF. §3.1, pp. 331–332, supplies the established integral ℓ-adic representations and their unramified Frobenius polynomials. Theorem 10 and §3.3, p. 335, concern image exceptions. These statements do not assert joint surjectivity for an arbitrary product of reductions.

[7] Jean-Pierre Serre, *Une interprétation des congruences relatives à la fonction τ de Ramanujan*, Séminaire Delange–Pisot–Poitou **9** (1967/68), no. 1, exposé 14, 1–17. NUMDAM PDF. §2.2, equation (21), p. 4, for modulo 691; §2.1, equation (16), p. 3, and §4.5, p. 11, for the stated prime congruences modulo 23 and 23^2. The representation existence is still conjectural in this 1968 text; its established version is cited from [6].

[8] Guo-Niu Han, *The Nekrasov–Okounkov hook length formula: refinement, elementary proof, extension and applications*, arXiv:0805.1398v1, 9 May 2008, Definition 1.1 and Theorems 1.1–1.2, p. 3. Published in Annales de l'Institut Fourier **60** (2010), 1–29, DOI: 10.5802/aif.2515. The checked specialization is $t = 5$, $z = 25$: shell $\sum v_i^2 = 10n$ and Vandermonde weight $D(v)/288$. Integrality and the sign consequences are separate project arguments.

[9] Frank Garvan, Dongsu Kim, and Dennis Stanton, *Cranks and t-cores*, Inventiones Mathematicae **101** (1990), 1–17. DOI: 10.1007/BF01231493. Checked author manuscript, §2, Bijections 1–2; §5, Theorem 4 and equation (5.3), printed p. 8. The exact five-core count is translated to V-coded packets using Han's bijection; it is not a sign assertion about the weighted sum.

[10] Tomoyoshi Ibukiyama and Brandon Williams, *Hermitian modular forms and algebraic modular forms on $SO(6)$*, arXiv:2505.23497v1, header 29 May 2025, internal date 12 June 2025. §3.4, printed pp. 18–19, gives vector-valued harmonic theta modularity and weight $\nu + n/2$. The scalar full-level sign-line descent is a project argument. The paper's conjectural Hermitian correspondences are not assumed.

[11] William Stein, *Modular Forms: A Computational Approach*, author online chapter *Modular Forms of Level 1*, checked 9 October 2026. Theorem 2.11 for valence; Theorem 2.14, Corollaries 2.15–2.16, and Theorem 2.17 for the level-one structure and dimensions. The chapter's Eisenstein series differ by nonzero scalars from the reader's constant-one normalization; its displayed normalized identity is $\Delta = (E_4^3 - E_6^2)/1728$.

[12] Tiago J. Fonseca, *Higher Ramanujan equations I: moduli stacks of abelian varieties and higher Ramanujan vector fields*, arXiv:1612.05081v1, header 15 December 2016, internal date 5 November 2018. Introduction, p. 2, and Remark 6.4, p. 29, for the normalized classical differential relations. The reader's polynomial residual identity is reconstructed from these inputs, rather than inferred from bounded coefficient tests.

[13] Douglas Niebur, *A formula for Ramanujan's τ-function*, Illinois Journal of Mathematics **19** (1975), 448–449. DOI: 10.1215/ijm/1256050746. The original proof could not be accessed through the check route. The universal convolution identity was checked in the primary-paper restatement [14]; the differential reconstruction in this reader uses [12, 11].

[14] B. K. Moriya and C. J. Smyth, *Index-dependent divisors of coefficients of modular forms*, author manuscript, dated 1 May 2013, accepted 28 April 2013. §7, printed pp. 10–11, restates Niebur's identity for every $n \geq 1$ and gives the symmetric half-sum convention. §2.1 explicitly retains the alternative $a(p) = 0$ in the size/divisibility hypothesis. The locators refer to the inspected manuscript.

[15] Nik Lygeros and Olivier Rozier, *A New Solution to the Equation $\tau(p) \equiv 0 \pmod{p}$*, Journal of Integer Sequences **13** (2010), article 10.7.4. Journal PDF, §§4–6, printed pp. 3–6, for the trace moments, Catalan inverse, and reduced-form convention. The Hurwitz count includes imprimitive forms and exceptional weights $1/2$ and $1/3$; $6H$ is the stored integer. The zeroth moment keeps half of the $t = 0$ term.

[16] Nik Lygeros and Olivier Rozier, *Odd prime values of the Ramanujan tau function*, Ramanujan Journal **32** (2013), 269–280. DOI: 10.1007/s11139-012-9420-8. Checked author PDF, p. 2, equations (3)–(11), as restatements of the sharper small-prime congruences, with their stated residue and unit hypotheses. The original earlier proofs of the entire congruence table were not separately audited.

[17] Peter Stevenhagen and Hendrik W. Lenstra Jr., *Chebotarëv and his density theorem*, The Mathematical Intelligencer **18** (1996), no. 2, 26–37. Checked author scan, p. 31 for the natural-density convention and p. 34 for the conjugacy-class theorem. The historical project URL failed; this author-hosted copy was inspected successfully. The theorem applies to one fixed finite quotient and a nonempty conjugacy class, giving density $|C|/|G|$.

[18] Pierre Colmez and Jean-Marc Fontaine, *Construction des représentations p-adiques semi-stables*, Inventiones Mathematicae **140** (2000), 1–43. DOI: 10.1007/s002220000042. Checked 37-page author manuscript: Theorem A, p. 2; §3.4, pp. 21–22; §4.1 and Corollary 4.3, pp. 23–24; the period-ring identity on p. 11. These are manuscript locators. Local realization does not identify the constructed object with the restriction of Deligne's global representation.

[19] David Kurniadi Angdinata et al., *ABC implies that Ramanujan's tau function misses almost all primes*, arXiv:2603.29970v3, 25 April 2026, internal date 28 April 2026. Only the contemporary introductory statement of Lehmer's conjecture and the classical inputs in §2.1, equation (2.1), and Proposition 6 were checked here. The paper's conditional prime-value theorem is not an input to this reader.

[20] Minjia Shi, Lu Wang, and Patrick Solé, *Proof of the Lehmer conjecture on Ramanujan's τ function*, arXiv:2503.23498v2, 1 April 2025, withdrawn. Only the explicit withdrawal metadata was checked. This title is not cited as an established proof or as a theorem input.