Inverse Galois

A visual guide to the realization boundary

Komorebi P.

READER EDITION 1.0 / 9 OCTOBER 2026

Twenty-two figures follow the passage from a finite group to a Galois extension, from an action to its invariant coordinates, and from a compatible lift to an arithmetic realization. The small examples are complete. The universal existence problem remains open.

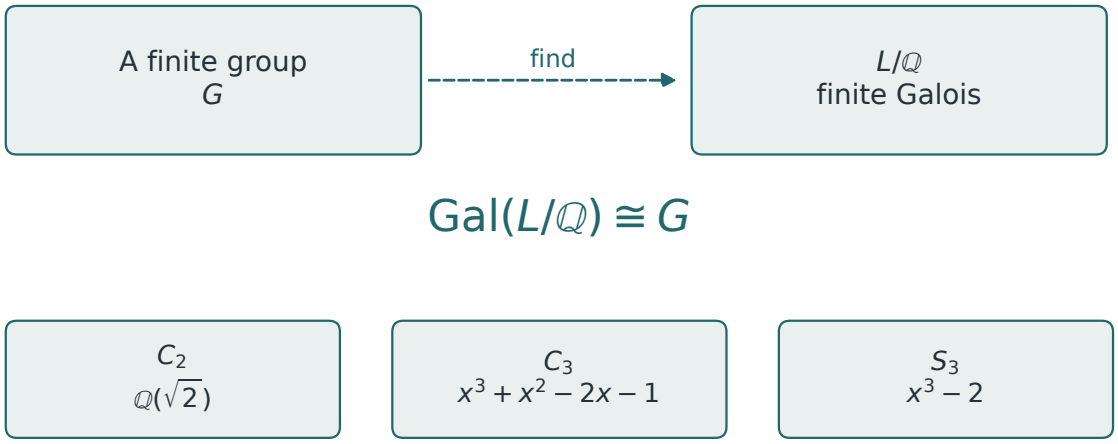
WORKED EXAMPLES / EXACT COORDINATES / OPEN PROBLEMS

01 / What must be realized?

OPEN QUESTION / one field for every finite group

The realization question

OPEN / a quantifier over every finite group



Examples witness particular groups. They do not settle the universal question.

READ THE FIGURE

The inverse Galois problem over $\mathbb{Q}$ asks whether every finite group G is isomorphic to $\mathrm{Gal}(L/\mathbb{Q})$ for some finite Galois extension L.

Each field at the bottom realizes one specific group. Such constructions teach us how an abstract symmetry becomes an arithmetic object.

The dashed arrow carries the universal quantifier. A family of examples, a faithful representation or a finite computation does not replace that quantifier with a proof.

The monograph develops the examples and then identifies the steps that the operator language still needs.

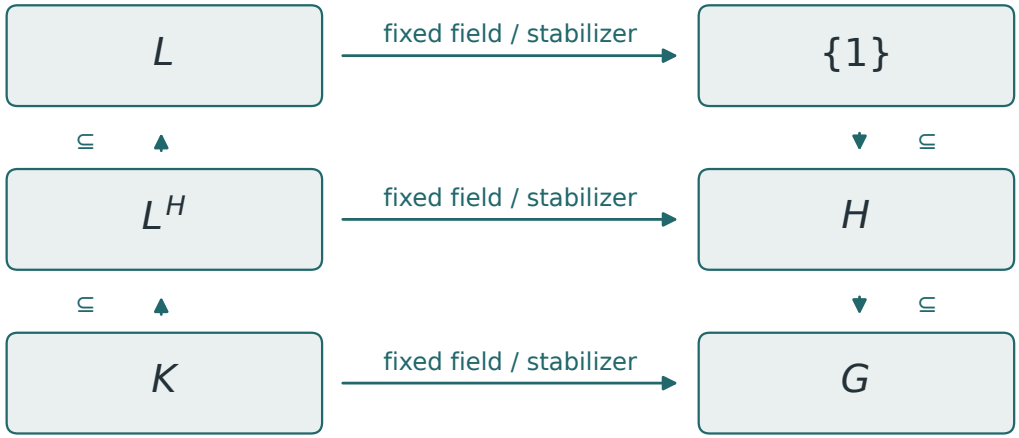
Carry forward. Keep the base field in view. A realization over an invariant function field and a realization over $\mathbb{Q}$ are different assertions.

02 / The correspondence reverses inclusion

SOURCE THEOREM / finite Galois correspondence

Subgroups and intermediate fields

SOURCE / finite Galois correspondence



$$[L : L^H] = |H|$$
$$[L^H : K] = [G : H]$$

READ THE FIGURE

Suppose L/K is finite Galois with group G. A subgroup $H \leq G$ fixes the intermediate field L^H. Larger subgroups impose more fixedness and therefore produce smaller fields. The arrows in the two columns point in opposite directions. The degree formulas keep the dictionary exact:

$$[L : L^H] = |H|, \qquad [L^H : K] = [G : H].$$

The correspondence is a theorem about an extension already given. It does not construct that extension from an arbitrary group. See Milne2022 and the monograph's source contracts.

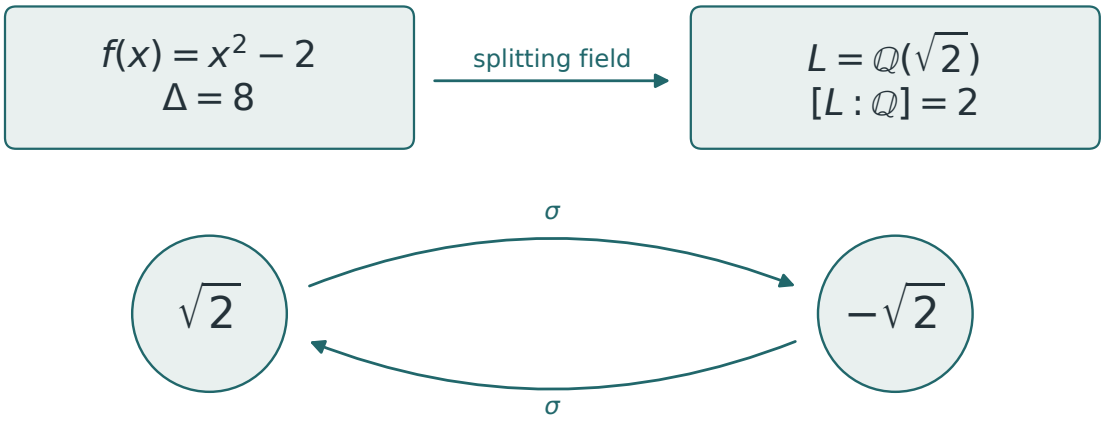
Carry forward. Subgroup order measures the degree down from L. Subgroup index measures the degree up from K.

03 / A quadratic group is completely visible

WRITTEN PROOF / the field $\mathbb{Q}(\sqrt{2})$

A complete quadratic realization

WRITTEN PROOF / Eisenstein at 2 and two conjugates



$$\sigma^2 = 1, \qquad \mathrm{Gal}(L/\mathbb{Q}) = \{1, \sigma\} \cong C_2$$

READ THE FIGURE

The polynomial $x^2 - 2$ is irreducible over $\mathbb{Q}$ by Eisenstein at 2. Adjoining one root also supplies its negative, so $L = \mathbb{Q}(\sqrt{2})$ is its splitting field. There are two automorphisms. The identity fixes the root, while σ changes its sign. Applying σ twice returns the root.

Thus $\mathrm{Gal}(L/\mathbb{Q}) \cong C_2$. The diagram represents exact conjugation rather than a measured trajectory. The discriminant 8 is an exact coefficient calculation.

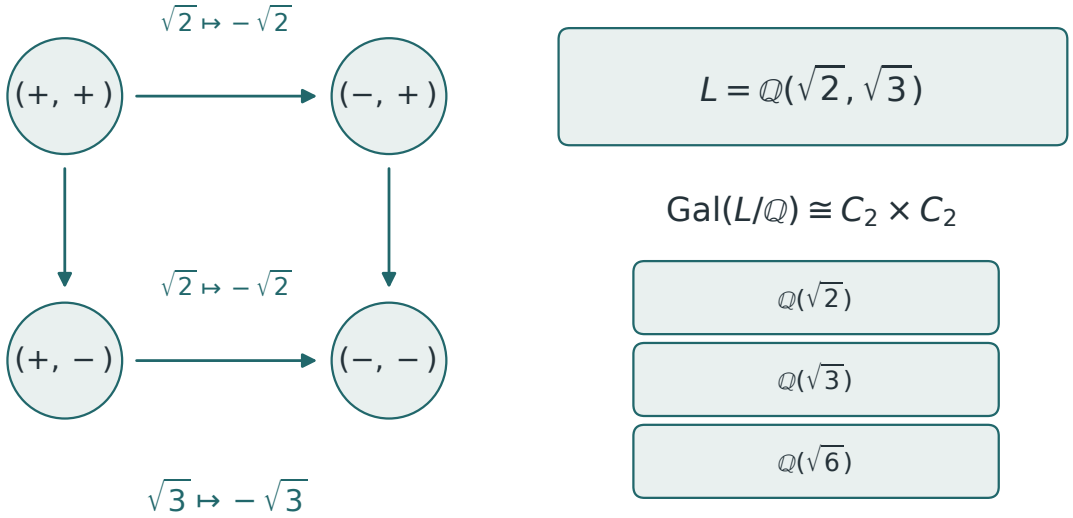
Carry forward. This is a closed realization: a polynomial, its splitting field, its degree and its complete automorphism group.

04 / Independent sign changes make a four-element group

WRITTEN PROOF / the biquadratic field

Two independent sign changes

WRITTEN PROOF / the biquadratic field



The three order-two subgroups give exactly three quadratic intermediate fields.

READ THE FIGURE

In $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, the signs of the two square roots can change independently. Neither square root belongs to the quadratic field generated by the other, so $[L : \mathbb{Q}] = 4$.

The four sign choices form $C_2 \times C_2$. Every non-identity element has order two.

Its three subgroups of order two fix exactly the three fields shown on the right. Each has degree two over $\mathbb{Q}$.

The square records group composition. It does not assert that every group is a product of sign changes.

Carry forward. A field lattice can be read through its subgroup lattice, once the Galois extension and its action have been established.

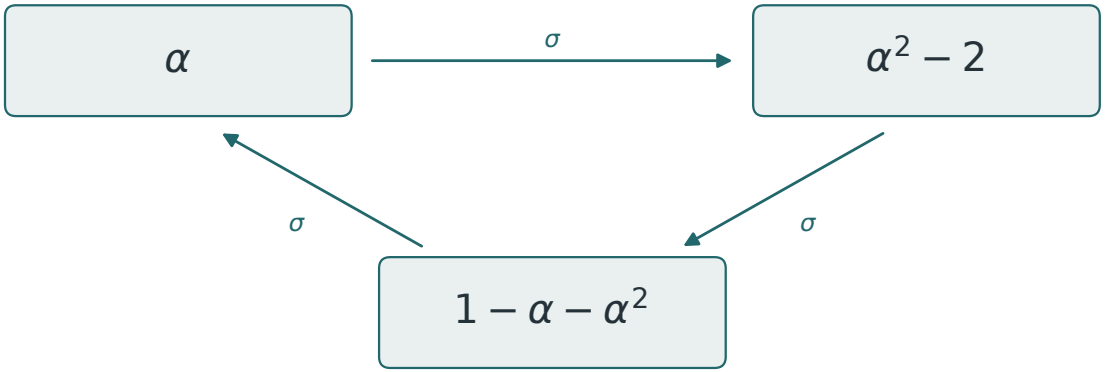
05 / An irreducible cubic can already be normal

WRITTEN PROOF / a cyclic cubic

A cubic whose three roots form one cycle

WRITTEN PROOF / an irreducible cubic with square discriminant

$$f(x) = x^3 + x^2 - 2x - 1, \qquad \Delta = 49$$



$$\sigma^3 = 1, \qquad L = \mathbb{Q}(\alpha), \qquad \mathrm{Gal}(L/\mathbb{Q}) \cong C_3$$

READ THE FIGURE

Let α be a root of $f(x) = x^3 + x^2 - 2x - 1$. The only possible rational roots are ± 1, and neither is a root. Thus f is irreducible.

Its discriminant is 49. An irreducible cubic has a transitive Galois group in S_3, and a square discriminant places that group inside A_3. The group is therefore C_3.

The other roots are $\alpha^2 - 2$ and $1 - \alpha - \alpha^2$. They already lie in $\mathbb{Q}(\alpha)$, and conjugation cycles the three expressions.

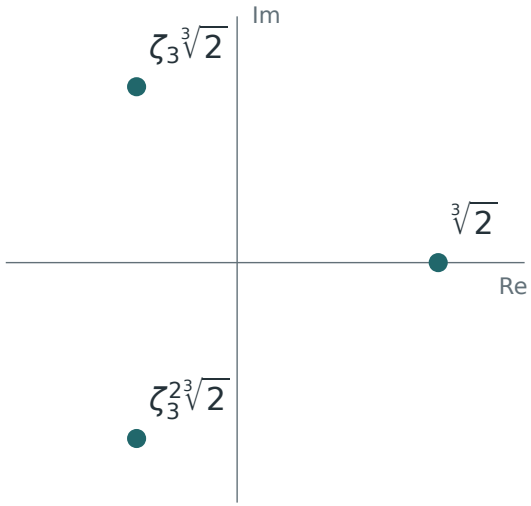
Carry forward. The square discriminant certifies an even-permutation group. Irreducibility supplies transitivity. Both facts are needed.

06 / A cubic root is not always the whole splitting field

EXACT EXAMPLE / approximate root positions

The splitting field needs the nonreal roots too

COMPUTATION / approximate positions, exact algebraic labels



$$L = \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$$

$$[L : \mathbb{Q}] = 6$$

$$\Delta(x^3 - 2) = -108$$

$$\mathrm{Gal}(L/\mathbb{Q}) \cong S_3$$

The root plot illustrates the field. Irreducibility and discriminant certify the group.

READ THE FIGURE

The root $\sqrt[3]{2}$ generates a cubic field, but its two nonreal conjugates require the cube root of unity ζ_3.

The splitting field is $\mathbb{Q}(\sqrt[3]{2}, \zeta_3)$. Its degree is six. Equivalently, $x^3 - 2$ is irreducible and its discriminant -108 is not a rational square, giving group S_3.

The dots use approximate complex coordinates. They are a visual aid. The polynomial, degree argument and discriminant establish the Galois group independently of those coordinates.

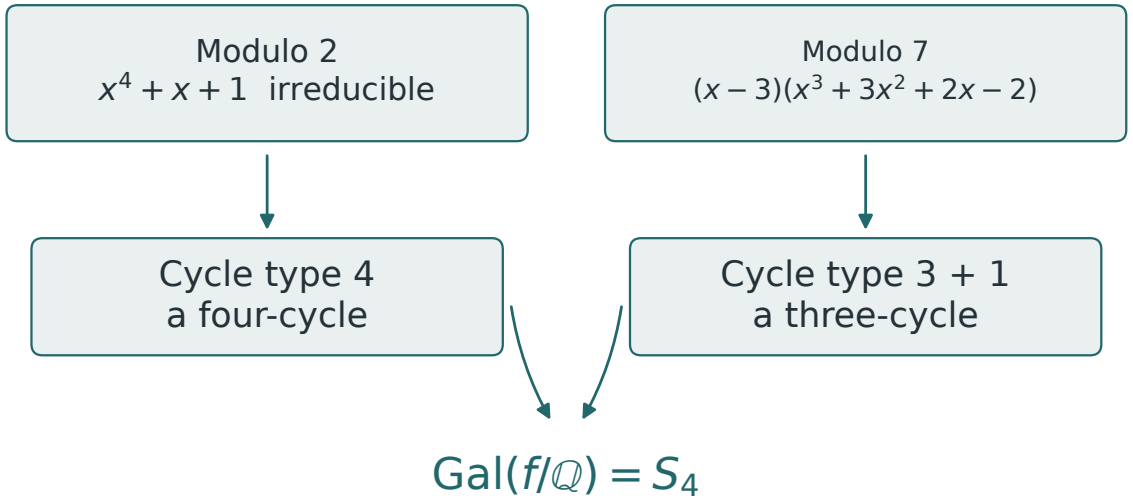
Carry forward. Always distinguish the field generated by one root from the field containing all roots. A root plot does not certify a Galois group.

07 / Reduction modulo a prime gives a cycle type

SOURCE INPUT + EXACT FACTORS / the quartic $x^4 - x - 1$

Finite factors reveal permutations

SOURCE + EXACT COMPUTATION / unramified primes for $x^4 - x - 1$



$\Delta = -283$; both primes are unramified. Finite factorization is not a universal realization.

READ THE FIGURE

At an unramified prime, the degrees of the irreducible factors record the cycle type of a Frobenius element in the Galois group.

Modulo 2, the quartic is irreducible. Modulo 7, it has irreducible factors of degrees 1 and 3. Since the discriminant is -283, both primes are unramified. The group contains a four-cycle and a three-cycle. Among the transitive subgroups of S_4, this forces S_4.

These are cycle types. The finite factors do not number the complex roots or identify specific permutations in a shared root labelling.

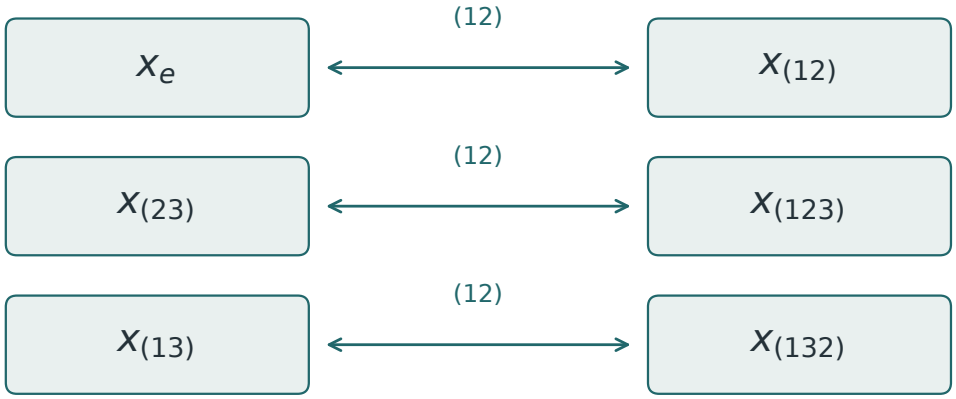
Carry forward. Exact finite calculations can support a complete example when a theorem translates their output into the required arithmetic statement.

08 / An abstract group has an exact variable action

WRITTEN PROOF / the left permutation action

Every group can act on independent variables

WRITTEN PROOF / faithful permutation action



$$h \cdot x_g = x_{hg}, \qquad F = \mathbb{Q}(x_g : g \in G)$$

READ THE FIGURE

For a finite group G, introduce algebraically independent variables x_g, one for each element. Define

$$h \cdot x_g = x_{hg}.$$

This gives a faithful action on $F = \mathbb{Q}(x_g : g \in G)$. The product convention is right to left: h acts by left multiplication on each index.

The picture shows the three swaps induced by (12) in S_3. It displays exact index changes rather than numerical coordinates.

Faithfulness means that different group elements induce different automorphisms. It still leaves the base field to be identified.

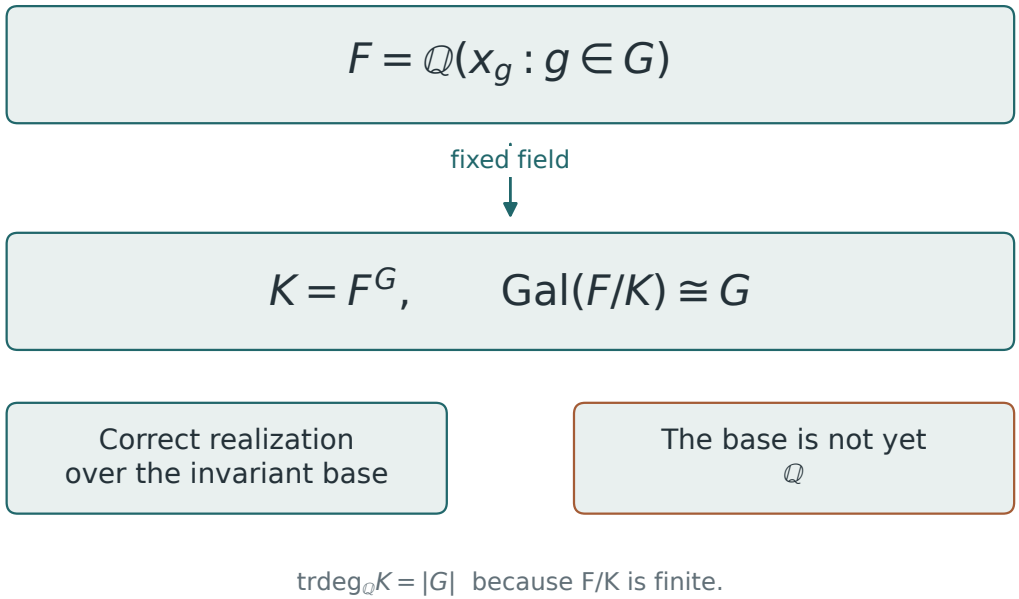
Carry forward. An action supplies symmetry. Its fixed field supplies a base. The two together produce a Galois extension over that base.

09 / The generic construction realizes the group over its fixed field

SOURCE THEOREM / Artin's theorem with the base stated

The invariant field is the base produced by the action

SOURCE / Artin fixed-field theorem



READ THE FIGURE

Set $K = F^G$. Artin's fixed-field theorem gives

$$\mathrm{Gal}(F/K) \cong G, \qquad [F : K] = |G|.$$

Thus every finite group has a realization over this invariant field.

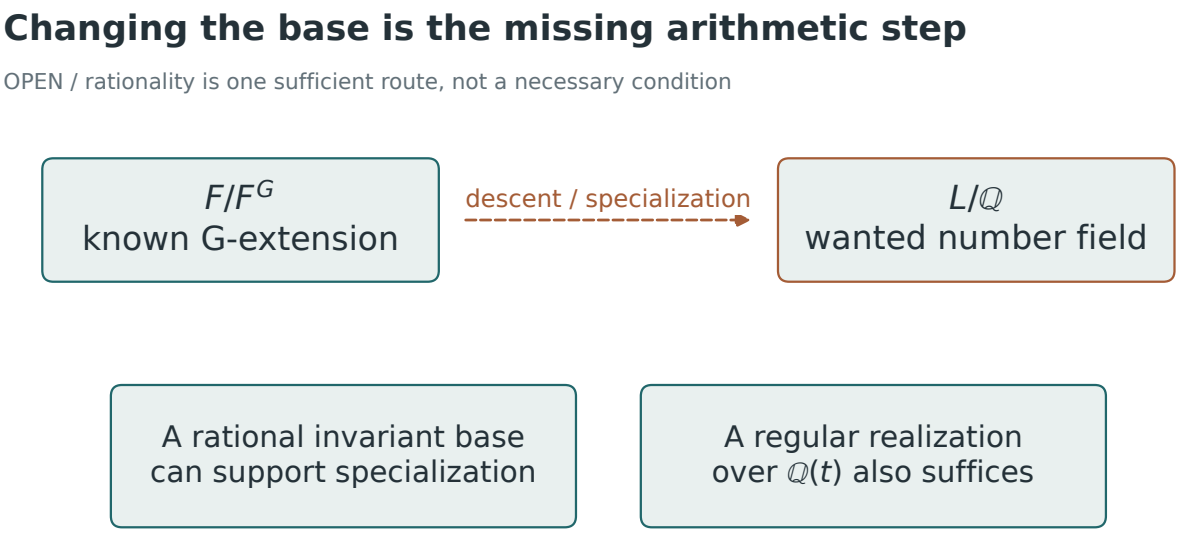
However, K is not $\mathbb{Q}$. Since F/K is finite, the two fields have the same transcendence degree over $\mathbb{Q}$, namely $|G|$.

Writing only the group isomorphism would hide the issue. The base F^G is part of the statement and part of its proof. See Milne2022.

Carry forward. The generic action closes the group-theoretic construction. It does not close the number-field existence problem.

10 / A route to another base needs its own theorem

OPEN BOUNDARY / sufficient routes are not equivalences



READ THE FIGURE

Rationality of the invariant field is a useful sufficient route. If a suitable invariant base is purely transcendental over $\mathbb{Q}$, specialization can bring the construction back to arithmetic.

Another route starts with a regular realization over $\mathbb{Q}(t)$ and applies Hilbert irreducibility.

Neither route should be read backwards. Failure of a particular rationality claim does not show that the group cannot occur over $\mathbb{Q}$.

The dashed arrow asks for an arithmetic construction with its hypotheses proved. It is not supplied by the action alone.

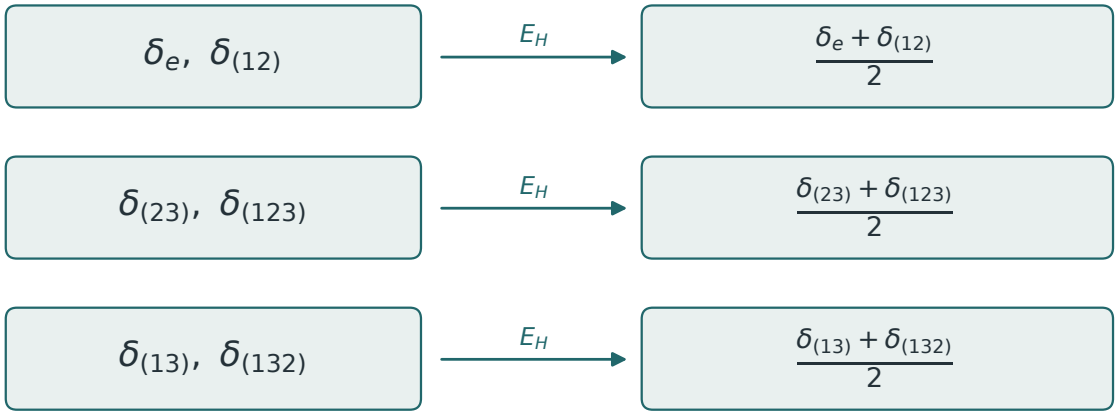
Carry forward. Noether's rationality problem is one route toward inverse Galois. It is not a necessary condition for every number-field realization.

11 / Averaging over a subgroup makes a projector

WRITTEN PROOF / exact invariant coordinates

An average turns orbits into invariant coordinates

WRITTEN PROOF / left regular action on Q[S₃]



$$H = \langle (12) \rangle, \quad E_H^2 = E_H, \quad \operatorname{rank}(E_H) = 6/2 = 3$$

READ THE FIGURE

Work in the regular representation $\mathbb{Q}[S_3]$, with basis δ_g and action $U_h \delta_g = \delta_{hg}$. For $H = \langle (12) \rangle$,

$$E_H = \frac{1}{|H|} \sum_{h \in H} U_h.$$

The three pairs are the left H-orbits Hg. Averaging replaces each pair by its invariant mean.
The image consists of vectors constant on each orbit. Hence $\operatorname{rank} E_H = 3$ and $E_H^2 = E_H$.
These identities concern a representation already constructed. They are not an arithmetic existence theorem.

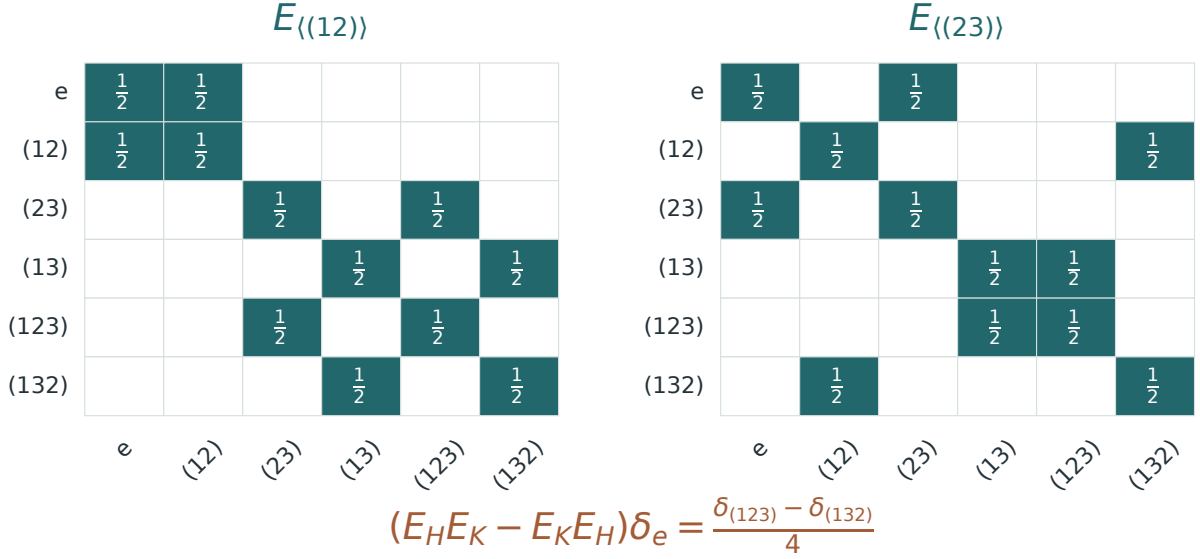
Carry forward. In the regular representation, the invariant dimension is the subgroup index $[G : H]$.

12 / Two subgroup projectors need not commute

EXACT COMPUTATION / written group-algebra argument

Two exact projector matrices

EXACT COMPUTATION / rational entries in the stated S_3 basis



Nonzero commutator. Both matrices are still self-adjoint idempotents.

READ THE FIGURE

The basis order is $e, (12), (23), (13), (123), (132)$. A column records the image of its basis vector. The two averages have rational entries 0 and $1/2$. Each is a self-adjoint idempotent. Yet their products occur in different orders:

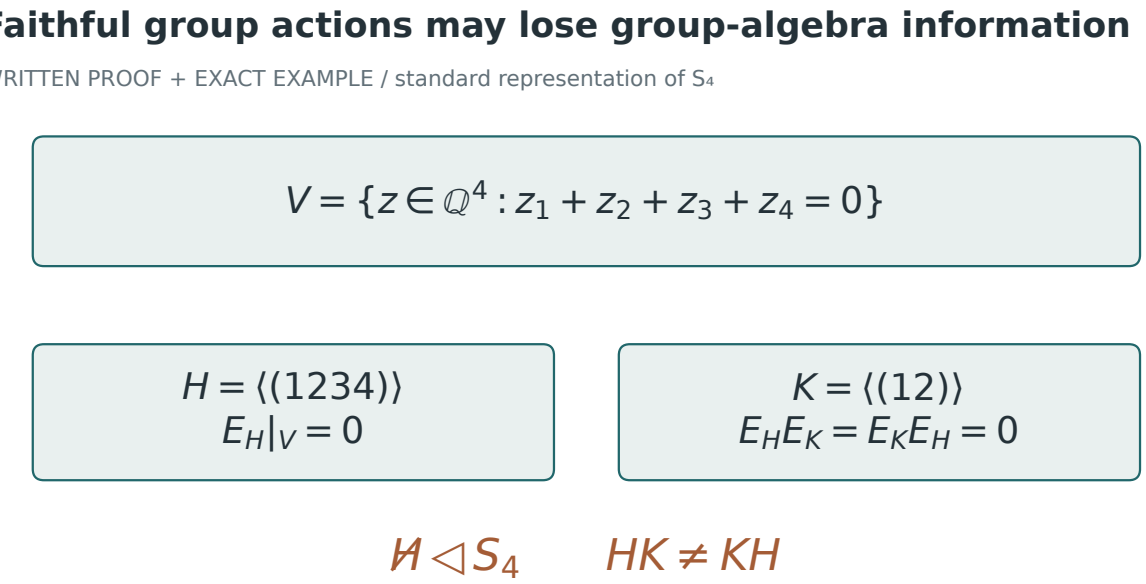
$$(E_H E_K - E_K E_H)\delta_e = \frac{\delta_{(123)} - \delta_{(132)}}{4}.$$

The exact calculation agrees with the written proof. In the group algebra, commuting subgroup averages correspond to the condition $HK = KH$. Here that condition fails.

Carry forward. Projectors can be individually valid while their compositions depend on order. The representation and coefficient domain must remain explicit.

13 / Faithfulness has two different meanings

WRITTEN PROOF + EXACT EXAMPLE / the standard S_4 representation



READ THE FIGURE

Let S_4 permute the four coordinates of its sum-zero subspace V. The action of the group on V is faithful.

The four-cycle subgroup H has no nonzero invariant vector in V, so $E_H = 0$. It commutes with every other operator. Nonetheless, H is not normal, and for $K = \langle (12) \rangle$ one has $HK \neq KH$.

The group-algebra representation loses information even though the group action is faithful. Therefore commuting averages in an arbitrary faithful group representation need not detect permutable subgroups.

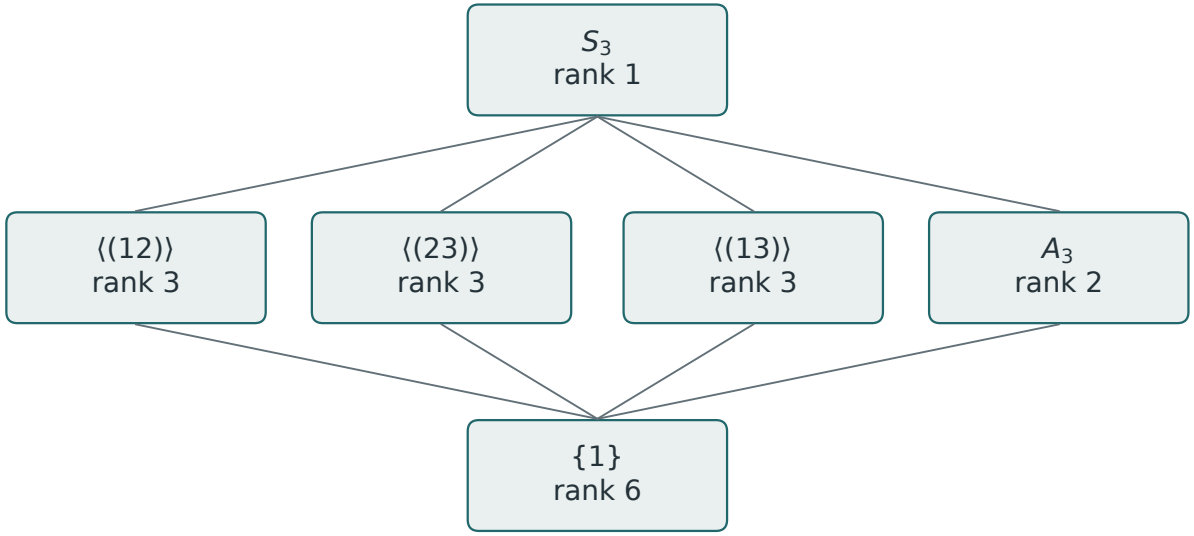
Carry forward. The regular representation faithfully sees the group algebra. An arbitrary faithful group representation need not do so.

14 / The regular representation reads the subgroup lattice

WRITTEN PROOF / six subgroups of S_3

The subgroup lattice also counts invariant dimensions

WRITTEN PROOF / Q[S₃] with the left regular action



$\dim \mathbb{Q}[S_3]^H = [S_3 : H]$ for every subgroup H.

READ THE FIGURE

The nodes are all subgroups of S_3. Lines record inclusion, with larger groups above smaller groups. For every subgroup H,

$$\dim \mathbb{Q}[S_3]^H = [S_3 : H].$$

The identity subgroup fixes the whole six-dimensional space. Each subgroup of order two fixes three dimensions. The subgroup A_3 fixes two, and S_3 fixes one.

This dimension count follows from left orbit coordinates. If a Galois extension with this group is given, its fixed-field lattice reverses these subgroup inclusions.

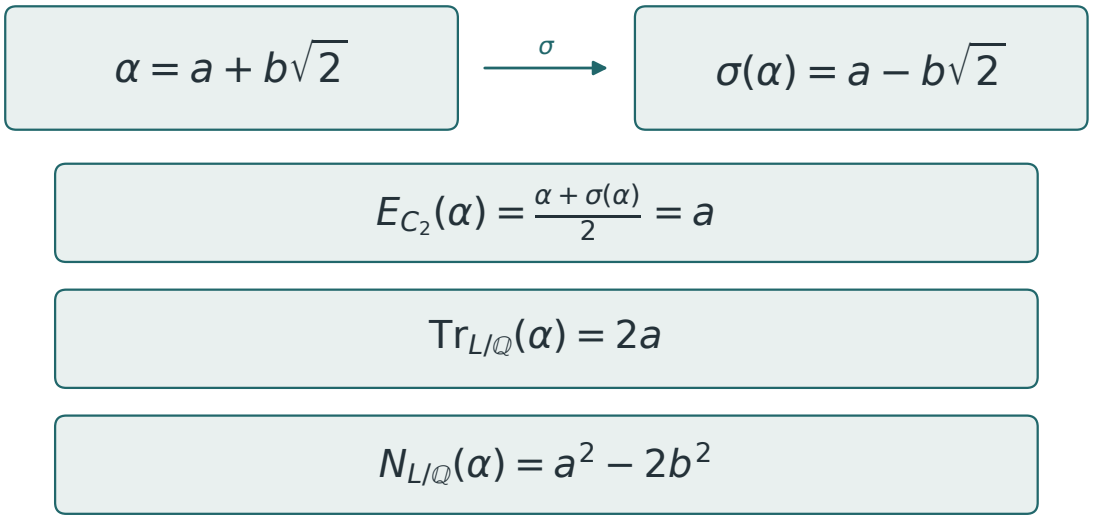
Carry forward. The same subgroup index counts regular invariant coordinates and the degree of the associated fixed field over the base. The two interpretations have different inputs.

15 / Trace and norm do not become the same map

WRITTEN PROOF / a quadratic calculation

A projector, a trace and a norm have distinct jobs

WRITTEN PROOF / quadratic conjugation



READ THE FIGURE

Write $\alpha = a + b\sqrt{2}$ with $a, b \in \mathbb{Q}$. Its conjugate is $a - b\sqrt{2}$.

The subgroup average gives the fixed component a. The trace is the sum of conjugates, $2a$. The norm is their product, $a^2 - 2b^2$.

Averaging and trace are linear. Norm is multiplicative and generally nonlinear as an additive coordinate map.

The shared conjugates explain their relation, but do not identify the three operations. General field constructions must preserve the distinction.

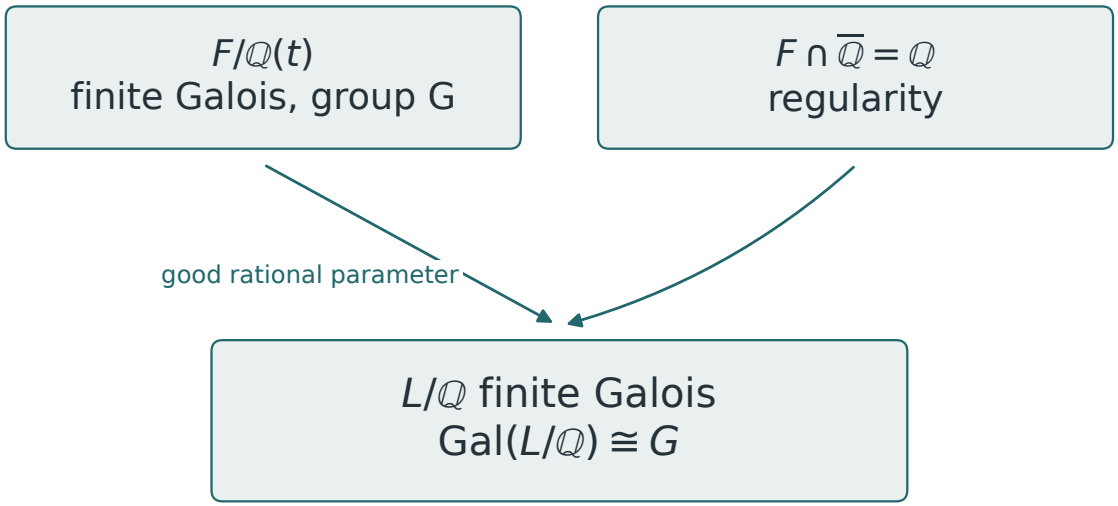
Carry forward. An average can model invariant extraction. A field norm requires the multiplicative structure of the extension.

16 / Regularity makes specialization an arithmetic bridge

SOURCE THEOREM / Hilbert irreducibility

A regular family can specialize to a number field

SOURCE / Hilbert irreducibility with the regularity hypothesis



The implication needs a regular family as input. Constructing it for every G remains open.

READ THE FIGURE

Suppose a finite Galois extension $F/\mathbb{Q}(t)$ has group G and is regular:

$$F \cap \overline{\mathbb{Q}} = \mathbb{Q}.$$

Hilbert irreducibility provides suitable rational specializations retaining the group, giving a finite Galois extension of $\mathbb{Q}$ with group G.

This is a sufficient implication with a substantial input. Constructing such a regular extension for every finite group is the regular inverse Galois problem.

The figure states the implication. It does not supply the universal regular family. See DebesHarbater2021 and DebesLectureNotes.

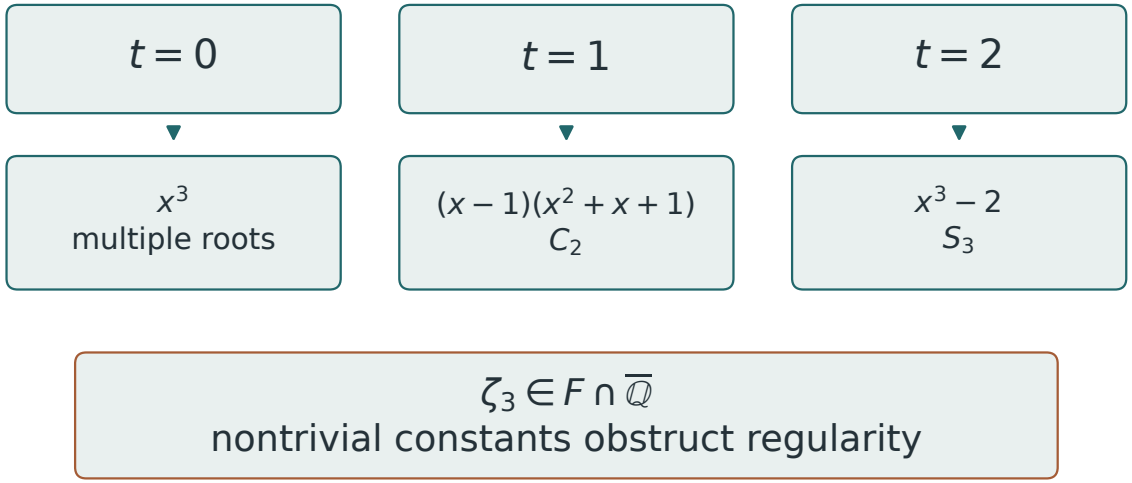
Carry forward. A regular realization over $\mathbb{Q}(t)$ implies a realization over $\mathbb{Q}$. The first existence statement still requires proof.

17 / A nonregular toy shows why hypotheses matter

EXACT EXAMPLES / specialization of $x^3 - t$

A specialization may change the group

EXACT TOY / x³ − t is not a regular splitting-field family over Q



Group changes in this toy do not supply the regular input in the previous figure.

READ THE FIGURE

At $t = 0$, the polynomial has a repeated root. At $t = 1$, its splitting field is $\mathbb{Q}(\zeta_3)$, with group C_2. At $t = 2$, the splitting field has group S_3.

Thus specialization can change both separability and the group.

The generic splitting field of $x^3 - t$ contains ζ_3. Its constant field is larger than $\mathbb{Q}$, so this family is not regular over $\mathbb{Q}$.

The toy cannot be inserted into the previous implication as a regular family. Its purpose is to expose that missing hypothesis.

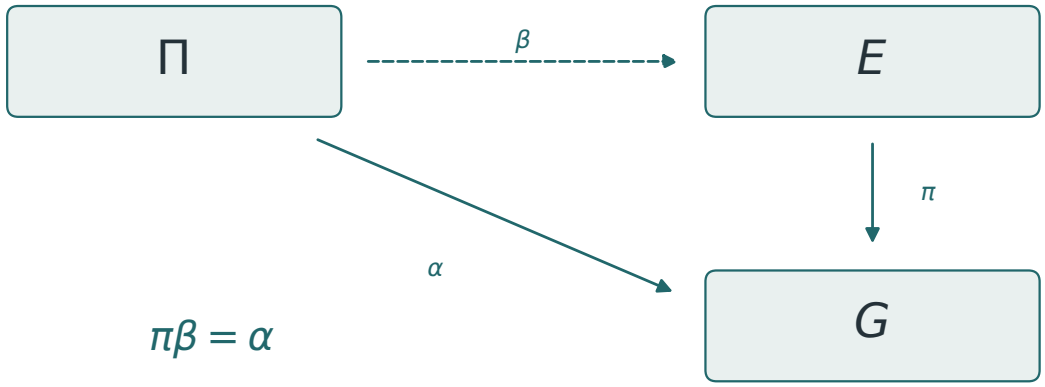
Carry forward. Check constants before invoking regular specialization. A parameterized polynomial is not automatically a regular Galois extension.

18 / A lift must exist in the specified domain

MODEL + FINITE OBSTRUCTION / embedding problems

An embedding problem asks for a compatible lift

MODEL / the domain and surjectivity requirements matter



Weak solution: a continuous lift. Proper solution: the lift is surjective.

Finite toy: Π=C₂, E=C₄, G=C₂, α=id has no lift through parity.

READ THE FIGURE

An embedding problem specifies a profinite group Π, a continuous epimorphism $\alpha : \Pi \to G$ and a finite-group epimorphism $\pi : E \to G$.

A weak solution is a continuous homomorphism $\beta : \Pi \to E$ satisfying $\pi\beta = \alpha$. A proper solution is also surjective.

For the abstract finite toy $\Pi = G = C_2$, $E = C_4$ and $\alpha = \mathrm{id}$, no lift exists through the parity quotient. An element of order at most two in C_4 has trivial parity.

This is not a counterexample involving an absolute Galois group.

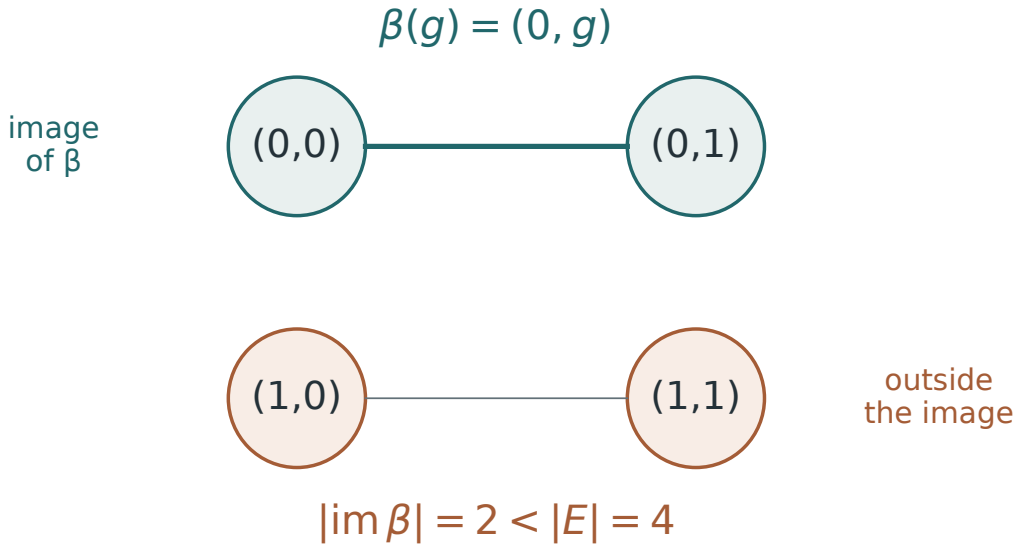
Carry forward. Compatibility, continuity and surjectivity are distinct requirements. The actual arithmetic domain cannot be replaced by an arbitrary finite toy.

19 / Splitting gives a weak lift, not automatic properness

EXACT FINITE TOY / a two-element image in a four-element target

A split weak lift can miss half the target

EXACT FINITE TOY / E=C₂×C₂, Π=G=C₂



This domain cannot surject onto E. The toy is not an absolute Galois impossibility.

READ THE FIGURE

Take $\Pi = G = C_2$, $E = C_2 \times C_2$, and let π project onto the second factor. The map

$$\beta(g) = (0, g)$$

is a compatible weak lift of the identity.
Its image is the upper row, with two elements. The target has four. Therefore the lift is not proper.
For this finite domain, no proper solution can exist by cardinality. An absolute Galois group is a different domain, so the toy gives no arithmetic impossibility statement. See SchmidtWingberg1998 for the general terminology.

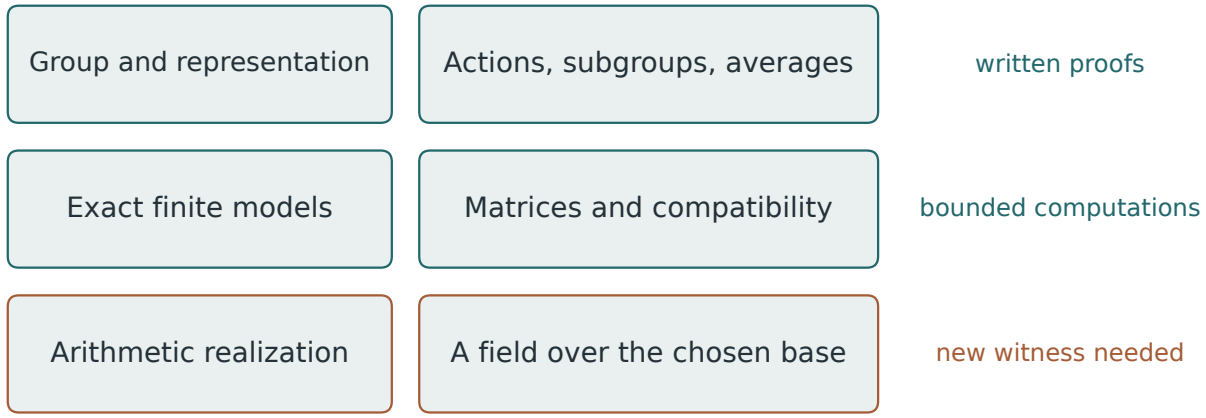
Carry forward. The section of a split extension produces a weak solution. Properness requires enough image to reach the kernel as well.

20 / Identify exactly what the calculus has established

CLAIM BOUNDARY / group algebra versus arithmetic existence

Separate the operator theorem from field realization

BOUNDARY / a correct reduction still needs its missing witness



An operator identity is not itself a polynomial, a splitting field, or a proper arithmetic lift.

READ THE FIGURE

The operator layer gives a precise language for actions, invariant coordinates, subgroup averages and compatibility.

Finite exact computations test these structures in stated examples. They can reveal a false implication and verify a concrete matrix calculation.

The arithmetic layer asks for an actual field over the chosen base, or for a proper solution in an absolute Galois domain. Neither layer is created by declaring an operator identity.

The useful reduction specifies that remaining witness. It must not erase the witness from the final claim.

Carry forward. An algebraic framework can sharpen an obstruction and make a missing implication explicit without resolving the universal existence problem.

21 / The status is part of the mathematical statement

PROOF / SOURCE / EXACT COMPUTATION / MODEL / OPEN

Read each result at its actual scope

STATUS / the guide does not claim a universal realization theorem

Classical explicit fields	C_2, $C_2 \times C_2$, C_3, S_3, S_4	closed examples
Generic invariant base	Every finite G over F^G	source theorem
Projector calculus	Regular and quotient actions	proved / exact
Regular realization for every G	Over Q(t)	open
Inverse Galois over Q	Every finite G	open
Lean certification of this set	No compiled certificate supplied	not claimed

Sources, written derivations and finite checks play different evidential roles.

READ THE FIGURE

The explicit small fields are complete examples. The generic invariant-base realization is a classical theorem. The projector statements have written derivations and exact finite checks.

The universal regular realization problem over $\mathbb{Q}(t)$ and the inverse Galois problem over $\mathbb{Q}$ remain open.

This publication supplies no compiled Lean certificate for its claims. Exact rational computations are not Lean proofs, and verified examples are not a theorem for every finite group.

Dated preprint context is recorded separately from established inputs in the source contracts.

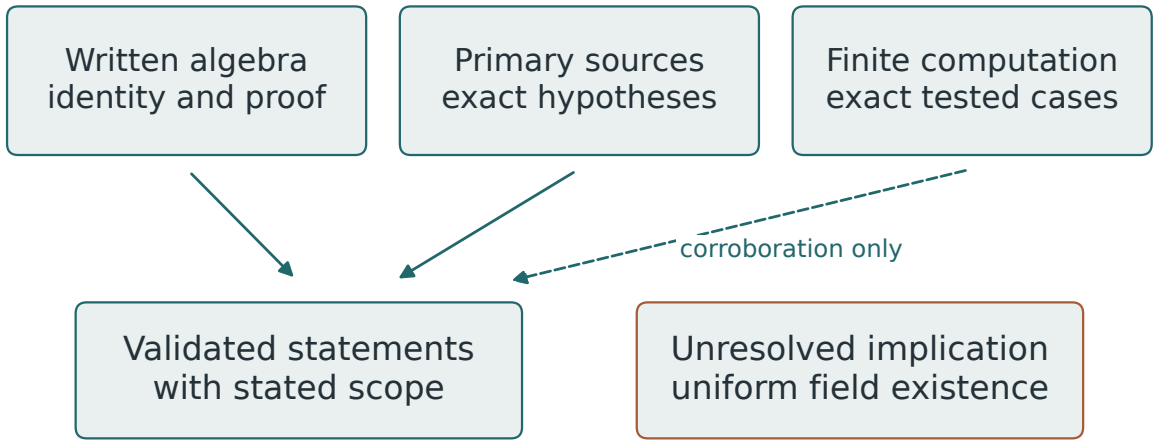
Carry forward. State a supported conclusion plainly, with its domain and hypotheses. Keep the unresolved universal step visible.

22 / Evidence must support the exact claim being made

DEPENDENCY MAP / how the publication can be checked

What supports the conclusions

DEPENDENCY / models illustrate; proofs and hypotheses carry the claim



No arrow in this diagram supplies a realization for an arbitrary finite group.

READ THE FIGURE

Written proofs establish the operator identities and the elementary examples. Primary sources supply the classical theorem inputs with their exact hypotheses.

The reproducible finite computation records polynomial data, modular factors, subgroup matrices and lifting toys. It has a stated finite scope.

The figures carry formulas and examples from those sources. Approximate root coordinates serve only as visual positions.

The remaining universal arithmetic existence question has no supporting proof in this set. The monograph identifies the gap rather than treating the diagram as its solution.

Carry forward. Follow each dependency to its evidence. A finite model may explain the open step, but cannot discharge it.

Reading the guide with the monograph

SOURCE MAP / REPRODUCTION / MATHEMATICAL SCOPE

A route through the figures

Figures 01–07 develop the problem and five complete small-group constructions. Figures 08–10 explain the generic invariant-base construction and the distinction between its base and $\mathbb{Q}$.

Figures 11–15 give the operator dictionary, including an exact noncommuting pair and a boundary example showing why faithful group actions need not faithfully represent the group algebra.

Figures 16–19 examine the arithmetic inputs: regularity, specialization and the difference between weak and proper lifts. Figures 20–22 summarize the claim boundary and its evidence.

Reproduce the plates

The source bundle contains `visual/figures.py`, the guide source, all PDF and PNG plates, and an asset hash record in `figure_provenance.json`. The exact computation data are stored separately in `computations/exact_examples.json`.

Run `python visual/figures.py` from the project environment. The `--only` option accepts one or more plate names. The generator checks its independently constructed subgroup matrices against the recorded exact input before rendering.

Primary reading and source contracts

Milne2022. J. S. Milne, *Fields and Galois Theory*. The source contracts locate fixed-field correspondence, finite automorphism groups and the standard polynomial criteria used here.

SchmidtWingberg1998. A. Schmidt and K. Wingberg, *Safarevic's theorem on solvable groups as Galois groups*, arXiv:math/9809211v1. The source distinguishes weak from proper solutions and proves the solvable-group realization theorem.

DebesHarbater2021. P. Dèbes and D. Harbater, *Pre-Galois theory*, Osaka J. Math. 58 (2021), 71–101. Together with Dèbes's *Arithmétique des revêtements de la droite*, it fixes the regularity and Hilbert specialization hypotheses.

HuangEtAl2026. X. Huang et al., *The Mathieu group M_{23} is a Galois group over* $\mathbb{Q}$, arXiv:2608.08538v2. This dated preprint is context, not a dependency of the elementary examples or a proof of the general problem.

The source bundle records exact titles, versions, primary links and theorem locators in `literature/source_contracts.md`. The monograph is the place to check the full proof and each source-dependent implication.

What remains open

This edition does not prove that every finite group occurs over $\mathbb{Q}$, or that every finite group has a regular realization over $\mathbb{Q}(t)$. It separates the written operator theorems, complete examples, external inputs and finite models from those universal assertions.

Edition boundary. Operation Komorebi, Conjecture Series 09. Author: Komorebi P. Reader edition 1.0, 9 October 2026. The source and evidence bundle preserves the recovered historical inputs alongside this publication.