Catalan’s constant

A visual guide to the irrationality question

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READER EDITION 1.0 / 9 OCTOBER 2026

The constant is easy to define and difficult to classify. Follow the exact identities, the rational approximants and the arithmetic cost of turning a small real error into a small integer linear form. The guide marks where the finite evidence ends and the open problem begins.

WORKED EXAMPLES / EXACT COORDINATES / OPEN PROBLEMS

A number we can locate

01 / THE QUESTION

$$G = \beta(2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

A defined real number. This edition does not resolve its rationality.



READ THE FIGURE

Catalan's constant is

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}.$$

The series converges. It defines a particular real number, about 0.9159655942.

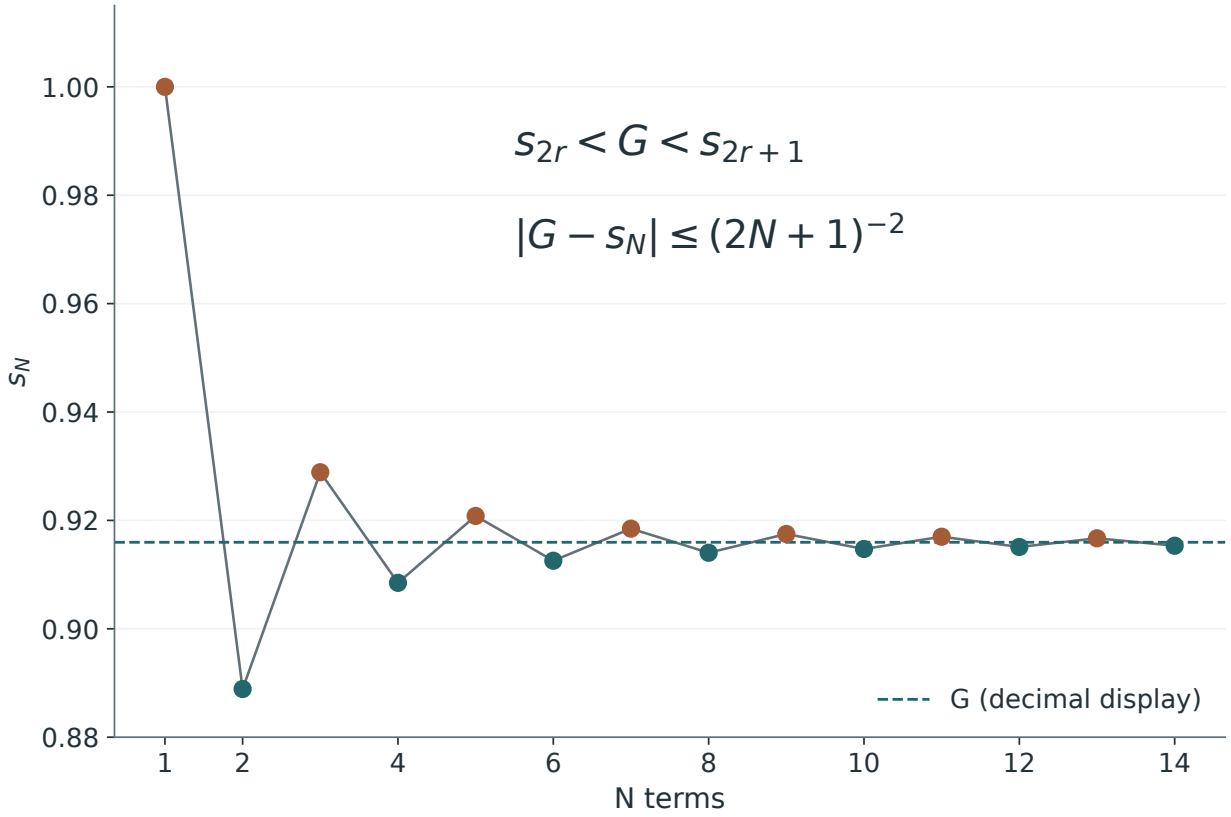
The conjectural assertion is $G \notin \mathbb{Q}$. Its decimal expansion, however many digits we compute, cannot decide that assertion.

This edition records research and audited computations. It does not establish irrationality.

Carry forward. A definition and an accurate decimal value leave the rationality question open.

Alternating terms bracket the constant

02 / AN ELEMENTARY CERTIFIED INTERVAL



READ THE FIGURE

Let s_N contain the first N terms. The odd partial sums lie above G. The even partial sums lie below it.

The next omitted term gives

$$|G - s_N| \le \frac{1}{(2N+1)^2}.$$

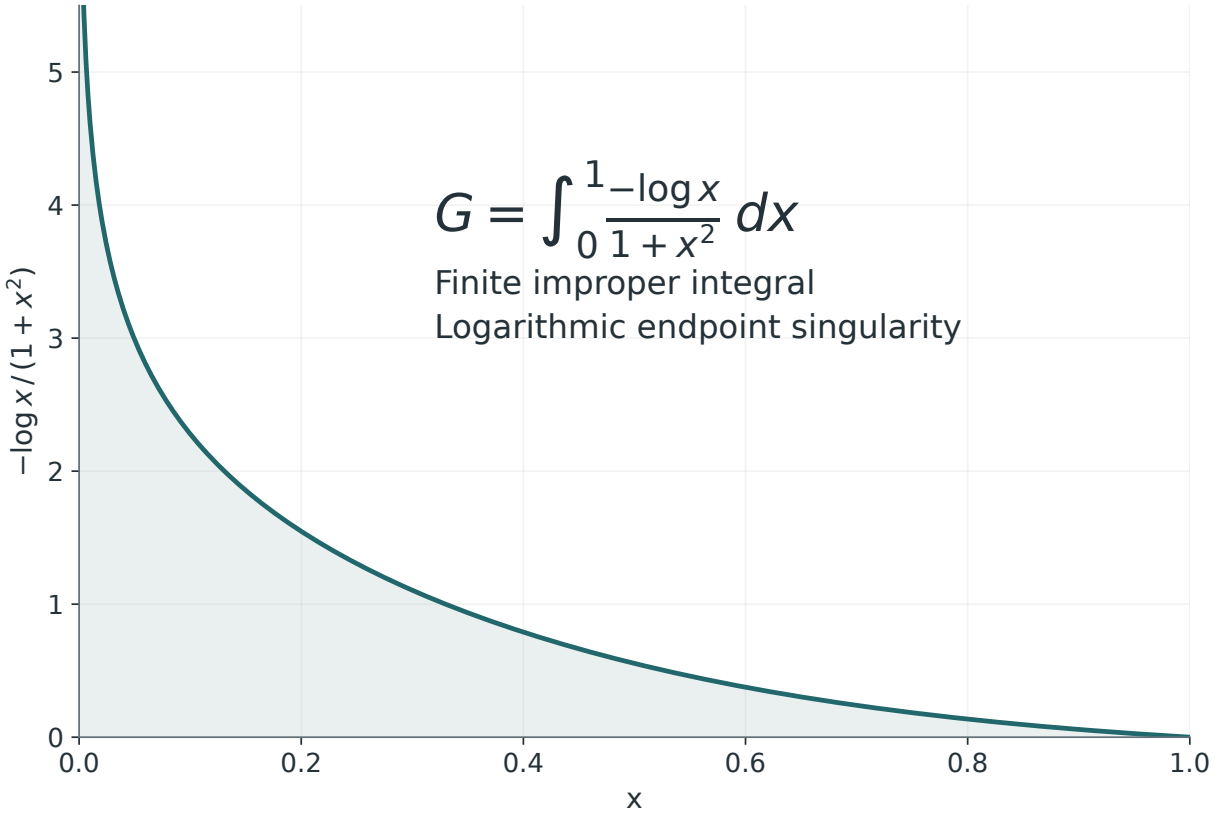
Every plotted partial sum is an exact rational number. The horizontal marker for G is a decimal display.

The bound certifies approximation. It gives no information about whether G is rational.

Carry forward. The series supplies exact rational enclosures with a transparent error bound.

The same constant as an area

03 / A SINGLE INTEGRAL



READ THE FIGURE

Expanding $(1+x^2)^{-1}$ and integrating the terms gives

$$G = \int_0^1 \frac{-\log x}{1+x^2}\,\mathrm{d}x.$$

The logarithm diverges at 0. Its integral over a short interval remains finite.

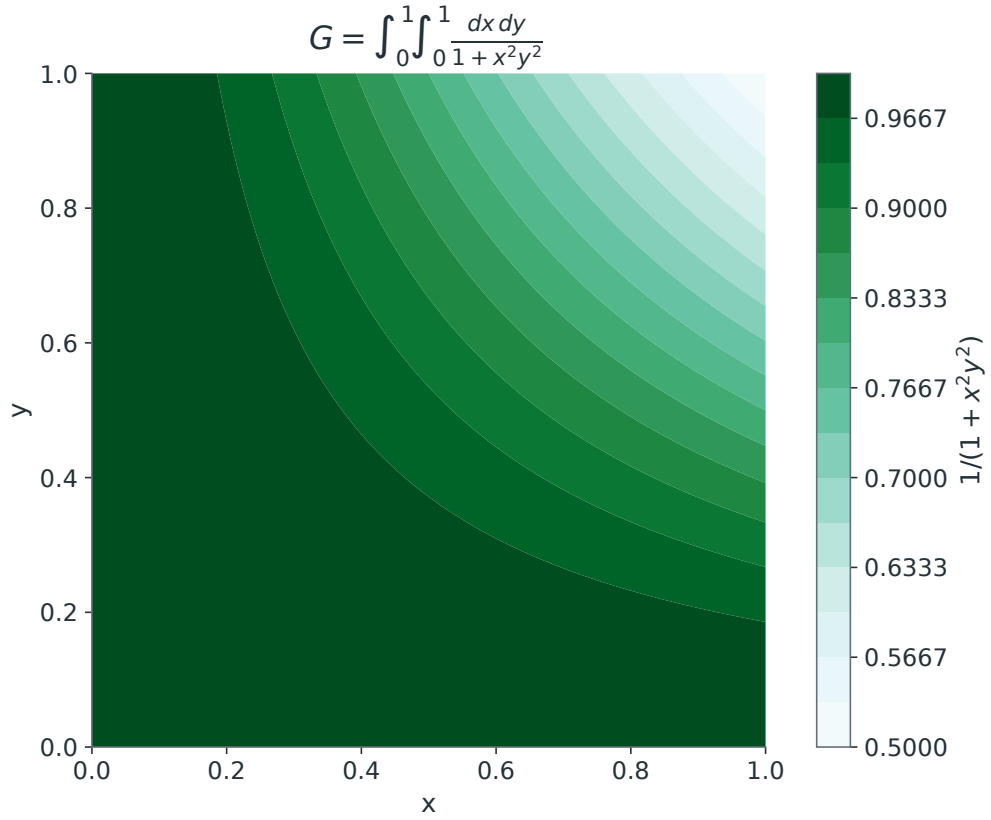
The graph clips the visible height at 5.5. The shaded display is not a numerical proof of the identity.

Integral identities can expose positivity and estimates that a decimal expansion hides.

Carry forward. An exact integral representation changes the available tools without resolving rationality.

A square instead of a line

04 / A DOUBLE INTEGRAL



READ THE FIGURE

On the unit square,

$$G = \int_0^1 \int_0^1 \frac{\mathrm{d}x\,\mathrm{d}y}{1+x^2y^2}.$$

The integrand stays between $1/2$ and 1.
Integrating each geometric term produces $(2k+1)^{-2}$. Absolute convergence of the integrated terms justifies that passage.
The contour colors show the integrand. They are floating display coordinates, not evidence from quadrature.

Carry forward. The squared odd denominators arise from integrating once in each coordinate.

A family with a difficult middle value

05 / DIRICHLET BETA

$$\beta(s) = \sum_{k \geq 0} (-1)^k (2k+1)^{-s}$$

$\beta(1) = \pi/4$
Known evaluation

$\beta(3) = \pi^3/32$
Known evaluation

$\beta(2) = G$
No proof established here

Exact identities at neighbouring arguments do not decide β(2).

READ THE FIGURE

The beta function packages these series as

$$\beta(s) = \sum_{k \geq 0} \frac{(-1)^k}{(2k+1)^s} \quad (s > 0).$$

At positive odd integers, familiar evaluations involve powers of π.
Catalan's constant is the positive even value $\beta(2)$.
The neighbouring formulas do not yield a comparable evaluation for it.
The checked sources establish neither its irrationality nor its transcendence. A September 2026 proof claim and a subsequent critical note remain a source dispute, recorded in the monograph.

Carry forward. A successful formula at $s = 1$ or $s = 3$ does not settle the value at $s = 2$.

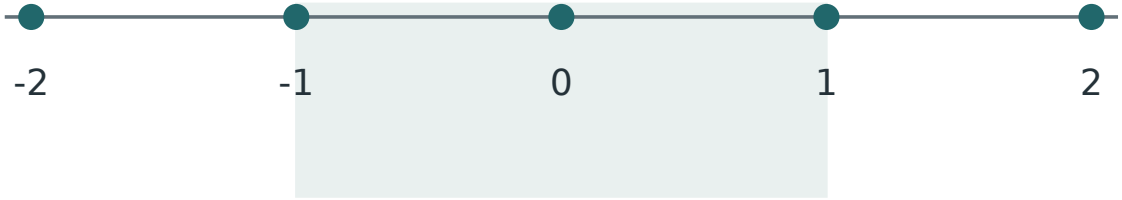
Why small integer forms would settle the question

06 / THE LATTICE GAP

$$G = a/b \quad \Longrightarrow \quad b(AG - B) = aA - bB \in \mathbb{Z}$$

$$0 < |AG - B| < 1/b$$

would place a nonzero integer in this gap



READ THE FIGURE

Suppose $G = a/b$ with integers a, b and $b > 0$.
For integer A, B,

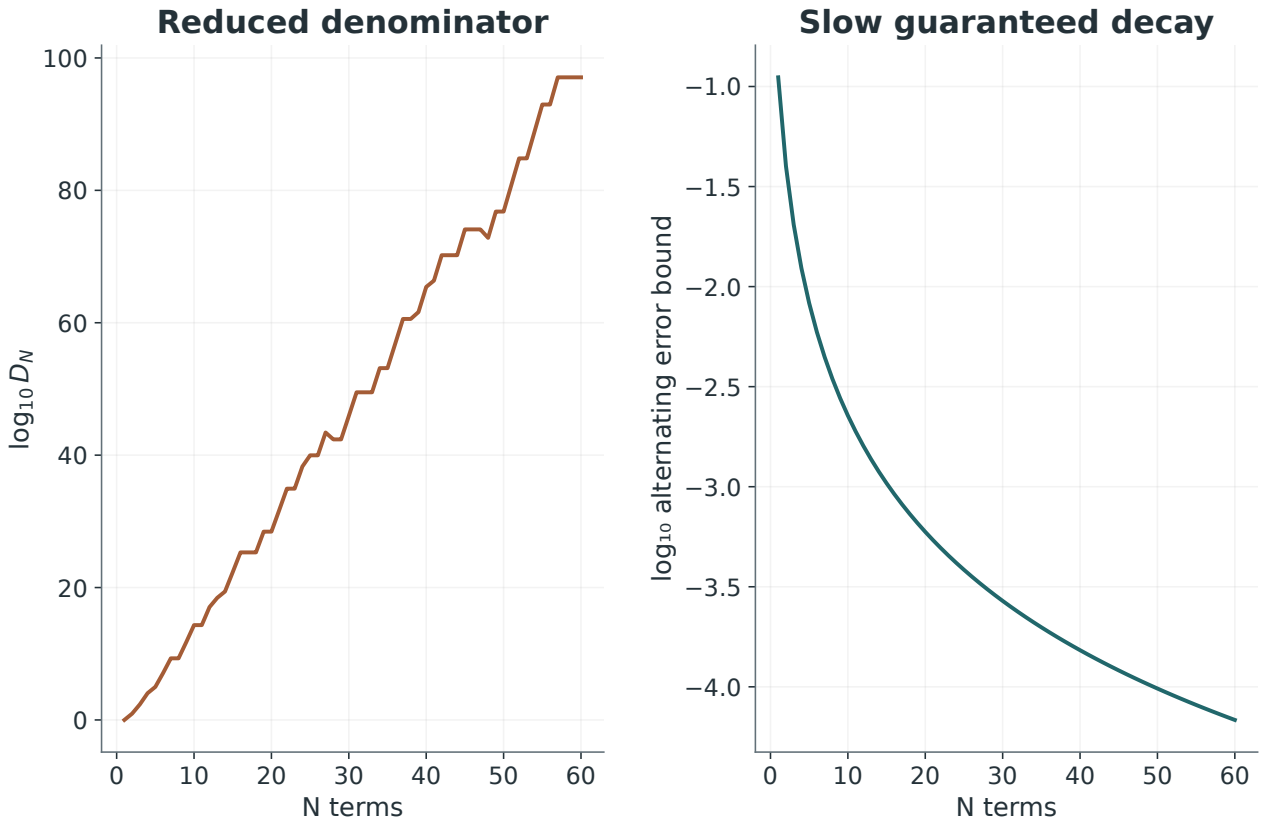
$$b(AG - B) = aA - bB \in \mathbb{Z}.$$

If the form is nonzero, its absolute value is at least $1/b$.
Thus an infinite sequence of nonzero integer forms tending to zero contradicts the assumed rationality.
All three words matter: integer, nonzero and vanishing.

Carry forward. The route is to force a nonzero integer into a gap too small to contain one.

A good decimal approximation has a denominator

07 / THE FIRST ARITHMETIC OBSTACLE



READ THE FIGURE

Write $s_N = P_N / D_N$ in lowest terms.

The left panel plots exact reduced denominators through $N = 60$. The right panel plots the guaranteed alternating error bound.

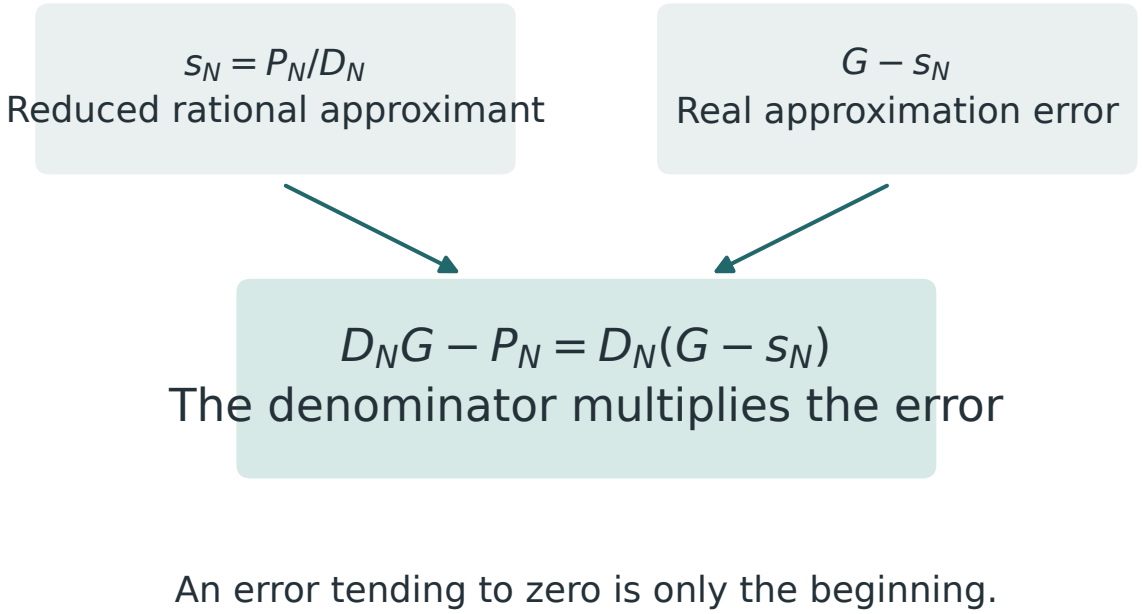
These quantities move on very different scales. Multiplying the approximation error by D_N can erase the advantage of the approximation.

The displayed error bound is an upper bound. Its failure to prove a small linear form does not prove that no sharper bound exists.

Carry forward. The relevant quantity is the error after the rational denominators are cleared.

Clearing is a multiplication, with a cost

08 / FROM A RATIONAL APPROXIMANT TO A LINEAR FORM



READ THE FIGURE

The identity is exact:

$$D_N G - P_N = D_N(G - s_N).$$

It produces integer coefficients. It also multiplies the error by the reduced denominator.

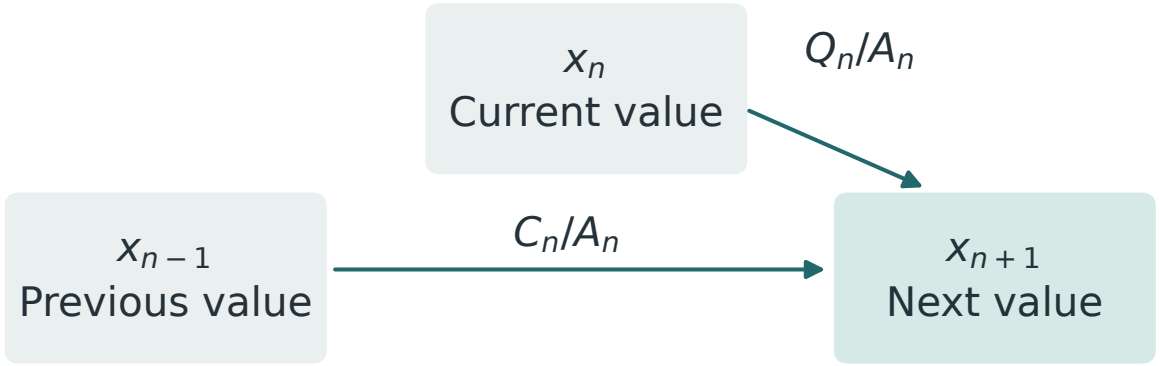
Fast approximation and controlled denominators must work together. A construction needs an estimate for their product.

Later common divisors can reduce that product. A chosen scale must be accounted for when measuring those divisors.

Carry forward. Track analytic decay and arithmetic growth in the same normalization.

A recurrence creates two coordinated sequences

09 / THE ZUDILIN RECURRENCE



$$A_n x_{n+1} = Q_n x_n + C_n x_{n-1}$$

Both u and v obey the same recurrence, from distinct initial data.

READ THE FIGURE

Set $p(n) = 20n^2 - 8n + 1$ and

$$A_n = (2n+1)^2(2n+2)^2 p(n),$$
$$C_n = (2n-1)^2(2n)^2 p(n+1).$$

The middle coefficient is

$$Q_n = 3520n^6 + 5632n^5 + 2064n^4 - 384n^3 - 156n^2 + 16n + 7.$$

Both sequences satisfy the displayed relation for $n \geq 1$. Their different initial values make their roles distinct.

Carry forward. The recurrence is exact algebra. Approximation and denominator claims require additional arguments.

Start with exact rational values

10 / INITIAL DATA FIX THE NORMALIZATION

Two rational sequences

n	u_n	v_n	v_n/u_n
0	1	0	0
1	7/4	13/8	13/14
2	649/64	10699/1152	10699/11682
3	19471/256	8025653/115200	8025653/8761950

Values shown are exact fractions from integer and rational arithmetic.

READ THE FIGURE

The historical audit uses

$$u_0 = 1, \quad u_1 = 7/4, \qquad v_0 = 0, \quad v_1 = 13/8.$$

With these initial values, the recurrence determines every later rational term.

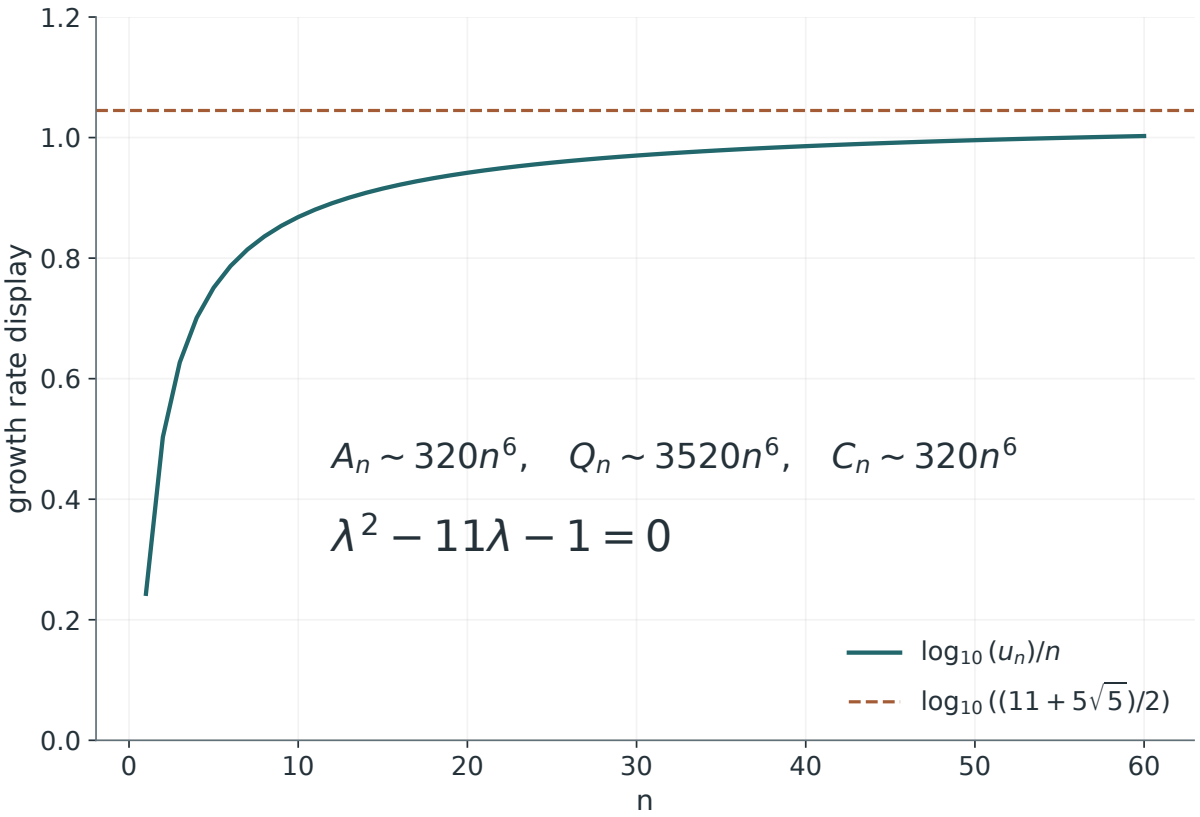
The quotient v_n/u_n is the approximant in this normalization. A source using an integral pair may multiply both sequences by a signed constant.

The source bundle records that translation explicitly. It must be applied before comparing equations or denominators.

Carry forward. Initial values are part of the mathematical definition, not an editorial detail.

The characteristic equation explains the scale

11 / GROWTH IS A SOURCE DEPENDENT CLAIM



READ THE FIGURE

The leading polynomial coefficients give

$$\lambda^2 - 11\lambda - 1 = 0, \qquad \lambda_\pm = \frac{11 \pm 5\sqrt{5}}{2}.$$

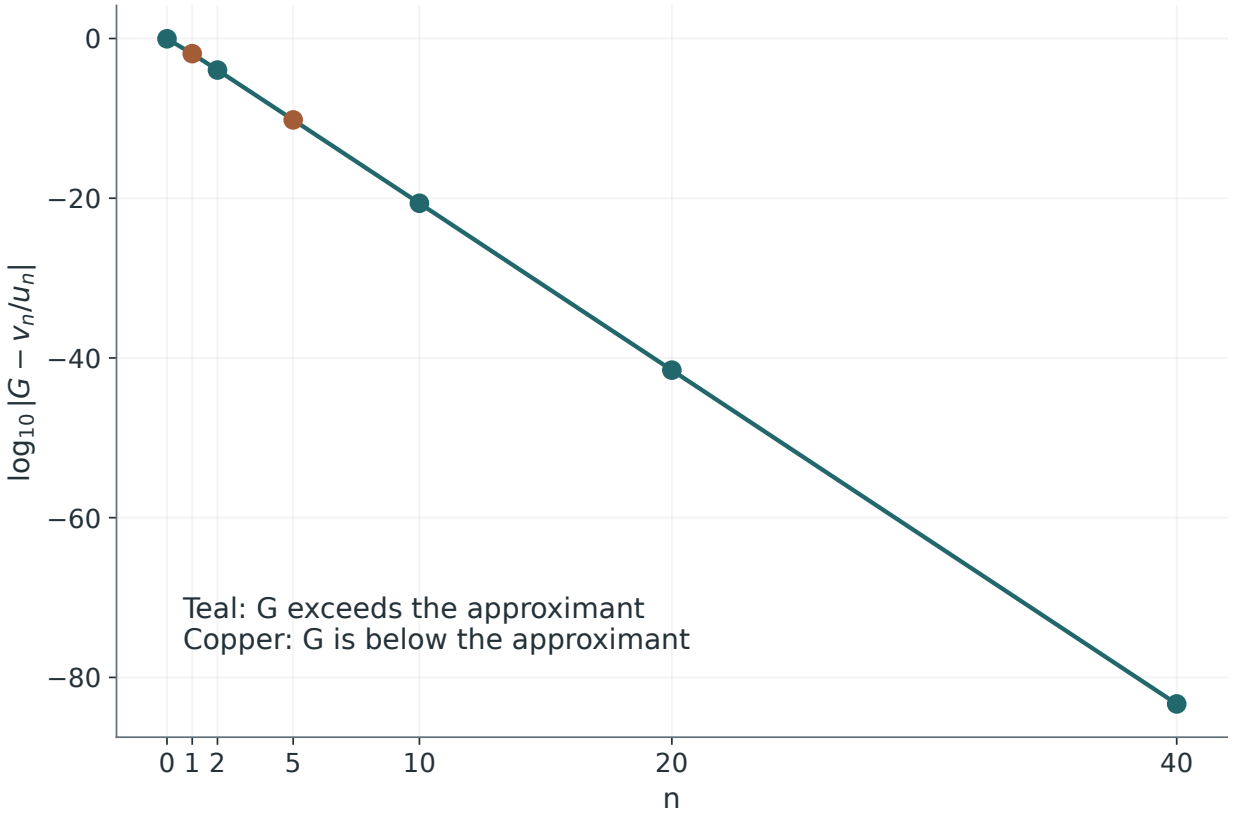
The finite curve plots $\log_{10}(u_n)/n$ through $n = 60$. The dominant positive root predicts the growth scale. Turning that prediction into a theorem about a selected solution requires the appropriate recurrence analysis and initial conditions.

This plot is a consistency display. It is not that theorem.

Carry forward. Leading coefficients identify a candidate scale. A rigorous asymptotic statement needs its hypotheses.

The approximants quickly approach the target

12 / CERTIFIED FINITE ERRORS



READ THE FIGURE

Each point uses exact u_n, v_n and a certified rational interval for G.

The interval determines the error's sign and brackets its magnitude. The plotted logarithm is a rounded display of that bracket.

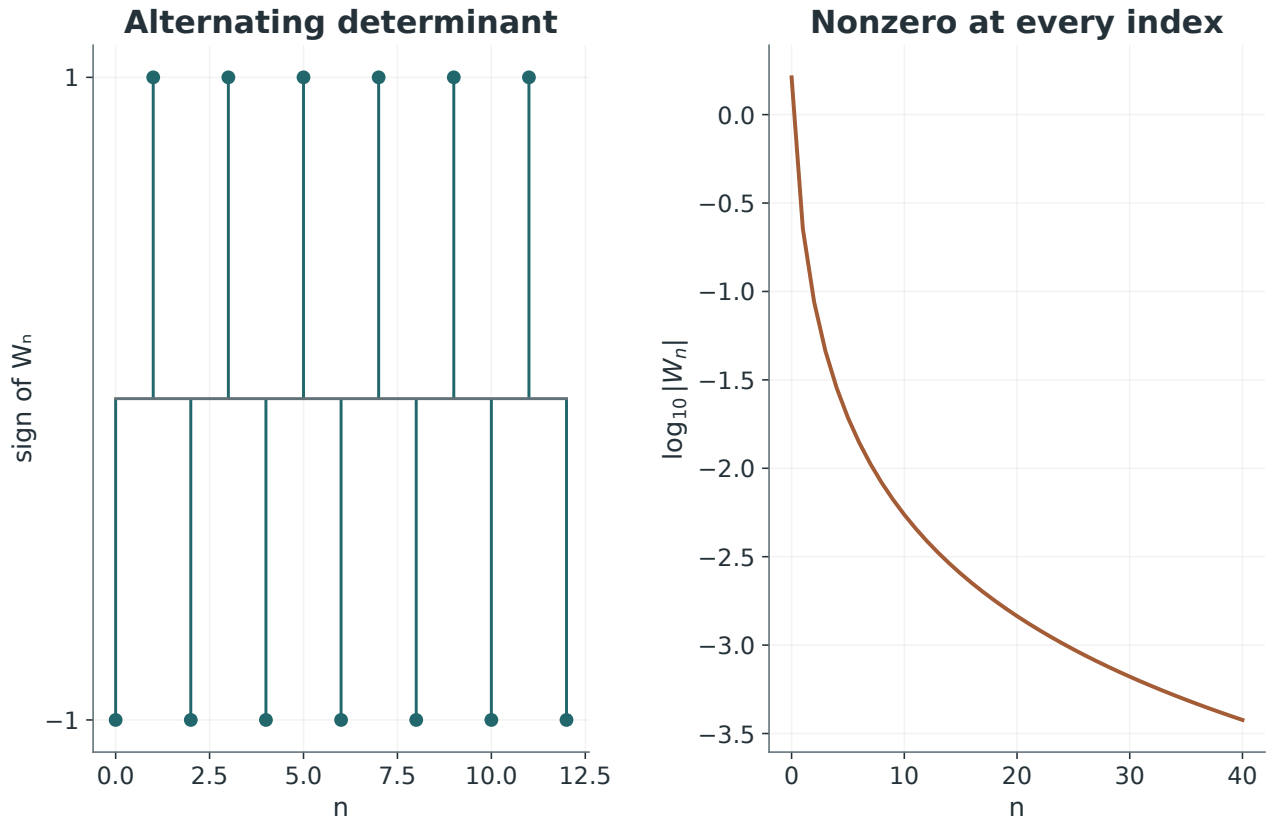
The selected indices are $0, 1, 2, 5, 10, 20, 40$. Teal and copper distinguish the two error signs.

This is much faster approximation than the elementary partial sums. The rational denominators still have to be tracked.

Carry forward. A tiny certified error is useful evidence about an approximant, not an irrationality proof by itself.

The determinant stays away from zero

13 / A WRONSKIAN IDENTITY



READ THE FIGURE

Define

$$W_n = u_{n+1}v_n - u_n v_{n+1}.$$

Its exact formula is

$$W_n = \frac{(-1)^{n+1}p(n+1)}{2(2n+1)^2(2n+2)^2}.$$

The numerator cannot vanish for $n \geq 0$. Adjacent approximants are distinct.

The left panel shows the alternating sign. The right panel shows the decreasing magnitude on a logarithmic scale.

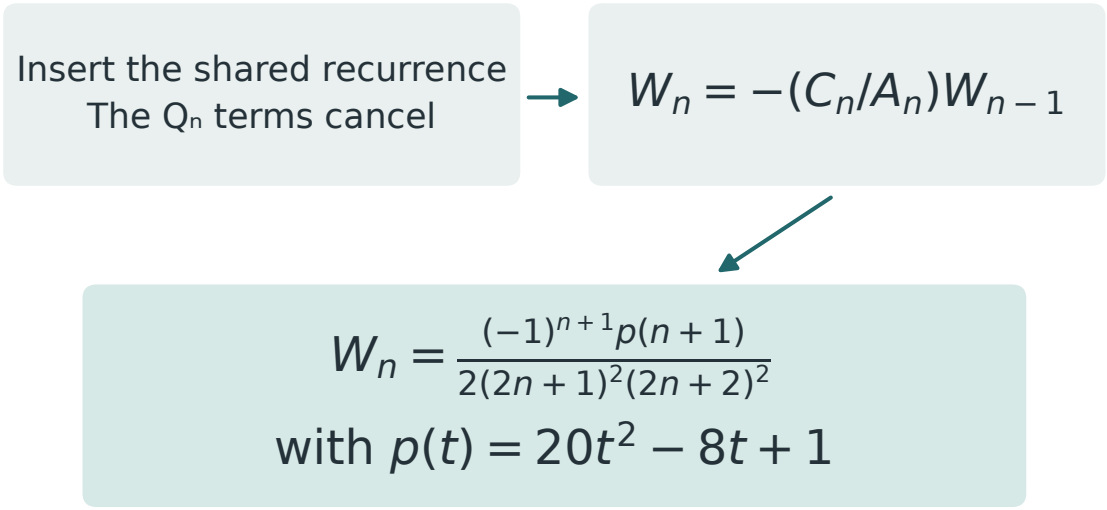
This determinant does not establish nonvanishing of an arbitrary filtered combination.

Carry forward. An exact determinant proves independence of the adjacent pairs. Its scope should remain explicit.

Cancellation makes the determinant tractable

14 / THE IDENTITY HAS AN ELEMENTARY PROOF

$$W_n = u_{n+1}v_n - u_n v_{n+1}$$



READ THE FIGURE

Substitute the recurrence into both terms of W_n. The contributions containing Q_n cancel. The remaining terms give

$$W_n = -\frac{C_n}{A_n}W_{n-1}.$$

The coefficient product telescopes. The initial determinant is $W_0 = -13/8$.

This establishes the formula for every nonnegative index. The finite check through $n = 200$ separately checks the implementation.

Carry forward. A universal written proof and a finite computational check are different kinds of evidence.

Scale the first coefficient before filtering

15 / INTEGER PACKET INPUTS

$$T_n = 16^n u_n$$

n	T_n
0	1
1	28
2	2596
3	311536
4	41759524
5	5953698928

Exact integer divisions were checked through T_{10000} in the default scan.

READ THE FIGURE

The packet construction writes

$$T_n = 16^n u_n.$$

The displayed values are exact integers. There is also a proof for every n.

Krattenthaler and Rivoal's coefficient formula gives

$$u_n = \sum_{j=0}^{n} \binom{n}{j} \binom{n-1/2}{j} \binom{n+j-1/2}{j}.$$

For integer $h \geq 0$, put $b_{h,j} = 4^j \binom{h-1/2}{j}$. These are integer coefficients of

$$(1+4z)^h (1+4z)^{-1/2},$$

since the second factor has coefficients $(-1)^j \binom{2j}{j}$.

Each summand of T_n is therefore $\binom{n}{j} 16^{n-j} b_{n,j} b_{n+j,j} \in \mathbb{Z}$.

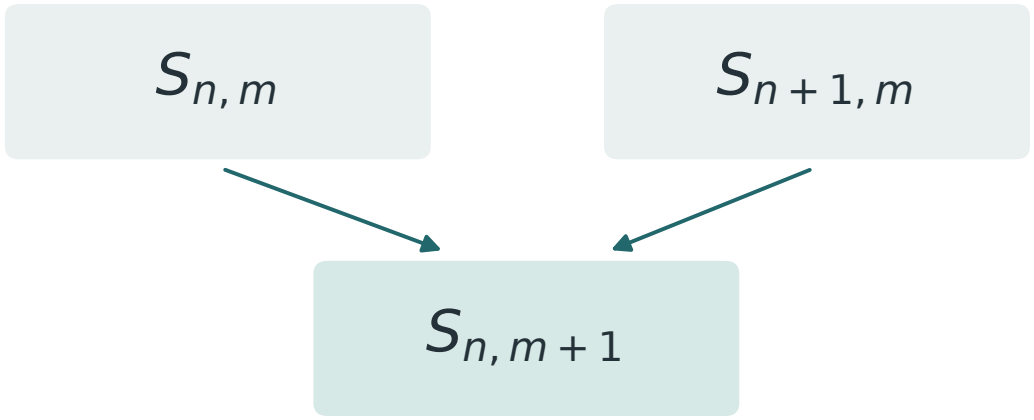
The separate finite audit checked every recurrence division through $n = 10000$.

Carry forward. A known coefficient formula supplies a universal proof. The exact division scan separately checks the implementation.

A binomial packet has a local update rule

16 / PASCAL ADDITION

$$S_{n,m} = \sum_{i=0}^{m} \binom{m}{i} T_{n+i}$$



$$S_{n,m+1} = S_{n,m} + S_{n+1,m}$$

READ THE FIGURE

Define

$$S_{n,m} = \sum_{i=0}^{m} \binom{m}{i} T_{n+i}.$$

Pascal's identity yields

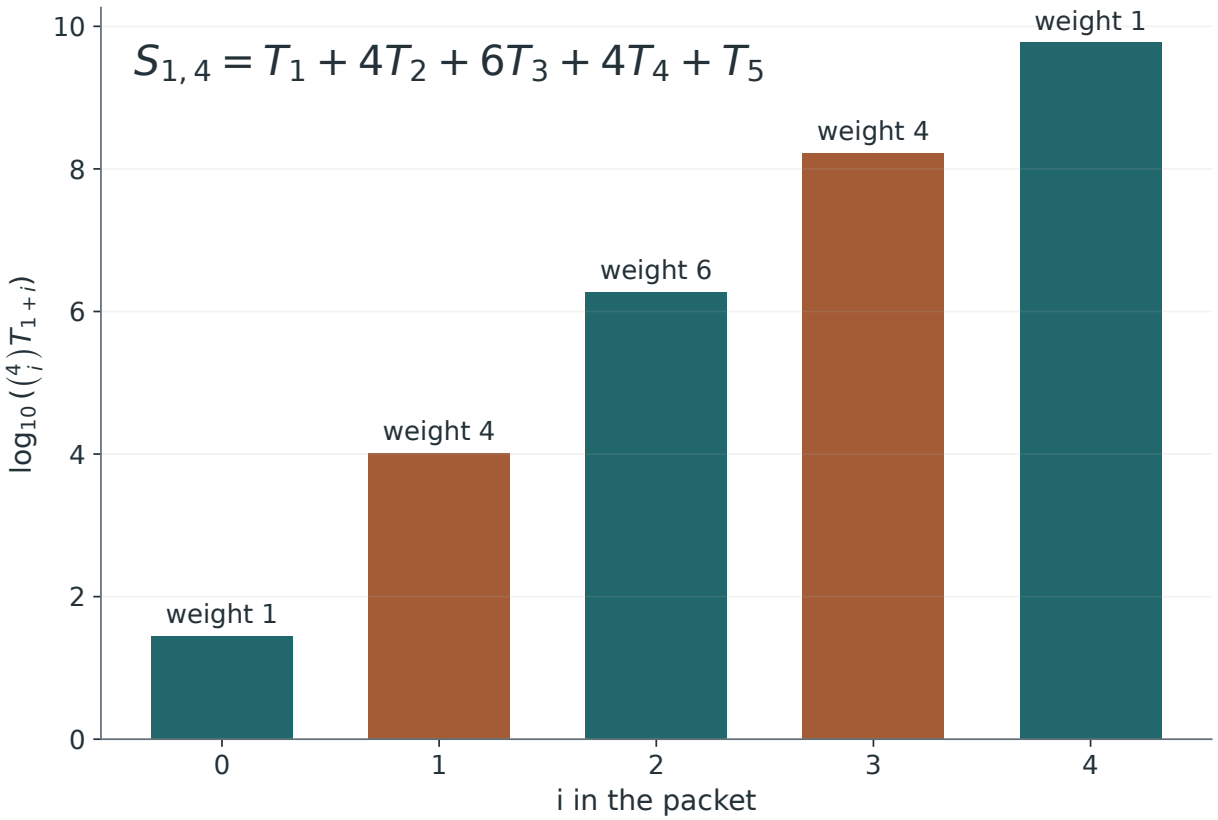
$$S_{n,m+1} = S_{n,m} + S_{n+1,m}.$$

The initial row is $S_{n,0} = T_n$.
This local recurrence allows a large triangular scan without rebuilding every binomial sum. It is also valid modulo 2^K.

Carry forward. The fast computation follows an exact binomial identity.

What is being added in one packet

17 / A WORKED FINITE EXAMPLE



READ THE FIGURE

For $n = 1, m = 4$, the weights are

$$1,\ 4,\ 6,\ 4,\ 1.$$

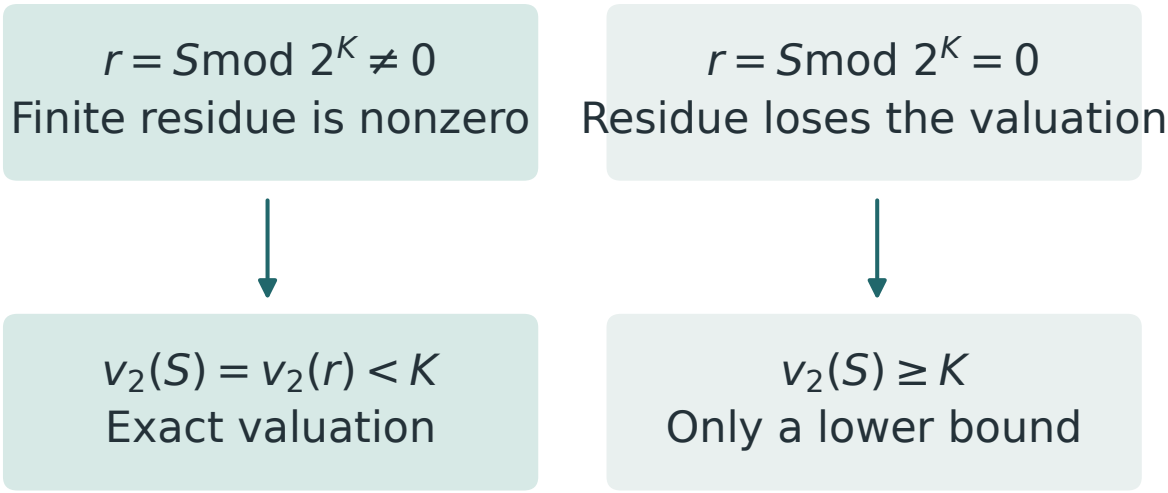
They multiply five consecutive terms $T_1, \ldots, T_5$. The bars show the positive summands on a logarithmic axis. The final sum is computed as an exact integer, with no floating summation.

Large summands do not directly determine the divisibility of their sum. Cancellation in residue classes is an arithmetic question.

Carry forward. Size and divisibility are separate properties of the packet sum.

A finite modulus can still give an exact valuation

18 / WHEN THE RESIDUE IS ENOUGH



Default run: K = 128. No zero residues. Maximum valuation = 31.

READ THE FIGURE

The valuation $v_2(S)$ counts the factors of 2 in a nonzero integer S.

If S mod 2^K is nonzero, its trailing zero bits give the exact valuation below K.

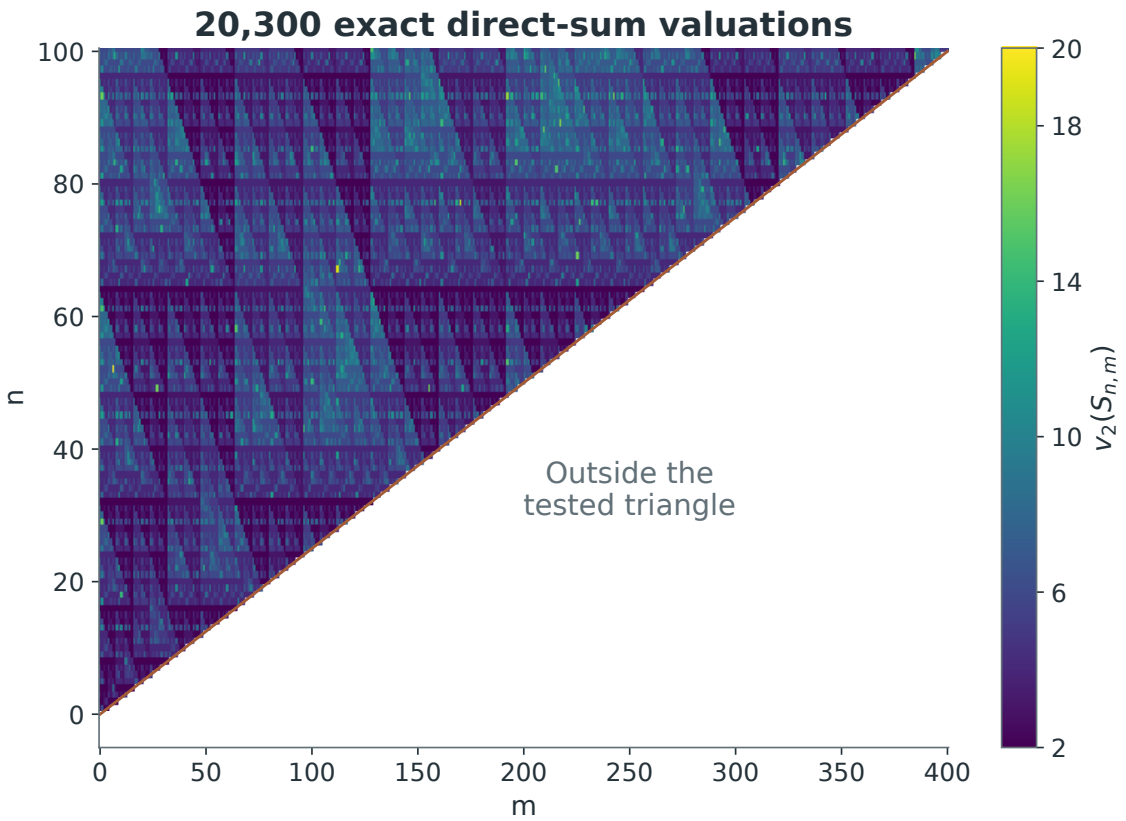
A zero residue would only imply $v_2(S) \geq K$. The audit counts such unresolved residues explicitly.

For $K = 128$ in the default run, none occurred. Every reported valuation in its finite domain is exact.

Carry forward. The word exact refers to the tested pairs. It does not extend the domain to infinity.

A small independent scan reveals the texture

19 / EXACT DIRECT SUMS



READ THE FIGURE

Every colored cell corresponds to one exact binomial sum with

$$1 \le n \le 100, \qquad 0 \le m \le 4n.$$

There are 20300 such pairs. White cells are outside that domain.

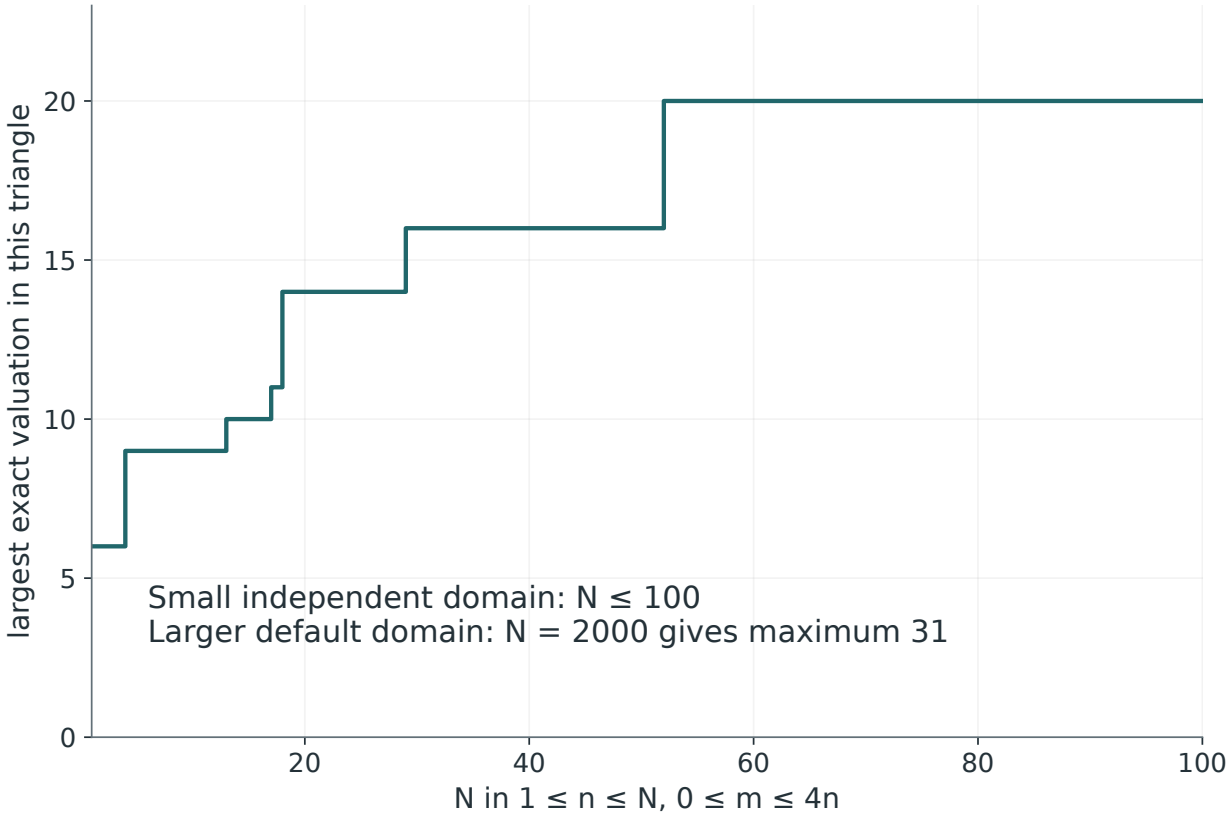
This implementation uses direct sums and arbitrary precision integers. It checks the faster Pascal method over the same smaller range.

The uneven pattern is finite data. A uniform asymptotic valuation bound needs an argument beyond the image.

Carry forward. Independent arithmetic methods check that the visible pattern is produced correctly.

The largest observed valuation grows in jumps

20 / PREFIX MAXIMA



READ THE FIGURE

For each N, maximize $v_2(S_{n,m})$ over the triangular domain up to N.

The plot shows the independently computed prefixes through $N = 100$. The maximum there is 20.

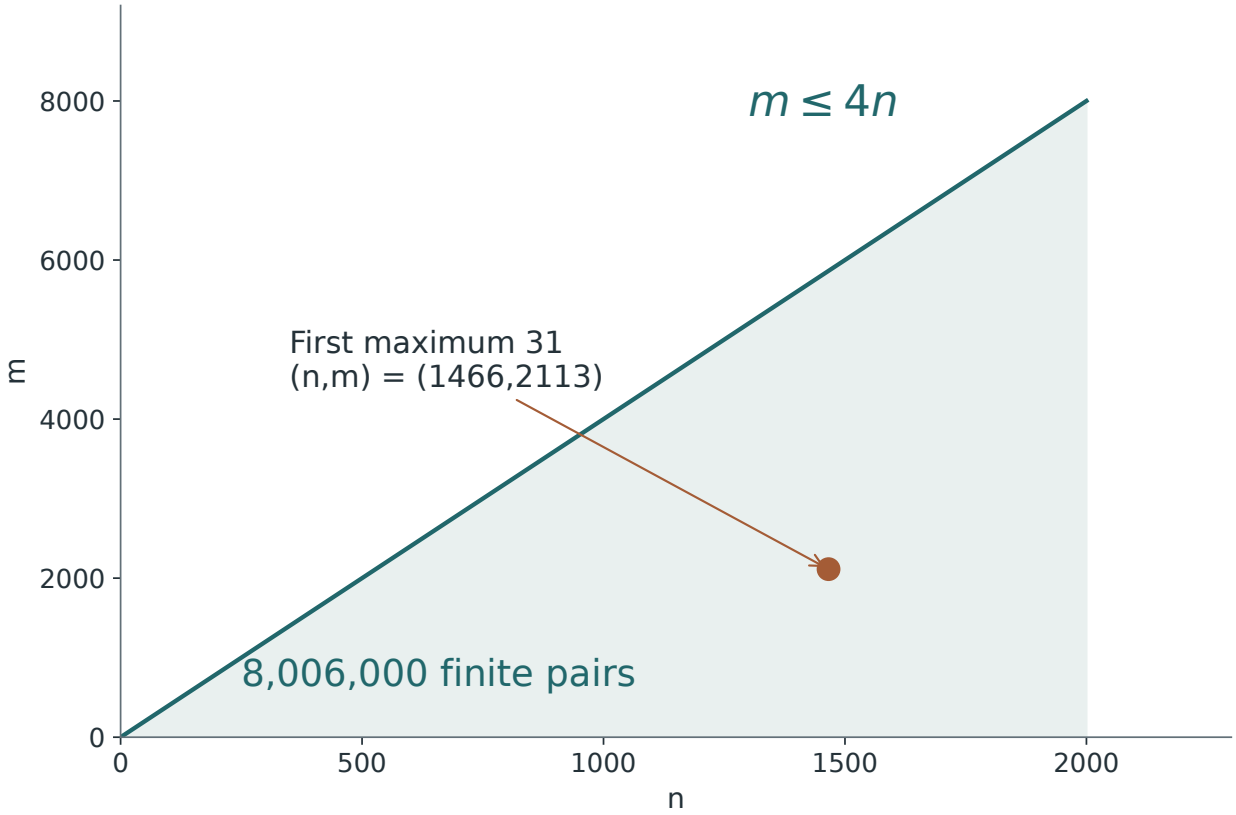
The larger default scan, up to $N = 2000$, reaches 31.

Neither observation supplies a universal upper bound. Rare large valuations could occur at later indices, and an asymptotic claim requires control of them.

Carry forward. A record value summarizes a finite search. It does not control the unsearched tail.

See the full domain of the large run

21 / EIGHT MILLION PAIRS, WITH A BOUNDARY



READ THE FIGURE

The default scan tested

$$1 \le n \le 2000, \qquad 0 \le m \le 4n.$$

The total is

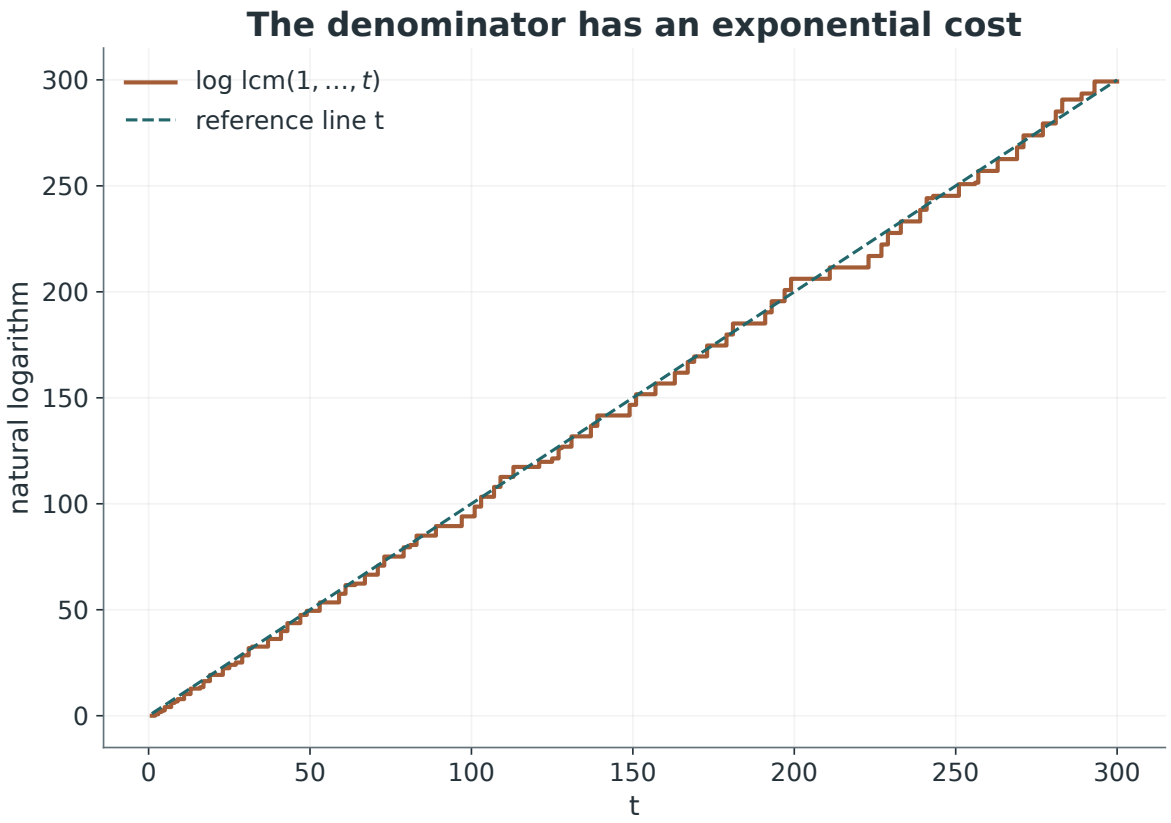
$$2N^2 + 3N = 8006000 \quad (N = 2000).$$

Its maximum valuation is 31. The first maximizer in the scan's order is $(1466, 2113)$.
The fresh run agrees exactly with the historical result. The unshaded region remains untested.

Carry forward. A large bounded search is strong implementation evidence and bounded mathematical evidence.

The least common multiple carries a real cost

22 / DENOMINATOR CLEARING



READ THE FIGURE

Let

$$L_t = \operatorname{lcm}(1, \ldots, t).$$

The historical cleared pairs use $Q = 8L^2_{2(n+m)}$.

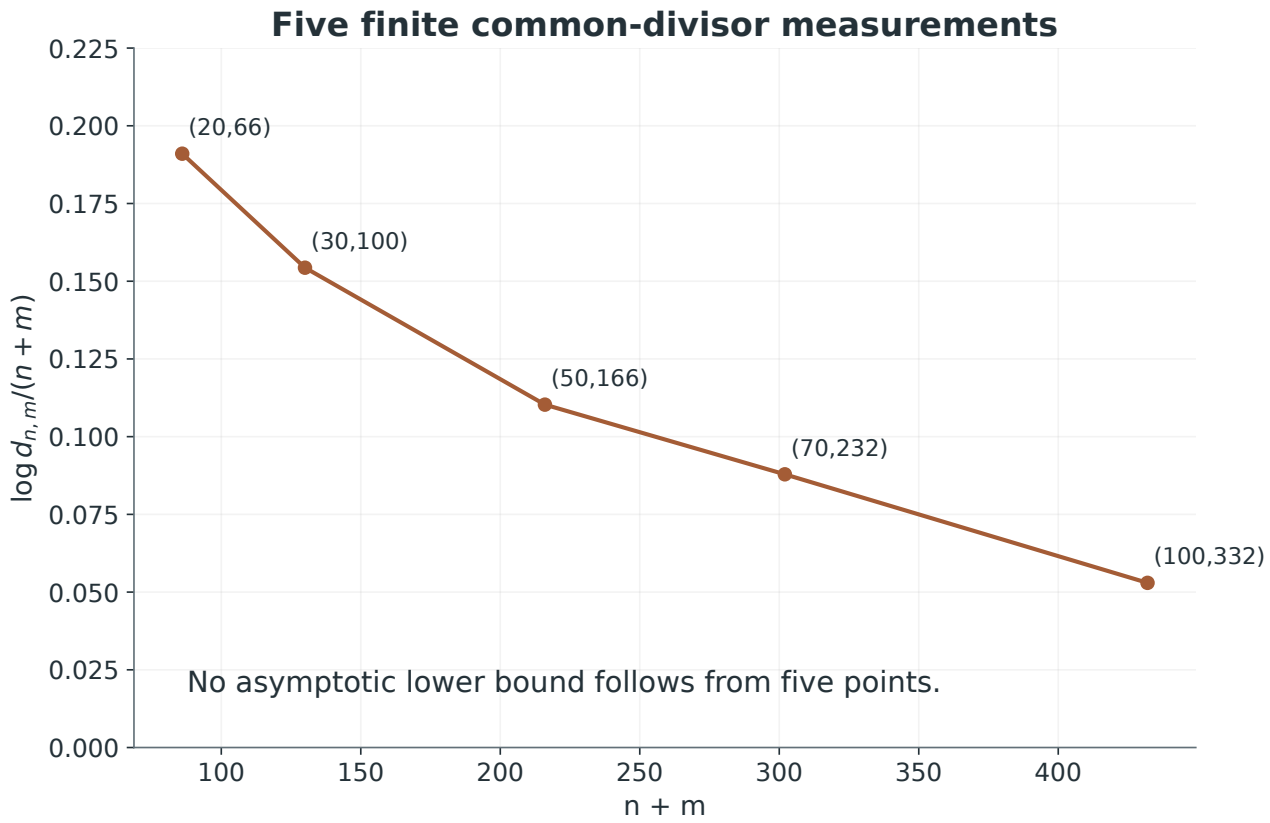
The step curve uses exact integer L_t through $t = 300$. Its logarithm shows the scale of the clearing factor.

The line t is a reference. The theorem $\log L_t \sim t$ is an external asymptotic input, equivalent to the prime number theorem in the standard formulation.

Carry forward. Clearing by an LCM converts rational coefficients to integers while multiplying the error.

Five common divisors do not establish a rate

23 / THE HISTORICAL CONTENT TABLE



READ THE FIGURE

For a cleared integer pair, write

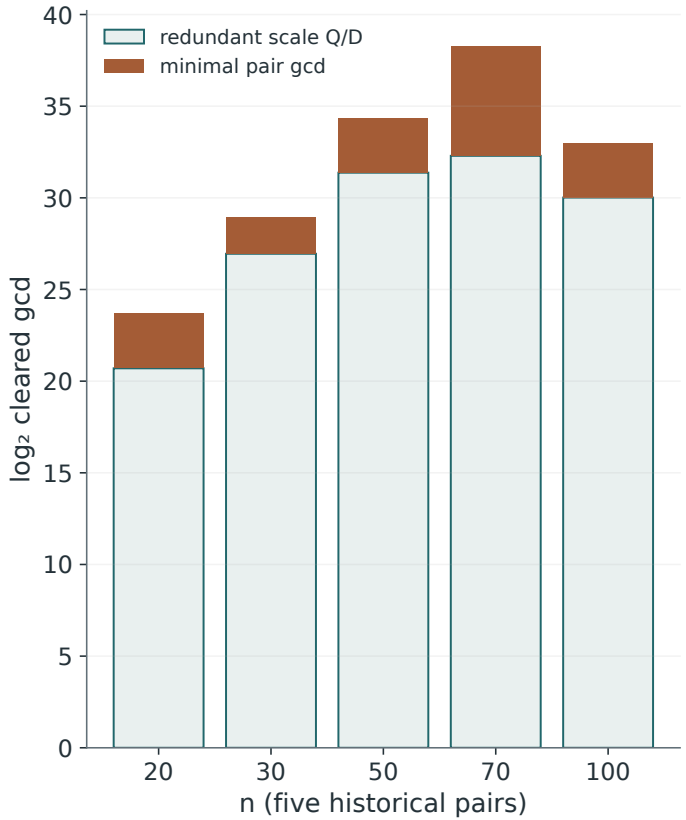
$$d_{n,m} = \gcd(|A_{n,m}|, |B_{n,m}|).$$

Dividing both coefficients by $d_{n,m}$ can improve the resulting integer form.
The five historical pairs were recomputed exactly.
The chart displays $\log d_{n,m}/(n+m)$.
These five values decrease. That is a statement about this table, not a proof of an asymptotic rate along an infinite sequence.

Carry forward. The useful question is the sustained divisor growth in a specified normalization.

Most of the large divisor comes from the chosen scale

24 / MINIMAL CLEARING CHANGES THE INTERPRETATION



$$d_{\text{cleared}} = R\,g$$

$$R = Q/D$$

Minimal pair gcds
8, 4, 8, 64, 8

The chosen clearing scale accounts for most of the reported common divisor.

READ THE FIGURE

Write the rational second packet coefficient as $H = N/D$ in lowest terms and the first as the integer S. Minimal clearing gives (DS, N). Historical clearing gives $R(DS, N)$, where $R = Q/D$ is an integer in these pairs.
Thus

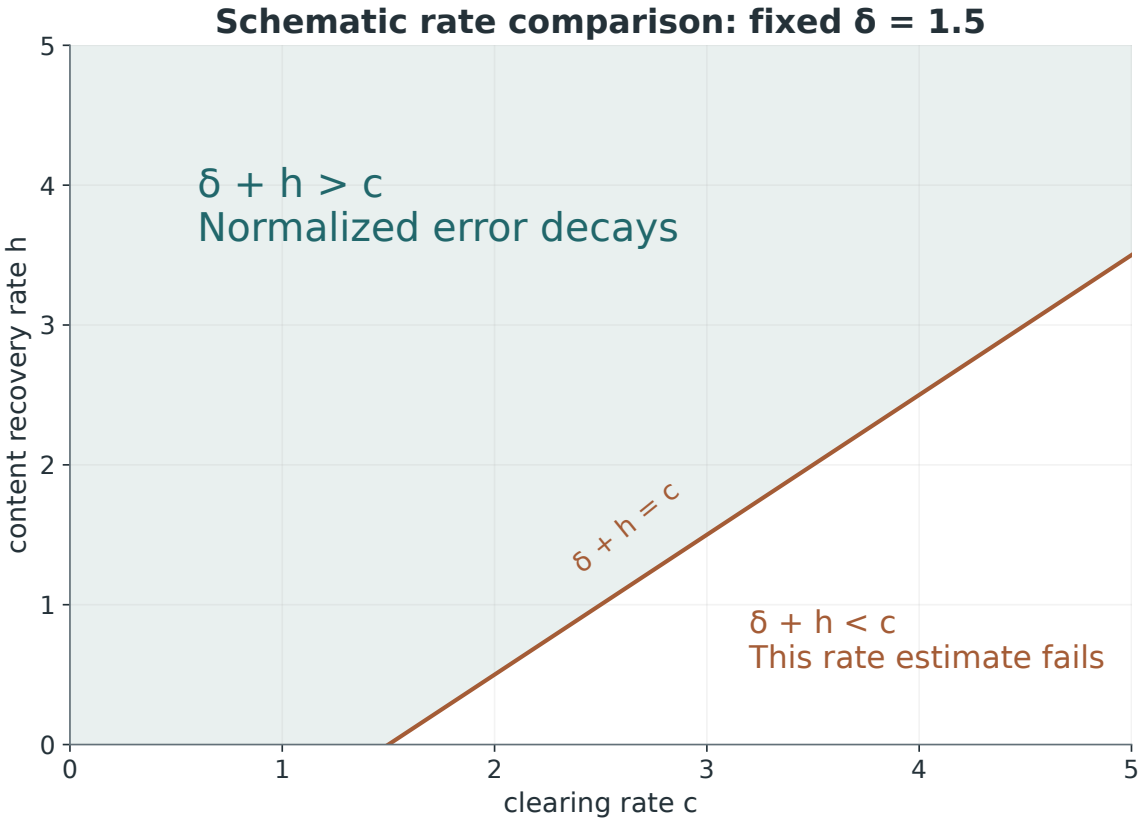
$$d_{\text{cleared}} = R\ \gcd(DS, N).$$

The five minimal pair gcds are $8, 4, 8, 64, 8$. A large cleared gcd can chiefly reflect a redundant clearing factor.

Carry forward. Separate recovered overclearing from intrinsic common divisibility.

The missing inequality is a comparison of rates

25 / A SCHEMATIC THRESHOLD MAP



READ THE FIGURE

Suppose, along a chosen sequence,

$$|F_n| = e^{-\delta n + o(n)},$$
$$Q_n = e^{cn + o(n)},$$
$$d_n = e^{hn + o(n)}.$$

Then

$$|Q_n F_n / d_n| = e^{(c-\delta-h)n + o(n)}.$$

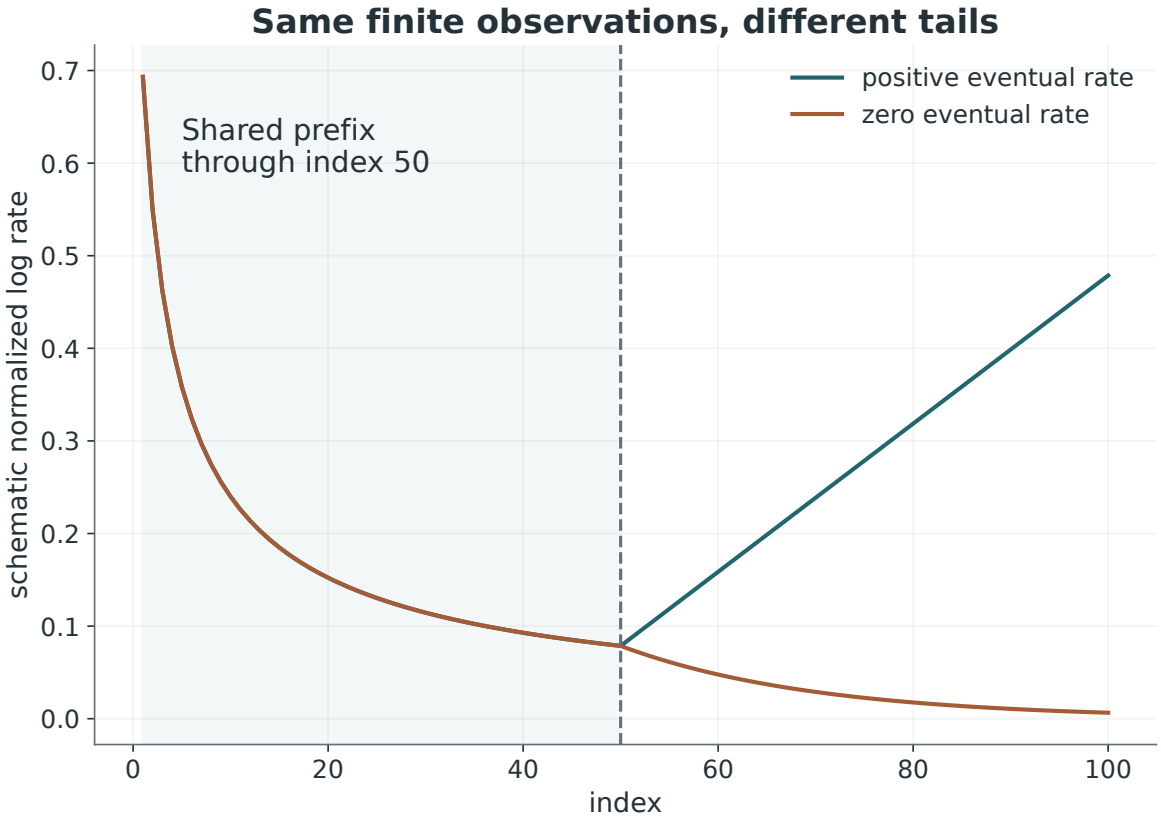
Strictly positive margin $\delta + h - c$ gives decay. Equality needs finer estimates.

The figure fixes the illustrative value $\delta = 1.5$. Its numerical boundary is not a Catalan theorem.

Carry forward. A successful proof needs a uniform estimate that crosses the boundary with a strict margin.

A finite prefix cannot decide an infinite tail

26 / COUNTERMODELS TO EXTRAPOLATION



READ THE FIGURE

The two drawn sequences agree through index 50 and have deliberately different tails.

They are synthetic examples. They are not fitted predictions for Catalan coefficients.

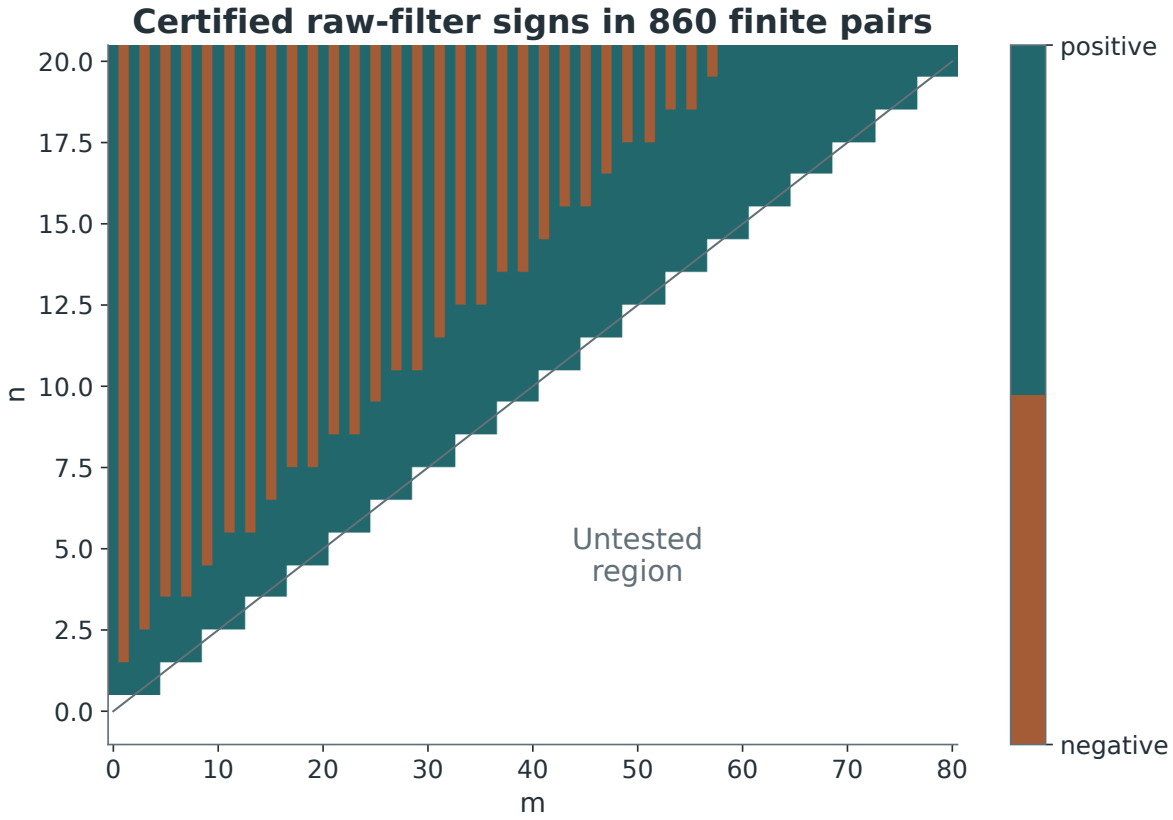
Any bounded table can be extended in many incompatible ways unless a theorem restricts the extension.

An eventual rate therefore needs a uniform inequality, a structural identity or an external result with matching hypotheses.

Carry forward. Agreement on every observed point cannot replace a theorem about all later points.

Even the sign deserves its own check

27 / FILTERED NONVANISHING



READ THE FIGURE

The raw audit form is

$$F_{n,m} = 8(-1)^n(S_{n,m}G - H_{n,m}),$$

where $H_{n,m} = \sum_i \binom{m}{i} 16^{n+i} v_{n+i}$.
Exact interval arithmetic certifies nonzero signs in 860 pairs through $n = 20$.
The guess $\operatorname{sgn} F_{n,m} = (-1)^m$ fails at $(1, 1)$:

$$F_{1,1} = \frac{173056}{9} - 20992G > 0.$$

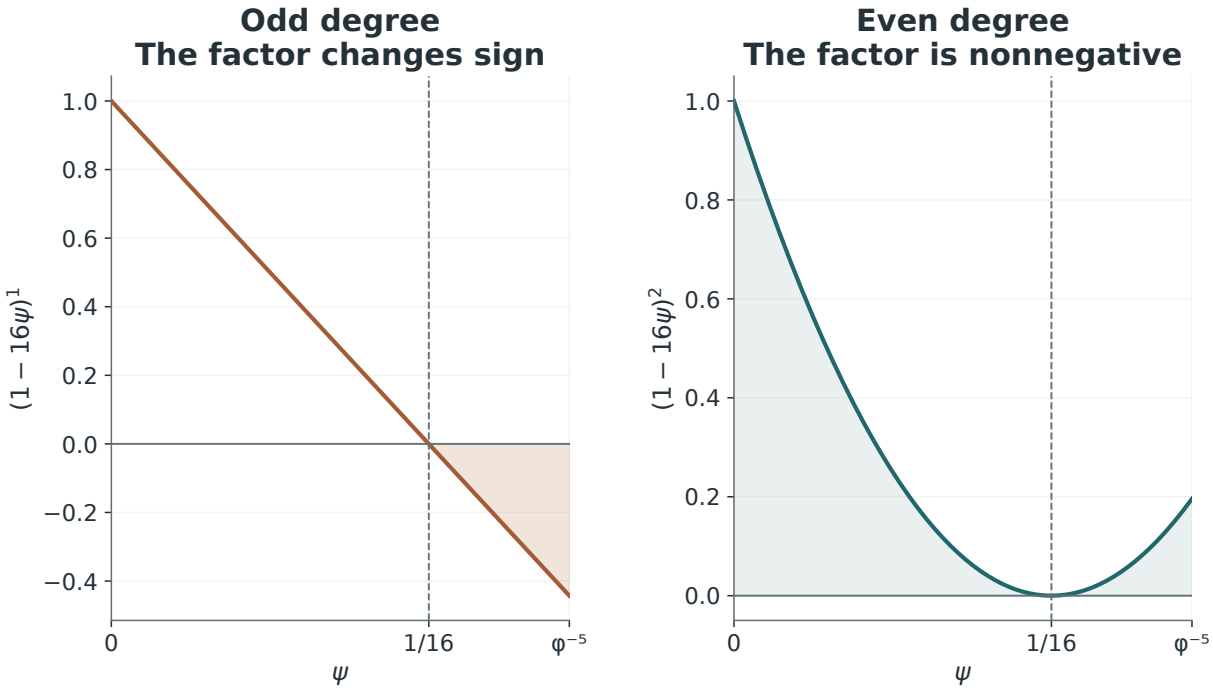
Under the corrected source normalization, $F_{n,m} = 16^n \widetilde{J}_{n,m}$. Even m gives a positive integral. Odd m has no comparable general sign rule here.

Carry forward. A failed simple sign rule is evidence that filtered nonvanishing needs a precise argument.

An even filter keeps the integral positive

27A / THE CORRECTED INTEGRAL

$$\widetilde{J}_{n,m} = \int_{[0,1]^2} \psi^n (1-16\psi)^m \, d\mu$$



READ THE FIGURE

Put

$$\psi = \frac{xy(1-x)(1-y)}{1-xy},$$

$$d\mu = \frac{dx\,dy}{\sqrt{x(1-y)}(1-xy)}.$$

The corrected source identity is

$$\widetilde{J}_{n,m} = \int_{[0,1]^2} \psi^n (1-16\psi)^m \, d\mu.$$

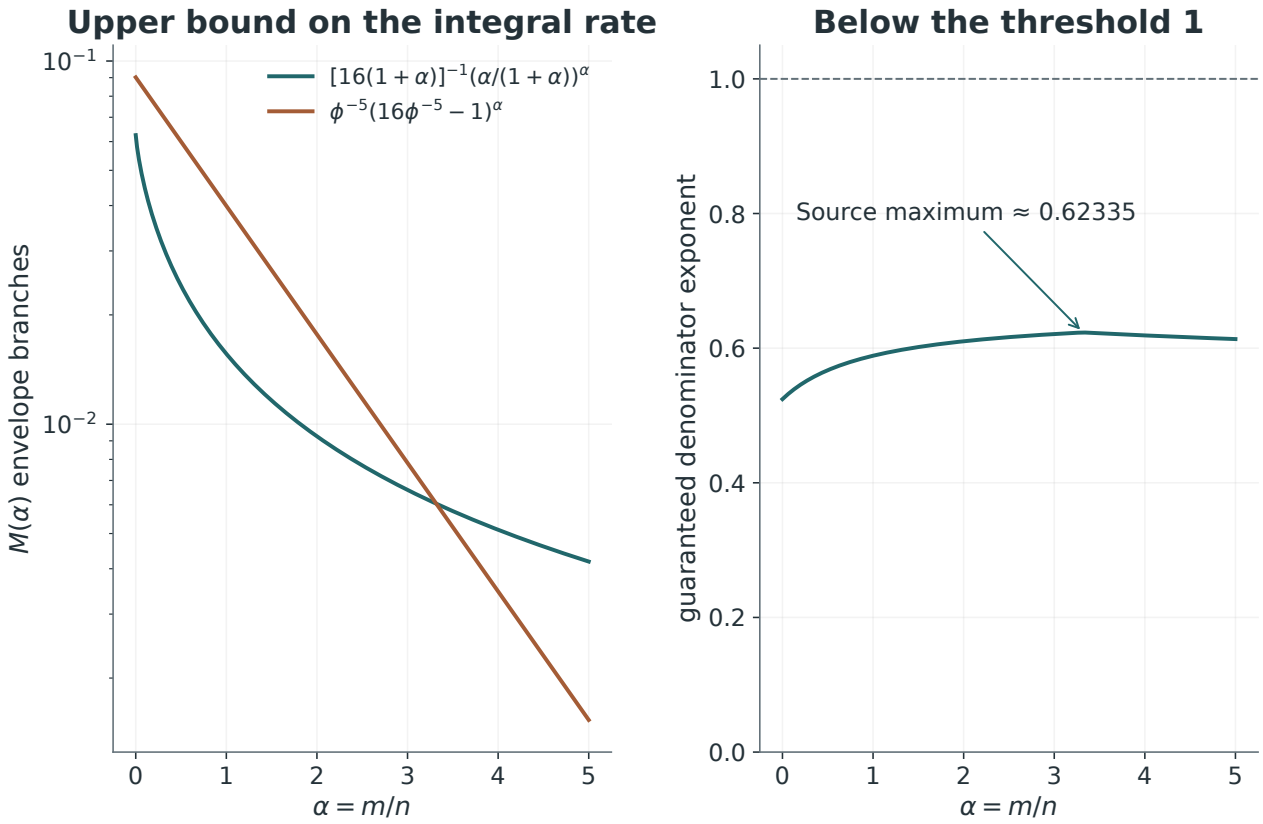
The square root does not cover $1-xy$. Also $J_0 = 8G$, which fixes the factor in $J_n = 8(-1)^n(u_nG - v_n)$.

The range ends at $\phi^{-5} > 1/16$, where $\phi = (1+\sqrt{5})/2$. The filter changes sign inside the square. For even m, the integrand is nonnegative and positive on an open set. Thus $\widetilde{J}_{n,m} > 0$ for every $n \geq 0$.

Carry forward. Choose even filter degrees to establish nonvanishing without extrapolating a sign table.

Better rational approximation still falls short

27B / THE SOURCE RATE BOUND



READ THE FIGURE

Take even $m = m(n)$ with $m/n \to \alpha \geq 0$. The source supplies

$$\limsup_{n\to\infty} |\widetilde{J}_{n,m}|^{1/n} \leq M(\alpha),$$

where $M(\alpha)$ is the larger of the two branches displayed in the figure. The first has its continuous value $1/16$ at $\alpha = 0$.

The reported reduced denominator exponent reaches about 0.62335. It remains below the irrationality threshold 1.

In the historical clearing, the exponential bound for the integer error is governed by

$$\kappa(\alpha) = 4(1+\alpha) + \log 16 + \log M(\alpha).$$

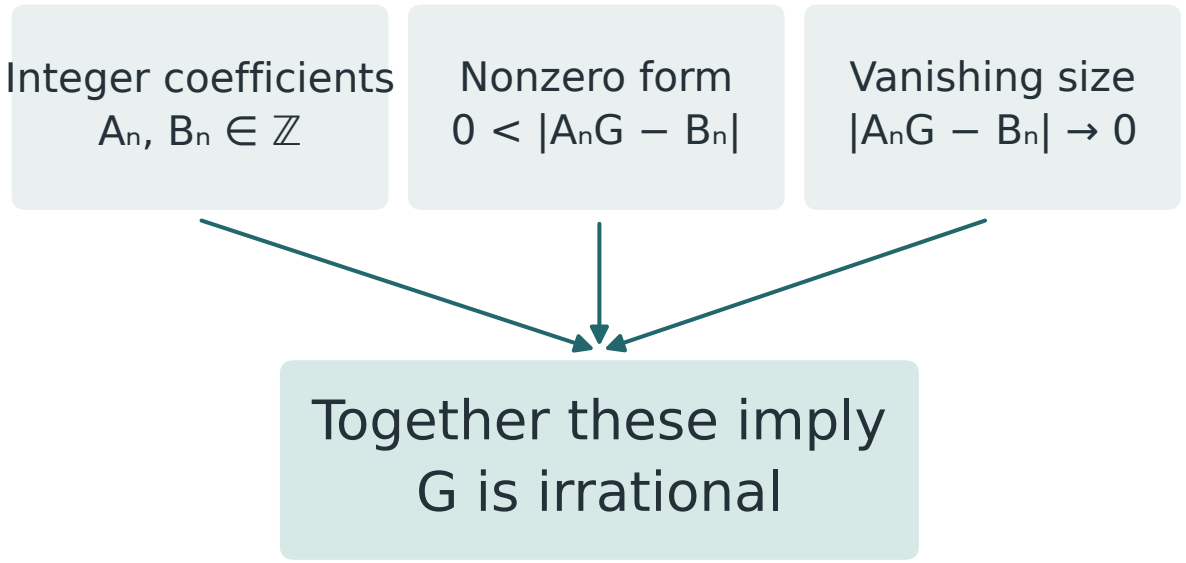
This bound does not force decay. Positive κ does not prove that the actual forms grow.

Further common divisibility or a sharper error estimate would need a uniform argument along infinitely many indices.

Carry forward. The source bound improves rational approximations. Its present arithmetic estimate does not establish irrationality.

The conditional proof route is completely clear

28 / WHAT WOULD FINISH THE ARGUMENT



Finite checks help audit the construction. They do not supply the third condition.

READ THE FIGURE

Construct integer pairs (A_n, B_n) such that the forms $A_nG - B_n$ are nonzero and tend to zero along an infinite sequence.

The elementary lattice argument then proves G irrational.

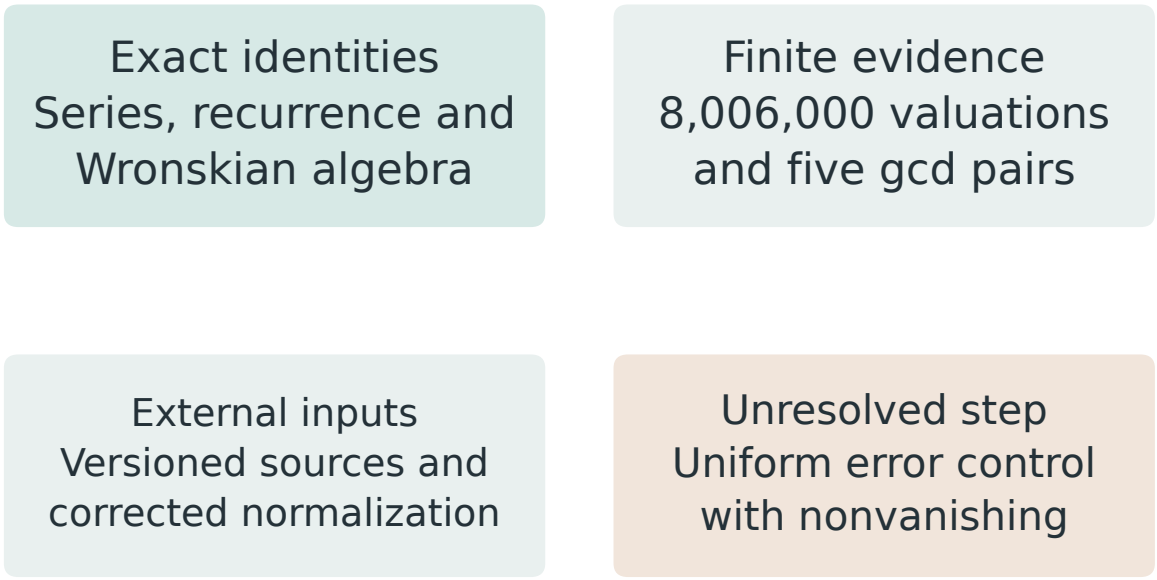
The research problem is to meet those hypotheses simultaneously. The source bundle separates identities, denominator results, finite checks and unresolved estimates.

No finite scan establishes the necessary global decay of the normalized integer forms.

Carry forward. The endpoint is a conditional theorem. Applying it to Catalan's constant remains unfinished.

Read the evidence with its proper scope

29 / SOURCES AND REPRODUCTION



READ THE FIGURE

The monograph gives proofs, notation translations and the detailed source contracts.

The evidence bundle preserves the historical audit and its outputs. It also contains the fresh run, independent diagnostics, figure source and figure provenance.

External sources include Zudilin's recurrence work and Eskandari's September 2026 preprint. The ledger records source versions, normalization repairs and open implications.

Reproduce the figures with `visual/figures.py`. Rounded logarithms and decimal coordinates are marked as display aids. Exact rational decisions remain in the computation files.

Carry forward. The release records audited progress and its limits. No proof of Catalan's irrationality is established here.