Beal

Arithmetic packets and the affine obstruction

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READER EDITION 1.0 / 8 OCTOBER 2026

A reader on primitive power equations, the recovered genus two research program, and the exact distinction between a finite linear operator and its arithmetic target.

EXACT IDENTITIES / FINITE EVIDENCE / OPEN CONJECTURE

How to read this piece

Beal asks a simple question about three powers. The research recorded here asks a more structured question about one difficult signature: what information survives when an integer power equation is transported to a curve, then to a Galois representation, then to local linear constraints? The strongest current reduction is an affine obstruction with seven or eight coordinates. Its linear operator has been computed. Its arithmetic target has not.

The distinction matters. A homogeneous kernel describes how a solution could vary after an origin is known. It cannot supply that origin. A local matrix satisfying the required identities is a useful positive control, but it is not an integer solution. A finite list of possible residual characters is a necessary cover within its stated conductor range, but it is not a list of existing Frey representations.

The first part develops the equation itself. It proves the exponent reduction, the exact primitive scaling criterion, and a complete local criterion for squareclass coefficient strata. These arguments explain why some very effective local tests nevertheless reach a saturation boundary. The middle part reconstructs the actual proof spine recovered in Blocks 47 through 80, including the earlier notes retained inside Block 47. It records how natural lattices, Kummer classes, periods, reciprocity, and same-representation Frobenius data tighten the surviving branch. The final part gives the restricted primitive theorem, the ordinary localization map, and the canonical affine obstruction in full detail.

Several layers of evidence accompany the reader. Exact polynomial identities and finite arithmetic are separated from written arithmetic applications. Lean statements are described according to their actual sources and replay reports. The inherited geometric inputs remain visible even where their consequences have exact matrix certificates. The source archive also preserves rejected attempts, because a failed strengthening often identifies precisely the information that a successful argument must retain.

The phrase “the bounded patterns” always refers to the stipulated depth one, all inert conductor regime where every rational prime ramifying in beta is at most 2000. It never means every possible Beal counterexample. None of the eight surviving tuples has been eliminated by the final affine computation, because no actual affine target has yet been evaluated.

This is a completed account of the recovered work and its current proof boundary. It is not a proof of Beal, a complete solution of the signature $(3,5,7)$, or an independent acceptance of every upstream arithmetic dependency.

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1 Three powers and one common prime

Conjecture 1.1 (Beal). *If positive integers A, B, C and integers $x, y, z > 2$ satisfy*

$$A^x + B^y = C^z, \tag{1.1}$$

then A, B, C have a common prime divisor.

The exponents are allowed to differ. That is the essential extra freedom beyond the equal exponent Fermat equation. The common prime belongs to all three bases, not merely to the product of two of them. The historical statement and its generalized Fermat context are discussed in Mauldin's account and its authenticated published translation [1, 2].

Definition 1.2. A positive solution of (1.1) is *primitive* when $\gcd(A, B, C) = 1$. Its ordered exponent triple (x, y, z) is its signature. The first two entries may be exchanged together with the first two bases.

Primitivity becomes stronger on an actual solution than it is on an arbitrary integer triple.

Lemma 1.3. *For a solution of* (1.1)*, every prime dividing two bases divides the third. Consequently a primitive solution has pairwise coprime bases.*

Proof. If a prime p divides A and B, the equation gives $p \mid C^z$, hence $p \mid C$. If p divides A and C, it gives $p \mid B^y$, hence $p \mid B$. The third case is the same. Thus a prime shared by any pair would divide the gcd of all three. □

Beal is therefore equivalent to the absence of primitive positive solutions for all signatures with entries greater than two. This reformulation is the one used throughout the research program. It makes the support of each base disjoint from the other two supports, which is useful when assigning local valuation sectors and conductor conditions.

Example 1.4. The equality $3^3 + 6^3 = 3^5$ is a valid power equation. Both terms on the left are positive, and all exponents exceed two. The common prime is 3, so this equality illustrates the conjecture rather than contradicting it. Its powered terms are $27, 216, 243$.

An ordinary division by $\gcd(A, B, C)$ is not a valid reduction to a primitive solution. If $A = dA'$, $B = dB'$, $C = dC'$, the three terms carry factors d^x, d^y, d^z. Unless the exponents agree, division by one of these powers changes the other coefficients. Later, a weighted scaling will preserve the equation. That operation has a precise valuation criterion and is not the same as dividing every base by d.

The positivity assumptions also carry information. They imply

$$A^x < C^z, \qquad B^y < C^z, \qquad C \geq 2. \tag{1.2}$$

The last inequality follows because $A^x + B^y \geq 2$. A base on the left may equal 1, so a finite classification of powered values need not be a finite classification of exponent labels. The equality $1^x = 1$ holds for every x. This small point is relevant when comparing a theorem for one fixed signature with a statement uniform in all exponents.

No bounded search can settle Conjecture 1.1 without a theorem covering its unsearched range. A search can exclude a stated finite region, identify local obstructions, and check an implementation. A reduction to finitely many residual patterns can be stronger than an ordinary integer search, but it still needs an exhaustive arithmetic cover and a proof that every covered pattern is impossible.

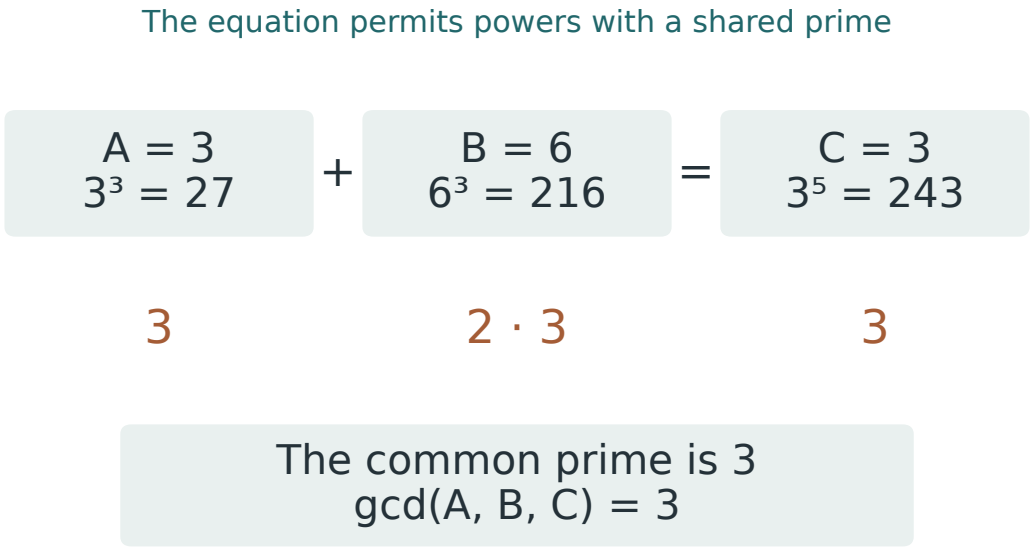


Figure 1: The equation in Example 1.4 has a common prime in its bases. Exact equality alone does not supply primitivity.

2 Which signatures a reduction must retain

Every exponent greater than two has a useful small divisor. The divisor need not be the same for the three exponents.

Proposition 2.1 (Prime or four reduction)**.** *If a primitive solution of* (1.1) *exists, there is a primitive solution whose three exponents each belong to*

$$\{4\} \cup \{p : p \text{ is an odd prime}\}.$$

Conversely, absence of primitive solutions for every ordered signature in this set implies Beal.

Proof. Let $n > 2$. If n has an odd prime divisor p, write $D^n = (D^{n/p})^p$. If it has no odd prime divisor, it is a power of two at least four, so write $D^n = (D^{n/4})^4$. Apply this operation to each term separately. The new bases have exactly the same prime supports as the old bases, so primitivity is preserved. The converse follows by reducing any hypothetical primitive solution in this way. □

The proposition retains infinitely many signatures. It does not reduce them all to $(3, 5, 7)$. A theorem for that signature also handles signatures $(3a, 5b, 7c)$ after absorbing the multipliers into the bases, and the symmetric version with the first two terms exchanged. It does not handle an unrelated prime signature such as $(5, 7, 11)$ merely because the same packet language can describe both.

If all three exponents have a common divisor $n > 2$, absorbing the remaining factors produces an equal exponent Fermat equation. Fermat's Last Theorem then excludes the primitive solution [3, 4]. This use requires a common divisor of all three exponents. Sharing a

divisor in only one pair is weaker. The general prime or four reduction must therefore retain mixed signatures even after the equal exponent cases are removed.

The fixed signature finiteness theorem of Darmon and Granville provides another useful perspective [8]. In the hyperbolic range

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}<1,$$

their theorem gives finiteness of primitive solutions for a fixed equation and fixed exponents. For $x,y,z>2$, the only nonhyperbolic possibility is $(3,3,3)$, where the reciprocal sum is one and Fermat already applies. Finiteness is valuable, but it is not absence. It also does not give a uniform finite enumeration over all signatures.

2.1 The conditional abc height bound

An abc estimate makes the role of the signature explicit. Let

$$H=C^z, \qquad R=\operatorname{rad}(ABC), \qquad s=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}.$$

For a primitive solution, the powered terms are pairwise coprime and their radical is R. Positivity gives $ABC<H^s$. Assume the all epsilon abc estimate

$$H\le K_\varepsilon R^{1+\varepsilon} \tag{2.1}$$

for every $\varepsilon>0$, with one constant K_ε independent of the triple. Then

$$H<K_\varepsilon H^{(1+\varepsilon)s}.$$

Whenever $(1+\varepsilon)s<1$, it follows that

$$H<K_\varepsilon^{1/(1-(1+\varepsilon)s)}. \tag{2.2}$$

This is a conditional bound, not an unconditional result obtained from the exponent equation alone.

For $(3,5,7)$, the reciprocal sum is $71/105$. Thus $0<\varepsilon<34/71$ gives

$$H<K_\varepsilon^{105/(34-71\varepsilon)}, \qquad C<K_\varepsilon^{15/(34-71\varepsilon)}. \tag{2.3}$$

At $\varepsilon=1/10$, the latter exponent is $150/269$. To turn this into a proof of absence would require the abc input, an explicit usable constant, and a complete verification below the resulting bound.

There is a stronger uniform bound across the Beal range. Apart from $(3,3,3)$, already excluded by Fermat, one exponent is at least four. Thus $s\le 1/3+1/3+1/4=11/12$. Take $\varepsilon=1/22$ and enlarge its constant to $K_{1/22}\ge 1$. Then $(1+\varepsilon)s\le 23/24$, and (2.2) gives

$$H<K_{1/22}^{24}. \tag{2.4}$$

This conditional constant is independent of the signature. It bounds the powered terms and yields finitely many possible primitive powered triples. It does not show that the finite set is empty. Since $2^z\le H$, it also bounds z and excludes solutions once the minimum exponent

is sufficiently large. The all epsilon hypothesis and the gap between finiteness and absence remain essential [8].

Unit bases can be disposed of separately by the classical theorem on consecutive perfect powers. If $A = 1$ and $B > 1$, then $C^z - B^y = 1$. Mihăilescu's theorem gives the only pair $3^2 - 2^3 = 1$, whose upper exponent is two and violates the Beal range [5, 6, 7]. The same argument handles $B = 1$. If both are one, $C^z = 2$ is impossible. Hence every actual Beal solution has all bases greater than one. After this accepted deep input, a powered height bound also bounds every exponent label. This is a use of a classical theorem, not a new proof of it.

2.2 Why a generic radical ceiling fails

The exponent equation gives $R < H^{71/105}$ for the signature $(3, 5, 7)$. It is tempting to reverse this relation by postulating $H \leq R^{105/71}$ for every primitive sum. That universal assertion is false. The exact equality

$$2 + 109 \cdot 3^{10} = 23^5 = 6\,436\,343$$

has radical $2 \cdot 3 \cdot 109 \cdot 23 = 15\,042$, and

$$(23^5)^{71} > 15\,042^{105}.$$

Its logarithmic quality is about 1.629911684. It is not a $(3, 5, 7)$ counterexample. It shows that an estimate strong enough to close the desired signature must retain its power conditions or contain the constants and exceptions that a genuine abc type estimate allows.

3 A rational parametrization and its integral defect

For the signature $(3, 5, 7)$, rational points are easy to manufacture. Primitive integer points are not. The following polynomial identity makes the difference exact.

Proposition 3.1 (Universal power identity)**.** *In any commutative ring, put $s = 1 - t$ and*

$$A_0 = t^{12}s^{28}, \qquad B_0 = t^7 s^{17}, \qquad C_0 = t^5 s^{12}.$$

Then $A_0^3 + B_0^5 = C_0^7$. Weighted scaling

$$(A, B, C) = (\lambda^{35} A_0, \lambda^{21} B_0, \lambda^{15} C_0) \tag{3.1}$$

preserves this equality.

Proof. The two terms on the left are $t^{36}s^{84}$ and $t^{35}s^{85}$. Their sum is $t^{35}s^{84}(t + s) = t^{35}s^{84}$, which is C_0^7. The three scaling weights multiply the powered terms by the same factor λ^{105}. □

Over a field, if t and $1 - t$ are nonzero, all three bases are nonzero and $A_0^3/C_0^7 = t$. Thus every regular parameter is locally populated. This works modulo arbitrary prime powers whenever t and $1 - t$ are units. A finite field test that asks only whether such a nonzero local solution exists cannot remove any of these parameter values.

For a nonzero rational solution, define $t = A^3/C^7$ and $\lambda = C^8/(AB^4)$. Substitution recovers (3.1). If the original bases are positive, then $0 < t < 1$ and $\lambda > 0$. The rational parametrization therefore captures the entire positive rational problem. Its missing condition is exactly primitive integrality.

Theorem 3.2 (Primitive scaling criterion)**.** *Fix a rational $0 < t < 1$. There is a positive rational λ for which (3.1) gives primitive positive integers if and only if, for every rational prime ℓ,*

$$\begin{aligned} v_\ell(t) > 0 &\implies 3 \mid v_\ell(t), \\ v_\ell(1-t) > 0 &\implies 5 \mid v_\ell(1-t), \\ v_\ell(t) < 0 &\implies 7 \mid -v_\ell(t). \end{aligned}$$

When it exists, λ is unique.

Proof. Write $m = v_\ell(t)$, $n = v_\ell(1-t)$ and $h = v_\ell(\lambda)$. Because $t + (1-t) = 1$, the pair (m, n) belongs to one of four sectors:

$$(m > 0, n = 0), \quad (m = 0, n > 0), \quad (m = n < 0), \quad (m = n = 0).$$

The output valuations are

$$\begin{aligned} a_\ell &= 35h + 12m + 28n, \\ b_\ell &= 21h + 7m + 17n, \\ c_\ell &= 15h + 5m + 12n. \end{aligned}$$

They must be nonnegative, and their minimum must be zero. Equivalently, the minimum of $3a_\ell, 5b_\ell, 7c_\ell$ must be zero.

In the first sector these weighted valuations are $105h + 36m, 105h + 35m, 105h + 35m$. Therefore $h = -m/3$, and the output is $(m/3, 0, 0)$. In the second sector they are $105h + 84n, 105h + 85n, 105h + 84n$. This forces $h = -4n/5$ and output $(0, n/5, 0)$. In the third sector write $m = n = -r$ with $r > 0$. The weighted valuations are $105h - 120r, 105h - 120r, 105h - 119r$. Thus $h = 8r/7$ and the output is $(0, 0, r/7)$. In the final sector $h = 0$.

An integral h exists in these four cases precisely under the stated divisibility conditions. Only finitely many primes occur in t or $1 - t$, so $\lambda = \prod_\ell \ell^{h_\ell}$ is a well defined positive rational number. The three output supports are disjoint, proving integrality and primitivity. Every h_ℓ was forced, so two positive choices of λ would have ratio a positive rational with zero valuation at every prime. That ratio is one. □

If $t = m/n$ is in lowest terms with $0 < m < n$, the theorem says that m must be a cube, $n - m$ a fifth power, and n a seventh power. The parametrization has not escaped the original power equation. It has isolated its defect in three valuation sectors.

Example 3.3. At $t = 1/2$, the denominator has valuation one at two, which fails the required divisibility by seven. No positive rational scaling produces a primitive integer triple. The choice $\lambda = 4$ nevertheless gives

$$(A, B, C) = (2^{30}, 2^{18}, 2^{13}),$$

and the power equation holds. Its common divisor 2^{13} makes it a negative control against interpreting rational parametrization as a primitive solution.

This is where the Komorebi packet viewpoint has concrete force. Each sector carries an exact linear valuation rule, a required divisibility residue, and a reconstruction weight. Keeping all three records prevents information loss. The records alone do not prove that no rational t satisfies every condition. A global closure still requires a relation linking the sectors, rather than additional names for the same local data.

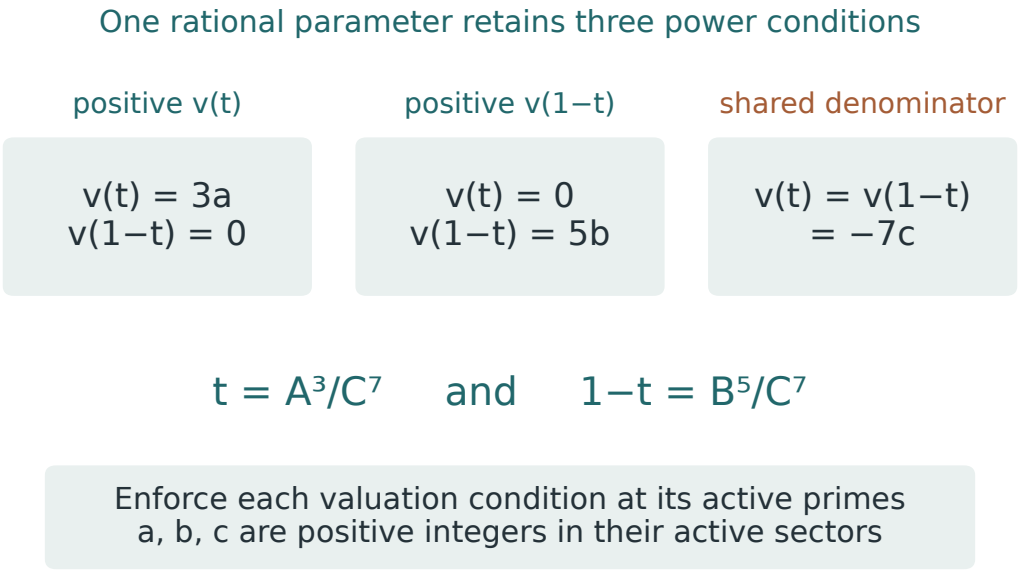


Figure 2: The three nontrivial valuation sectors retain different divisibility conditions. A residue projector can preserve their labels without making the conditions hold.

4 The complete local squareclass test

Write a hypothetical primitive solution as

$$A = au^2, \qquad B = bv^2, \qquad C = cw^2,$$

where a, b, c are positive pairwise coprime squarefree integers. The equation becomes

$$a^3u^6 + b^5v^{10} = c^7w^{14}. \tag{4.1}$$

At a prime p, a local solution is called primitive when at most one base is divisible by p. Roots and bases are required to be nonzero in $\mathbb{Z}_p$. The roots may depend on p. This last freedom is exactly why local existence can be weaker than an integer solution.

Theorem 4.1 (Local coefficient criterion)**.** *For fixed a, b, c as above, primitive local solvability of (4.1) is determined by the following conditions.*

1. *At two, at least one row in the table must be admissible. An even coefficient forces its own base to be the even base.*
2. *At an odd prime dividing a, b, c, respectively, the condition is*

$$\left(\frac{bc}{p}\right) = 1, \qquad \left(\frac{ac}{p}\right) = 1, \qquad \left(\frac{-ab}{p}\right) = 1.$$

3. *At every odd prime not dividing abc, a primitive local solution exists.*

Even base	Required odd coefficients	Relation modulo eight
A	b, c	$b \equiv c$
B	a, c	$a \equiv c$
C	a, b	$a + b \equiv 0$

The proof is worth retaining in full, because it establishes the absence of an untested prime power tail. The criterion is not an extrapolation from residue sweeps.

4.1 Odd prime lifting

For the polynomial on the left of (4.1) minus its right side, the partial derivatives are

$$6a^3u^5, \qquad 10b^5v^9, \qquad -14c^7w^{13}.$$

In a primitive residue solution, at least two bases are units. At an odd prime, at most one of $6, 10, 14$ is divisible by p. Thus at least one unit variable has unit derivative. Lift the other variables to nonzero integers, using p for a zero residue if necessary. Successive correction in the selected variable by multiples of p^n kills one new digit at each stage. The derivative remains a unit, so the corrections converge to a nonzero root with the original primitive pattern. Every primitive residue solution therefore lifts to $\mathbb{Z}_p$, including at $3, 5, 7$.

If $p \mid a$, primitivity forces the other two bases to be units. Reduction gives $b^5v^{10} = c^7w^{14}$. The quadratic character of this equality proves the necessary symbol condition. Conversely, $v^{10}w^{-14}$ ranges over the squares of $\mathbb{F}_p^\times$, because $\gcd(10, 14, p-1) = 2$. The symbol condition lets us choose the two unit roots, then lift. The other two cases use $\gcd(6, 14, p-1) = 2$ and $\gcd(6, 10, p-1) = 2$. Moving a term across the equality gives the minus sign when $p \mid c$.

4.2 Odd primes away from the coefficients

Let α, β, γ be the quadratic characters of a, b, c, and let $e = (-1/p)$. A primitive residue solution with one zero base exists if

$$\beta = \gamma, \qquad \alpha = \gamma, \qquad \alpha = e\beta,$$

respectively. The same cyclic group argument proves this, and the lifting construction replaces the zero root by a nonzero multiple of p.

If $p \equiv 1 \pmod 4$, two of the three signs agree, so one of these conditions holds. If $p \equiv 3 \pmod 4$ and all three fail, then $\alpha = \beta = -\gamma$. The two remaining sign patterns can be handled by the universal identity of Proposition 3.1.

There are $(p+1)/4$ elements t for which both t and $1-t$ are nonsquares. Here is a direct count. Extend the quadratic character χ_p by zero. Counting $(x-y)(x+y) = d$ for $d \neq 0$ gives

$$\sum_x \chi_p(x^2 - d) = -1.$$

Completing the square then gives $\sum_t \chi_p(t(1-t)) = -\chi_p(-1) = 1$. After omitting $0, 1$, both single character sums are -1. Expanding the indicator $(1 - \chi_p(t))(1 - \chi_p(1-t))/4$ over the remaining points gives $(p+1)/4 > 0$.

For such a t, the three bases in Proposition 3.1 have characters $(1, 1, -1)$. Multiplication by the weighted powers of a nonsquare λ changes all three signs and gives $(-1, -1, 1)$. In either

needed sign pattern, the ratios of these bases to a, b, c are squares. Choosing their square roots gives a solution of (4.1) with all bases units. The lifting argument completes the proof at these primes.

4.3 The dyadic criterion

Exactly one base is even in a primitive two adic solution. An even powered base is zero modulo eight, while an odd squared root is one modulo eight and an odd coefficient has the same third, fifth, and seventh power as itself modulo eight. This proves the necessity of the table.

For sufficiency, fix the selected even root to be one when its coefficient is even, and two when the coefficient is odd. Fix one odd root to be one. After moving the terms and dividing by the remaining coefficient power, the required remaining root power is a unit congruent to one modulo eight. Such a unit is a square in $\mathbb{Z}_2$, and every odd power map on $\mathbb{Z}_2^\times$ is bijective. It consequently has the required sixth, tenth, or fourteenth root.

The square assertion has a direct lifting proof. For $n \geq 3$, changing an odd trial square root by 2^{n-1} changes its square by 2^n modulo 2^{n+1}, allowing the next digit to be selected. Odd power maps have odd derivative and lift uniquely. Thus the dyadic criterion also covers every higher precision.

4.4 Two families that survive every local refinement

The coefficients $(a, b, c) = (3, 5, 2)$ pass the even C row because $3 + 5 \equiv 0 \pmod 8$. At three, $bc = 10$ is a square residue. At five, $ac = 6$ is a square residue. Hence

$$27u^6 + 3125v^{10} = 128w^{14}$$

has a primitive solution at every prime. Likewise $(2, 3, 11)$ passes the even A row since $3 \equiv 11 \pmod 8$. At three, $ac = 22 \equiv 1$, and at eleven, $-ab = -6 \equiv 5 = 4^2$. Thus

$$8u^6 + 243v^{10} = 11^7 w^{14}$$

also has a primitive solution at every prime. Both have positive real solutions. Neither is asserted to have an integer solution.

For either coefficient triple and any modulus M, there are positive integer roots giving pairwise coprime bases and satisfying the equation modulo M. Enlarge M to contain the coefficient primes, choose primitive local residues, and combine them by the Chinese remainder theorem. Choose u in its positive progression. Choose v in its progression with $v \equiv 1$ at each prime dividing u outside the enlarged modulus. Choose w with the analogous conditions at primes dividing uv. The auxiliary moduli are coprime to the original one. Inside the modulus, local primitivity prevents shared base primes. Outside it, the avoidance conditions do so. This proves the finite congruence assertion, while exact equality remains unproved.

Joint labels still recover real exclusions. Among the 64 all odd coefficient triples modulo eight, the three row criterion allows 36. Every individual coordinate marginal is full. Replacing the joint relation by independent marginals restores 28 excluded triples. Packet refinement therefore has two distinct lessons: preserve correlations that already contain arithmetic information, and recognize when the projected local problem has become completely soluble.

5 A curve that retains the exponent seven

The signature route uses the integral curve

$$y^2 + (X^3 + B)y = 2BX^3 + 3AX + B^2. \tag{5.1}$$

Completing the square with $Z = 2y + X^3 + B$ gives

$$Z^2 = f_{A,B}(X) = X^6 + 10BX^3 + 12AX + 5B^2. \tag{5.2}$$

This is the genus two family used in the recovered source. The comparison with the family of Pacetti and Villagra Torcomian is a rational change of coordinates, with their variables translated as $(a, b, c, p) = (B, -C, A, 7)$ [10]. Explicitly, put $t = -B^5/A^3$, $d = -B^2/A$, and substitute $x_{\mathrm{PV}} = d/X$, $y_{\mathrm{PV}} = tZ/X^3$ into their equation $y^2 = 5x^6 - 12x^5 + 10tx^3 + t^2$. Multiplication by X^6/t^2 gives (5.2). The map is rational for $AB \neq 0$. It does not on its own verify a twist, integral frame, or stable model. Their general theorems whose hypotheses exclude seven cannot be applied at seven merely because the family is the same. The versioned theorem hypotheses are recorded in the literature report.

Proposition 5.1 (Polynomial discriminant)**.** *The sextic in* (5.2) *satisfies*

$$\operatorname{disc}(f_{A,B}) = 2^{12}3^6 5^5 (A^3 + B^5)^2.$$

For a hypothetical $(3, 5, 7)$ *solution this becomes* $2^{12}3^6 5^5 C^{14}$*.*

Proof. Give X, A, B weights $1, 5, 3$. Then

$$f_{\lambda^5 A, \lambda^3 B}(\lambda X) = \lambda^6 f_{A,B}(X).$$

The six roots scale by λ, so their fifteen squared differences give discriminant weight thirty. The only nonnegative integer solutions of $5i + 3j = 30$ are $(i, j) = (6, 0), (3, 5), (0, 10)$. The discriminant therefore has the form $uA^6 + vA^3B^5 + wB^{10}$.

Exact Sylvester determinants at $(A, B) = (1, 0), (0, 1), (1, 1)$ give respectively $K, K, 4K$, where $K = 9\,331\,200\,000 = 2^{12}3^6 5^5$. These values force $u = w = K$ and $v = 2K$. This proves the identity in the entire polynomial ring. The finite calculation determines three coefficients in a proved three dimensional monomial space, rather than extrapolating a formula from a sample. □

This is the discriminant of a displayed polynomial. It has not been identified with a minimal discriminant, a field discriminant, or the conductor of the Jacobian. What it does show directly is that, outside $2, 3, 5$, every possible bad reduction prime divides C. At a prime at least seven not dividing C, the sextic has distinct roots and supplies a good reduction model.

At a variable prime dividing C, primitivity makes A, B units. In the residue field put $u = -B^2/A$. Then $u^3 = B$, $u^5 = -A$, and

$$f_{A,B}(uX) = u^6(X^2 + X - 1)^2(X^2 - 2X + 5). \tag{5.3}$$

The discriminants of the two quadratics are 5 and -16, and their resultant is 45. Away from $2, 3, 5$, the special fiber has exactly two ordinary nodes, rational normalization, and smooth infinity. This factorization is the local geometric entry point for the period and monodromy calculations.

5.1 A characteristic seven cancellation

In characteristic seven, the Cartier coefficient calculation comes from f^3. Its relevant coefficients are

$$[X^6]f^3 = 1575B^4, \qquad [X^5]f^3 = 4320A^2B,$$
$$[X^{13}]f^3 = 36A, \qquad [X^{12}]f^3 = 315B^2.$$

The diagonal lanes vanish modulo seven and the off diagonal lanes become

$$M = \begin{pmatrix} 0 & A^2B \\ A & 0 \end{pmatrix}, \qquad M^2 = A^3BI_2. \tag{5.4}$$

These are polynomial and matrix identities. Their geometric interpretation requires the correct Cartier and Hasse–Witt conventions. On a good reduction fiber at seven they imply ordinarity when AB is nonzero modulo seven and seven rank zero when AB is zero. There is no rank one stratum in this calculation. The retained sources and the formal report distinguish these algebraic statements from the associated finite flat and Frobenius applications.

The cancellation has genuine arithmetic value: it sharply reduces the possible local Frobenius behavior. It does not provide residual irreducibility by itself. Nor does a vanishing diagonal coefficient make the original integer equation vanish. A transport theorem must retain both the correct lattice and the labelled Frobenius action.

5.2 What the direct irreducibility argument proves

The canonical recovered result is absolute irreducibility of the associated residual rank two RM representation when $14 \nmid C$, subject to its declared realization, conductor, local weight, and finite flat inputs. Its two branches use different information.

When C is odd, the integral model has good reduction at two. Over the norm four prime of $K = \mathbb{Q}(\sqrt{5})$, the full Frobenius polynomial is either $(T^2+T+4)^2$ or $(T^2+4)^2$. A hypothetical degree one constituent, after the conductor and global unit argument, must have eighth power one. Neither quadratic has a root with eighth power one modulo seven. For T^2+4, a root has square three, hence eighth power four. For T^2+T+4, reduction of powers gives fourth power five, again eighth power four. The contradiction uses the full polynomial and the same global constituent.

The global unit step uses $\phi = (1+\sqrt{5})/2$, whose residue has order sixteen in $\mathbb{F}_{49}^{\times}$. The local weight possibilities at seven are $0, 1, 7, 8$, up to a common reciprocity sign. Applying reciprocity to the eighth power of each constituent gives $\overline{\phi}^{8e} = 1$ and rules out the mixed weights $1, 7$. The determinant norm relation $\delta^8 = \chi^2$ then leaves one constituent unramified at seven, without assuming an unproved rank two determinant normalization in this argument. Its eighth power is everywhere unramified, and class number one makes it trivial. This is why the local quadratic contradiction applies to that constituent.

When C is even and seven does not divide C, the two adic torus calculation replaces good reduction at two. The global character has Frobenius value -1 at two. Reciprocity on ϕ^4 and $2/7$ transports that value to seven. The finite flat comparison at good seven, together with (5.4), gives the value A^3B. Thus $A^3B = -1$ modulo seven. Since both bases are nonzero, $B = -A^3$ and $A^3 + B^5 = 0$ modulo seven, contradicting $7 \nmid C$.

The conclusion removes an obstruction to a modular method. It does not eliminate the modular candidates. Its uncovered parity and divisibility branch is precisely $14 \mid C$. The later

reducible research concerns a further stipulated branch with $3990 \mid C$, retaining the same curve and its specified twist.

6 Modular labels and information that a norm forgets

The recovered modular comparison keeps a candidate form, a prime of its coefficient field above seven, and a residue embedding. These labels must remain compatible across auxiliary primes. Taking norms or minimal polynomials can produce a sound necessary condition while forgetting those labels.

Suppose a Hecke eigenvalue $a_{\mathfrak{q}}$ is replaced by its minimal polynomial. A resultant against a possible curve trace polynomial asks whether some conjugate can match some allowable trace. Repeating the test at a different prime can choose a different conjugate. If the original representation requires one residue embedding for every prime, this relaxed sequence of tests may preserve candidates that the labelled comparison would exclude. An exclusion obtained from the relaxation remains valid. Survival becomes less informative.

The Block 47 audit identified this mechanism in the pinned Pacetti–Villagra Torcomian code. It also identified a parameter dictionary issue at eleven: the unlabelled multisets agreed, but individual labels matched after $t \mapsto t^{-1}$. A refinement attached to the wrong labels could incorrectly combine local constraints. The new reader therefore distinguishes direct point counts on (5.2) from imported hypergeometric parameter tables.

Two exact local restrictions were established under the stated modular and multiplicative prime comparisons. A rank two trace zero comparison at a prime above eleven forces $11 \mid A$. At a prime above nineteen it forces $19 \mid B$. In either case the full four dimensional reduction to check is

$$(T^2 + q)^2 \pmod 7,$$

which is stronger than zero four dimensional trace. At bad q, the multiplicative comparison gives $\pm(q+1)$, nonzero modulo seven for these two primes. At good q, the direct curve table selects the stated base zero stratum. These are divisibility restrictions, not contradictions.

6.1 From point counts to a common polynomial

For a smooth genus two curve over $\mathbb{F}_q$, let N_1, N_2 count its points over $\mathbb{F}_q, \mathbb{F}_{q^2}$, including infinity. Write

$$s_1 = q + 1 - N_1, \qquad s_2 = \frac{s_1^2 - (q^2 + 1 - N_2)}{2}.$$

Its Frobenius polynomial is

$$T^4 - s_1 T^3 + s_2 T^2 - q s_1 T + q^2. \tag{6.1}$$

In the RM setting, the rank two traces are the roots of

$$X^2 - s_1 X + (s_2 - 2q).$$

The two roots belong to one common quartic. Choosing one root from one fiber and another root from an unrelated fiber can enlarge a necessary locus, but does not give a compatible simultaneous reduction of the original curve. The later Frobenius sieves explicitly retain this common polynomial condition.

6.2 The remaining modular branches

Block 48 and the accumulated Block 49 notes retain conditional local type and divisibility reductions, including the branch $3990 \mid C$. They do not certify every Hilbert modular candidate as eliminated. The imported authors' transcripts have their own index conventions and some aggregate exception lists do not identify a complete seven specific set. A form index in such a transcript is not an intrinsic arithmetic identifier.

The surviving CM branch also requires care. CM can force a trace to vanish at suitable inert primes, which produces base divisibility constraints such as the one at eleven. It does not imply that every representation with a compatible residual trace is the CM representation itself. A congruence modulo seven is weaker than equality of characteristic zero representations.

The packet interpretation here is exact and limited. A packet consists of a candidate, its coefficient field, the chosen prime above seven, the residue embedding, and the paired local polynomial. Transfer to an unlabelled norm is a projection of that packet. Recovering the forgotten labels is a legitimate strengthening. The required exports were not present in every recovered modular data set, so no completed all form elimination is inferred from the proposed strengthening.

7 The natural lattice and the depth of an opposite extension

The reducible branch uses two different kinds of radical. The base field is

$$K = \mathbb{Q}(\sqrt{5}), \qquad \phi = (1+\sqrt{5})/2.$$

The coefficient ring and field are

$$\mathcal{O} = \mathbb{Z}_7[w]/(w^2-5), \qquad k = \mathbb{F}_7[w]/(w^2-5) = \mathbb{F}_{49}.$$

The base unit ϕ defines Kummer classes. The coefficient w acts on a rank two representation. Although both involve a square root of five, they have different roles and are never identified by an implicit residue substitution.

The inherited characteristic zero representation is the specified RM factor of the twisted Jacobian on the retained $3990 \mid C$ branch, with determinant the cyclotomic character χ. Its natural residual orientation is

$$0 \longrightarrow k(\chi) \longrightarrow \overline{V} \longrightarrow k \longrightarrow 0.$$

The opposite endpoint has a nonzero class $\beta \in H^1(K, k(\chi^{-1}))$. A stable lattice change can alter residual inertia. Therefore properties of the natural Jacobian lattice cannot automatically be transferred to an arbitrary lattice supplied by Ribet's lemma [22].

7.1 Why a new eligible prime is necessary

The opposite orientation forces an additional prime divisor of C. Here is the conditional argument, retaining the class group and local geometric inputs. Suppose a characteristic zero local toric line has reduction χ, while the opposite residual extension has subline one and quotient χ. Intersect the actual toric line with its stable lattice. The intersection is saturated, so it reduces to a nonzero local χ line. If the local character χ is nontrivial, that line is distinct from the global subline one. They split the residual extension locally. This proof survives stable lattice changes because it starts with the characteristic zero line.

At seven, the specified twist makes the toric characters χ and one, and cyclotomic inertia is nontrivial. At a variable bad place with $Q \not\equiv 1 \pmod 7$, the same argument applies. At the places over two, three and five, a separate local cohomology calculation excludes opposite extensions. For a non seven local field of residue cardinality Q, an inertia cocycle with coefficients $k(\chi^{-1})$ can be nonzero only when $Q^2 \equiv 1 \pmod 7$: the tame Frobenius relation multiplies its inertia value by Q, while the coefficient action multiplies it by Q^{-1}. The unramified quotient is zero when $Q \not\equiv 1$. Thus $Q \not\equiv \pm 1$ forces local H^1 to vanish. Here the relevant norms are four, nine and five, all outside those two residue classes.

If no variable C prime had $Q \equiv 1 \pmod 7$, beta would restrict to an everywhere unramified additive class over $L = K(\zeta_7)$. Restriction is injective because $[L : K] = 6$ is prime to seven. The inherited Masley class number result gives $h(\mathbb{Q}(\zeta_{35})) = 1$ [16]. The field L is a subfield of index two. Extension and norm on ideal classes compose to multiplication by two, so the seven primary class group of L is zero. An everywhere unramified nonzero additive seven class is therefore impossible by class field theory.

At least one variable prime above a divisor of C must consequently have norm one modulo seven. A rational prime splits in K when it is one or four modulo five, giving residue degree one. At an inert prime the degree is two. Combining $q^f \equiv 1 \pmod 7$ with these splitting possibilities gives exactly

$$q \pmod{35} \in \{1, 8, 13, 22, 27, 29\}.$$

None of $2, 3, 5, 7, 19$, the primes already in 3990, supplies this new variable prime. The conclusion asks for an eligible additional prime, not divisibility by a particular fixed prime. Once such primes are allowed, their ray class columns can support beta. Ordinary class number one does not make that ramified character space vanish.

7.2 An actual finite segment of stable lattices

The inherited split toric calculation at three supplies Frobenius eigenvalues nine and one. Their difference eight is a seven adic unit. The projectors

$$E_9 = \frac{\rho(g_0) - I}{8}, \qquad E_1 = \frac{9I - \rho(g_0)}{8}$$

are complementary integral idempotents. Choose generators of their images. In this basis, $\rho(g_0) = \operatorname{diag}(9, 1)$ and every lower entry is divisible by seven. Natural upper monodromy at three has a unit entry. Absolute irreducibility in characteristic zero forces some lower entry to be nonzero.

Define

$$n = \min_{g : \rho(g)_{21} \neq 0} v_7(\rho(g)_{21}). \tag{7.1}$$

This positive integer is attained. It is an invariant of the fixed natural frame, not a truncation selected for a computation. Every $\mathcal{O}$ lattice stable under ρ is preserved by the two projectors, so up to homothety it has the form

$$\Lambda_j = \mathcal{O}e_1 + \mathcal{O}7^j e_2.$$

The unit upper monodromy forces $j \geq 0$. The attained lower entry minimum forces $j \leq n$. Conversely every j in that interval is stable. Thus the stable lattices form a finite segment with the natural and opposite endpoints at $0, n$.

At the opposite endpoint, the attained minimum gives a nonzero lower residual entry. A split complement would have to be the first coordinate line, because g_0 has distinct residual eigenvalues. That line is not stable. Hence the opposite residual extension is nonsplit. This constructs the relevant class on a compatible endpoint rather than importing natural inertia through an unjustified lattice change.

7.3 A local depth theorem

Here is the matrix argument controlling where this opposite class can ramify. Let $\mathcal{O}$ be a discrete valuation ring with uniformizer π, let $F \in \mathrm{GL}_2(\mathcal{O})$, $Q \in \mathcal{O}^\times$ with $Q \neq 1$, and let $n \geq 1$. Suppose $N \in M_2(\mathcal{O})$ is a nonzero nilpotent matrix of the form

$$N = \mu \begin{pmatrix} -xy & x^2 \\ -y^2 & xy \end{pmatrix},$$

where $\mu \in \mathcal{O} \setminus \{0\}$, $(x, y) \in \mathcal{O}^2$ is primitive, $t = v(\mu) \geq 0$, and

$$FN = QN, \qquad NF = N, \qquad F_{21} \in \pi^n \mathcal{O}, \qquad v(N_{21}) = n.$$

Set $h = v(Q - 1)$. Then

$$n = t + 2s, \qquad s = v(y), \qquad 0 \leq s \leq h. \tag{7.2}$$

Proof. The lower nilpotent entry is $-\mu y^2$, so its valuation gives $n = t + 2s$. The common image and kernel line of N is generated by $(x, y)^T$. The two identities for F show that $F - I$ has image in this line and kills no extra quotient direction. Write

$$F = I + \begin{pmatrix} x \\ y \end{pmatrix} (a, b), \qquad ax + by = Q - 1.$$

The lower entry is ya, so $a \in \pi^{n-s}\mathcal{O} = \pi^{t+s}\mathcal{O}$. Both ax and by are divisible by π^s. Their sum $Q - 1$ is therefore divisible by π^s, giving $s \leq h$. □

The square on y explains the factor two. A bound $n \leq t + h$ would omit the geometry of the rank one nilpotent and is false. The examples

$$N = 7^t \begin{pmatrix} -7^s & 1 \\ -7^{2s} & 7^s \end{pmatrix}, \qquad F = \begin{pmatrix} 1 & (Q-1)/7^s \\ 0 & Q \end{pmatrix}$$

realize every allowed s when $v_7(Q - 1) = h$. They are integral local matrix controls. They are not automatically global Galois representations or Jacobians.

For the natural curve, the variable node thickness is $7v_q(C)$. The natural monodromy content is consequently

$$t = 1 + v_7(v_q(C)).$$

This inherited geometric bridge uses the ordinary node smoothing parameter and the identification of the graph pairing with the period valuation [20]. Applied to an actually ramified beta prime, (7.2) becomes

$$n = 1 + v_7(v_q(C)) + 2s, \qquad 0 \leq s \leq v_7(q^f - 1). \tag{7.3}$$

Here f is the residue degree in K. If $7 \nmid v_q(C)$ at every eligible prime, the actual depth is odd. If in addition every eligible residue norm is nontrivial modulo forty nine, the only possible depths are one and three. These are necessary conditions within the inherited arithmetic contract, not a uniform depth bound for arbitrary C.

7.4 The local cup endpoint

At an eligible beta prime, $t \geq 1$ and $h \geq 1$, so χ is trivial modulo seven. Natural inertia is trivial modulo seven, while opposite inertia is nonzero. Below the upper local endpoint $n = t + 2h$, the same matrix calculation forces natural Frobenius $\alpha(F)$ to vanish. At the endpoint it is nonzero.

If $\alpha \cup \beta$ were a coboundary in the endpoint case, the two adjacent entries would extend to a representation in the three by three upper unitriangular group over k. That group has exponent seven. The tame relation $FTF^{-1} = T^Q$ with $Q \equiv 1 \pmod 7$ would force the images to commute. Their central commutator is $\alpha(F)\beta(T) - \alpha(T)\beta(F) = \alpha(F)\beta(T) \neq 0$. This contradiction proves local nonvanishing at the endpoint. The zero below the endpoint and nonzero at it are statements about the same labelled local representation.

8 Global coefficients, diagonal characters, and Frobenius precision

The natural class has a fixed global expansion

$$\alpha = a_\phi[\phi] + a_3[3] + a_5[\sqrt{5}], \qquad a_3 a_5 \neq 0. \tag{8.1}$$

This lies in a Kummer group tensored with k. It does not mean that an element of $\mathbb{F}_{49}$ is an exponent of a base field unit. The three coefficients are normalized together, up to one common scalar. Using a different scalar at each completion would discard the global information that makes the reciprocity tests effective.

The actual matrices have the form

$$\rho(g) = \begin{pmatrix} a(g) & b(g) \\ 7^n c(g) & d(g) \end{pmatrix}.$$

Modulo 7^n, the diagonal entries are characters. Determinant χ lets us write them as $\chi\eta_n$ and η_n^{-1}. The natural monodromy restrictions and fixed local traces constrain the same η_n at every place. A locally chosen ray character with the right residue does not prove it occurs as this actual diagonal character.

The sequence of Blocks 52 through 58 tightened this coupling. Scalar Frobenius at good thirteen first forced $a_\phi = 0$ on that subcase. Later, torsion exponent information at eighty three forced $a_\phi \neq 0$ in the retained actual branch. Hence thirteen must be bad there. The conclusion is not a contradiction from two incompatible coefficients. The first coefficient statement assumed good thirteen, so the combined result is $13 \mid C$.

At good auxiliary primes, the actual rank two trace obeys a precision condition of the form

$$\tau \equiv Q + 1 \pmod{7^r \mathcal{O}}.$$

The exponent r is supplied by the depth and local diagonal estimates. It is not inferred merely from a rank four congruence. This stronger comparison yields a particularly useful universal lemma.

Lemma 8.1 (Discriminant collapse)**.** *Suppose a good Frobenius polynomial has integral coefficients and factorization*

$$P(T) = (T^2 - \tau T + Q)(T^2 - \sigma(\tau)T + Q) = T^4 - ST^3 + C_2T^2 - QST + Q^2,$$

where $\mathcal{O}/\mathbb{Z}_7$ is unramified quadratic, σ is its nontrivial automorphism, all complex roots have absolute value $\sqrt{Q}$, and $\tau \equiv Q+1 \pmod{7^r\mathcal{O}}$. Then the integer

$$D = S^2 - 4(C_2 - 2Q)$$

satisfies $7^{2r} \mid D$ and $0 \le D \le 16Q$. If $7^{2r} > 16Q$, then

$$P(T) = (T^2 - aT + Q)^2, \quad a \in \mathbb{Z}, \quad |a| \le 2\sqrt{Q}, \quad a \equiv Q+1 \pmod{7^r}.$$

Proof. Expansion gives $S = \tau + \sigma(\tau)$, $C_2 - 2Q = \tau\sigma(\tau)$, and $D = (\tau - \sigma(\tau))^2$. The congruence is preserved by σ, so this difference is divisible by 7^r. Unramifiedness makes its square give ordinary integer divisibility by 7^{2r}.

The two traces are algebraic roots of $X^2 - SX + (C_2 - 2Q)$. Under their complex embeddings, the corresponding quadratic factors have roots of product Q and magnitude $\sqrt{Q}$. Their sums are real and lie in $[-2\sqrt{Q}, 2\sqrt{Q}]$. Their difference therefore has magnitude at most $4\sqrt{Q}$, proving the bound. If the divisibility exceeds the bound, $D = 0$. The common trace is rational and integral, hence an integer a, with the stated congruence and magnitude. □

For $Q = 2549$ and $r = 3$, one has $7^6 = 117649 > 40784 = 16Q$. The allowed integral trace would lie in $[-100, 100]$ and be congruent to 149 modulo 343. Neither nearest representative, 149 or -194, lies in the interval. Therefore the inherited precision theorem forces $2549 \mid C$ whenever it applies at depth at least three.

The rank four congruence alone is weaker. The retained negative control

$$P(T) = T^4 + 45T^3 + 2603T^2 + 114705T + 6497401$$

is a generic reciprocal Weil polynomial at 2549 satisfying $(T-1)^2(T-Q)^2$ modulo 343. Its trace discriminant is $5 \cdot 49^2$, not divisible by 7^6. It does not satisfy the actual rank two precision condition. It is not claimed to be a Frey reduction. Its role is to falsify the shortcut that would replace the stronger hypothesis by an unlabelled quartic congruence.

Other fixed primes forced into C at stated depths include the good reduction tests at $41, 59, 197, 491$. Combining these bad primes with reciprocity rows eventually trivialized the diagonal character and reduced depth on declared finite high support classes. The finite support hypotheses remain essential: a new conductor outside the declared universe changes the global ray character presentation.

9 Marked periods and a complete local torsion criterion

The natural monodromy valuation is only one part of a toric period. The unit part can contain the global obstruction. Blocks 59 and 60 computed the marked period constants rather than replacing them by unspecified units.

At a variable place away from $2, 3, 5, 7$, write $C = q^c C_0$ with C_0 a unit. In the marked two node basis, the actual leading period coordinates are

$$\begin{aligned}
\Pi_{aa} &= -\frac{\varepsilon\phi^2}{675\sqrt{5}}(1+\mathfrak{m}),\\
\Pi_{bb} &= \frac{\varepsilon\phi^{-2}}{675\sqrt{5}}(1+\mathfrak{m}),\\
\Pi_{ab} &= \Pi_{ba} = -\phi^{-4}(1+\mathfrak{m}),
\end{aligned}$$

where $\varepsilon = C^7/A^3$ and the notation means that the quotient by the displayed leading term is a principal unit. The analytic uniformization and marked smoothing comparison are inherited written inputs, with precise sources in the literature report [17, 18]. The principal unit qualification is essential at seven, where those units may carry higher information. Away from seven they are divisible by every power of seven.

Put $M = 7^j$ and write $[u]$ additively in $F^\times/F^{\times M}$ for the local field $F = K_v$. Let

$$X = [\phi], \qquad Z = 7[C] - 3[A] - 3[3] - 5[\sqrt{5}].$$

The entire marked period matrix becomes

$$([\Pi_{ij}]) = \begin{pmatrix} Z + 2X & -4X \\ -4X & Z - 2X \end{pmatrix} = ZI + 2XR, \qquad R = \begin{pmatrix} 1 & -2 \\ -2 & -1 \end{pmatrix}, \quad R^2 = 5I. \tag{9.1}$$

Since four is invertible modulo M, every marked period coordinate vanishes if and only if $X = Z = 0$. The matrix R belongs to this marked local period calculation. It is not silently identified with a global endomorphism or with an integral semilinear descent operator.

Proposition 9.1 (Full torsion and period powers)**.** *For the stipulated split uniformization $J^{\mathrm{an}} = \mathbf{G}_m^2/\Lambda$ with rational period lattice, full $J[M]$ is F rational if and only if $\mu_M \subset F$ and every coordinate of a basis of Λ is an Mth power in F.*

Proof. The uniformization gives

$$0 \longrightarrow \mu_M^2 \longrightarrow J[M] \longrightarrow \Lambda/M\Lambda \longrightarrow 0.$$

If all torsion points are rational, so are μ_M^2. For $\lambda \in \Lambda$, choose an Mth root u in the torus. Rationality of its torsion class gives $g(u)/u \in \Lambda$ for every $g \in G_F$. Since λ is rational, this quotient also lies in μ_M^2. The lattice is torsion free, so the intersection is trivial. Thus u is rational. Every period coordinate is an Mth power.

Conversely, take rational Mth roots of each period basis vector. Their classes, together with μ_M^2, generate all M torsion by the exact sequence. Every generator is rational. □

At residue characteristic different from seven, a unit is an Mth power precisely when its residue is, by simple root lifting. A general element also needs valuation divisible by M. If Q is the residue cardinality, $\mu_M \subset F$ precisely when $M \mid Q - 1$. Applying Proposition 9.1 to (9.1) gives the complete local test

$$\begin{gathered} 7^j \mid Q - 1, \qquad 7^j \mid 7c, \\ \phi \in (\kappa_v^\times)^{7^j}, \qquad \frac{C_0^7}{3^3 A^3 (\sqrt{5})^5} \in (\kappa_v^\times)^{7^j}. \end{gathered} \tag{9.2}$$

The valuation condition on c is not lost when replacing C by its unit part C_0.

At level seven, the last unit condition reduces to $75A$ being a seventh power. At level forty nine, exact multiplicative identities reduce it to

$$3 \cdot 5^9 A/C_0^{35} \in (\kappa_v^\times)^{49}.$$

Thus full forty nine torsion is rational if and only if $49 \mid Q - 1$, $7 \mid c$, ϕ is a forty ninth power residue, and the displayed rational unit is as well. These are two way local criteria for the stipulated local Jacobian.

They do not produce one global beta class or determine its actual depth. The archive retains positive local Frey families at $197, 883, 1373, 1667$ satisfying the full criterion. Their survival is compatible with the later exclusion of a global beta supported only at those primes.

That exclusion illustrates why the common global coefficients matter. Correct inverse cyclotomic reciprocity on this four prime support leaves only inertia multiples of $(1, 0, 0, 3)$. A nonzero class must therefore ramify at 197 and 1667. The natural rows in $(\phi, 3, \sqrt{5})$ coordinates are $(0, 6, 6)$ and $(0, 3, 4)$. They force both

$$6a_3 + 6a_5 = 0, \qquad 3a_3 + 4a_5 = 0.$$

The determinant is six modulo seven, so $a_3 = a_5 = 0$, contradicting (8.1). No value of a_ϕ can repair the contradiction because both unit symbols vanish. This is a global common coefficient exclusion. It does not retract any local positive family or prohibit an individual prime from appearing in a larger conductor.

10 Twisted reciprocity and actual trace products

The opposite class has coefficient action χ^{-1}. Its ray relations therefore use the inverse cyclotomic eigenspace, not the ordinary character rows that will appear later for differences of primitives.

In the descended all inert regime, let $M = \mathbb{Q}(\zeta_7)$ and let $\gamma \in H^1(\mathbb{Q}, \mathbb{F}_7(\chi^{-1}))$ restrict to the relevant beta class. Over M, the restriction is additive and satisfies

$$\gamma(\delta_a g \delta_a^{-1}) = a^{-1}\gamma(g).$$

At a conductor prime $q \equiv 1 \pmod 7$, there are six degree one primes in M. At $q \equiv -1 \pmod 7$, there are three degree two primes. If z_q is a chosen seventh root of unity and ℓ_q is the corresponding seventh power logarithm, the local functional at the embedding $\zeta_7 \mapsto z_q^b$ is $bs_q\ell_q$.

For degree two, the indices b and $-b$ represent the same prime. Frobenius sends a residue x to x^q, and $\ell_q(x^q) = -\ell_q(x)$. Therefore

$$(-b)\ell_q(x^q) = b\ell_q(x).$$

The functional is well defined on prime ideals, not just on the six embeddings. Principal reciprocity gives, for a test prime generator π_l outside the conductor,

$$R_{l,q} = -\sum_{b \in \mathrm{reps}_q} b\ell_q(\pi_l(z_q^b)) = -f_q^{-1}\sum_{b=1}^{6} b\ell_q(\pi_l(z_q^b)), \tag{10.1}$$

where $f_q = 1$ or two. Using all six terms without dividing by two in the second case would double count each prime. Constant weights or inverse b weights give different modules.

10.1 The same representation trace product

At depth one and an unramified place of residue norm $Q \equiv 1 \pmod 7$, write the actual Frobenius in the fixed natural frame as

$$F = \begin{pmatrix} a & b \\ 7c & d \end{pmatrix}, \qquad \det F = Q.$$

Put $d = 1 + 7D$ and $\mathrm{tr}(F) = Q + 1 + 7T$. Then $a = Q + 7(T - D)$, and the determinant equation gives

$$bc = T + (Q-1)D + 7TD - 7D^2. \tag{10.2}$$

Modulo seven this becomes

$$\frac{\mathrm{tr}(F) - Q - 1}{7} = \alpha(F)\beta(F) \quad \text{in } k. \tag{10.3}$$

This is the coefficient field rank two trace of the actual representation. It is not half the rational four dimensional trace. The hypothesis $Q \equiv 1 \pmod 7$ makes the extra diagonal term disappear. At other residue norms, (10.2) must be retained.

At good places, the right side must equal one of the corrections extracted from the common quartic (6.1). At bad toric places, exact trace $Q + 1$ makes the product zero. The zero product condition does not always imply $\beta(F) = 0$, because $\alpha(F)$ may already vanish. This is the error corrected in the Block 70 draft. It was consequential: imposing the stronger zero beta condition would falsely remove valid necessary system data.

If the actual polynomial is

$$P(X) = X^4 - SX^3 + BX^2 - QSX + Q^2,$$

the correction $u = (\tau - Q - 1)/7$ has

$$\mathrm{Tr}(u) = \frac{S - 2(Q+1)}{7}, \qquad \mathrm{Nm}(u) = \frac{(B - 2Q) - (Q+1)S + (Q+1)^2}{49}$$

modulo seven, under the stated divisibility assumptions. The two implementations extract these values using different methods, including an integer trace discriminant and an enumeration of k by trace and norm. Neither assigns a Frobenius value at a beta conductor prime by pretending beta is unramified there. Such conductor values require the separate self unit calculation of Section 21.

10.2 Coefficient rigidity without radical identification

The recovered coefficient theorem, initially in the descended depth one regime and later in the all depth semilinear formulation, normalizes the natural class as

$$\alpha = vw[\phi] + r[3] + [\sqrt{5}], \qquad v, r \in \mathbb{F}_7^{\times}. \tag{10.4}$$

Its mechanism retains the actual joint finite quotient. The three Kummer classes are independent after cyclotomic restriction. Distinct cyclotomic eigenspaces separate the natural and opposite images. Chebotarev then realizes joint vectors at good split primes, where the same quartic at the paired K places forces products to be coefficient conjugate. The resulting conditions are $u^7 = -u$ and $r^7 = r$. In the descended subcase beta has an $\mathbb{F}_7$ representative.

This is a cohomological coefficient statement. It does not identify the base radical with w and does not descend an arbitrary Tate lattice. In the more general semilinear formulation, beta may decompose as a descended plus component and a minus component twisted by the quadratic character of $K/\mathbb{Q}$. The bounded all inert computation uses the stipulated plus regime. The minus component remains a separate arithmetic obligation.

11 Eight patterns inside a declared conductor range

The conductor sieve proceeded by adjoining correctly normalized columns and retaining both good and bad alternatives at each test prime. The predecessor stages are useful because they show which data each strengthening removed. They should not be read as percentages of Beal solved.

The early twenty five prime depth one system, using inert rational conductors congruent to one modulo seven and bounded by 2000, retained 112 normalized tuples. Further same polynomial tests reduced these to $38, 12, 6, 2$, then a test at 1051 removed that restricted conductor class. Degree two conductor primes $307, 797, 1483$, congruent to minus one modulo seven, could not be discarded. Their natural Kummer rows vanish, so they supply genuine extra columns in the necessary system.

The enlarged calculation retained 94 tuples after the tests at $701, 911, 1009$. Further tests at $1289, 1429, 1471, 1499, 1709$ reduced that list to 18. Finally the tests at $2129, 2269, 2311$ left eight. These are historical exact finite necessary system counts. The fresh reproduction scope and input pins are reported separately in Section 23. A count quoted from an immutable report is not silently called a new run.

Every final tuple has natural ratio $r = 6$, nonzero unit coefficient, and exactly nine ramifying rational primes. Six primes are mandatory:

$$307, 337, 547, 883, 1483, 1933.$$

The other three belong to $197, 673, 743, 797$ in the four support patterns below. Each row has two coefficient conjugate tuples, according to the sign of u.

Pattern	u	Ordered beta support and inertia vector
A	$\pm 3w$	$307, 337, 547, 673, 743, 797, 883, 1483, 1933$ $(4, 4, 2, 4, 3, 6, 3, 1, 3)$
B	$\pm 3w$	$197, 307, 337, 547, 673, 797, 883, 1483, 1933$ $(5, 1, 4, 2, 6, 4, 1, 4, 2)$
C	$\pm w$	$197, 307, 337, 547, 673, 743, 883, 1483, 1933$ $(1, 5, 4, 1, 4, 1, 6, 5, 4)$
D	$\pm w$	$197, 307, 337, 547, 743, 797, 883, 1483, 1933$ $(4, 5, 5, 6, 2, 4, 2, 1, 6)$

The table is a necessary cover under the inherited geometry, actual depth one, all inert regime, and the requirement that every rational prime ramifying in beta is at most 2000. It is not a finite cover of all primitive integer triples. Additional divisors of C unramified in beta remain allowed. New beta ramification is outside the cover and must add its columns everywhere, including in the Frobenius evaluations.

11.1 Why a homogeneous finite sieve can saturate

Block 70 proved a limitation for the specified zero or good test family. Chebotarev supplies arbitrarily many eligible inert conductor primes where the natural Kummer rows all vanish. On N such primes, the inverse cyclotomic ray character space has dimension N over $\mathbb{F}_7$ under the declared class and unit inputs. Each prescribed test zero value adds at most one linear equation. If N exceeds the number of equations, a nonzero kernel remains. Every trace product at those tests is zero, so the bad alternative accepts the abstract data.

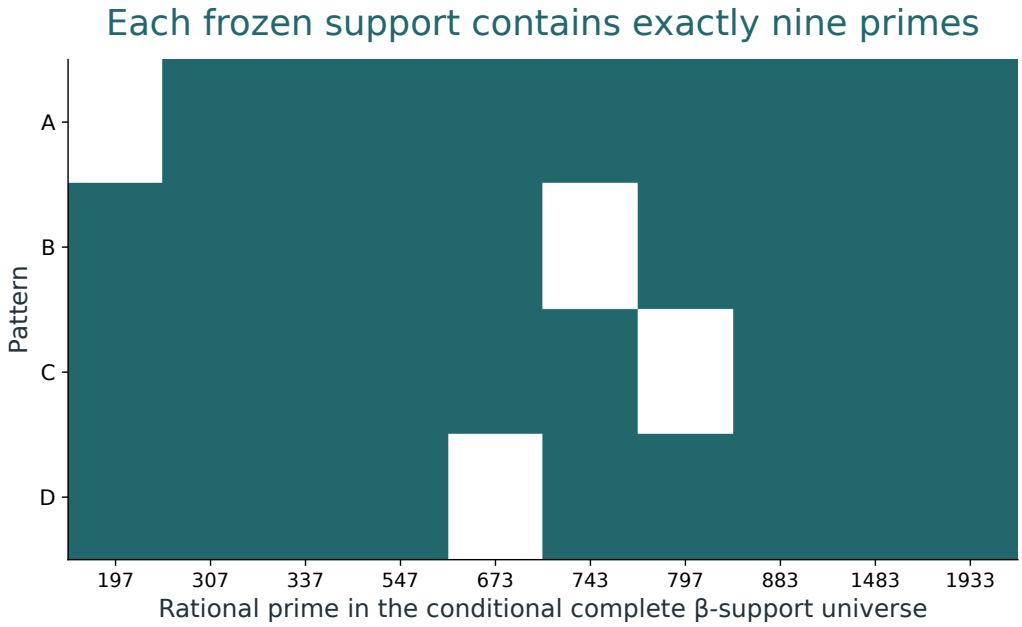


Figure 3: Four supports account for the eight coefficient conjugate tuples in the frozen bounded regime. Additional beta ramification changes the column system.

The archive contains an explicit twenty four row witness on twenty five conductor primes above 3000, with a kernel vector having twenty one nonzero inertia coordinates. It is not a global Frey representation. Its purpose is to show that enlarging a fixed list of these homogeneous tests alone cannot give an unrestricted support theorem.

The conclusion is scoped. It does not rule out a finite certificate that uses stronger facts. A necessarily good reduction condition, a nonhomogeneous local value, a justified conductor restriction, or an integral relation from the same actual representation can escape this saturation. The later fixed seven and affine stages seek precisely such information.

12 The fixed seven period target

At seven, principal units are not automatically seventh powers. The residue class of $8 = 1 + 7$ is a simple warning. The away from seven period simplification cannot be copied without an error bound. The inherited integral marked family supplies an error in $1 + 7^{7c}\mathcal{O}_F$, where $c = v_7(C) \geq 1$ and $F = \mathbb{Q}_7(\sqrt{5})$. That specific error is sufficiently deep for the stated period window.

Let $X = [\phi]$ and $H = [3]$ in the mod seven local Kummer group. Exact logarithmic first digits give

$$L(\phi) = 5\overline{\sqrt{5}}, \qquad L(3) = 6, \qquad L(\sqrt{5}) = 3, \quad L(u) = \frac{u^{48} - 1}{7} \pmod 7.$$

The two unit coordinates X, H are independent, and $[\sqrt{5}] = 4H$. For a rational seven adic unit A, put $q_7(A) = (A^6 - 1)/7$ modulo seven. The local marked period comparison has

$$P = ZI + 2XR, \qquad Z = (3q_7(A) + 5)H,$$

with R from (9.1).

The actual coefficient action must intertwine the toric submodule and quotient period matrices. Their valuation matrix is exactly $7cI$, so the two action matrices agree after cancellation over $\mathbb{Z}_7$. Their reduction commutes with R and has characteristic polynomial $T^2 - 5$. The retained centralizer argument then gives one orientation $\epsilon \in \{1, -1\}$ and local scalar class

$$\kappa_7 = 2\epsilon w X + (3q_7(A) + 5)H. \tag{12.1}$$

Merely knowing $R^2 = 5I$ would not identify this as the actual coefficient action. The period intertwiner is the extra geometric input.

The globally normalized natural class (10.4) restricts to $vwX + (r + 4)H$. Comparing it with (12.1) allows one common nonzero graded scalar $g = v/(2\epsilon)$. The remaining coordinate gives

$$2r + 1 = \epsilon v(3q_7(A) + 5),$$

and hence

$$(2r + 1)^2 = v^2(3q_7(A) + 5)^2 \quad \text{in } \mathbb{F}_7. \tag{12.2}$$

The scalar is not independently adjustable in the two unit coordinates.

For the final bounded patterns $r = 6$. Patterns A and B, with $u = \pm 3w$, require $q_7(A) \in \{0, 6\}$ and $A^6 \equiv 1$ or 43 modulo 49. Patterns C and D, with $u = \pm w$, require $q_7(A) \in \{1, 5\}$ and $A^6 \equiv 8$ or 36. Each tuple still permits twelve rational unit residues A modulo 49. None is eliminated.

12.1 Higher precision retains a valuation coordinate

For $1 \le j \le 7c - 1$, the deep period errors vanish in $F^\times / F^{\times 7^j}$. This follows because seventh powering takes $1 + 7^s\mathcal{O}_F$ bijectively onto $1 + 7^{s+1}\mathcal{O}_F$ for $s \ge 1$. The guaranteed window does not assert that actual errors become nontrivial at the next precision.

Use the three coordinate basis

$$X_j = [\phi], \qquad H_j = [3], \qquad T_j = [7].$$

The third coordinate is valuation. The actual local scalar class in the window is

$$\kappa_j = 2\epsilon w X_j + \Lambda_j H_j + 7cT_j, \tag{12.3}$$

where

$$\Lambda_j = 7\ell_j(C_0) - 3\ell_j(A) - 3 - 5d_j, \quad \ell_j(a) = \frac{\log(a^6)}{\log(3^6)} \pmod{7^j}, \quad d_j = \frac{\ell_j(5)}{2}.$$

At level 49, with $Q_A = (A^6 - 1)/7$ modulo 49 and $q_0 = (C_0^6 - 1)/7$ modulo seven,

$$\Lambda_2 = 47 + 38Q_A + 14Q_A^2 - 7q_0 \pmod{49}. \tag{12.4}$$

Both C_0 and the coordinate $7c[7]$ disappeared modulo seven. Neither may be omitted modulo forty nine.

At higher actual lattice depth, put $m = \lfloor n/2 \rfloor + 1$. The inherited diagonal theorem supplies an actual global upper extension α_j for $1 \le j \le m$. In the common range $1 \le j \le \min(m, 7c - 1)$, its local coordinates satisfy $U_j = 2\epsilon w g_j$, $V_j = \Lambda_j g_j$, $W_j = 7c g_j$ for one unit

g_j. The exact valuation matrix preserves the valuation identity at every legitimate triangular precision $1 \le j \le m$, even beyond the proved unit period window. Consequently

$$v_7^{\mathrm{trunc}}(W_j) = \min(j, 1 + v_7(c)).$$

At depth one, only the residual global triangular precision is supplied. Replacing the actual mod forty nine representation by an arbitrary triangular Kummer lift would erase its lower 7β entry. The local higher class exists, but its global comparison must retain the nontriangular deformation.

The archive contains exact seven adic fifth root positive controls for all admitted local residues. Their B values solve $B^5 = C^7 - A^3$ in $\mathbb{Q}_7$. They are not asserted rational or ordinary integers. The higher local period criterion therefore provides real conditions while still leaving every bounded tuple locally possible.

13 The coupled adjoint equations

Choose one reference Kummer representation over $\mathcal{O}/49$,

$$\rho_0(g) = \begin{pmatrix} \chi(g) & b_0(g) \\ 0 & 1 \end{pmatrix}, \tag{13.1}$$

whose reduction has the same residual α as the actual natural representation. The reference is built by lifting the coefficients of the same global classes $[\phi]$, $[3]$, $[\sqrt{5}]$. It has no seven valuation coordinate at the fixed place. It is an auxiliary representation, not the actual one.

Every determinant χ framed lift of this residual representation has a unique expression

$$\rho(g) = (I + 7X(g))\rho_0(g), \qquad X(g) = \begin{pmatrix} p(g) & q(g) \\ \beta(g) & -p(g) \end{pmatrix} \in \mathfrak{sl}_2(k). \tag{13.2}$$

The determinant condition makes X traceless. Multiplication shows that ρ is a representation if and only if

$$X(gh) = X(g) + \mathrm{Ad}(\overline{\rho}_0(g))X(h).$$

For $\overline{\rho}_0(g) = \left(\begin{smallmatrix} x & a \\ 0 & 1 \end{smallmatrix}\right)$, direct conjugation sends (p, q, b) to

$$(p + ax^{-1}b,\ xq - 2ap - a^2x^{-1}b,\ x^{-1}b).$$

Thus the three coupled equations are

$$\beta(gh) = \beta(g) + \chi(g)^{-1}\beta(h), \tag{13.3}$$

$$p(gh) = p(g) + p(h) + \alpha(g)\chi(g)^{-1}\beta(h), \tag{13.4}$$

$$q(gh) = q(g) + \chi(g)q(h) - 2\alpha(g)p(h) - \alpha(g)^2\chi(g)^{-1}\beta(h). \tag{13.5}$$

All coefficients in these equations lie in k. The reference itself is a representation over $\mathcal{O}/49$, so no characteristic seven carry term has been silently discarded in deriving them.

Throughout, α and beta are normalized continuous cocycles with their displayed coefficient actions. Their cup has trivial target action and is explicitly $(\alpha \cup \beta)(g, h) = \alpha(g)\chi(g)^{-1}\beta(h)$. Define the ordinary differential by

$$dp(g, h) = p(g) + p(h) - p(gh)$$

and the twisted differential by $d_\chi q(g,h) = q(g) + \chi(g)q(h) - q(gh)$. Then

$$dp = -\alpha \cup \beta, \tag{13.6}$$

$$d_\chi q(g,h) = 2\alpha(g)p(h) + \alpha(g)^2\chi(g)^{-1}\beta(h). \tag{13.7}$$

The right side of (13.7) is a two cocycle once the preceding equations hold. For a general fixed p, its vanishing in cohomology is the next bare existence problem. Solving the p equation alone does not formally solve the q equation. The project does not establish that this bare upper class is nonzero in the actual branches. Prescribed upper geometry and integral lifting retain additional conditions.

The actual trace is

$$\operatorname{tr}\rho(g) = \chi(g) + 1 + 7\big((\chi(g)-1)p(g) + \alpha(g)\beta(g)\big) \pmod{49}. \tag{13.8}$$

The upper cochain q is invisible to trace at this precision. On $\chi(g) = 1$, the diagonal cochain p is also invisible and only the product $\alpha\beta$ remains. This explains why a homogeneous Frobenius sieve can leave essential deformation coordinates undetermined.

Strict conjugation by $I + 7Y$ changes X by $Y - \operatorname{Ad}(\overline{\rho}_0)Y$. Independent local changes can simplify representatives but do not create a common global integral frame. The affine target in the last part of this reader is formulated after one frame and one reference have been fixed.

13.1 A delayed tame obstruction

Suppose at a beta support prime the actual inertia and Frobenius have

$$I + 7M \pmod{49}, \qquad I + 7A \pmod{49}, \qquad Q = 1 + 7h \pmod{49}.$$

The tame relation is automatically insensitive to some first digit changes modulo forty nine. One precision later it forces

$$[A, M] = hM \quad \text{over } k. \tag{13.9}$$

A trace invisible correction modulo forty nine need not pass (13.9). The archive retains a seven valuation correction that fails this next relation at 883 when complete inertia is held fixed. This excludes that strengthening inside the stipulated deformation fiber. It does not prove that the actual valuation coordinate is zero in every candidate, because equality of a finite trace list is weaker than equality of the entire framed inertia and trace data used by the fiber theorem.

14 A mixed trace that detects the valuation

The valuation coordinate can be read from a joint word in the same representation. For $T(g) = \operatorname{tr}_{\mathcal{O}}\rho(g)$, define

$$D(\tau, g) = T(\tau g) - T(\tau) - T(g) + 2.$$

The elementary identity

$$D(\tau, g) = \operatorname{tr}\big((\rho(\tau) - I)(\rho(g) - I)\big) \tag{14.1}$$

shows why the product label matters. Two lists of individual traces do not determine the trace of the product.

The local Kummer basis $[\phi], [3], [7]$ supplies a unit annihilating probe τ_j after adjoining the required 7^j roots of unity. It kills the unit coordinates and evaluates the uniformizer coordinate to one. The inherited construction uses the cyclotomic action to show restriction remains injective. At higher cyclotomic levels this is not a blanket prime to seven degree argument, because the extension also has a seven primary part.

To define the valuation functional used here, strictly align the actual local toric cyclotomic line at K_7 with the reference line. The aligned local representative of X has $p = \beta = 0$ and upper entry q a $k(\chi)$ Kummer cocycle. Define $\nu_7(X)$ to be its $[7]$ valuation coordinate. Remaining aligned strict gauges change q by a coboundary and a scalar multiple of residual α, both of valuation zero. Thus the coordinate is well defined under these strict equivalences. The reference b_0 has valuation zero and the actual difference is $b_7 - b_0 = 7q$, giving the fixed value below. This local alignment never erases beta from the global deformation.

The actual period valuation matrix is $7cI$, so this probe has

$$\rho(\tau_j) = I + 7cPE_{12}P^{-1} \pmod{7^j},$$

where the same local toric line has the required alignment with the natural line. At actual depth one and a detecting g in the residual kernel, (14.1) gives

$$D(\tau, g) = 49\nu_7(X)\beta(g) \pmod{343}. \tag{14.2}$$

The actual fixed seven period comparison requires

$$\nu_7(X) = c\frac{v}{2\epsilon} \quad \text{in } \mathbb{F}_7.$$

If $\beta(g) \neq 0$, the mixed trace determines this value.

The all depth formula uses $t = 1 + v_7(c)$, precision $j \geq n + t + 1$, the probe τ_j, and a detecting $g \in \ker \overline{\rho}$ with $\beta(g) \neq 0$. It is

$$v_7(D(\tau_j, g)) = n + t, \qquad \frac{D(\tau_j, g)}{7^{n+t}} = \frac{c}{7^{t-1}}\frac{v}{2\epsilon}\beta(g) \pmod{7}.$$

The coefficient on the right is nonzero in the detecting case. This is an exact valuation statement under the probe and actual geometric hypotheses. It does not bound n or c.

Example 14.1. Let $U_a = \left(\begin{smallmatrix} 1 & 7a \\ 0 & 1 \end{smallmatrix}\right)$ and $V = \left(\begin{smallmatrix} 1 & 0 \\ 7 & 1 \end{smallmatrix}\right)$. Both have trace two and determinant one, for every a. Their product has trace $2 + 49a$. Thus $a = 1, 2$ give product traces $51, 100$ modulo 343, while every individual trace and every trace modulo forty nine loses this distinction.

These matrices are exact controls for the information loss. They are not claimed to be the actual labelled probe and detecting element. The current eight tuples do not supply $T(\tau)$, $T(g)$, and $T(\tau g)$ from one numerical global realization. The required mixed value is a target, not an observed contradiction.

For instance, $c = 1, v = 3, \epsilon = -1, \beta(g) = 1$ requires $D \equiv 98 \pmod{343}$. Writing that number does not evaluate the trace of an unknown candidate. A future global calculation must identify the words and the representation together.

15 Integral semilinearity and the unlabelled image

The integral semilinear theorem supplies more than equality of residual cohomology classes. Under the retained geometric inputs, an actual map between the conjugate curves induces an

integral operator $U = C\sigma$ on the Tate lattice, with

$$\rho(jgj^{-1}) = C\sigma(\rho(g))C^{-1}, \qquad C\sigma(C) = -\rho(j^2), \qquad C_{21} \in 7^n\mathcal{O}.$$

It preserves the natural endpoint, the opposite endpoint, and the stable lattice segment. The minus sign in the square comes from the actual conjugate curve map. It cannot be replaced by a plus sign simply to obtain a preferred descent datum.

The proof separates the ordinary coefficient linear part of the actual geometric operator from its semilinear part. An ordinary intertwiner would make the normalized natural classes $(u, r, 1)$ and $(-u, r, 1)$ proportional. The final coordinate fixes the scalar to one, forcing $u = 0$, contrary to the retained nonvanishing theorem. The ordinary part must vanish. The actual integral origin and endpoint uniqueness then retain a semilinear comparison on the lattices, not just over their fraction field.

With a compatible local choice of j, one has $\rho(j^2) = I$. The coefficient unit $(1 + w)/2$ has norm minus one, so dividing U by it gives a semilinear involution. An adapted basis of its plus lattice produces a global endpoint preserving integral frame with

$$\rho(jgj^{-1}) = \sigma(\rho(g)).$$

This is an existential normal form. The unknown candidate frame has not been numerically constructed. Its plus lattice need not be G_K stable, so the statement is not ordinary rank two Q_7 descent.

15.1 The depth one image theorem

Let K_∞ be the cyclotomic tower. Write B for the additive image of beta on its Galois group and $d = \dim_{\mathbb{F}_7} B$. The inherited group theorem at actual depth one identifies

$$H = \rho(G_{K_\infty}) = \left\{ M \in \mathrm{SL}_2(\mathcal{O}) : \begin{array}{l} M \bmod 7 = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right),\ x \in k, \\ (M_{21}/7) \bmod 7 \in B \end{array} \right\}. \tag{15.1}$$

In particular it contains the full kernel of reduction modulo forty nine. Its finite quotient orders are

$$|H \bmod 7^j| = 7^{6j-6+d} \quad (j \geq 2), \qquad |H \bmod 7| = 49.$$

For the bounded patterns $d = 1$, the orders at forty nine and 343 are 7^7 and 7^{13}. The latter is 96 889 010 407. This is a structural count, not enumeration of that many matrices or arithmetic representations.

The abelianization is $k \oplus B$. The independent global Kummer class $[7]$ cannot factor through that abelianization with its required cyclotomic weight, giving the stated global noncontainment of the uniformizer Kummer extension in the torsion field over K_∞. This field statement does not imply local noncontainment or zero local period valuation.

The image group contains upper and lower elements T_a, G_b with

$$\mathrm{tr}(T_a G_b) - \mathrm{tr}(T_a) - \mathrm{tr}(G_b) + 2 = 49ab \pmod{343}.$$

All leading residue values occur as the elements vary. This flexibility does not allow a fixed local uniformizer probe to be replaced by any convenient upper element. Preimages in the Galois group may have different local membership and different arithmetic labels.

Knowledge of the unlabelled image is therefore weaker than the quotient map and compatible local embeddings required by the mixed trace problem.

The theorem also retains its depth hypothesis and full residual upper image. A single $\mathbb{F}_7$ upper direction does not force the same congruence subgroup. At depth greater than one, the lower ideal may be too small to contain a desired element at the previous precision. The archive supplies explicit controls against both invalid strengthenings.

16 Cup vanishing and the secondary obstruction

The first equation $dp = -\alpha \cup \beta$ has an existence obstruction in $H^2(K, k)$. Once it vanishes, a further obstruction remains for the upper equation. The distinction is algebraic and can be proved without assuming an arithmetic realization.

Put $s(g) = \alpha(g)\beta(g)$. Expansion of the two cocycle laws gives

$$ds = -(\alpha \cup \beta + \beta \cup \alpha). \tag{16.1}$$

For a chosen primitive p, set $D_{12} = -p$ and $D_{23} = p - s$. They form a defining system with differentials $\alpha \cup \beta$ and $\beta \cup \alpha$. The associated twisted Massey type cocycle is

$$\mu_p = \alpha \cup (p - s) - p \cup \alpha. \tag{16.2}$$

Explicitly it equals

$$\mu_p(g, h) = \alpha(g)(p(h) - \alpha(h)\beta(h)) - p(g)\chi(g)\alpha(h).$$

Let R_p denote the right side of (13.7). Direct expansion of $f(g) = \alpha(g)p(g)$ gives

$$R_p + d_\chi f = \mu_p. \tag{16.3}$$

Thus $[R_p] = [\mu_p]$ in $H^2(K, k(\chi))$. For this fixed p, the upper equation is soluble if and only if this class is zero.

Every other first primitive is $p + z$ with $z \in Z^1(K, k)$ an ordinary character. Formula (16.2) gives

$$\mu_{p+z} - \mu_p = \alpha \cup z - z \cup \alpha.$$

Graded commutativity in odd characteristic makes its cohomology class $2[\alpha] \cup [z]$. Therefore the choice independent object at this stage is the affine coset

$$[\mu_p] + 2[\alpha] \cup H^1(K, k). \tag{16.4}$$

There exists some first primitive for which the upper equation is soluble precisely when zero belongs to this coset. There is not one canonical class independent of the primitive without taking this quotient by indeterminacy. With restricted ramification or prescribed Stage P local conditions, only the corresponding allowed character subspace may vary p. After the final Stage P reduction this is $\ker A$, so the larger absolute indeterminacy cannot discharge those affine conditions.

16.1 An exact control separating the two stages

Take the finite group $U_3(\mathbb{F}_7)$ with coordinates (a, b, c) and multiplication

$$(a, b, c)(A, B, C) = (a + A, b + B, c + C + aB).$$

Use trivial cyclotomic action, $\alpha = a$, $\beta = b$, and $p = c$. Then $dp = -\alpha \cup \beta$, so the first obstruction vanishes. The elements $x = (1, 0, 0)$ and $z = (0, 0, 1)$ commute, but

$$R_p(x, z) = 2, \qquad R_p(z, x) = 0.$$

For every ordinary cochain q, commuting arguments give $dq(x, z) = dq(z, x)$. Hence no q solves the second equation. This is a finite group control, not an absolute Galois group or a Frey representation. It proves that cup zero alone does not formally imply upper solvability.

16.2 The fixed primitive local to global theorem

For $K = \mathbb{Q}(\sqrt{5})$, localization on $H^2(K, k(\chi))$ is injective. To see the coefficient translation, restrict to $L = K(\zeta_7)$, whose degree over K is six. Restriction followed by corestriction multiplies by six, which is invertible on seven primary cohomology. Over L, the module is two trivial $\mathbb{F}_7$ copies and μ_7 is present. Kummer theory identifies each copy of $H^2(L, \mathbb{F}_7)$ with $\mathrm{Br}(L)[7]$. Brauer–Hasse–Noether gives local to global injectivity for the Brauer group. A class zero at every completion over K is therefore zero over L and then over K [13]. The exact leaves are II.1.22 and II.1.30 for Kummer and restriction with corestriction, and VIII.4.2 for the Brauer sequence.

At a fixed place where beta is the coboundary of λ_v and $p = p_v^{\mathrm{fix}}$, the explicit local upper primitive

$$q_v^{\mathrm{fix}}(g) = -\lambda_v \chi(g)^{-1} \alpha(g)^2$$

satisfies $d_\chi q_v^{\mathrm{fix}} = R_p$ by direct expansion. At support, $\alpha = 0$ also kills the local upper right side. These facts prevent treating a formal secondary obstruction as an established nonzero obstruction in the actual local branches. Full prescribed upper values and geometric compatibility remain separate.

The local to global theorem uses the same global p at every place. Independently chosen local primitives can have vanishing secondary classes without supplying one such global primitive. Even once q exists globally, its local values must match the actual toric, endpoint, semilinear, and fixed seven conditions. Those conditions form another affine localization problem.

General Massey vanishing theorems do not erase these qualifications. The all prime theorem of Mináč and Tân concerns ordinary products over absolute Galois groups with trivial $\mathbb{F}_p$ coefficients [24]. The JEMS theorem cited in some older discussion has a mod two statement [25]. The 2026 strong vanishing theorem likewise concerns its stated ordinary coefficient setting and may introduce additional ramification [26]. None is a theorem that the one twisted defining system prescribed here, on a fixed ramification set and with actual affine local targets, has a compatible integral lift.

17 Unrestricted local matching and its exact limitation

Finite local cohomological conditions need not obstruct a global character when its ramification is unrestricted. Neukirch's criterion, in its authenticated original form and the

formulation recorded by Neftin, applies to both modules relevant to the two adjoint stages [28, 29].

For a finite module A, let $A' = \mathrm{Hom}(A, \mu_7)$ and let G' be the finite group of the field cut out by that dual. The cited sufficient criterion is $H^1(G', A') = 0$. For $A = k$ with trivial action, $A' \simeq k(\chi)$ and $G' = C_6$. Averaging kills its positive cohomology because six is invertible modulo seven. For $A = k(\chi)$, the source action cancels the cyclotomic target action, so the dual is trivial and $G' = 1$. Both satisfy the criterion.

Consequently the unrestricted maps

$$H^1(K, k) \longrightarrow \prod_{v \in S} H^1(K_v, k), \qquad H^1(K, k(\chi)) \longrightarrow \prod_{v \in S} H^1(K_v, k(\chi))$$

are surjective for every finite set S under the cited localization theorem. The precise source formulation and its fixed additional finite set refinement are recorded in the literature audit. This statement does not keep the character unramified at every other place.

If p_0 is one global bare primitive and p_v are desired finite local primitives, their differences $p_v - p_0|_v$ are ordinary characters. For trivial coefficients there are no nonzero coboundaries. Surjectivity supplies one global character matching every prescribed finite difference exactly. Then $p = p_0 + z$ meets those local equations, but may ramify at new primes.

For the upper equation, once one global q_0 exists, prescribed local differences are $k(\chi)$ cocycles. Their cohomology classes can likewise be matched at finitely many places. Local coboundaries correspond to strict upper conjugation. Thus conditions invariant under such conjugation cannot alone create an unrestricted finite localization obstruction when each is locally attainable.

17.1 The local toric hyperplane

At a beta support prime, use a pro seven tame generator T and arithmetic Frobenius F. Put

$$c = \beta(T) \neq 0, \qquad b = \beta(F), \qquad h = (Q - 1)/7 \pmod 7.$$

The local first digit normal form is

$$M = c \begin{pmatrix} \theta & -\theta^2 \\ 1 & -\theta \end{pmatrix}, \qquad A = \begin{pmatrix} \theta b & \theta h - \theta^2 b \\ b & h - \theta b \end{pmatrix}.$$

Relative to the chosen reference,

$$p(T) = \theta c, \qquad p(F) = \theta b - h.$$

Eliminating θ gives the invariant affine equation

$$c p(F) - b p(T) = -hc. \tag{17.1}$$

Conversely any pair satisfying this equation determines $\theta = p(T)/c$ and reconstructs the displayed first order packet. The packet converse does not construct a global curve or establish every higher integral relation.

Changing F to $T^m F$ adds mc to b and $m p(T)$ to $p(F)$, so the left side of (17.1) is unchanged. A consistent Frobenius lift change cannot remove the affine right side.

The apparent freedom from unrestricted localization is exactly why a justified ramification set is needed. Any useful obstruction must retain restrictions that the arbitrary global

correction is not allowed to evade. These include the actual bad prime set, the one integral frame, and the next precision tame relation. Block 79 supplied such a set for the diagonal first cochain. It did not supply it for every upper correction.

18 A restricted first stage and an unrestricted upper boundary

We now state the inherited arithmetic contract used by the final affine reduction. These are actual geometry inputs, not consequences of a finite matrix scan.

H1.
The actual primitive Frey family, its specified twist and integral $\mathcal{O}$ action, determinant χ, and natural depth one endpoint conventions are fixed.

H2.
At every variable bad place the actual inertia has $I + N$ with $N^2 = 0$ and N in $7M_2(\mathcal{O})$. Its primitive line factorization and same frame Frobenius obey the toric equations. Good inertia is trivial.

H3.
Outside the four fixed rational primes $2, 3, 5, 7$, every possible bad place divides C. The reference is unramified there.

H4.
At each fixed place over $2, 3, 5, 7$, the actual rank two trace is $\chi + 1$, and a local element has $\chi - 1$ a unit modulo seven.

H5.
The four frozen beta supports, its stated all inert descent, and local natural class zero at every beta support place are retained.

H6.
The complete level seven period criterion and natural node thickness are used for the local Frey positive control only.

The exact original contracts remain in the source archive. Preservation of their bytes is not independent mathematical acceptance of the geometric chain.

Proposition 18.1 (No auxiliary diagonal inertia)**.** *Under H1–H3, the first cochain p is unramified outside the fixed primes and the complete beta ramification support. It factors as a cochain through the corresponding restricted Galois group.*

Proof. At a variable bad place, the reference inertia is identity. Reduce $N/7$ modulo seven. Its traceless matrix is

$$\begin{pmatrix} p(T) & q(T) \\ \beta(T) & -p(T) \end{pmatrix},$$

and nilpotence gives $p(T)^2 + q(T)\beta(T) = 0$. If beta is unramified, its value is zero on every inertia element. Thus $p(T)^2 = 0$ in the field k, implying $p(T) = 0$. This square zero inference is in a field, not in the ring $\mathcal{O}/49$, which has nilpotents. At good places both representations are unramified, giving the same conclusion.

Let I be an omitted inertia subgroup. The residual classes and p all vanish on I. The multiplication law (13.4) yields $p(gi) = p(g) = p(ig)$ for $i \in I$. It also gives vanishing on

conjugates and products of omitted inertia elements. Continuity extends this to their closed normal closure. The cochain descends to the restricted quotient. □

This result permits a finite Stage P computation even if C has many extra divisors unramified in beta. The set of all those divisors need not be bounded. The theorem controls only the diagonal first cochain.

If beta is unramified and $q(T)$ is nonzero, the first digit inertia is purely upper. It can survive in the actual toric local category. The archive retains an exact local Frey positive control at 5167, with $A = 3238$, $C = 5167$ and a local fifth root B satisfying the equation. Its leading residue is 3658, and simple root lifting supplies the root to every precision. This B is a local number, not an ordinary integer solution.

18.1 Upper values at beta support are fixed after p

At beta support, the natural class is identically zero locally, and the coefficient action is trivial modulo seven. Relative to the same reference, put $B_0(F) = b_0(F)/7$ modulo seven. The local packet yields

$$cp(F) - bp(T) = -hc, \tag{18.1}$$

$$cq(T) = -p(T)^2, \tag{18.2}$$

$$c(q(F) + B_0(F)) = -p(T)p(F). \tag{18.3}$$

The reference term in the last equation is retained. It is not reset to zero by an independent local basis choice. Once p and beta are fixed, all upper values on the full beta supporting local group are fixed at this precision. Two upper corrections with the same data therefore have zero local difference there.

At beta unramified bad places, upper inertia may still occur when the local beta class splits. The exact matrices $N = 7uE_{12}$ and $F = \left(\begin{smallmatrix} Q & R \\ 0 & 1 \end{smallmatrix}\right)$ provide all precision toric controls. Hence the attempted theorem confining *every* adjoint ramification to beta support is false. The finite restricted set is justified for Stage P. Stage Q retains its own auxiliary arithmetic problem.

19 A bare primitive on the justified ramification set

Fix one of the nine prime beta supports S in Section 11. Let Σ consist of S, the places above $2, 3, 5, 7$, and the two real places. The residual classes factor through $G_{K,\Sigma}$. At every beta support place, χ is trivial modulo seven and the natural local class is zero. Hence its normalized cocycle is identically zero there. At the fixed places, beta is a local coboundary under the inherited contract.

There are three questions to keep distinct:

1. Does the bare restricted equation $dp_0 = -\alpha \cup \beta$ have a solution?
2. Does one such global solution satisfy all prescribed first trace and toric conditions?
3. Does a matching first solution extend to the upper and higher integral geometric stages?

The first question can now be answered using finite set duality. Its answer does not assume an actual higher representation in order to construct the first primitive.

Lemma 19.1 (The dual Kummer kernel)**.** *For every finite Σ containing the prime above seven and all real places,*

$$\operatorname*{Sha}^{1}_{\Sigma}(K,\mu_7) = \ker\left(H^1(G_{K,\Sigma},\mu_7) \longrightarrow \prod_{v\in\Sigma} H^1(K_v,\mu_7)\right) = 0.$$

Proof. The discriminant of K is five. Minkowski's ideal bound is $\sqrt{5}/2 < 2$, so every ideal class has a representative of norm one. Thus $\mathrm{Cl}(K) = 1$ [14].

Its units are $\{\pm\phi^e : e \in \mathbb{Z}\}$. For an elementary verification, scale a unit by a power of ϕ and by a sign so that its first real embedding lies in $[1,\phi)$. If its value a were strictly between the endpoints, a norm minus one unit would have integral trace $a - a^{-1}$ strictly between zero and one. A norm plus one unit would have trace $a + a^{-1}$ strictly between two and $\sqrt{5} < 3$. Both are impossible. The normalized unit is therefore one, proving the assertion.

A Kummer class in the restricted group has valuations divisible by seven outside Σ. Class number one lets us remove those valuations by a global seventh power and leave a Σ unit. If its local class is zero at every place in Σ, the remaining prime valuations are also divisible by seven. Remove these as well. Since minus one is its own seventh power, the class is represented by ϕ^e.

At $F = \mathbb{Q}_7(\sqrt{5})$, exact arithmetic modulo 49 gives

$$\frac{\phi^{48}-1}{7} = 5\overline{\sqrt{5}} \neq 0 \quad \text{in } \mathbb{F}_{49}. \tag{19.1}$$

If a unit is a seventh power, its forty eighth power is one modulo forty nine. Indeed, any unit has forty eighth power in $1 + 7\mathcal{O}_F$, and the seventh power of this group is contained in $1 + 49\mathcal{O}_F$. Formula (19.1) shows that ϕ is not a seventh power in F.

If e is nonzero modulo seven and ϕ^e is a seventh power, a Bezout identity between e and seven makes ϕ a seventh power too. Therefore local triviality forces $e \equiv 0 \pmod 7$, and the global Kummer class is zero. □

The lemma is uniform in finite Σ. It does not use a prime sample to test arbitrary global units. The single seven adic symbol is sufficient only after the class number and unit reductions have removed every other possible coordinate.

Theorem 19.2 (Conditional restricted bare primitive)**.** *Assume the stated residual and local properties for $\alpha \in Z^1(G_{K,\Sigma}, k(\chi))$ and $\beta \in Z^1(G_{K,\Sigma}, k(\chi^{-1}))$. Then there is a normalized continuous cochain $p_0 : G_{K,\Sigma} \to k$ satisfying*

$$dp_0 = -\alpha \cup \beta.$$

No auxiliary ramification outside Σ is needed.

Proof. The Cartier dual of trivial $\mathbb{F}_7$ is μ_7. The nondegenerate $\mathbb{F}_7$ trace pairing on k identifies the dual of trivial k with $k(\chi)$. Lemma 19.1 in two $\mathbb{F}_7$ coordinates therefore gives $\mathrm{Sha}^1_\Sigma(K, k(\chi)) = 0$.

Finite set Poitou–Tate duality pairs this group perfectly with $\mathrm{Sha}^2_\Sigma(K,k)$. Every prime dividing the module order belongs to Σ, as required by the cited theorem. Odd primary real Tate terms vanish. Thus localization

$$H^2(G_{K,\Sigma},k) \longrightarrow \bigoplus_{v\in\Sigma} H^2(K_v,k) \tag{19.2}$$

is injective [12].

At beta support, α is identically zero locally, so its cup with beta is zero. At a fixed place, beta is a coboundary, so the local cup class is zero. The real terms vanish. Injectivity of (19.2) gives $[\alpha \cup \beta] = 0$ globally in the restricted group. A continuous one cochain primitive exists by the definition of cohomological vanishing. Evaluating the primitive equation at the identity gives $p_0(1) = 0$. □

It would be invalid to replace (19.2) by an assertion about unrestricted absolute cohomology and infer the same ramification control. The dual kernel has been killed inside the restricted problem itself.

For the frozen supports, each supporting local field has residue norm one modulo seven and contains μ_7. Local duality gives one k dimensional H^2 term there. The four fixed fields do not contain μ_7 and have zero $H^2(K_v, k)$. Global dual invariants also vanish. The final portion of the Poitou–Tate sequence then identifies the restricted H^2 with k^9. The dimension nine specializes the frozen support, whereas Lemma 19.1 held for arbitrary finite Σ.

The theorem proves existence of an origin in the bare primitive torsor. It does not compute values of p_0 on labelled Frobenius or inertia, a defining field for p_0, or its affine compatibility class. Those numerical values are exactly what the final evaluator still lacks.

19.1 The unique local fixed trace primitive

At a fixed place v above $2, 3, 5, 7$, write

$$\beta(g) = (\chi(g)^{-1} - 1)\lambda_v.$$

The local separating element makes λ_v unique. Define

$$p_v^{\mathrm{fix}}(g) = \lambda_v \chi(g)^{-1} \alpha(g). \tag{19.3}$$

Direct cocycle expansion yields

$$dp_v^{\mathrm{fix}}(g, h) = -\alpha(g)\chi(g)^{-1}\beta(h), \tag{19.4}$$

$$(\chi(g) - 1)p_v^{\mathrm{fix}}(g) + \alpha(g)\beta(g) = 0. \tag{19.5}$$

The second equation is precisely the first trace condition in (13.8).

If another local primitive satisfies the same trace condition, their difference z is an ordinary character and $(\chi(g) - 1)z(g) = 0$ for every g. If $\chi(g) \neq 1$, this already gives $z(g) = 0$. For $\chi(g) = 1$, choose h with $\chi(h) \neq 1$. Both $z(h)$ and $z(hg)$ vanish, so additivity gives $z(g) = 0$. Thus (19.3) is the unique local cochain satisfying both equations in this fixed frame. It is not uniqueness of a global compatible primitive.

For a bare global p_0, the difference $p_v^{\mathrm{fix}} - p_0|_v$ is an ordinary local character. Only that difference has the class field coordinates used below. Neither inhomogeneous cochain can be treated individually as an ordinary character.

20 The complete ordinary localization map

Let z be an ordinary k valued character of $G_{K,\Sigma}$. It is a difference of bare primitives, not beta and not either primitive itself. Local class field theory gives the following coordinate order:

$$F_2, F_3, F_5, \quad 7 : \phi, 3, 7, \quad I_{q_1}, F_{q_1}, \ldots, I_{q_9}, F_{q_9}. \tag{20.1}$$

We use the local Artin convention sending a selected uniformizer to arithmetic Frobenius. At five the uniformizer is $\sqrt{5}$. At support primes it is the rational prime q. Every sign in the reciprocity rows and every value of $h = (Nv - 1)/7$ below uses this convention [13].

At $2, 3, 5$, the residue norms are $4, 9, 5$, whose unit orders have no seven factor. Only the unramified coordinate remains. At seven, the two independent pro seven unit coordinates are $\phi, 3$, whose normalized logarithmic first digits are $5\sqrt{5}, 6$, and the third coordinate is the uniformizer seven. Each of the nine inert support primes has one seventh power residue coordinate and one unramified coordinate. Real odd primary characters vanish. The total local dimension is

$$3 + 3 + 2 \cdot 9 = 24 \quad \text{over } k.$$

The chosen $g_q = \phi + a_q$ generating the residue seventh power quotient supplies I_q. It need not generate the whole multiplicative group. Only its nonzero seventh power symbol and specified normalization are required.

20.1 Fourteen principal relations are exhaustive

The Σ unit group modulo seventh powers has basis

$$\phi, 2, 3, \sqrt{5}, 7, q_1, \ldots, q_9. \tag{20.2}$$

Prime valuations prove independence of the thirteen prime coordinates, and (19.1) proves the unit coordinate is nontrivial. Class number one removes extra ideal class variables. Principal reciprocity pairs these fourteen units with the twenty four local characters and gives a matrix L of size 14 by 24.

There are two complementary reasons why this matrix has rank fourteen. Lemma 19.1 proves that global Kummer localization is injective. Under the local perfect pairings, L is the transpose of the global Kummer localization matrix, so the fourteen global Kummer coordinates give fourteen independent functionals. The finite arithmetic then reconstructs their declared coordinate entries and verifies this rank with explicit certificates. This order avoids treating a computational rank as the only reason a global reciprocity presentation is complete.

The finite set Poitou–Tate sequence identifies the entire ordinary localization image with $\ker L$. Equivalently, the ray class presentation says that a compatible family of local characters extends exactly when it annihilates the principal Σ units [12, 13]. This is a restricted image theorem. It is not unrestricted finite localization surjectivity.

Ordinary global localization is injective here. A character with zero local classes at every allowed ramification place is everywhere unramified. Odd real terms vanish, and the ordinary class group is trivial. Thus it is zero. The ordinary global space is therefore ten dimensional, since $24 - 14 = 10$.

20.2 An explicit image parametrization

Put $a = z_7(3)$ and $x_i = z(I_{q_i})$. Let ℓ_i be the seventh power logarithm normalized by $\ell_i(g_{q_i}) = 1$. The supporting symbols of ϕ vanish. For rational seven adic units define

$$\ell_7(t) = \frac{\log(t^6)}{\log(3^6)} \pmod 7 = 6\frac{t^6 - 1}{7} \pmod 7.$$

Principal reciprocity determines the remaining coordinates:

$$z_7(\phi) = 0, \tag{20.3}$$
$$z(F_2) = -5a - \sum_i \ell_i(2)x_i, \tag{20.4}$$
$$z(F_3) = -a - \sum_i \ell_i(3)x_i, \tag{20.5}$$
$$z(F_5) = -4a - \sum_i \ell_i(\sqrt{5})x_i, \tag{20.6}$$
$$z_7(7) = -\sum_i \ell_i(7)x_i, \tag{20.7}$$
$$z(F_q) = -\ell_7(q)a - \sum_{i:q_i \neq q} \ell_i(q)x_i. \tag{20.8}$$

At its own conductor place, the rational prime q is the chosen uniformizer with unit part one. Its self residue logarithm in the ordinary character presentation is zero, not the undefined logarithm of zero. Every other support prime contributes simultaneously.

These equations define a 24 by 10 matrix J. The ten free parameters occur in their declared local coordinates, so J is injective. Direct substitution gives $LJ = 0$. Since both $\operatorname{im} J$ and $\ker L$ have dimension ten,

$$\operatorname{im} J = \ker L. \tag{20.9}$$

Every parameter vector gives a global ordinary character. This is not merely an overapproximation by necessary equations.

The matrices have entries in $\mathbb{F}_7$ and are extended to k. Their use does not assume coefficient descent. A smaller common pointwise semilinear problem can be imposed afterward, but it requires the same j, frame, and reference for the compared cochains.

21 Frobenius at a conductor prime

The toric row $cz(F) - bz(T)$ needs $b = \beta(F)$ even where beta ramifies. An unramified test prime formula is insufficient. The final calculation includes the self conductor unit and converts the old twisted inertia normalization to the ordinary Artin generator of Section 20.

Let $M = \mathbb{Q}(\zeta_7)$. For a rational conductor $q \equiv 1 \pmod 7$, choose a generator π_q of norm q for the selected degree one prime. The source gives exact coefficient vectors in $1, \zeta_7, \dots, \zeta_7^5$ for $q = 197, 337, 547, 673, 743, 883, 1933$. For example,

$$\pi_{197} = \zeta_7 + 2\zeta_7^2 + 2\zeta_7^4,$$

and

$$\pi_{337} = -1 + \zeta_7 - \zeta_7^2 + 3\zeta_7^3 - \zeta_7^4 + \zeta_7^5.$$

The norm equality identifies one prime ideal. It does not leave an unspecified product of several primes.

At the chosen q place, write $\pi_q = qu_q$ with u_q a unit. The same norm identity gives

$$\ell_q(u_q) = -\sum_{b=2}^{6} \ell_q(\pi_q(z_q^b)). \tag{21.1}$$

Principal reciprocity on π_q therefore supplies the own conductor column

$$D_{q,q} = \sum_{b=2}^{6} (1-b)\ell_q(\pi_q(z_q^b)). \tag{21.2}$$

For $r \neq q$, the column is

$$D_{q,r} = -\sum_{b \in \mathrm{reps}_r} b\ell_r(\pi_q(z_r^b)). \tag{21.3}$$

The degree two representative convention is the one proved in (10.1). These formulas use the same arithmetic Frobenius principal idele convention as the ordinary localization rows.

The direct independent check lifts the chosen root of Φ_7 modulo q^2 by Hensel, then computes $\pi_q(\widetilde{z}_q)/q$ modulo q. It verifies the self unit rather than assigning its missing value from the norm relation. The norm calculation and Hensel calculation share the input generator but use different arithmetic routes.

Because each support prime is inert in $K/\mathbb{Q}$, restriction at a rational $q \equiv 1 \pmod 7$ multiplies the compatible rational Frobenius value by two:

$$b_q = 2\sum_r D_{q,r} s_r. \tag{21.4}$$

For $q \equiv -1 \pmod 7$, the compatible squared rational Frobenius has beta value zero, from the cocycle factor $1 + \chi(q)^{-1} = 0$. This does not say beta is unramified there.

The ordinary inertia generator differs from the frozen twisted residue convention. Its beta value is

$$c_q = s_q \ell_q(\mathrm{Nm}_{\mathbb{F}_{q^2}/\mathbb{F}_q} g_q) \quad (q \equiv 1 \pmod 7),$$

and $c_q = s_q \ell_q(g_q)$ for $q \equiv -1 \pmod 7$. The conversion factors are

197 : 5	307 : 2	337 : 5	547 : 4	673 : 1
743 : 1	797 : 4	883 : 3	1483 : 1	1933 : 1

Setting $c_q = s_q$ without this conversion mixes two local generators and changes the problem.

Finally, the expression $c_q z(F_q) - b_q z(I_q)$ is invariant under a compatible Frobenius lift change. Both b_q and $z(F_q)$ acquire their respective inertia multiples, and the extra terms cancel. This invariance does not permit omitting the self conductor unit or changing only one normalization.

22 The canonical affine obstruction

The completed linear operator acts on ordinary differences. It remains to supply the arithmetic origin on which those differences act.

On the local space k^{24}, let E have fifteen rows. Its first six rows select all fixed local character coordinates in (20.1). Its remaining nine rows are

$$z \longmapsto c_q z(F_q) - b_q z(I_q).$$

Define

$$A = EJ : k^{10} \longrightarrow k^{15}. \tag{22.1}$$

The self conductor values and inertia conversions in Section 21 are part of this operator. Dropping them would reproduce a different matrix.

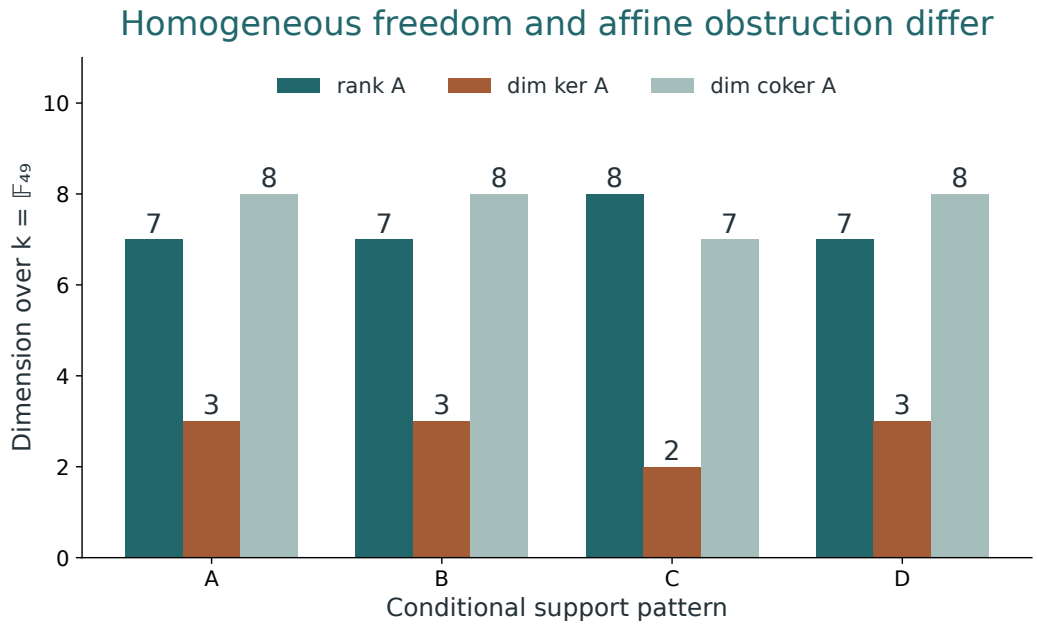


Figure 4: The kernel measures ambiguity after an affine solution exists. The cokernel measures possible first stage incompatibility. Neither dimension evaluates the actual target.

22.1 The homogeneous results

The complete exact matrices, row reductions, null bases, and rank minors give

Pattern	rank A	$\dim_k \ker A$	$\dim_k \operatorname{coker} A$	Remaining difference support
A	7	3	8	$307, 797, 1483$
B	7	3	8	$307, 797, 1483$
C	8	2	7	$307, 1483$
D	7	3	8	$307, 797, 1483$

In the global parameter order $(a, x_{q_1}, \ldots, x_{q_9})$, the kernel is exactly the span of the unit vectors at the listed minus congruence primes. After all toric rows are imposed, the plus congruence inertia differences vanish. This is a theorem about differences. It does not imply that the absolute inertia of a bare primitive vanishes at those plus primes.

Pattern C improves the previous homogeneous dimension from four to two. Its two former plus directions, evaluated on the added toric rows at $197, 337$, give

$$\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}, \qquad \det = 1 \quad \text{in } \mathbb{F}_7.$$

Thus both directions are removed. Nonzero minors prove rank lower bounds, while full null bases prove the matching upper bounds. Exact arithmetic, not numerical singular value thresholds, determines the ranks.

22.2 The arithmetic target

Choose a bare global primitive p_0 from Theorem 19.2. This is one continuous global cochain. Its values have not been computed. Define $t(p_0) \in k^{15}$ as follows. Its first six coordinates are the ordinary local character coordinates of $p_v^{\mathrm{fix}} - p_0|_v$. At each support prime, put

$$t_q(p_0) = -h_q c_q - c_q p_0(F_q) + b_q p_0(T_q), \qquad h_q = (q^2-1)/7 \pmod 7. \tag{22.2}$$

At support, α is zero and χ trivial, so the restriction of p_0 is an ordinary character. At fixed places only its difference from the prescribed primitive is ordinary. The target therefore has the declared coordinates without mistyping either inhomogeneous cochain as an ordinary character.

Theorem 22.1 (Exact Stage P criterion)**.** *Fix the same residual cocycles, full support, natural frame, and reference, together with the accepted inputs used above. Let N be any full left null matrix of A, with independent rows and*

$$NA = 0, \qquad \operatorname{rank} N = 15 - \operatorname{rank} A.$$

There is a global restricted primitive meeting all fixed first trace and support toric conditions if and only if

$$Nt(p_0) = 0. \tag{22.3}$$

The class $[t(p_0)] \in \operatorname{coker} A$ is independent of the chosen bare primitive.

Proof. Two bare primitives have the same differential, so their difference is an ordinary global character. Conversely adding any such character preserves the differential. By Section 20, every character has a unique parameter vector $z \in k^{10}$ with local vector Jz.

A candidate $p = p_0 + z_{\mathrm{char}}$ meets the fixed local condition precisely when the selected local character coordinates equal those of $p_v^{\mathrm{fix}} - p_0|_v$. At support, substitution in (17.1) gives

$$c_q z_{\mathrm{char}}(F_q) - b_q z_{\mathrm{char}}(T_q) = t_q(p_0).$$

Together, these fifteen equations are exactly $Az = t(p_0)$. Thus Stage P is soluble if and only if $t(p_0) \in \operatorname{im} A$.

Since $NA = 0$, the image is contained in $\ker N$. Rank nullity gives

$$\dim \ker N = 15 - \operatorname{rank} N = \operatorname{rank} A = \dim \operatorname{im} A.$$

Hence $\ker N = \operatorname{im} A$, proving the equivalence with (22.3). Merely having some left null rows would prove necessity, not sufficiency. Completeness of N is part of the theorem.

Now replace the origin by $p_0' = p_0 + z_{0,\mathrm{char}}$, where its localization is Jz_0. At the fixed places, the target differences decrease by the selected coordinates of this character. At support, (22.2) decreases by $c_q z_{0,\mathrm{char}}(F_q) - b_q z_{0,\mathrm{char}}(T_q)$. Therefore

$$t(p_0') = t(p_0) - Az_0. \tag{22.4}$$

The quotient class and Nt are unchanged. The coordinate basis for N itself is optional, but its vanishing criterion and the quotient class are canonical. □

The result concerns the stated first stage. It does not identify its solution set with actual representations, solve the upper equations with their prescribed geometry, or prove higher integral liftability.

Corollary 22.2 (Affine fibers)**.** *If Stage P has one solution with parameter z_*, its complete solution set for these linear conditions is $z_* + \ker A$. It has $49^3, 49^3, 49^2, 49^3$ choices in patterns A through D, respectively. If the target is not in the image, the set is empty.*

Proof. $Az = t$ is equivalent to $A(z - z_*) = 0$. The exact kernel dimensions give the counts over k. □

An additional common pointwise semilinear j, frame, and reference condition makes the relevant minus conductor coefficients lie in $w\mathbb{F}_7$. Under that extra hypothesis, and only after compatible nonemptiness is known, the corresponding first stage ambiguity bounds are $343, 343, 49, 343$. They are neither representation counts nor integer solution counts. The bare theorem does not prove the semilinear affine fiber nonempty.

22.3 A small model of the mistake to avoid

Consider $A : k \to k^2$ given by $A(z) = (z, 0)$. Its kernel is zero, its rank is one, and it has a full left null functional $N(u, v) = v$. For target (a, b), a solution exists precisely when $b = 0$. The map has a one dimensional image and a zero kernel, but this gives no information about the target's second coordinate. Substituting $(0, 0)$ because it is convenient manufactures solvability of a different problem.

The final Beal operator has the same logic with fifteen target coordinates. Synthetic image targets are useful tests of an evaluator. They cannot be substituted for the arithmetic values of p_0.

22.4 What the evaluator actually returns

The supplied program takes a pattern, a field basis, fifteen encoded values, a kind tag, and provenance. It returns the obstruction coordinates, a character correction when one exists, and a homogeneous kernel basis. Field values use the encoding $a + 7b$ for $a + bw$, with $0 \le a, b < 7$. The program records arithmetic target authentication as false. It verifies linear arithmetic on supplied data. It neither constructs a defining extension for p_0 nor authenticates its values.

No actual $t(p_0)$ or $Nt(p_0)$ has been supplied for any of the eight patterns. Some obstruction coordinates may vanish for further arithmetic reasons, but none has been evaluated here. The next arithmetic obligation is therefore concrete: construct a defining extension or a duality evaluation of this same canonical class. A nonzero result excludes the first stage branch under its inputs. A zero result supplies a first stage family for the prescribed upper and integral compatibility problem.

23 What the evidence checks

The source and evidence archive separates original recovered payloads, fresh computational records, formal sources and logs, literature leaves, and the new publication sources. This makes it possible to check a claim against the kind of evidence that actually supports it.

23.1 Exact finite arithmetic

The final Block 80 assertions were replayed in the current session, including the immediate Block 79 predecessor. Primary and separate programs agree on their full scientific records.

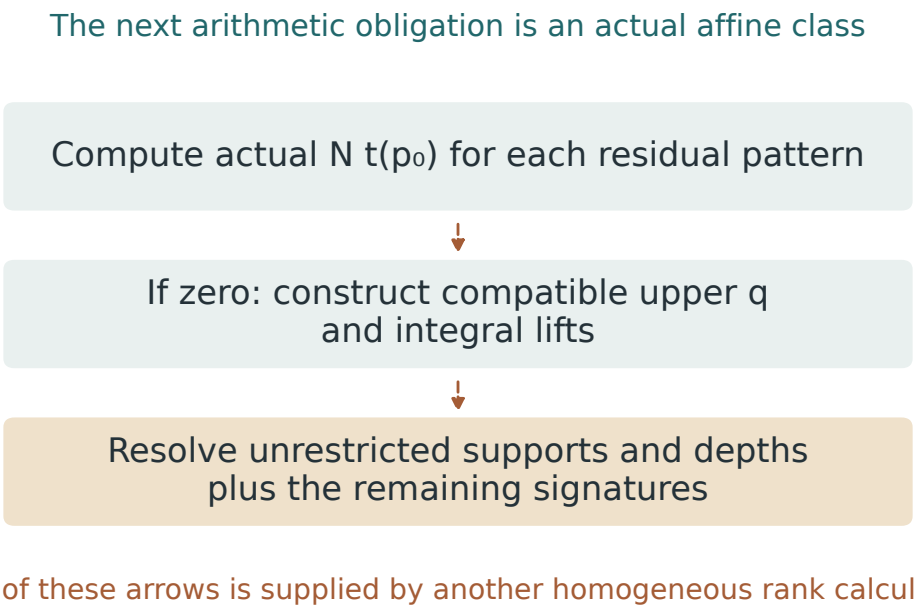


Figure 5: The canonical affine class is the immediate target. Passing it supplies a first stage family, after which upper geometry and integral lifting remain. Other conductors, depths, components, and signatures require their own coverage.

Their field bases, norms, self conductor unit calculations, and rank certificates differ. The final scope includes 592 residue symbol evaluations per arithmetic implementation, seven direct self uniformizer checks, sixteen rank minor certificates, 12 348 Frobenius lift identity cases, and 12 348 fixed primitive identity cases. The evaluator exercises eighty four image and nonimage targets, forty origin changes, and sixteen malformed inputs. A separate synthetic image family contains 560 tests.

The additional audit independently checks twenty matrices by exact prime field ranks and integer minors, all 2401 products across the two quadratic field bases, 2940 basis field targets, 1960 image reconstructions, 1960 origin changes, thirty six invalid scalar encodings, and three symbolic identities. These numbers describe their declared finite domains. They are not independent arithmetic targets or Galois realizations.

Selected historical computations also have fresh replay records. The portable scientific replays for Blocks 67 through 80 all passed at their unchanged documented bounds. Their precise source pins, options, output comparisons, resource limits, and failures are recorded in the computation report. The existence of many nested archives is not a claim that every historical wrapper was freshly run to completion. Where a portable scientific runner is used, its result is distinguished from the original packaging wrapper.

Two current failures deserve an explicit place in the reader. The archival Block 69 wrapper reached an aggregate 840 second resource timeout without a completed certificate. That attempt is inconclusive, not a mathematical exclusion or a passed replay. Its separate portable scientific run subsequently passed and matched all seventeen result JSONs at the unchanged bounds. The unchanged original Block 67 verifier completed with status `PASS` and exit zero, comparing its eight scientific outputs and rejecting nine corruptions. A new generic

wrapper then incorrectly required the different status string `PASS_DECLARED_CHECKS` and rejected that successful outcome. The failed wrapper record is retained. A separate pinned interpretation check identifies the actual original process result without overwriting it, and the portable scientific replay independently passed. The correction is in interpreting the current wrapper, not in silently repairing historical source bytes.

The audit also found a legacy Block 71 display metadata error. One primary record said the unit domain modulo 49 contained 2304 elements, but its loop checks $49^2 - 7^2 = 2352$ units. The independent record already reports 2352. The mod seventh power quotient size forty nine and its forty eight element fibers are unchanged. The original record is preserved, while the corrected domain count is stated here.

Other failures remain visible in the history. Block 70 originally required beta itself to vanish in a bad alternative, instead of requiring both trace products to vanish. The rejected source and outputs are retained. Block 80 initially used a too strong full primitive residue generator requirement, although only a generator of the seventh power quotient was needed. A wrapper syntax failure stopped another attempt. Corrected executions have separate successful records.

No actual affine target is authenticated by any finite test in this release. Exactly zero residual patterns are excluded by the final affine calculation. The finite ranks describe the operator, and the synthetic cases verify its evaluator.

23.2 The formal layer

Forty named public Lean statements have fresh successful checks in three scopes. The original Block 47 has twenty one statements: seven generic lane certificate results, five Boolean correlation results, and nine exact mod seven polynomial or matrix identities. The original Block 48 has eight characteristic seven algebra and elementary bounded dimension statements. These unchanged historical sources were checked with Lean 4.34.0-rc2 and their exact mathlib revision. The Block 47 project module also passed its separate kernel replay, with three deliberately invalid controls rejected.

The new reader module has eleven statements checked with Lean 4.19.0 and its exact mathlib revision. They prove ordinary differential additivity, primitive difference additivity, preservation under adding a character, the explicit fixed primitive differential and trace identity, affine solvability translation, cokernel origin invariance, a necessary null test, the complete null criterion under its explicit completeness hypothesis, and the kernel translate description of a fiber. They passed strict elaboration and axiom inspection. No separate optional checker binary is claimed for this new module.

The successful axiom reports use only subsets of Lean's standard extensionality, classical choice, and quotient axioms, with no admitted proof or custom axiom. Failed import, API, runtime layout, and resource attempts are retained separately. They are not included in the forty passed statement count.

The formal sources do not encode finite set Poitou–Tate duality, the actual Galois group, continuity of a primitive, the fifteen by ten arithmetic matrix, its full null basis, or the arithmetic target. Generic exclusion statements explicitly assume a complete lane cover and sound attachment to one fixed candidate. Compilation proves those implications. It does not supply their Diophantine instantiation.

23.3 Written proofs and independent scope review

The reader supplies complete elementary proofs for the exponent reduction, valuation scaling, local squareclass criterion, depth matrix relation, discriminant collapse, period torsion equivalence, restricted unit kernel, fixed local primitive, and affine obstruction criterion. External applications retain their named source hypotheses. A separate internal reviewer checked the new proof text and its load bearing boundaries. This is useful adversarial review of the stated arguments, not independent human peer review or a certificate for every historical analytic geometric input.

The final target remains mathematical rather than editorial. A defining arithmetic origin or equivalent duality calculation must evaluate $[t(p_0)] \in \operatorname{coker} A$. Successful linear algebra cannot replace this missing input. After a compatible first primitive is obtained, the prescribed upper geometry and higher integral realization remain to be solved. Broader conductors, higher depths, the minus component, other modular branches, and other signatures require their own exhaustive coverage.

24 Sources and the literature boundary

The literature check distinguishes a theorem statement, an exact hypothesis leaf, a source dependent application, and a current reported claim. The accompanying source and leaf manifests identify the versions and inspected locations. Original project reports retain their historical source status, including unsuccessful retrievals and later corrections.

The source basis for the final restricted theorem is Milne's finite set Poitou–Tate duality and local duality [12]. Class field conventions, ray reciprocity, restriction with corestriction, and Brauer localization use the authenticated course notes [13]. The principal ideal and unit arguments have been written out in the reader rather than deferred to a computed class number. Marked periods use the stated degeneration and uniformization results [17, 18, 19]. Their application to the particular natural family remains an explicit inherited geometry input.

For the wider signature context, Darmon–Granville gives fixed signature finiteness in its declared hyperbolic range [8]. The Pacetti–Villagra Torcomian preprint supplies the family and specific modular results with hypotheses that must be read individually [10]. In particular, a large prime irreducibility theorem whose cutoff excludes seven is not justified at seven by an unrelated resultant being nonzero modulo seven. The direct project proof has its own arithmetic dependencies.

A current source check also found Chocian's preprint of 22 September 2026, which reports a computer assisted proof for the primitive equation $x^3 + y^5 = z^7$ [11]. Its existence and reported scope are recorded here. This release does not independently audit that claimed proof or import it as a theorem closing the recovered project route. Earlier project descriptions of the route as incomplete should therefore be read as descriptions of this project's evidence, not as an exhaustive assertion about all current literature.

The formal source count, exact finite tests, and historical block numbers do not establish novelty. No novelty claim is inferred from a limited source search. The value of this release is a reproducible account of the actual constructions, proved reductions, corrections, and the precisely isolated arithmetic input that the final affine operator still requires.

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