PAPER XI / RESEARCH COMPANION

Toward Arithmetic Packet Realization

Restricted constructions, failed routes, and the remaining obligations of Conjecture 66

Cumulative research reader

Komorebi P.

Working paper v0.4.0 / Publication Block 04 / 10 October 2026

Edition map

This working edition preserves the first two parts and adds an independently editable third part. It does not replace the original Paper XI main, visual or Lean readers.

Part I. Restricted arithmetic constructions. The complete 33-page v0.2.0 reader is retained, including odd-prime and weighted dyadic recursion, full-character induction, native quotient obstructions and low-rank enlarged-carrier recovery. It occupies PDF pages 3–35.

Part II. Monomial routes, Clifford descent and exact obstructions. The complete 19-page Block 03 continuation is retained. Sections 14–21 separate admissible induction towers, genuine multiplicities, character occurrence and self-twist failure, compact binders and exact corner obstructions. It occupies PDF pages 36–54.

Part III. Conductors, native markings and the equal-capacity boundary. The new 23-page continuation contains Sections 22–30. It audits conductor normalization and candidate filters, separates pair invariants from quotient capacity, and gives the fixed-refinement criterion with explicit conditional finite corrections. It occupies PDF pages 55–77.

Source continuity. Part I remains a preserved PDF because the complete editable Block 02 archive was not recovered. Parts II and III, this edition map, the assembly programs, exact suites, source maps and versioned ledger are editable. The compact checkpoint contains selected supporting research sources, not the entire historical archive.

Mathematical boundary. The general realization conjecture, same-candidate H5 and corrected-character occurrence retain their separate open statuses. No canonical native quadratic marking has been inferred from the host dimension. The finite correction theorems require their supplied actions and permitted changes. No fresh Lean build or external peer review occurred in Publication Block 04.

Restart. Resume at Publication Block 05 using the ledger and source queue. The next publication task is the dedicated conjecture visual companion, followed by synchronization of the original XI readers. The chosen candidate's actual native marking and phases remain a distinct arithmetic task.

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Conjecture 66 progress paper

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Working paper v0.2.0 / Publication Block 02 / 10 October 2026

Reading scope

This paper collects the mathematical progress toward the arithmetic realization endpoint of Paper XI. Its central object is not a numerical completion score. It is the proposed passage from a supplied finite packet, with its character values, phases and transport maps, to an actual supercuspidal representation carrying that same packet.

The general passage remains Conjecture 66. That fact does not erase the constructions obtained along the way. The available record contains finite comparison theorems, arithmetic dictionaries, explicit parameter constructions, full character arguments, exact obstructions to particular hosts, and a low-rank dyadic repair using larger arithmetic inducing groups. The purpose of a separate paper is to develop these results in their own right, without confusing a restricted construction with a universal realization theorem.

This second working edition expands the actual parameter, full-character, and low-rank recovery arguments. It makes the corestriction normalization, native traces, depth endpoints, and noncompact inducing data explicit. The principal Bushnell–Kutzko inputs have been checked directly against the retained book. Their classical status remains visible. No new Lean compilation or external peer review is claimed. Earlier records described as independent reviews are internal reviews, not external peer review.

A versioned progress ledger accompanies the paper. It separates manuscript coverage, ordinary proofs, classical inputs, exact computation, inherited formal evidence, and unresolved research obligations. The original three reader volumes remain unchanged during this content-expansion block.

The main Paper XI monograph, visual guide and Lean guide are retained as inherited reading editions. They are not silently replaced by this working companion. The archive includes Markdown files containing detailed proofs, construction notes, audits and research records. A machine-readable source map identifies the exact files behind each chapter and distinguishes newly checked finite examples from inherited evidence.

Not yet an exhaustive synthesis. The recovered archive includes the earlier monomial and Clifford-descent branches, the formal continuation history, and intermediate binder work. This edition identifies those branches but does not claim to have integrated every argument from every retained snapshot. Their full reconciliation is a named task for subsequent blocks.

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1 The realization problem and its quantifiers

1.1 Three different mathematical tasks

Paper XI separates finite comparison, arithmetic existence and local-factor characterization. These tasks interact, but they have different inputs. A finite Heisenberg representation can be constructed from a nondegenerate alternating pairing. A compatible family of intertwiners can be constructed after its source and target representations are supplied. Neither statement, by itself, constructs the required representation of a general linear group over a local field.

Fix the reciprocity, additive-character, local-factor and phase conventions of Paper XI. Let $\mathcal{A}$ denote the declared class of irreducible wild Weil atoms. For $\rho \in \mathcal{A}$, write d_ρ for its *supplied* assembled finite datum. The historical names H1–H4 refer to the following data: an intrinsic finite break decomposition, equivariant symplectic bridges to the supplied breakwise type quotients, compatible central-extension and phase data including the declared markings, and a coherent transport system whose matrix holonomy obstruction vanishes. These are assumptions on a specified source, not a theorem that every wild Weil representation already comes equipped with a canonical instance [1, 3].

Definition 1.1 (Arithmetic realization witness)**.** A witness $R(d_\rho, \pi)$ consists of actual arithmetic inducing data for an admissible supercuspidal representation π of $\mathrm{GL}_{\dim \rho}(F)$, together with comparison and recovery maps carrying the prescribed assembled finite packet. Those maps must preserve the declared stabilizer action, supplied character values, central-extension and phase data, and compatible transports. Recovery returns the same π, with the fixed central character and noncompact completion.

An enlarged actual inducing carrier is permitted by this contract. The entire assembled packet need not be the original maximal-simple-type quotient. Conversely, an independently fixed native marking, an original matrix subgroup embedding, or a prescribed numerical depth is not automatically preserved merely because the scalar extension is preserved. Such requirements must be stated and proved separately.

1.2 The conjecture and the surviving comparison theorem

The endpoint is the following statement, with its historical identifier retained:

> **Conjecture 66.** For every $\rho \in \mathcal{A}$ with its supplied H1–H4 datum d_ρ, there exist an actual admissible supercuspidal representation π of $\mathrm{GL}_{\dim \rho}(F)$ and a witness $R(d_\rho, \pi)$.

This is the realization requirement isolated from historical Theorem 16.2, original proof obligation 66, and row 95 of the all-records register. It is not row 66 of that register. It asks for one witness per admitted source. It does not ask that every proposed host work, or that mutually inconsistent noncompact completions be reconciled.

Proposition 1.2 (Comparison of the same realized candidate)**.** *Suppose $R(d_\rho, \pi)$ is supplied for one actual admissible supercuspidal π of the required rank. Suppose H5 holds for this π: its required pair local factors agree with the factors prescribed for ρ on the stipulated characterizing family of twists, in the fixed normalizations. Assume the applicable local-factor characterization for that family and class. Then $\pi \simeq \mathrm{LLC}(\rho)$.*

Proof. The characterization applies to the two actual representations π and the reference representation $\mathrm{LLC}(\rho)$. H5 provides equality of the characterizing factors, so their isomorphism classes agree. The

witness R identifies this same candidate as the carrier of the prescribed packet. The argument neither constructs π nor derives H5 from the finite datum. □

Evidence and scope. This is the pointwise conditional implication in the current paper, not a new local converse theorem. The local-factor characterization is a stated external input.

The distinction can be written as

$$\forall \pi\, [R(d,\pi) \Rightarrow H5(\pi)] \qquad \text{versus} \qquad \exists \pi\, [R(d,\pi) \wedge H5(\pi)].$$

The first statement is true when there are no realizers. The second supplies a realizer. Likewise, a witness of $R(d,\pi_1)$ and a verification of H5 for an unrelated π_2 cannot be combined. This elementary point is important because the general formal interface can quantify over several realizers, while the publication comparison uses one fixed candidate.

1.3 What a host obstruction does and does not refute

Let $\mathcal{H}(d)$ be the class of admissible hosts for a datum d. A failed construction establishes that one chosen H does not realize d, or perhaps that no H in a particular subclass does. To refute Conjecture 66 for that datum would require excluding every allowed arithmetic witness. The capacity and inner-normalizer obstructions below are therefore substantive negative results without being counterexamples to the conjecture. They identify constraints a successful construction must avoid.

2 What the formal continuation actually contributes

2.1 The original obligation count is not a measure of distance

The inherited formal record organizes the original paper around 66 obligations. Its retained 65/66 checkpoint means that the final general realization obligation has not been closed under the original scope. It does not mean that one short lemma is left, or that the remaining task is one sixty-sixth of the mathematical work.

The final Lean guide selects a union of 423 module paths and 4,342 explicitly named lexical declaration commands. It starts with the 302-module v0.44 source tree, adds checked normalized-Frobenius material and retains the arithmetic/type branch through base108 and continuations110–113. Repeated identical source bytes are collapsed in that catalog. These are historical source-census numbers, not numbers of independent discoveries, and no new kernel acceptance is claimed here [2].

2.2 From abstract dictionaries to arithmetic attachments

The continuation sequence matters because it repeatedly strengthens the interpretation of a formal object. A module can prove a theorem about a lattice index without identifying that lattice with the canonical lattice of a supplied arithmetic type. A structure can contain an existence field without constructing an inhabitant of that field. An arbitrary character with the correct values on a tested subgroup need not be the original full character.

Continuation 110 develops Morita-lattice indices, exact one-plus comparisons, and same-carrier dictionaries. Its remaining task is not to rename an abstract parameter as canonical. It is to attach the arithmetic groups, the original character, the canonical extension, and the residue factor to the exact formal data. Continuation 111 moves the character and index attachment further and separates a valid one-break

conditional comparison from the general common-host problem. Separate actual break types do not automatically assemble into one arithmetic type of the total rank [4, 5].

Continuations 112 and 113 make the boundary more concrete. The former supplies complex scalar-cochain transport and honest cyclic-action machinery. Its degree obstruction concerns a whole maximal-type carrier, rather than refuting every permissible assembly carrier. The latter records a cubic non-occurrence test: the first proposed simple character does not occur in the original automorphic-induction candidate. A corrected wild-unit proposal passes the stated self-twist test, but that positive test alone does not prove occurrence [6, 7].

2.3 The retained contract for formal evidence

A Lean theorem certifies its displayed conclusion from its displayed binders and imported declarations. It does not silently instantiate a local field or produce an arithmetic representation that was supplied as a field of a structure. In the general comparison interface, `Realization.exists_atom` is an existence input. The source can still prove nontrivial descent, comparison and uniqueness results using that input. The existence obligation remains distinct.

The recorded environment is Lean 4.34.0-rc2 with Mathlib revision `aeb72f1e9981b6d35b426b16dca7759be3e7ba06`. This working edition preserves those records but does not invoke Lean, Lake or leanchecker. Same-kernel replay is also not an independently implemented proof checker. These distinctions remain attached to every inherited formal claim.

3 Central products remove one proposed finite obstruction

The extension diagnosis of Block 163 isolates a useful positive fact. Once the full factor extensions and their orthogonality are genuinely supplied, a further mysterious mixed scalar extension class is not automatically left over. The following proofs reconstruct the elementary core of that diagnosis [8].

Proposition 3.1 (Orthogonal factors form a balanced central product)**.** *Let $1 \to Z \to P \to A \to 1$ be a central extension, with $A = \bigoplus_{i=1}^{d} A_i$ with $d \geq 1$. Let P_i be the inverse image of A_i. Assume elements of P_i and P_j commute for $i \neq j$. Then multiplication induces an isomorphism*

$$\Big(\prod_i P_i\Big)\Big/\{(z_i) \in Z^d : \prod_i z_i = 1\} \;\simeq\; P.$$

Proof. Commutation makes multiplication a homomorphism. Given $x \in P$, decompose its image in A as $\sum_i a_i$ and choose a lift $x_i \in P_i$ of each a_i. The difference between x and $\prod_i x_i$ is central and can be absorbed into one factor. Thus multiplication is onto. If $\prod_i x_i = 1$, its image in the direct sum gives $a_i = 0$ for every i. Hence each x_i lies in Z, and the only remaining condition is $\prod_i x_i = 1$. This is exactly the displayed kernel. □

This proposition uses the actual factor extensions, not just a list of dimensions or break labels. Distinct break numbers do not, by themselves, prove the required independence or commutation.

Proposition 3.2 (Same commutator over a divisible scalar center)**.** *Let A be a finite abelian group. Two central extensions of A by $\mathbb{C}^\times$ with the same alternating commutator are isomorphic by a map inducing the identity on A and on $\mathbb{C}^\times$. The isomorphism need not preserve additional independently fixed lifts or finite-center markings.*

Proof. Choose sections and write the two normalized cocycles as a_1, a_2. Their ratio $b = a_2/a_1$ has trivial commutator, so its extension is abelian. Write $A = \bigoplus_j \mathbb{Z}/n_j\mathbb{Z}$ and choose a lift u_j of each cyclic generator

in that abelian extension. Its n_j th power is a scalar. Since $\mathbb{C}^\times$ has n_j th roots, rescale u_j so that $u_j^{n_j} = 1$. The adjusted commuting lifts define a splitting. Thus b is a coboundary and the two original extensions are equivalent. The rescaling choices need not fix any prescribed lift. □

Automorphisms inducing the identity on quotient and center form $\mathrm{Hom}(A, \mathbb{C}^\times)$: they multiply a lift of a by a character value. This residual freedom is exactly why an unmarked scalar-extension equivalence is not a canonical identification of all source data.

3.1 Quadratic information survives only when its marking travels

In characteristic two, let $V = \mathbb{F}_2^2$ and

$$q_+(x,y) = xy, \qquad q_-(x,y) = x + y + xy.$$

They have the same polarization $b((x,y),(x',y')) = xy' + x'y$. Their Gauss sums are different:

$$\sum_{v \in V} (-1)^{q_+(v)} = 2, \qquad \sum_{v \in V} (-1)^{q_-(v)} = -2.$$

The numbers of zeros are three and one. Thus the alternating form alone cannot recover the quadratic refinement.

For a literal representation, put

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Both pairs (X, Z) and (iX, iZ) have commutator $-I$. In the first pair the generating lifts square to I; in the second they square to $-I$. They generate the usual dihedral and quaternion groups of order eight. After the center is extended to $\mathbb{C}^\times$, the scalar groups generated together with all scalar matrices coincide. Their distinguished finite subgroups and lift-square functions do not coincide.

This example explains the dyadic attachment rule. An extension map F_0 preserves a chosen square relation when the whole marked finite subgroup is carried along, because $F_0(a)^2 = F_0(a^2)$. It does not prove that this transported marking equals an independently fixed native marking of the host [19]. A kernel already killed by the scalar pushout cannot be recovered afterwards by an extension isomorphism.

4 Abstract stabilizer lifting is not an arithmetic normalizer

Two elementary arguments remove finite scalar obstructions under a coprimality hypothesis. They must not be misread as constructing matrices in the required arithmetic normalizer.

Proposition 4.1 (Coprime lifting of an extension action)**.** *Let A be a finite abelian p-group, and let P be a central extension by $\mathbb{C}^\times$. Let a finite group C of order m prime to p act on A, preserving the extension class. Then the action lifts to an action on P fixing the center.*

Proof. Choose a lift q_c for each $c \in C$. The kernel of the group of extension automorphisms over the quotient and center is $M = \mathrm{Hom}(A, \mathbb{C}^\times)$, a finite abelian p-group. Conjugation defines the C-module structure on M. Write the defect of the chosen lifts additively as

$$q_c q_d = f(c,d) q_{cd}, \qquad f(c,d) \in M.$$

Associativity gives

$$f(c,d) + f(cd,e) = cf(d,e) + f(c,de).$$

Let $S(c) = \sum_{e \in C} f(c,e)$. Summing the identity over e gives

$$mf(c,d) = S(c) + cS(d) - S(cd).$$

Multiplication by m is invertible on M. Put $b(c) = m^{-1}S(c)$ and replace q_c by $(-b(c))q_c$. Their new multiplication defect is zero. The adjusted lifts form the required action. □

The hypothesis that each quotient automorphism preserves the extension class supplies the initial individual lifts. Proposition 3.2 supplies that hypothesis when the quotient action preserves the commutator in the finite abelian setting. Neither proposition realizes those lifts by conjugation in $\mathrm{GL}_n(F)$.

Proposition 4.2 (Coprime linearization of projective intertwiners)**.** *Let C be finite of order m. Suppose U_c are invertible operators on a complex vector space of dimension D, with $U_c U_d = a(c,d)U_{cd}$ for a scalar cocycle a. If $\gcd(D,m) = 1$, the operators can be rescaled to a genuine representation of C.*

Proof. Taking determinants shows that a^D is the coboundary of $c \mapsto \det U_c$. Set $A(c) = \prod_{x \in C} a(c,x)$. The cocycle identity, multiplied over x, gives $a(c,d)^m = A(c)A(d)/A(cd)$. Choose integers r,s with $rD + sm = 1$. Then

$$a = \delta\big((\det U)^r A^s\big).$$

Rescaling by the inverse of this cochain eliminates the cocycle. □

For a Heisenberg carrier of dimension p^k and a prime-to-p group C, the coprimality condition holds. Two honest linearizations can still differ by a character. Thus this argument does not select a canonical beta-extension phase. Nor does it cover a stabilizer with p-torsion without further work [8, 23].

5 The compact binder theorem and the missing solvability input

The binder idea asks whether finite compatibility conditions can be tied together into one global coefficient choice. Block 162 proves a precise compactness theorem. Its value is to locate the missing input, not to conceal it under a new name [9].

Theorem 5.1 (A fixed compact coefficient binder)**.** *Let F be a nonarchimedean local field, $\mathcal{O}$ its compact ring of integers, and ϖ a uniformizer. Let $X \subseteq \mathcal{O}^m$ be closed and let $f_j : X \to \mathcal{O}$ be continuous for $j \geq 1$. Define*

$$C_n = \{x \in X : f_j(x) \in \varpi^n \mathcal{O} \text{ for every } j \leq n\}.$$

There is an $x \in X$ with $f_j(x) = 0$ for all j if and only if every C_n is nonempty.

Proof. An exact zero belongs to every C_n. Conversely, each C_n is a closed subset of the compact space X, and $C_{n+1} \subseteq C_n$. Nonempty nested closed subsets of a compact space have nonempty intersection. Choose x in that intersection. For any fixed j, one has $f_j(x) \in \varpi^n \mathcal{O}$ for all $n \geq j$. The intersection of these ideals is zero, so $f_j(x) = 0$. □

The theorem does not require surjectivity of a chosen sequence of reduction maps. It does require a single compact coefficient space and simultaneous finite solvability within that space. Allowing the rank, pole order or number of coefficients to grow with precision changes the space and invalidates this argument unless a separate compact model is supplied.

5.1 When a finite empty level is an actual certificate

For polynomial equations with integral coefficients, or rational equations whose denominators remain units, a fixed finite residue template gives a finite test at each precision. If the template exactly describes X modulo ϖ^n, then emptiness of its complete solution set certifies failure of that fixed template. If there is no exact solution in the fixed compact model, Theorem 5.1 guarantees some empty level.

This is a semidecision procedure for nonexistence within that model. It provides neither a uniform bound on the first empty level nor a proof that a partial search was exhaustive. A failed random search, a resource cutoff and an actual empty finite level have different logical meanings.

Example 5.2 (Individual equations are not simultaneous solvability)**.** Over $\mathbb{Z}_2$, the equations $x = 0$ and $x = 1$ are each soluble to every precision. They have no common solution even modulo two. The nonempty sets needed in Theorem 5.1 contain joint solutions to the first n equations, not a separate solution for each equation.

Example 5.3 (A nonclosed admissibility condition can disappear)**.** The elements $x_n = 2^n$ are nonzero in $\mathbb{Z}_2$ and satisfy $x_n \equiv 0 \pmod{2^n}$. Their limit is zero. A requirement such as $x \neq 0$, or an unspecified nonzero determinant, is not automatically a closed condition. A fixed unit leading coefficient or a prescribed nonzero residue class is a different condition and can survive the limit.

5.2 The actual matrix binder obligation

The retained matrix template uses a square-zero variable x and expressions of the form

$$Q(x) = -s(xb_0x), \qquad \beta(x) = \gamma' + [x, b_0] - xb_0x, \qquad x^2 = 0.$$

The purpose is to force desired corestriction and leading-lane behavior inside actual matrix algebras. Compactness can assemble solutions only after the source-to-equation encoding is faithful, the admissibility conditions define a closed fixed template, and every finite collection of equations is jointly soluble. The general H1–H4 assumptions have not been proved to imply those three facts. The binder theorem therefore sharpens the realization problem without closing it.

6 Profiles, twists and the identity of the candidate

6.1 Compact restrictions forget noncompact information

Proposition 6.1 (Unramified twists are invisible on compact subgroups)**.** *Let J be a compact subgroup of $\mathrm{GL}_n(F)$. Then $v_F \circ \det$ vanishes on J. For any $a \in \mathbb{C}^\times$, the unramified character $\nu_a(g) = a^{v_F(\det g)}$ is trivial on J.*

Proof. The continuous image of the compact group J in the discrete group $\mathbb{Z}$ is a finite subgroup. The only finite subgroup of $\mathbb{Z}$ is zero. The character formula now gives the assertion. □

Consequently $\pi|_J$ and $(\pi \otimes \nu_a)|_J$ agree, while the central character on a scalar uniformizer changes by a^n. Choosing $a^n \neq 1$ makes this a visible difference. Compact packet recovery therefore has to retain the noncompact extension, not just its compact restriction.

6.2 Information tests do not replace full pair factors

Block 164 contains a more specific inherited information test. Its rank-nine examples, conditional on the pinned source construction used there, have the same recorded determinant, Frobenius matrix, slope, Swan conductor, projective Weil map and self-pair data. They differ in an actual scalar quotient map and are separated by a stated twist test. In the recorded test the Artin conductors are 33 and 45. The source does not claim equality of H5, and these examples are not counterexamples to local-factor characterization [10].

Twist transport has a positive use as well. Once a compatible base realization is genuinely supplied, its character and inducing data can be transported under the stated twist. That transports an existing witness. It does not supply the missing base witness. The distinction is the same one encountered in the formal existence field.

6.3 Equal numerical profiles do not determine quotient capacity

The later dyadic record gives an exact arithmetic comparison of two host patterns. At rank eight with $(h_1, h_2, h_3) = (1, 3, 5)$, the full defining tower has levels $(60, 18, 3)$, Swan conductor 60, torsion number 1 and self-pair conductor 342. Its symplectic type quotient has dimension 6. A skipped-degree pattern $8 \to 2$ can have the same three numerical invariants with quotient dimension 4 [18].

The formula makes the comparison transparent. Put $L = 12h_3$ and $c = h_1 + 4h_2$. For the stated skipped-degree construction,

$$a_{\mathrm{skip}} = 63 + 4L + 3c = 63 + 48h_3 + 12h_2 + 3h_1 = a_{\mathrm{full}}.$$

Yet the two capacity formulas remain different. The fresh checks in this working edition verify the displayed arithmetic identities. The assertions that the two patterns arise from actual simple strata remain attached to the ordinary arithmetic proof in the retained source.

7 The direct odd-prime construction

Evidence and scope. This section expands the written construction of Block 166. The parameter and character steps are ordinary mathematical arguments relative to the explicit Bushnell–Kutzko inputs recorded in Section 13. The new publication pass checks those cited statements and their principal hypotheses. It does not supply new Lean proofs of simple-character theory or a proof of H5.

7.1 Notation and the construction contract

Let p be odd and put $F = \mathbb{F}_p((t))$. Choose

$$0 < h_1 < \cdots < h_d, \qquad p \nmid h_i, \qquad q_i = (p+1)h_i, \qquad N = p^d.$$

The construction produces a literal matrix $\beta \in M_N(F)$, rather than a formal symbol for an unknown arithmetic parameter. Its generated field is maximal and totally ramified, and it normalizes a complete lattice chain. A subsequent, separate argument constructs a full simple character and an extended inducing representation. Only after those steps is there an actual supercuspidal candidate.

At an intermediate local field L, write $\mathcal{O}_L$ for its ring of integers and t_L for its chosen uniformizer. For a hereditary order $\mathfrak{A}$ with radical $\mathfrak{P}$, put $U^r_{\mathfrak{A}} = 1 + \mathfrak{P}^r$ for $r \geq 1$. The valuation $v_{\mathfrak{A}}(x)$ is the largest integer r with $x \in \mathfrak{P}^r$. On a complete period-N chain, scalar valuations satisfy $v_{\mathfrak{A}}(a) = N v_L(a)$. We use ordinary matrix trace over the displayed coefficient field. No trace is divided by a matrix size.

7.2 A separable degree-p step with a native trace identity

Lemma 7.1 (Artin–Schreier parameter and uniformizer)**.** *Let L have characteristic p and residue field $\mathbb{F}_p$, and let $q > 0$ with $p \nmid q$. The equation*

$$\alpha^p - \alpha = t_L^{-q}$$

defines a separable totally ramified extension E/L of degree p. With normalized valuation on E, one has $v_E(\alpha) = -q$. Choose $1 \le b < p$ and $a \in \mathbb{Z}$ with $qb \equiv -1 \pmod{p}$ and $pa = qb + 1$. Then $z = t_L^a \alpha^b$ is a uniformizer and $E = L(z)$. For odd p, $\mathrm{Tr}_{E/L}(\alpha) = 0$.

Proof. If $x \in L$ has negative valuation, then $v_L(x^p - x) = pv_L(x)$ is divisible by p. If its valuation is nonnegative, so is the valuation of $x^p - x$. Thus the right side is not an Artin–Schreier coboundary. All roots differ by constants in $\mathbb{F}_p$. The splitting field obtained by adjoining one root has Galois group a subgroup of the additive group of $\mathbb{F}_p$, so its degree is either one or p. The absence of a root in L forces degree p.

The derivative is -1, which proves separability. If $e = e(E/L)$, the valuation equation gives $pv_E(\alpha) = -eq$. Since $p \nmid q$, one has $p \mid e$. The extension has degree p, hence $e = p$ and $v_E(\alpha) = -q$. The residue degree is one. Finally

$$v_E(z) = pa - qb = 1, \qquad \mathrm{Tr}_{E/L}(\alpha) = \sum_{c \in \mathbb{F}_p} (\alpha + c) = 0$$

for odd p. An element of valuation one cannot belong to L, whose value group in E is $p\mathbb{Z}$, so $L(z) = E$. □

The minimal polynomial $f(X) = \sum_{k=0}^{p} a_k X^k$ of z, with $a_p = 1$, is Eisenstein. It is explicitly computable from

$$f(X) = \prod_{c \in \mathbb{F}_p} \left(X - t_L^a (\alpha + c)^b\right),$$

reducing by the Artin–Schreier relation. This instruction uses only finite field operations. It does not call an existence oracle for the desired final field.

7.3 The corestriction normalization and its filtered right inverse

In the basis $1, z, \dots, z^{p-1}$ of E/L, let T_{ij} be the coefficient matrix unit. The trace-dual basis is

$$z_j^* = \frac{\sum_{k=j+1}^{p} a_k z^{k-j-1}}{f'(z)}, \qquad 0 \le j < p.$$

The trace corestriction pr satisfies $\mathrm{pr}(T_{ij}) = z^i z_j^*$. Set

$$\kappa = -\frac{f'(z)}{z^{p-1}}, \qquad s = \kappa\,\mathrm{pr}.$$

The leading term of $s(T_{ij})$ is $-z^{i-j}$. Every other term has strictly higher valuation. Indeed $v_E(a_k) \ge p$ for $0 \le k < p$, and the term indexed by $k \ge 1$ has valuation at least k above the leading term. This is the normalization used below, not an arbitrary scalar multiple whose effect on the character is left unspecified.

Let P_0 extract the constant coefficient in each E-coordinate of E^m. The equation $f(z) = 0$ gives

$$z_0^* = -\frac{a_0}{zf'(z)}, \qquad u_0 = s(P_0) = \frac{a_0}{z^p} \in \mathcal{O}_E^\times.$$

For $C \in M_m(E)$ define the L-linear operator

$$\mathrm{Lift}_E(C) = \nu_0^{-1} C P_0. \tag{1}$$

Here the order of composition matters. In a wildly ramified extension, replacing P_0 by an E-linear commuting perturbation would destroy the intended derived contribution.

Proposition 7.2 (Exact corestriction and trace identities)**.** *For the complete interleaved chains on E^m, the operator in* (1) *satisfies*

$$\begin{aligned} s(\mathrm{Lift}_E(C)) &= C, \\ v_{\mathfrak{A}}(\mathrm{Lift}_E(C)) &= v_{\mathfrak{B}}(C) \quad (C \neq 0), \\ \mathrm{tr}_L(\mathrm{Lift}_E(C)) &= \mathrm{Tr}_{E/L}\big(\kappa^{-1} \mathrm{tr}_E(C)\big). \end{aligned}$$

Here $\mathfrak{B}$ is the complete E-order and $\mathfrak{A}$ the corresponding interleaved L-order. In particular $s(\mathfrak{P}^r) = \mathfrak{Q}^r$ at every integer grade.

Proof. The corestriction is an $M_m(E)$-bimodule map. Thus its value on $\nu_0^{-1} C P_0$ is $\nu_0^{-1} C \nu_0 = C$.

For the filtration assertion, take the basis $w_{mi+u} = z^i v_u$, with $0 \le i < p$ and $0 \le u < m$. Multiplication by the unit ν_0^{-1} preserves the E-chain valuation. A term $c_i z^i E_{uv}$ in an entry of $\nu_0^{-1} C$ contributes to a constant-coefficient column of the lift at grade

$$pm\, v_L(c_i) + mi + u - v.$$

This is precisely its E-chain grade. For a fixed entry, the valuations of different powers z^i are distinct modulo p, so a minimum cannot disappear by cancellation. The minimum over all entries proves equality of grades. The leading-term formula for $s(T_{ij})$ gives filtration preservation for s itself, while the explicit lift gives surjectivity on every grade.

Finally the L-trace of the rank-one operator yP_0 is its constant coefficient, namely $\mathrm{Tr}_{E/L}(z_0^* y)$. Apply this entrywise to the diagonal of the lift and use $z_0^* \nu_0^{-1} = \kappa^{-1}$. □

The proposition is not a bimodule splitting of a wildly ramified integral corestriction sequence. The lift is a chosen L-linear coefficient construction. In particular it need not commute with the embedded field. This distinction agrees with the warning following BK 1.4.10 and with BK 2.2.6 [25].

7.4 Recursion, native trace, and literal arithmetic fields

For $d = 1$, let $B(L, t_L; h_1)$ be multiplication by the Artin–Schreier element for q_1. At $d \ge 2$, use the largest q_d for the outer extension E/L, put $m = p^{d-1}$, and set

$$B(L, t_L; h_1, \ldots, h_d) = \alpha I_m + \mathrm{Lift}_E\big(B(E, z; h_1, \ldots, h_{d-1})\big). \tag{2}$$

Proposition 7.3 (Native trace zero)**.** *Every native matrix in* (2) *has ordinary trace zero over its own coefficient field. Every identity amplification, and every difference of two consecutive compatible native approximations, also has trace zero.*

Proof. The base trace is zero by Lemma 7.1. If C is the inner matrix, induction gives $\mathrm{tr}_E(C) = 0$. Proposition 7.2 then gives

$$\mathrm{tr}_L(B) = m\,\mathrm{Tr}_{E/L}(\alpha) + \mathrm{Tr}_{E/L}(\kappa^{-1}\,\mathrm{tr}_E(C)) = 0.$$

Amplification multiplies a trace by its multiplicity. Both traces in each consecutive difference already vanish at their native ranks, so the difference has trace zero as well. □

This proof is stronger than observing that a multiplicity divisible by p kills a trace. The latter observation says nothing about the native matrix before amplification. Native trace is the datum used when the next full character is constructed.

Theorem 7.4 (Literal parameter field, relative to BK refinement)**.** *Assume the standard simple-stratum definitions and BK Remark 2.2.7. The displayed rank-N matrix $\beta = B(F, t; h_1, \ldots, h_d)$ generates a totally ramified maximal field K/F, normalizes the complete chain, and gives a simple stratum $[\mathfrak{A}, n, 0, \beta]$ with*

$$n = p^{d-1}q_d, \qquad k_0(\beta, \mathfrak{A}) = -q_1.$$

The literal approximation fields have degrees $p, p^2, \ldots, p^d$, with defining levels

$$n_j = p^{d-j}q_{d-j+1} \qquad (1 \le j \le d).$$

Proof. The degree-p case is minimal because $p \nmid q_1$ and its residue extension is trivial. At the next step write $\gamma = \alpha I_m$, $b = \mathrm{Lift}_E(C)$ and

$$n = p^{d-1}q_d, \qquad R = p^{d-2}q_{d-1}.$$

The outer minimal parameter has $k_0(\gamma, \mathfrak{A}) = -n$. By induction, $E[C]$ is a maximal field in $M_m(E)$, its derived stratum at level R is pure, and $k_0(C, \mathfrak{B}) = -q_1$. The lift supplies $s(b) = C$ and $v_{\mathfrak{A}}(b) = -R$. Since $q_d > q_{d-1}$,

$$n > pR > 2R, \qquad -2R > -n - q_1.$$

These are the actual inequalities required by BK Remark 2.2.7. Its initial stratum is $[\mathfrak{A}, n, R, \gamma]$, its perturbation is $b \in \mathfrak{P}^{-R}$, and its derived field is the supplied maximal $E[C]$. The remark gives the field generated by the actual sum $\gamma + b$, not merely by a conjugate or by a separate model. Its ramification index is p^d, and its critical exponent is $\max(-n, -q_1) = -q_1$. Its valuation remains $-n$ because $n > R$.

For the approximation statement, truncate the same recursion after j native steps and amplify by $I_{p^{d-j}}$. All truncations use the same largest parameters and the same chosen uniformizers at shared outer steps. The lift commutes with identity amplification. Its exact grade preservation gives the level n_j for the corresponding consecutive correction. The shorter native field has degree p^j, completing the defining sequence. □

The derived field $E[C]$ is an input to the refinement theorem. It is not thereby identified with the final field $F[\gamma + b]$. In particular this proof makes no assertion that the approximation fields are nested as literal subfields of K. Likewise, trace zero alone does not prove separability of an arbitrary final field in characteristic p. The source result used here is the maximal, totally ramified field statement. The outer Artin–Schreier fields are separately known to be separable.

7.5 A compatible full character is more than a deep trace calculation

The extension step is elementary once it is placed at the correct maximal parameter field.

Lemma 7.5 (Extension with a fixed trivial subgroup)**.** *Let A be a compact abelian group, $B \subseteq A$ an open subgroup, and $C \subseteq A$ a closed subgroup. A smooth character $\chi : B \to \mathbb{C}^\times$ has a smooth extension to A that is trivial on C if and only if χ is trivial on $B \cap C$.*

Proof. Necessity follows by restriction. For sufficiency, define $\chi'(bc) = \chi(b)$ on BC. Intersection compatibility makes this well defined. Its kernel contains the open subgroup $\ker \chi$ of A, so it factors through a finite subgroup of $A/\ker \chi$. A character of a subgroup of a finite abelian group extends to the whole group: adjoin a generator whose multiple belongs to the subgroup and choose the required root of its prescribed character value in $\mathbb{C}^\times$, then repeat. The resulting character is smooth because it factors through that finite quotient. □

We apply the lemma with $A = U_K^1$, $B = U_K^{m+1}$ and $C = U_F^1$. It does not authorize an arbitrary extension through a nonmaximal matrix centralizer. The determinant condition and the normalization in BK 3.2.5 still have to hold.

For a minimal native parameter α at level q, put $m = \lfloor q/2 \rfloor$. The deep character is ψ_α on $U_{\mathfrak{A}}^{m+1}$. Its restriction to scalar units is trivial because $\mathrm{tr}_F(\alpha) = 0$. Its field-unit restriction therefore extends to U_E^1 trivially on U_F^1 by Lemma 7.5. The minimal simple-character definition, BK 3.2.1, joins these agreeing characters on

$$H^1(\alpha) = U_E^1 U_{\mathfrak{A}}^{m+1}.$$

At a nonminimal native step, the following order of operations is essential.

Transfer the whole preceding character. Let θ_j be the full native character already constructed at rank p^j, trivial on U_F^1. Amplify its parameter by multiplicity p and transfer θ_j to that actual parameter γ. BK 3.6.13(ii) and 3.6.14 preserve the whole beta-determinant unit character χ_j, not just its restriction to deep units. On a scalar $u \in U_F^1$, the transferred value is

$$\Theta_j(uI) = \chi_j\big(\det_{M_p(K_j)}(uI_p)\big) = \chi_j(u^p) = 1. \tag{3}$$

The determinant exponent p cannot be silently dropped. In the transfer theorem the centralizer orders may have different periods, but the full-character transfer uses $m_1 = m_2 = 0$. Thus both normalized indices $\lfloor m_i/e_i \rfloor$ are zero, exactly as required.

Apply the actual correction at the permitted depth. At the next native matrix $\beta = \gamma + \delta$, let $r = q_{d-j}$ and $m = \lfloor r/2 \rfloor$. The two strata are equivalent at level r, and their groups at depth $m+1$ agree. Since

$$\lfloor r/2 \rfloor \le m \le r - 1, \qquad r < n,$$

BK 3.3.18 gives the character

$$\vartheta = \Theta_j|_{H^{m+1}(\beta)}\, \psi_\delta \quad \text{in} \quad \mathcal{C}(\mathfrak{A}, m, \beta).$$

The sign is fixed by the source bijection $\theta \mapsto \theta\psi_\delta^{-1}$. Proposition 7.3 makes ψ_δ trivial on scalar units in its domain, so ϑ is trivial on its intersection with U_F^1.

Extend at the new maximal native field. Now $K = F[\beta]$ is maximal at this native rank. Its centralizer is K itself, so the relevant determinant is the identity. The restriction of ϑ to U_K^{m+1} has the intersection compatibility required by Lemma 7.5. Choose its extension χ_K to U_K^1, still trivial on U_F^1. BK 3.2.5, used at full index zero, gives

$$\theta(kh) = \chi_K(k)\vartheta(h), \qquad k \in U_K^1, \quad h \in H^{m+1}(\beta). \tag{4}$$

Agreement on the intersection makes the formula well defined. The normalization and determinant hypotheses are supplied by the simple-character theorem. This proves full shallow-unit triviality inductively, not merely the cancellation of one leading coefficient.

7.6 Central completion without changing the canonical extension

Retain the chosen canonical beta extension κ_θ. The maximal residue factor is a character of $k_K^\times = \mathbb{F}_p^\times$. Where that factor is free to be chosen, it adjusts the compact central character to a prescribed smooth ω trivial on U_F^1. An independently fixed incompatible residue character is not replaced.

The group $\mathcal{N} = K^\times J$ has quotient $\mathcal{N}/J \simeq \mathbb{Z}$. Choose an extension Λ_0 of the fixed compact type. Twisting it by the character that takes a chosen generator to $s \in \mathbb{C}^\times$ changes its value on a scalar uniformizer by s^N. Choose s with

$$s^N = \omega(t)/\omega_{\Lambda_0}(t).$$

This does not assume that a uniformizer of K has literal Nth power t. Their quotient is a unit already handled by the fixed compact data. Once chosen, s becomes part of the candidate and is retained in every recovery argument.

BK 6.2.2 applies to the resulting maximal simple type and its actual extension. It yields

$$\pi = \mathfrak{c}\text{-Ind}_{\mathcal{N}}^{\mathrm{GL}_N(F)} \Lambda$$

as an irreducible supercuspidal representation. The current proof uses this classical theorem. It does not obtain supercuspidality from a finite Heisenberg dimension count.

7.7 Capacity and the precise packet admitted by this construction

The profile formulas retained from Block 166 are

$$t(\pi) = 1, \qquad \mathrm{Sw}(\pi) = p^{d-1}(p+1)h_d,$$
$$\dim_{\mathbb{F}_p}(J^1/H^1) = N-1, \qquad \dim \eta = p^{(N-1)/2},$$
$$a(\pi^\vee \times \pi) = N^2 - 1 + \sum_{i=1}^{d}(p^{2i} - p^{2i-2})h_i.$$

The conductor identities retain their separately pinned arithmetic inputs. They are not consequences of the trace lemmas alone.

There is a useful capacity argument that does not need every lower conductor. Put $n = p^{d-1}q_d$ and $M = n/2$. The leading minimal approximation has degree p. The concrete subgroup $U_{\mathfrak{A}}^M$ survives in $J^1(\beta)$ through all lower corrections. The recursive BK definitions keep it there because every lower critical level is less than $2M$. They also identify

$$H^M(\beta) = H^M(\gamma), \qquad S \simeq \mathfrak{P}^M/(\mathfrak{Q}^M + \mathfrak{P}^{M+1}).$$

The complete-chain grades have dimensions N and N/p, respectively. Therefore $\dim_{\mathbb{F}_p} S = (p-1)p^{d-1}$. On this subgroup, $\beta - \gamma \in \mathfrak{P}^{1-n}$ and commutators lie in $1 + \mathfrak{P}^n$, so the two character commutators agree. BK 3.4.1 makes the resulting form nondegenerate.

Now assume a one-vertex packet with trivial stabilizer and d actually supplied, labelled, full character extensions, each with a whole plane quotient. Their balanced central product has quotient dimension $2d$ and Heisenberg carrier dimension $p^d = N$. Since

$$(p-1)p^{d-1} \geq 2d,$$

a labelled symplectic embedding of the source into S exists. Proposition 3.2 attaches the full scalar extension, including its supplied character values. A line in the complementary Heisenberg factor gives the rank-N sector. The retained induction and idempotent argument recovers the same extended type and the same π [14].

For $p = 3$, the retained native cubic factors provide plane quotients. For $p > 3$, a native minimal rank-p quotient normally has dimension $p - 1$, not two. The plane admission hypothesis must be verified. Neither this capacity proof nor scalar-extension equivalence supplies a nontrivial arithmetic stabilizer action, independently fixed native markings, or literal placement at previously specified numerical depths.

For $p = 3$, $d = 2$ and $(h_1, h_2) = (1, 2)$, the rank is nine, the defining levels are $(24, 4)$, the Swan conductor is 24, the quotient dimension is eight, $\dim \eta = 81$, and the self-pair conductor is 232. The preserved sparse Laurent program has been replayed for this matrix. Its exact identities check the implementation, while the field and character conclusions above use the written proof and its classical inputs.

8 The weighted dyadic construction

Evidence and scope. The dyadic construction uses a different trace mechanism, not a formal substitution $p = 2$ in the odd-prime trace-zero proof. This section reconstructs the native trace, depth, and full-character arguments from Block 167. The general parameter and simple-character steps remain relative to the classical inputs listed in Section 13.

8.1 Trace one can be invisible at the required depth

Lemma 8.1 (Depth-weighted trace visibility)**.** *Let ψ_F be an additive character trivial on $\mathfrak{p}_F$. Suppose $u \in U_F^s$, with $s \geq 1$, and a matrix δ satisfies*

$$v_F(\operatorname{tr}_F \delta) + s \geq 1.$$

Then $\psi_F((u-1)\operatorname{tr}_F \delta) = 1$.

Proof. The product has valuation at least $s + v_F(\operatorname{tr}_F \delta)$, so it belongs to $\mathfrak{p}_F$, where the character is trivial. □

For $F = \mathbb{F}_2((t))$, choose the additive character reading the constant coefficient. It is trivial on $t\mathcal{O}_F$ and takes the value -1 on 1. A trace of 1 is therefore invisible when multiplied by $u - 1$ for $u \in U_F^1$. A trace of t^{-1} is not: $u = 1 + t$ gives $(u-1)t^{-1} = 1$.

The Euler work motivated replacing literal zero by a weighted visibility condition. It is not an arithmetic input to this proof. The weights here are valuation depths. Multiplication by the integer two in characteristic two would erase the trace for an entirely different reason and would not prove the native assertion.

8.2 The literal quadratic step and exact trace recursion

Take odd integers $0 < h_1 < \cdots < h_d$, put $q_i = 3h_i$ and $N = 2^d$. At a recursive base L, use

$$\alpha^2 - \alpha = t_L^{-q}, \qquad a = (q+1)/2, \qquad z = t_L^a \alpha.$$

The valuation argument of Lemma 7.1 applies with $p = 2$, but the trace is now one. Direct substitution gives

$$f(z) = z^2 + t_L^a z + t_L = 0, \qquad \mathrm{Tr}_{E/L}(\alpha) = 1. \tag{5}$$

The polynomial is Eisenstein and its derivative t_L^a is nonzero. Its corestriction normalization is

$$\kappa = \frac{t_L^a}{z} = \alpha^{-1}, \qquad s(P_0) = \frac{t_L}{z^2}, \qquad \mathrm{Lift}_E(C) = \frac{z^2}{t_L} C P_0.$$

Consequently

$$s(\mathrm{Lift}_E(C)) = C, \qquad \mathrm{tr}_L(\mathrm{Lift}_E(C)) = \mathrm{Tr}_{E/L}(\alpha\, \mathrm{tr}_E(C)). \tag{6}$$

These are the same coefficient and dual-trace computations as Proposition 7.2, with the signs evaluated in characteristic two.

Proposition 8.2 (Three distinct trace statements)**.** *Every native dyadic matrix in the recursion has trace one over its own coefficient field. A preceding parameter amplified by multiplicity two has trace zero. The consecutive correction at the next native rank has trace one.*

Proof. At rank two the trace is $\mathrm{Tr}_{E/L}(\alpha) = 1$. At greater native rank write $B = \alpha I_m + \mathrm{Lift}_E(C)$, with $m = 2^{d-1}$ even. Inductively $\mathrm{tr}_E(C) = 1$. Thus

$$\mathrm{tr}_L(B) = m\, \mathrm{Tr}_{E/L}(\alpha) + \mathrm{Tr}_{E/L}(\alpha) = 0 + 1 = 1.$$

An identity amplification of multiplicity two has trace $2 \cdot 1 = 0$. Subtracting it from the next native matrix gives correction trace one. □

At full rank eight for $(h_1, h_2, h_3) = (1, 3, 5)$, the distinction is visible without any asymptotic notation:

Native rank	Native trace	Multiplicity in rank eight	Ambient trace	Defining level
2	1	4	0	60
4	1	2	0	18
8	1	1	1	3

The ordinary trace used for a new character is the native one. The table does not identify the three trace operations or infer a character from the zeros in its fourth column.

8.3 Refinement, fields, and a separability consequence

For a step of length $d \geq 2$, the outer level and the derived level are

$$n = 2^{d-1} q_d, \qquad R = 2^{d-2} q_{d-1}.$$

Since $q_d > q_{d-1}$, one has $n > 2R$. Hence $-2R > -n - q_1$, the exact inequality in BK Remark 2.2.7. The pure derived maximal field, filtered lift, and critical exponent are supplied as in the odd case. The

theorem gives the actual maximal totally ramified field generated by the displayed sum. It also gives the literal defining levels

$$n_j = 2^{d-j} q_{d-j+1}, \qquad k_0(\beta, \mathfrak{A}) = -q_1.$$

No equality between the final field and its derived centralizer model is used.

Proposition 8.3 (Trace one forces separability after maximality)**.** *Suppose the rank-N dyadic matrix β generates a field K/F of degree $N = 2^d$. Its native trace-one identity implies that K/F is separable.*

Proof. Since the generated field has degree equal to the matrix size, the minimal polynomial of β equals its characteristic polynomial. The coefficient of X^{N-1} is $-\operatorname{tr}_F(\beta) = 1$. Its derivative contains the nonzero term $(N-1)X^{N-2} = X^{N-2}$ in characteristic two. An irreducible polynomial with nonzero derivative is separable. □

This is an elementary consequence of two already explicit hypotheses. Trace one alone does not prove that the matrix generates a field of full degree. The maximality step still uses the refinement theorem. In particular this observation is not an independent replacement for the simple-stratum construction.

8.4 The floor in the correction depth is essential

Every native critical level $r = q_i$ is odd. Put $m = (r-1)/2$, so $r = 2m+1$. For $x, y \in \mathfrak{P}^{m+1}$ and $\delta \in \mathfrak{P}^{-r}$,

$$\delta x y \in \mathfrak{P}^{2(m+1)-r} = \mathfrak{P}. \tag{7}$$

The level-one matrix additive character is trivial there. Thus the error between evaluating the additive expression at $(1+x)(1+y)$ and multiplying its values at $1+x$ and $1+y$ vanishes. This is the depth calculation behind the correction character on the relevant congruence subgroup.

Replacing $m+1$ by m would give $2m - r = -1$, not a positive depth. Writing "approximately half the level" is therefore inadequate here. The endpoint $m = \lfloor r/2 \rfloor$ lies in the closed interval permitted by BK 3.3.18, and the character is evaluated on H^{m+1}.

8.5 The full shallow-character induction

For a minimal rank-two parameter, the deep character ψ_α is trivial on its scalar intersection with U_F^1, since $\operatorname{tr}_F(\alpha) = 1$ and Lemma 8.1 applies. Extend its field-unit character to U_E^1 trivially on U_F^1 using Lemma 7.5. The minimal BK definition then gives the full character on H^1.

Assume a full native character θ_j, trivial on U_F^1, has already been constructed. Transfer it to the multiplicity-two parameter γ. Preservation of the beta-determinant character gives $\Theta_j(uI) = \chi_j(u^2) = 1$ on scalar units. This uses the entire native unit character. It does not transfer a guessed deep approximation and then assume its shallow values.

At the next actual native parameter $\beta = \gamma + \delta$, Proposition 8.2 gives $\operatorname{tr}_F \delta = 1$. Put $r = 2m+1$ as above. The strata and depth range meet BK 3.3.18, so

$$\vartheta = \Theta_j|_{H^{m+1}(\beta)}\, \psi_\delta$$

is a simple character at that depth. Lemma 8.1 gives $\vartheta = 1$ on $U_F^1 \cap H^{m+1}(\beta)$. The actual final field at this native step is maximal. Its forced unit character on U_K^{m+1} and the trivial character on U_F^1 therefore agree on the intersection. Lemma 7.5 extends them to U_K^1, and BK 3.2.5 yields the full formula (4).

The residue field is $\mathbb{F}_2$, so $k_K^\times$ is trivial and $J = J^1$. The compact type is the Heisenberg representation η, with no nontrivial residue character to choose. A prescribed central character trivial on U_F^1 is unramified here. Its remaining value is fixed by the noncompact extension exactly as in the odd case. No division by two takes place in the local coefficient field.

8.6 What the dyadic parameter theorem does and does not realize

The construction supplies a parameter and full character for the stated arbitrary lengths, with

$$t(\pi) = 1, \qquad \mathrm{Sw}(\pi) = 3 \cdot 2^{d-1} h_d,$$
$$\dim_{\mathbb{F}_2}(J^1/H^1) = N - 2, \qquad \dim \eta = 2^{N/2-1},$$
$$a(\pi^\vee \times \pi) = N^2 - 1 + \sum_{i=1}^{d} 3 \cdot 4^{i-1} h_i.$$

The precise conductor inputs remain identified separately. A full source packet of d planes has quotient dimension $2d$ and carrier dimension 2^d. The original maximal-type quotient has enough capacity only for $d \geq 3$. Thus even a valid parameter and full-character construction does not settle attachment at ranks two and four.

Section 9 retains that genuine original-carrier obstruction. Section 10 changes the actual inducing group and proves recovery of the unchanged candidate. Neither operation weakens the dimension theorem for the original quotient. The rank-eight sparse Laurent construction has been replayed, including native and amplified traces and the grades $(60, 18, 3)$. Those finite identities are checks of the displayed matrix algorithm, not substitutes for the full arithmetic proof.

9 Capacity, markings and the low-rank obstruction

9.1 The two dimensions that must not be conflated

For a finite Heisenberg quotient W of dimension $2r$ over $\mathbb{F}_p$, the irreducible carrier with its fixed nontrivial central character has dimension p^r. Quotient dimension and representation dimension measure different objects. A rank-N Weil source assembled from d planes has quotient dimension $2d$ and carrier dimension $N = p^d$. The type carrier can be substantially larger than N.

In the dyadic full tower, the graded contribution at stage j is

$$(2^{d-j+1} - 2^{d-j})\, \mathbf{1}_{n_j \text{ even}}.$$

The last level is odd and the earlier levels are even. Thus

$$\dim W = \sum_{j=1}^{d-1} 2^{d-j} = 2^d - 2.$$

The source quotient fits in this maximal-type quotient precisely when $2^d - 2 \geq 2d$, or $d \geq 3$.

d	N	Host dim W	Host dim η	Source carrier
1	2	0	1	2
2	4	2	2	4
3	8	6	8	8
4	16	14	128	16

These exact values explain why the original rank-two and rank-four maximal-type carriers cannot contain the whole source packet. The later enlargement does not change these numbers. It changes the arithmetic assembly carrier.

9.2 Transported markings versus a fixed native refinement

For $d \geq 3$, the retained scalar-pushout attachment embeds the source symplectic quotient and carries the entire declared marked finite extension. It preserves the squares of those transported lifts and the corresponding refinement data. If an independent native quadratic form q_W is also prescribed, that is an additional compatibility problem [19].

At $d = 3$, source and host have equal dimension six. A native quadratic embedding must be an isometry, so the Arf invariants must agree. For $d \geq 4$, there is at least one complementary hyperbolic-size plane available at the level of quadratic-space classification. When the embedding can be chosen, a complementary Arf contribution permits either source type. For an already fixed embedding f, however, the actual requirement is pointwise:

$$q_W(f(v)) = q_{\mathrm{src}}(v) \quad \text{for every } v.$$

A dimension count or an equality of total Arf invariants does not establish this identity for that fixed map. These assertions about native embeddings remain distinct from preserving the source marking under a constructed extension map.

9.3 Why profile adjustments did not resolve the capacity issue

The capacity record considers changes of parameters and scalar twists while retaining the prescribed numerical profile. Its low-rank negative conclusions concern the original maximal-simple-character carriers in that profile. They do not show that no larger compact subgroup of the same $\mathrm{GL}_N(F)$ can carry the packet. This distinction makes the next construction possible without contradicting the earlier obstruction.

10 Enlarged arithmetic carriers recover the same representation

Evidence and scope. This section expands the rank-two and rank-four ordinary constructions in Block 167. The groups are actual subgroups of the same general linear group. Their character and filtration inputs are retained explicitly. A general finite-index recovery proposition then keeps the full inducing representation, including its noncompact extension. No new candidate is selected through the reference correspondence.

10.1 Rank two: enlarge the group rather than the original quotient

Let $F = \mathbb{F}_2((t))$, $G = \mathrm{GL}_2(F)$ and $[\mathfrak{A}, q, 0, \beta]$ be the actual minimal dyadic stratum, with $q = 2m + 1 \geq 3$ and $K = F[\beta]$. Its complete period-two order has

$$J = J^1 = H^1 = U_K^1 U_{\mathfrak{A}}^{m+1}.$$

The full simple character θ is one-dimensional on J. Fix its actual extension Λ on $\mathcal{N} = K^\times J$ and the candidate $\pi = \mathfrak{c}\text{-}\mathrm{Ind}_{\mathcal{N}}^{G} \Lambda$. The central character and the value of the chosen noncompact generator are fixed data throughout the argument.

Set

$$P = U_K^1 U_{\mathfrak{A}}^m. \tag{8}$$

The field normalizes the complete order, so this product is a compact open group. The filtration commutator estimates give $[P, J] \subseteq U_{\mathfrak{A}}^{m+1}$, and hence $J \triangleleft P$. On the grade $\mathfrak{P}^m/\mathfrak{P}^{m+1}$, the ambient order contributes dimension two over $\mathbb{F}_2$ and the field line contributes dimension one. Thus

$$[P : J] = 2, \qquad P \cap \mathcal{N} = J.$$

For the intersection, an element $kj \in P$ with $k \in K^\times$ has $k \in U_{\mathfrak{A}}^1$, hence $k \in U_K^1 \subseteq J$.

The maximal minimal-character intertwining formula gives $I_G(\theta) = \mathcal{N}$. Choose $g \in P \setminus J$. It does not intertwine θ, so $\theta^g \neq \theta$. Consequently

$$\lambda(j) = \theta(gjg^{-1})/\theta(j)$$

is nontrivial. Its values lie in $\{1, -1\}$ because its commutators lie in the deep subgroup where θ is the additive character of a characteristic-two matrix trace. The entire shallow character θ need not have order two.

Since $g^2 \in J$ and θ is a character, $\lambda^g = \lambda^{-1} = \lambda$. Thus $H_0 = \ker \lambda$ is normal in P, with $[P : H_0] = 4$. Finite-index induction gives

$$\Xi = \mathrm{Ind}_J^P \theta,$$

an irreducible two-dimensional representation. Its restriction to J is the sum of the two distinct characters θ and θ^g, and g exchanges their lines.

10.2 The rank-two quotient and the retained character

In a two-coset model, the represented action has the form

$$j \longmapsto \theta(j) \begin{pmatrix} 1 & 0 \\ 0 & \lambda(j) \end{pmatrix}, \qquad g \longmapsto X = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}, \qquad a = \theta(g^2) \neq 0.$$

For an element with $\lambda(j) = -1$, the nonscalar part is $Z = \mathrm{diag}(1, -1)$. One has

$$X^2 = aI, \qquad Z^2 = I, \qquad XZX^{-1} = -Z.$$

The scalar preimage is exactly H_0, so $P/H_0 \simeq \mathbb{F}_2^2$ is the actual projective quotient. Its commutator is nondegenerate. This is more information than the dimension of Ξ alone.

The scalar a remains the value of the original character. It need not equal one, and the matrices do not automatically match an independently prescribed native square marking. A source scalar-pushout identification can carry its marked finite extension, but equality with a fixed native quadratic refinement requires the separate compatibility condition from Section 9.

The carrier occurs in $\pi|_P$ in the functions supported on $\mathcal{N}P$. Explicitly, a function f in $\mathrm{Ind}_J^P \theta$ extends there by $f(np) = \Lambda(n)f(p)$. The equality $\mathcal{N} \cap P = J$ proves that this is well defined. The construction uses the same Λ and does not change its noncompact root [21].

10.3 Rank four: the actual final-field approximation

Use the rank-four dyadic constructor with odd $h_1 < h_2$. Put

$$q_i = 3h_i, \qquad n = 2q_2 = 2a, \qquad r = q_1 = 2m+1, \qquad m \geq 1.$$

Then $a > r$ and $n > 2r$. Let γ be the degree-two outer minimal approximation, $E = F[\gamma]$, $B = \mathrm{End}_E(F^4) = M_2(E)$, and $\mathfrak{B} = \mathfrak{A} \cap B$. Its radical is $\mathfrak{Q}$. Write $\beta = \gamma + \delta$ and $c = s(\delta)$. The derived parameter c is minimal of degree two over E and has level r. The actual final field is $K = F[\beta]$ of degree four. We do not assume $E \subseteq K$.

The recursive group definitions give

$$H = H^1(\beta) = U_K^1 U_{\mathfrak{B}}^{m+1} U_{\mathfrak{A}}^{a+1},$$
$$J = J^1(\beta) = U_K^1 U_{\mathfrak{B}}^{m+1} U_{\mathfrak{A}}^{a}.$$

Thus J/H has dimension two and its Heisenberg representation η has dimension two. The existing θ on H is not replaced.

Lemma 10.1 (Approximation by the outer centralizer)**.** *For $j \geq 1$ and $k \in K \cap \mathfrak{P}^j$, the minimal corestriction exact sequence gives*

$$k = b + e, \qquad b \in \mathfrak{Q}^j, \qquad e \in \mathfrak{P}^{j+n-r}. \tag{9}$$

These leading $\mathfrak{B}$-grades form precisely the centralizing line of the leading derived parameter c.

Proof. Since k commutes with β, one has $[\gamma, k] = -[\delta, k] \in \mathfrak{P}^{j-r}$. The exact commutator sequence for the minimal parameter γ, BK 1.4.10, removes this commutator by an error in grade $j+n-r$. The remainder centralizes γ and lies in B, proving (9).

Apply s to $[\beta, k] = 0$. The corestriction kills $[\gamma, e]$, and its bimodule property gives $s([\delta, b]) = [c, b]$. The remaining error has grade at least $j+n-2r$, so

$$[c, b] \in \mathfrak{Q}^{j+n-2r}.$$

Because $n > 2r$, the leading grade of b centralizes that of c. In the complete period-two E-chain, choose the shift R with $R^2 = zI$. The leading parameter is R^{-r}, with r odd. Its centralizer in grade j is the line spanned by R^j. The actual maximal totally ramified field K contributes a one-dimensional grade. Estimate (9) is injective on that grade, so its image is exactly the line. □

This is where a literal arithmetic attachment is needed. A dimension comparison between two abstract graded spaces would not identify the line belonging to the actual final field. The use of the minimal exact sequence is recorded separately in the input audit.

10.4 An actual index-two enlargement and a nontrivial character orbit

Set $P = JU_{\mathfrak{B}}^m$. The field approximation implies

$$[U_{\mathfrak{B}}^m, U_K^1] \subseteq U_{\mathfrak{B}}^{m+1} U_{\mathfrak{A}}^{m+n-r+1} \subseteq H.$$

Together with the ordinary filtration estimates, this proves that P is a compact open group and $J \triangleleft P$. The grade of $U_{\mathfrak{B}}^m$ has dimension two. Its intersection with J has exactly the one-dimensional centralizing

line of Lemma 10.1. The converse containment uses $m + n - r > a$, which puts the approximation error inside $U^a_{\mathfrak{A}} \subseteq J$. Hence $[P : J] = 2$.

Take $D = \operatorname{diag}(1, 0)$ and $u = 1 + R^m D$. Its leading grade is outside that line, so it represents the nontrivial coset. Also $u^2 \in U^{2m}_{\mathfrak{B}} \subseteq H$. The character

$$\lambda(h) = \theta(uhu^{-1})/\theta(h), \qquad h \in H,$$

is J-invariant and has values in $\{1, -1\}$. To see the value rather than guess it, use the defining-sequence formula $\theta = \theta_\gamma \psi_\delta$ on the relevant commutator domain. The outer character is invariant under $U^1_{\mathfrak{B}}$, leaving the actual correction ψ_δ.

Lemma 10.2 (The leading dyadic commutator)**.** *Let* $R = \left(\begin{smallmatrix} 0 & z \\ 1 & 0 \end{smallmatrix}\right)$ *over* $\mathbb{F}_2[z, z^{-1}]$. *For* $m \geq 1$, $X = R^m \operatorname{diag}(A, B)$ *and* $Y = R^{m+1} \operatorname{diag}(C, D)$ *with* $A, B, C, D \in \mathbb{F}_2$, *one has*

$$\operatorname{tr}\big(R^{-2m-1}(XY - YX)\big) = (A + B)(C + D).$$

Proof. If $m = 2k$, factor z^k from both matrices and use $R^2 = zI$. After multiplication by R^{-2m-1}, the trace reduces to $D(A + B) + C(A + B)$. If $m = 2k + 1$, the roles of the two diagonal entries are exchanged and the same expression results. In characteristic two the subtraction is addition. Both cases give the displayed product. □

Choose $X = R^m D$ and $Y = R^{m+1} D$. The two sums in the lemma are one. The group-commutator remainder has $\mathfrak{B}$-grade at least $3m + 1$. Multiplication by the leading parameter of grade $-2m - 1$ leaves grade at least $m \geq 1$, so the remainder is invisible. The difference between c and R^{-r} also contributes only positive grade. The normalized character satisfies

$$\psi_E(1) = \psi_F(\operatorname{Tr}_{E/F} \alpha) = \psi_F(1) = -1.$$

Thus $\lambda(1 + Y) = -1$. In particular $\theta^u \neq \theta$ and $\eta^u \not\simeq \eta$. Finite-index induction gives the irreducible four-dimensional carrier $\Xi = \operatorname{Ind}_J^P \eta$.

10.5 The rank-four scalar preimage and quotient

Put $H_0 = \ker \lambda$ inside H. In the two-coset model, $h \in H$ acts by

$$\operatorname{diag}\big(\theta(h) I_2, \theta^u(h) I_2\big).$$

Since $u^2 \in J$ and λ has order two, $\lambda^u = \lambda$, so $H_0 \triangleleft P$. The common character $\theta_0 = \theta|_{H_0}$ is P-invariant.

The depth estimates give $\lambda = 1$ on $U^{a+1}_{\mathfrak{A}}$ because $a + 1 + m - r = a - m > 0$. Write $J = HU^a_{\mathfrak{A}}$. The character λ kills $[H, H]$ and $[H, U^a_{\mathfrak{A}}]$ by J-invariance. It kills $[U^a_{\mathfrak{A}}, U^a_{\mathfrak{A}}]$ and squares from that subgroup because they lie in $U^{2a}_{\mathfrak{A}}$. It kills squares from H because it has order two. Finally $[u, U^a_{\mathfrak{A}}] \subseteq U^{a+m}_{\mathfrak{A}}$ and $\lambda([u, H]) = 1$, while $\lambda(u^2) = 1$ since u commutes with its own square. Consequently

$$[P, P] \subseteq H_0, \qquad g^2 \in H_0 \ (g \in P), \qquad [P : H_0] = 2 \cdot 4 \cdot 2 = 16.$$

An element of J acting scalarly on Ξ must act scalarly on its first η block. Nondegeneracy of the original J/H forces it into H, and then scalar equality of the two blocks forces it into H_0. An element outside J exchanges the two blocks and cannot act scalarly. Therefore H_0 is the entire scalar preimage.

It follows that $P/H_0 \simeq \mathbb{F}_2^4$ with actual commutator form $(gH_0, kH_0) \mapsto \theta_0([g,k])$. If a class belongs to the radical, its represented operator commutes with the irreducible image and is scalar by Schur's lemma. It therefore belongs to H_0. The form is nondegenerate.

The finite scalar center is $\theta_0(H_0)$, which can have order larger than two. The result is an actual Heisenberg central extension with a four-dimensional binary projective quotient. It is not automatically a literal extraspecial group of order 32, nor an identification with an independently marked native quadratic space [22].

10.6 The final field normalizes the enlarged rank-four group

To obtain an inducing datum rather than only compact occurrence, one must prove $kPk^{-1} = P$ for every $k \in K^\times$. Units of K lie in $U_K^1 \subseteq J$ because the residue field is $\mathbb{F}_2$, so they normalize P. Let k be a uniformizer of K. Lemma 10.1 at grade one gives

$$k = b + e, \qquad b = Rv \quad (v \in 1 + \mathfrak{Q}), \qquad e \in \mathfrak{P}^{n-r+1}.$$

Hence $k = b(1+\varepsilon)$ with $\varepsilon \in \mathfrak{P}^{n-r}$. Conjugation by b preserves the two chain filtrations. For $w \in U_{\mathfrak{B}}^m$, the error from $1+\varepsilon$ lies in

$$U_{\mathfrak{A}}^{n-r+m} \subseteq U_{\mathfrak{A}}^a \subseteq J, \qquad n - r + m = 2a - m - 1 > a.$$

Thus $kU_{\mathfrak{B}}^m k^{-1} \subseteq P$. Also $kJk^{-1} = J$ by normalization of the actual simple-character groups. The subgroup kPk^{-1} contains J with the same index two as P, so it equals P. This proof does not insert the unjustified assumption $E \subseteq K$.

10.7 One finite-index recovery theorem for both low ranks

Proposition 10.3 (Enlargement retaining the full inducing representation)**.** *Let $G = \mathrm{GL}_n(F)$, let K/F be a maximal field in $M_n(F)$, and suppose $J \triangleleft P$ are compact open subgroups with $[P : J] = 2$. Assume $K^\times$ normalizes both groups, $K^\times \cap P \subseteq J$, and*

$$\mathcal{N} = K^\times J, \qquad \pi = \mathfrak{c}\text{-}\mathrm{Ind}_{\mathcal{N}}^G \Lambda, \qquad \Lambda|_J = \eta$$

with η irreducible. If $\eta^u \not\simeq \eta$ for $u \in P \setminus J$, then $M = K^\times P$ is open and compact modulo the center,

$$\mathcal{N} \triangleleft M, \quad [M : \mathcal{N}] = 2, \quad \Sigma = \mathrm{Ind}_{\mathcal{N}}^M \Lambda \text{ is irreducible},$$

and

$$\Sigma|_P \simeq \mathrm{Ind}_J^P \eta, \qquad \mathfrak{c}\text{-}\mathrm{Ind}_M^G \Sigma \simeq \pi. \tag{10}$$

The central character and the noncompact extension in Λ are unchanged.

Proof. Normalization makes M a group. The action of $K^\times$ on P/J is trivial because that quotient has order two. Thus $[K^\times, P] \subseteq J$, proving normality of $\mathcal{N}$ in M. The intersection hypothesis gives index two. The group is open because it contains J, and compact modulo $F^\times$ because both $K^\times/F^\times$ and P are compact.

The restriction of Σ to $\mathcal{N}$ is the sum of Λ and Λ^u. They are irreducible and nonisomorphic, since their restrictions to J already differ. Any $\mathcal{N}$-subrepresentation is a sum of these distinct irreducible summands. The element u exchanges them, leaving no nonzero proper M-stable subrepresentation. This proves irreducibility without applying a finite-group semisimplicity assertion to an arbitrary noncompact group.

There is a single $\mathcal{N}$–P double coset in M and $\mathcal{N} \cap P = J$, giving the first isomorphism in (10). Finite-index induction from $\mathcal{N}$ to M is compact induction. Induction in stages proves the second. No character in Λ has been rescaled. □

For rank two, $K^\times$ normalizes $P = U_K^1 U_{\mathfrak{A}}^m$ directly. For rank four the preceding subsection provides the missing arithmetic normalization. The resulting Σ has dimension two or four, respectively, and induces to the original candidate. This consolidates the rank-four induction note and makes the same finite-index argument explicit for rank two.

The word "same" in this result is literal. It refers to the previously fixed Λ, not merely to an unramified-twist class or to a representation with matching conductor. It supplies the enlarged actual inducing data required by this restricted realization contract. It does not prove that their source packet has a nontrivial prescribed stabilizer action or the full same-candidate H5 factors.

10.8 A conditional induction statement with actual groups

Let $G = \mathrm{GL}_n(F)$, and fix

$$\pi = \mathfrak{c}\text{-}\mathrm{Ind}_{\mathcal{N}}^{G} \Lambda, \qquad \eta = \Lambda|_J,$$

where J is compact open and normal in $\mathcal{N}$, η is irreducible and $\mathcal{N}/J$ is infinite cyclic. Suppose P is an actual compact open subgroup containing J, with $P \cap \mathcal{N} = J$, and put $\Xi = \mathrm{Ind}_J^P \eta$.

Proposition 10.4 (Occurrence and compact recovery)**.** *The subspace of $\pi|_P$ supported on $\mathcal{N}P$ is isomorphic to Ξ. Moreover,*

$$\mathfrak{c}\text{-}\mathrm{Ind}_P^G \Xi \simeq \mathfrak{c}\text{-}\mathrm{Ind}_J^G \eta.$$

If the actual intertwining set of η in G is $\mathcal{N}$, then Ξ is irreducible.

Proof. Restriction to the support $\mathcal{N}P$ identifies its functions with induction from $\mathcal{N} \cap P = J$, using $\Lambda|_J = \eta$. The displayed isomorphism is induction in stages. For irreducibility, the finite Mackey endomorphism formula has no contribution from nontrivial double cosets in P, since every intertwiner lies in $\mathcal{N}$ and $P \cap \mathcal{N} = J$. Its endomorphism space is therefore $\mathrm{End}_J(\eta) = \mathbb{C}$. Finite-dimensional smooth representations of compact P factor through finite groups and are semisimple over $\mathbb{C}$. A semisimple representation with scalar endomorphisms is irreducible. □

The original Λ, its central character and its noncompact root are unchanged. The equality above does not yet recover the chosen π from the compact carrier alone. It recovers the larger compactly induced representation from J.

10.9 The noncompact relation is a quotient before central normalization

Choose $u \in \mathcal{N}$ generating $\mathcal{N}/J$. Use left-covariant functions $f(jg) = \eta(j)f(g)$ with right translation by G, and write $X = \mathfrak{c}\text{-}\mathrm{Ind}_J^G \eta$. Define

$$(Tf)(g) = \Lambda(u)^{-1} f(ug).$$

Normality and the actual intertwining identity imply that T preserves covariance. The normalization remembers the chosen noncompact extension.

Proposition 10.5 (Recovery of the fixed noncompact extension)**.** *There is a natural isomorphism*

$$X/(T-1)X \simeq \pi.$$

Proof. Define

$$(Qf)(g) = \sum_{k\in\mathbb{Z}} \Lambda(u)^{-k} f(u^k g).$$

The sum is finite at each argument because f has compact support modulo J, while the cosets indexed by k are discrete. Reindexing shows $Qf(ug) = \Lambda(u)Qf(g)$, and the J-covariance gives full $\mathcal{N}$-covariance. The support is compact modulo $\mathcal{N}$, so $Qf \in \pi$. Choosing representatives of the finitely many relevant $\mathcal{N}$-cosets proves surjectivity.

On each $\mathcal{N}/J$-orbit, the kernel statement is the augmentation identity for finitely supported sequences $(v_k)_{k\in\mathbb{Z}}$. If $\sum_k v_k = 0$, the partial sums are finitely supported and write (v_k) as consecutive differences. These differences are exactly $(T-1)X$ in the normalized coordinates. Finally $Q(Tf) = Qf$, so the induced quotient map is the required isomorphism. □

There need not be a splitting $\pi \to X$ before fixing the center. In this general-linear setting, determinant valuations of the support of a nonzero vector in X form a finite set. A scalar uniformizer shifts that set by n, so such a vector cannot be an eigenvector for the scalar uniformizer. Every vector of π has the fixed central character. Thus a nonzero G-map $\pi \to X$ is impossible in this setting [20].

10.10 After retaining the center, a finite projector is available

Let $Z = F^\times I_n$, $J_Z = ZJ$, and suppose $\mathcal{N}/J_Z$ is cyclic of order ℓ. Retain $\Lambda_0 = \Lambda|_{J_Z}$. On $Y = \mathfrak{c}\text{-Ind}_{J_Z}^{G} \Lambda_0$, the same normalized shift satisfies $T^\ell = 1$. Hence

$$e_\Lambda = \frac{1}{\ell} \sum_{j=0}^{\ell-1} T^j$$

is an idempotent. Its image is the space of $\mathcal{N}$-covariant functions, which is precisely the fixed π. Division by ℓ takes place in $\mathbb{C}$, not in the residue field. Choosing another noncompact root changes T and can select another extension.

This projector is a finite Hecke/induction projector after central normalization. It is not the older image-algebra projector onto a sector of the original η. The distinction is part of the result, not a cosmetic change of notation. The low-rank enlargement repairs a particular assembly failure while preserving the obstruction for the original carrier.

11 The native stabilizer obstruction and a scalar positive case

11.1 The obstruction belongs to a specified mechanism

For the totally ramified degree-$n = p^r$ host in the stated residue-rank-one setting, the retained native character-preserving normalizer calculation gives

$$\mathcal{N} = K^\times J^1, \qquad \mathcal{N}/(F^\times J^1) \simeq \mathbb{Z}/n\mathbb{Z}.$$

This is an arithmetic input of the source, not a fact about every possible host. Its immediate finite consequence is decisive [23].

Proposition 11.1 (Prime-to-p inner actions vanish in this host)**.** *If a finite group C of order prime to p acts on the host symplectic quotient through the displayed native-normalizer quotient, its action is trivial. If the action on the full scalar extension fixes the scalar center pointwise and lies over that trivial quotient action, that action is also trivial.*

Proof. The image in the cyclic p-group has order dividing both $|C|$ and p^r, so it is trivial. The remaining automorphisms over quotient and center form $\mathrm{Hom}(W, \mathbb{C}^\times)$, a finite p-group. The same coprimality argument makes the remaining image trivial. □

Honest operators on an irreducible Heisenberg carrier are then scalar by Schur's lemma. Matching the source therefore requires the source extension action to be trivial and its scalar character to equal the host character. A general source with a nontrivial prescribed symplectic action cannot be forced into this native mechanism by adjusting a scalar cochain.

The source record gives a nontrivial prime-to-p symplectic action on separate admitted break targets. That is consistent with the original H2 input and inconsistent with the stronger claim that the current totally ramified host can implement every such action by its own inner normalizer. The obstruction refutes the stronger proposed mechanism, not the original finite hypothesis and not Conjecture 66.

11.2 A genuine scalar inertia phase can descend from a cover

Block 166 also proves a restricted positive extension beyond $C = 1$. Assume C is prime to p, acts trivially on the source scalar extension, and its prescribed honest operators on the entire rank-N Weil carrier are $\chi(c)I_N$ in the actual inertia image. Fix the same-source determinant central character under the specified reciprocity convention.

Let the residue cardinality be q and write $B = \omega(\mathrm{Teich}(k^\times))$. The stated actual-inertia and determinant hypotheses give $\chi(C)^N \subseteq B$. Since N is a power of p and the order of $\chi(C)$ is prime to p, taking N th powers permutes $\chi(C)$, so $\chi(C) \subseteq B$. Choose $k_0 N \equiv 1 \pmod{q-1}$ and set

$$\delta(a) = \omega(\mathrm{Teich}(a))^{k_0}, \qquad \widetilde{C} = \{(c, a) \in C \times k^\times : \chi(c) = \delta(a)\}.$$

The projection $\widetilde{C} \to C$ is onto. The central matrix $\mathrm{Teich}(a^{k_0})I_N$ acts by $\chi(c)$. Its kernel acts trivially on the entire candidate, so the represented action descends to C [24].

A direct matrix homomorphism from C itself need not exist. An arbitrary scalar phase on a multiplicity subspace is not the required actual-inertia phase on the whole source carrier. The argument also does not rescale a fixed canonical beta extension to manufacture a desired character. These qualifications are essential to the positive result.

12 What has been gained and what remains

12.1 The restricted positive result is substantial

The recovered ordinary constructions now address arbitrary odd-prime and dyadic lengths in their stated families. They build actual parameter fields, control native traces or trace visibility, extend full characters through the required shallow intersections, and retain central and noncompact completion choices. For the one-vertex trivial-stabilizer scalar-pushout contract, the dyadic low-rank repair allows an actual enlarged assembly carrier where the original maximal-type quotient is too small. These are mathematical results with explicit hypotheses, not merely numerical suggestions.

At the same time, these families do not supply an actual witness for every d_ρ admitted by the general conjecture. The same-candidate H5 comparison also remains a separate conditional step. The new paper should make both facts easy to see without reducing all progress to either "finished" or "unfinished."

12.2 The remaining obligations are different kinds of problems

Admission and canonical source data. Broader source classes need intrinsic envelopes, the declared equivariant bridges, and the requisite odd or dyadic phase and marking data. Conjecture 66, as stated, starts after H1–H4 have been supplied. Extending the supply of those inputs is related work, but it is not the same quantified task.

A common arithmetic host. Separate breakwise targets do not yet give one actual inducing system of the total rank. The direct families establish some hosts. Their existence does not prove that the host can realize every admitted source action, transport graph or numerical layer placement.

Nontrivial stabilizer actions. The native inner-normalizer obstruction forces a change of host or a different arithmetic action mechanism for genuinely nontrivial prime-to-p symplectic actions. Abstract cohomological lifting has already removed some finite scalar defects. Repeating that lifting does not solve the missing arithmetic attachment.

Independent native markings and fixed embeddings. Transporting a marked source extension is not the same as matching a separately specified native quadratic refinement. A proposed match must respect the actual fixed embedding, rather than merely an abstract isometry class.

General transport diagrams. A one-vertex packet has no nontrivial global transport constraint. Finite holonomy theorems compare supplied source and target systems. A realization theorem for arbitrary admitted diagrams must construct an arithmetic system and show that its maps recover those particular transports.

Same-candidate factors. H5 remains about the actual candidate that realizes the packet. Conductor, torsion and self-pair identities are useful tests, but they do not supply the full characterizing family. A program seeking an independent factor comparison must construct that proof rather than invoke the reference correspondence through an implicit definition.

12.3 The next publication blocks

This edition has expanded the parameter recursion, normalized corestriction, native trace argument, ordered full-character construction, and both low-rank enlarged carriers. The actual finite-index inducing data are retained alongside the compact quotient and central-projector explanations. Section 13 gives the principal classical input audit.

The next content block should reconcile the earlier monomial, Clifford-descent, and intermediate binder branches with this later construction. It should retain both failed mechanisms and their surviving positive results, distinguishing source-defined arithmetic induction from the direct constructor. The independent conductor and marking audits remain explicit ledger tasks rather than implied blanket approvals.

After that reconciliation, the main monograph, visual guide and Lean guide can be updated together. The progress paper also needs its own worked visual route. Final proofreading and release verification belong after those steps. This is a second working checkpoint, not the final Paper XI sign-off.

13 Classical inputs and the proof boundary

13.1 What was read directly

The primary reference in this block is Bushnell and Kutzko, *The Admissible Dual of GL(N) via Compact Open Subgroups* [25]. The retained book and its extracted text were available, and the cited passages were checked against page images. The PDF page number is twelve greater than the printed page number for the passages listed below. This locator offset is recorded rather than guessed from a search hit.

The audit establishes which published statement is used and how the current construction meets its stated interface. It does not turn a citation into a Lean proof or reprove the entire classical theory. Nor does it certify unrelated source-dependent conductor formulas simply because they occur in the same archive.

Input	Printed pages	Application and restriction
BK 1.4.10	38	Minimal commutator exactness used for the actual rank-four field approximation. It is not an integral bimodule splitting in wild ramification.
BK 2.2.6–7	55	Noncommuting refinement. A pure maximal derived field and the strict inequality $-2R > k_0(\gamma) + k_0(C)$ are supplied. The conclusion concerns the literal final sum.
BK 3.1.7–10	90–94	Recursive H and J groups, with distinct floor and ceiling thresholds. These identify the actual congruence subgroups and surviving high layer.
BK 3.2.1–5	98–100	Minimal and nonminimal character definitions, then extension from depth $\lfloor r/2 \rfloor + 1$. The determinant and normalization conditions are retained.
BK 3.3.2	105	Full-character intertwining. At the minimal rank-two full level it reduces to $K^{\times} J$, so the new coset has a genuine nontrivial character orbit.
BK 3.3.18	112–113	Correction at index $m = \lfloor r/2 \rfloor$ with $r < n$. The map is $\theta \mapsto \theta\psi_{\delta}^{-1}$, so the new character is $\Theta\psi_{\delta}$.
BK 3.4.1	114–116	Nondegenerate alternating form for the actual simple-character quotient. It does not assign a native quadratic refinement.
BK 3.6.13(ii), 3.6.14	136–137	Full-character transfer preserves the beta-determinant character. Multiplicity p gives $\chi(u^p)$. Both normalized full indices are zero.
BK 6.2.2–3	203–204	An actual extended maximal simple type induces to the supercuspidal candidate. The compact type alone fixes only an unramified-twist class.
BK 6.2.5	204–205	Torsion number and principal-order normalization. The period must be attached to the same actual type, not to an abstract lattice with the same index.

13.2 The exact refinement substitution

The refinement step is worth displaying as a substitution table. It is the arithmetic bridge between the explicit matrix recursion and a maximal field.

Published requirement	Value in the recursive construction
Initial simple stratum	$[\mathfrak{A}, n, R, \gamma]$
Perturbation	$b = \mathrm{Lift}_E(C) \in \mathfrak{P}^{-R}$
Derived parameter	$s(b) = C$
Pure maximal derived field	$E[C] \subseteq M_{p^{d-1}}(E)$ by induction
Outer critical exponent	$k_0(\gamma, \mathfrak{A}) = -n$
Inner critical exponent	$k_0(C, \mathfrak{B}) = -q_1$
Strict inequality	$-2R > -n - q_1$
Literal output field	$K = F[\gamma + b]$

For odd p, $n > pR > 2R$. For $p = 2$, $n > 2R$. Both imply the required strict inequality. Maximality of the derived field is a genuine induction hypothesis. It is not implied merely by counting its matrix entries.

13.3 Inputs that remain separately audited or deferred

The conductor formulas in Sections 6, 7 and 8 retain their previously pinned arithmetic sources. This block rechecks displayed numerical specializations where indicated, but does not claim a fresh end-to-end audit of every conductor theorem. The source-defined Bushnell–Henniart arithmetic induction/descent branch is also separate from the direct matrix constructor. It must not be inserted as a hidden existence step in the direct proof.

The marked source attachment remains a scalar-pushout contract. Transporting the whole source marking does not identify it with an independently fixed host marking. The full finite-image center in low rank can exceed order two. Finally the native stabilizer obstruction and the same-candidate factor requirement are unchanged. These boundaries remain visible even though the parameter, character, and enlarged-induction arguments are now developed at greater length.

13.4 A ledger instead of an ambiguous completion percentage

The companion files `PROGRESS_LEDGER.md` and `PROGRESS_LEDGER.json` give stable task identifiers, evidence classes, source paths, checks, unresolved inputs and restart instructions. Publication blocks are numbered separately from historical research Blocks 166 and 167. A statement can be fully written and internally reviewed while its classical inputs remain external, or while its Lean compilation has not been replayed. The ledger records those different states rather than collapsing them into one completion score.

A Source and evidence map

The references below use local research identifiers so that each statement can be traced without changing historical block names. The inherited map is retained at `evidence/CLAIM_SOURCE_MAP.json`. New source pins and audit records are in `evidence/block02/SOURCE_INPUT_AUDIT.json`. Full retained source material

is in the inherited directory of the working archive. Titles containing “completion” remain historical filenames, not new claims of unrestricted completion.

Chapter	Principal source layer	Status in this working edition
1	Main paper, section 16; contract 162	Current conjecture and conditional comparison retained.
2	Final Lean guide; continuations 110–113	Historical formal results summarized, no fresh compiler run.
3–4	Extension diagnosis 163	Elementary proofs reconstructed with explicit finite and coprime hypotheses.
5	Compact binder theorem 162	Compactness proof reconstructed, arithmetic finite solvability not supplied.
6	Witness 164; capacity 167	Information tests distinguished from H5 and universal realization.
7	Proof 166; trace, character and capacity components	Parameter and full-character proofs expanded, with direct primary-input checks.
8	Proof 167; tower and full-character components	Native trace-one and depth proofs expanded, including the separability consequence.
9	Capacity167; markings 167	Whole-quotient obstruction and transported/native marking distinction.
10	Rank2/rank4 enlargement; recovery contract 167	Both actual low-rank groups expanded, with common finite-index inducing recovery.
11	Stabilizer contract 166; scalar attachment 166	Host-specific obstruction and restricted scalar-cover construction.
12	All preceding boundaries	Research and publication obligations, not newly proved endpoints.
13	Primary BK text and progress ledger	Exact source interfaces and remaining audit scope.

B Fresh finite checks and their limits

Four bounded gates were replayed in this block. The inherited Block 01 suite still passes its twelve test methods. A new standard-library suite passes eleven methods covering uniformizer identities, native and amplified traces, correction-depth endpoints, character extension, determinant transfer, low-rank commutators, filtration inequalities, and noncompact recovery. The original exact rank-nine odd and rank-eight dyadic matrix scripts also pass unchanged in isolated working directories.

The new suite includes 256 binary leading-commutator cases, 153 low-rank normalization parameter pairs, 2,211 finite cyclic character-extension cases, and 972 finite Laurent-kernel cases. These are distinct finite model families, not a count of universal theorems. Other bounds and negative controls are listed separately in the JSON receipts. No aggregate assertion count is used as a measure of progress toward Conjecture 66.

The two matrix scripts use sparse Laurent arithmetic in positive characteristic. They check actual re-

cursive matrix identities and grades, not merely floating-point approximations. They do not prove field maximality, simple-character existence, source attachment, the H5 factor comparison, or a universal realization statement. A proof reconstruction in prose is likewise not a Lean compilation.

C Publication provenance and incomplete coverage

The immediate predecessor is `Paper_XI_Working_Block01_2026-10-10.zip`, pinned by its SHA-256 in the progress ledger. Its manifest was freshly rechecked in this block: all 8,600 listed files matched and no extra files were present. The older source-tree census remains a historical record. Scientific file counts are not theorem counts.

The inherited core PDFs remain the 109-page monograph, 51-page visual guide and 234-page Lean guide. They are preserved as the baseline, not newly signed-off readers. The prior 21-page progress paper and its editable source remain available beside this revision so that changed exposition can be traced. This block does not reinterpret earlier publication checks as fresh Lean acceptance or external review.

The manuscript now has a detailed arithmetic proof route, but earlier monomial and Clifford-descent documents, intermediate binder work, independent conductor-source audits, and the dedicated visual companion are still separate tasks. The ledger marks their coverage explicitly. A checkpoint is useful only if the next block can resume from its actual boundary without repeating completed work or silently discarding a failed route.

References

[1] Komorebi P. *Paper XI, Traditional Paper*. Standardized reading edition, 8 October 2026. Sections 16 and 19 and construction appendices.

[2] Komorebi P. *Paper XI, Lean and Evidence Guide*. Standardized reading edition, 8 October 2026. Chapters 1,13–14 and source catalog.

[3] *Original Contract162*. Retained source in Closure Attempt162, also preserved among Block 167 dependencies.

[4] *Continuation 110: Arithmetic Bridge*. Retained continuation and its handoff, source contracts and evidence.

[5] *Continuation 111: Canonical Attachment*. Retained README, source contract and conditional proof audit.

[6] *Continuation112: Realization Test*. Retained complex-cochain and whole-carrier obstruction records.

[7] *Continuation113: Arithmetic Descent*. Retained cubic occurrence and corrected-character test records.

[8] *Finite Extension Test*. Extension Diagnosis163. Orthogonal central products, scalar extensions and coprime actions.

[9] *Compact Binder Theorem*. Closure Attempt162, compact branch. Fixed templates, joint finite solvability and exact limits.

[10] *Two Input Witness164*. Status, wild and unramified pair records, and twist-transport notes. Conditional source dependencies retained.

[11] *Proof166*. Direct trace-zero realization in arbitrary odd-prime length. Ordinary proof synthesis, 5 October 2026.

[12] *Trace Zero Artin–Schreier Tower166* and *Recursive Tower166*. Parameter, corestriction and defining-sequence arguments.

[13] *Trace Zero Character Induction166*. Full native characters, transfer, correction and shallow-unit extension.

[14] *Leading Layer Capacity166*. Host subspace and scalar-pushout attachment. Its separate classical induction application is not an input to the direct constructor.

[15] *Proof167*. Weighted dyadic realization and low-rank enlargement. Final retained synthesis.

[16] *Dyadic Artin–Schreier Tower167*. Recursive matrices, native trace one and defining levels.

[17] *Dyadic Full Character167*. Depth-weighted compatibility and full-character extension.

[18] *Dyadic Capacity167*. Graded quotient counts, low-rank obstruction and skipped-degree profile comparison.

[19] *Dyadic Markings167*. Whole finite extensions, transported lifts and independent native quadratic refinements.

[20] *Enlarged Carrier Contract167*. Actual compact carriers, quotient recovery and central-normalized finite projectors.

[21] *Rank2 Enlarged Carrier167*. Actual rank-two compact group and Heisenberg enlargement.

[22] *Rank4 Enlarged Carrier167* and *Rank4 Induction by Stages167*. Actual rank-four carrier, normalization and same-candidate induction.

[23] *Stabilizer Contract166*. Original source hypotheses and the stronger native inner-normalizer obstruction.

[24] *Scalar Native Attachment166*. Genuine whole-carrier inertia phases and finite-cover descent.

[25] Colin J. Bushnell and Philip C. Kutzko. *The Admissible Dual of GL(N) via Compact Open Subgroups*. Annals of Mathematics Studies 129. Princeton University Press, 1993. Exact theorem and page locators are given in Section 13.

PAPER XI / RESEARCH COMPANION

Monomial Routes, Clifford Descent, and Exact Obstructions

Toward Arithmetic Packet Realization
Part II of the Conjecture 66 progress paper

Publication Block 03

Komorebi P.

Working continuation v0.3.0 / 10 October 2026

Reading this continuation

The first part of this research companion records the direct odd-prime and weighted dyadic constructions, including the low-rank carrier repairs. This continuation returns to the earlier monomial and Clifford branches. Their conclusions are not interchangeable. A genuine arithmetic candidate may have the right dimension and conductor while failing to contain a prescribed full simple character. A finite carrier may have enough dimension while failing the required action or marking. A compatible formal interface may still await its actual arithmetic input.

The purpose here is to make those distinctions mathematically usable. We extract the precise restricted implications, explain the negative results, and retain the surviving routes. The presentation includes elementary proofs where the argument is short enough to expose completely. Deeper arithmetic statements are identified as inherited ordinary arguments with named classical inputs. No theorem in this continuation is newly certified by Lean.

Publication continuity. The 33-page Block 02 reader and its ledger were recovered. Its complete editable source archive was not. Consequently, the cumulative v0.3.0 reader preserves those 33 pages and appends this independently editable continuation. It is not a claim that the unavailable Block 02 source has been rebuilt. The complete Block 01 source archive supplies the research records examined here. The ledger and source map document this recovery boundary.

Mathematical continuity. Conjecture 66 remains the general arithmetic realization obligation, with the same-candidate H5 comparison kept separate. The positive constructions below are restricted statements. The negative examples refute specified shortcuts or prescribed candidates, not the entire conjecture.

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14 The monomial branch and its actual scope

14.1 Three statements that must remain separate

The recovered monomial materials contain three different layers. The oldest manuscript develops a classical-input route from rank-one characters through automorphic induction. The finite formal core describes packet and group-theoretic interfaces. The later Block 45 module connects a degree-induced representation to actual residue and Frobenius data. None of these layers, merely by existing, constructs every arithmetic witness demanded by Conjecture 66.

A useful way to read the branch is to keep its input on the left:

rank-one character and admissible induction tower $\quad\Longrightarrow\quad$ the corresponding induced candidate.

This is not the implication from an arbitrary packet diagram to an admissible tower. The tower and its compatibility are substantial data. Nor does an equality between finite conductor profiles identify the full local factors of a separately specified candidate.

14.2 Actual Weil pullback and inertia invariants

The Block 45 source uses an actual residue-action pullback and a degree homomorphism

$$\delta : W \longrightarrow \mathbf{Z}.$$

The subgroup for an induction of degree n is $\delta^{-1}(n\mathbf{Z})$. A finite cyclic inertia map is explicitly required to have image $\ker \delta$. This equality is a hypothesis to verify for the actual pullback, not an arbitrary identification of an abstract cyclic group with inertia.

Lemma 14.1 (Fixed vectors depend on the actual image)**.** *Let $\rho : G \to \mathrm{GL}(V)$ and let $\iota : H \to G$ be a homomorphism with image I. Then*

$$V^{\rho\circ\iota(H)} = V^{\rho(I)}.$$

Proof. A vector fixed by every $\rho(\iota(h))$ is fixed by every element of $\rho(I)$ because every element of I has a preimage. The reverse inclusion follows immediately from $\iota(H) \subseteq I$. □

The hypothesis that the first positive ramification group acts trivially places the representation in the tame sector. In the degree-induced examples, triviality of the inertia character gives an n-dimensional fixed space. A nontrivial inertia character, together with its Frobenius conjugates, gives no inertia-fixed coordinate. Thus the Artin conductor is respectively 0 or n, while the Swan conductor is zero. The local polynomial is formed on V^I, not automatically on the entire induced vector space.

This is meaningful arithmetic comparison at a finite interface. It is not a formal definition of an automorphic representation and does not import every analytic identity attached to one. The finite Gauss phase also does not become a fully normalized local epsilon factor without the relevant additive character and measure conventions.

Source and status. [M45] supplies the production module and binder conditions. The fixed-image lemma is proved here. The full module's terminal compilation receipt was not recovered. Earlier checked finite-tame comparison is not thereby invalidated.

14.3 Admissible towers rather than a blanket solvability claim

Let Ω be a finite group and $\Lambda \leq \Omega$. For the induction route in the older manuscript, the required condition is a chain

$$\Lambda = \Lambda_r \triangleleft \Lambda_{r-1} \triangleleft \cdots \triangleleft \Lambda_0 = \Omega, \qquad \Lambda_{i-1}/\Lambda_i \text{ cyclic of prime order.} \tag{14.1}$$

For a finite Galois extension with group Ω, the reversed fixed-field chain has cyclic prime-degree steps. The rank-one character can then be processed through the stated cyclic automorphic-induction inputs. Induction in stages identifies the resulting Weil parameter with the original induced character. Cuspidality requires the appropriate regularity and stabilizer conditions. Without them, an isobaric conclusion must not be relabeled as a cuspidal one.

Proposition 14.2 (Subnormality is the subgroup requirement)**.** *For a finite solvable group Ω, a subgroup Λ admits a chain of the form* (14.1) *if and only if it is subnormal in Ω.*

Proof. One direction is the definition of subnormality. Conversely, start with a subnormal chain. Each quotient of two successive terms is finite and solvable. A composition series of that quotient has cyclic prime-order factors. Pulling it back refines the original step into the required chain. Repeat for all steps. □

The statement does not say that every subgroup of a solvable group is subnormal. In S_3, a subgroup generated by a transposition has order two. There is no proper intermediate subgroup between it and S_3, and it is not normal in S_3. It cannot serve as the bottom of (14.1). Yet S_3 is solvable through its normal subgroup of order three. The exact finite suite includes this countercontrol.

The recovered manuscript has valid additional sectors, including nilpotent groups and suitable abelian-by-cyclic constructions. Their usefulness comes from producing the needed subgroup chain, not from deleting that requirement. Similarly, a virtual character identity with negative coefficients is not itself a construction of an actual automorphic representation. Dividing products of L-functions does not, by that operation alone, prove the required holomorphy.

14.4 What cyclic Clifford extension really supplies

Proposition 14.3 (Extension across a cyclic quotient)**.** *Let $H \triangleleft G$ be finite, with G/H cyclic of order d. Let τ be an irreducible complex representation of H whose isomorphism class is G-invariant. Then τ extends to G. Any two extensions differ by a character of G/H.*

Proof. Choose $t \in G$ lifting a generator. Invariance supplies an invertible operator A with

$$A\tau(h)A^{-1} = \tau(tht^{-1}).$$

The operator $A^d\tau(t^d)^{-1}$ commutes with $\tau(H)$ and is scalar by irreducibility. The same conjugation identity shows that A commutes with $\tau(t^d)$. Rescale A by a complex dth root so that $A^d = \tau(t^d)$. Sending t to this rescaled operator and retaining τ on H respects the defining conjugation and power relations, hence defines an extension.

After identifying the two restrictions with τ, the ratio between two possible operators at t commutes with $\tau(H)$ and is scalar. Its dth power is one. These scalars are precisely the characters of the cyclic quotient. □

This finite representation argument removes a cyclic projective extension obstruction. It does not prove automorphic descent for an arbitrary local-field construction. The latter needs an actual descent theorem and its hypotheses. This distinction explains why the monomial branch supplies a strong restricted route while leaving the general assembly problem intact.

Source and status. [M] contains the admissible tower and classical-input completion. [BH99, §2.3] supplies a characteristic-zero monomial route through cyclic prime-degree steps. The finite group proofs above are ordinary proofs, independent of any unverified compilation claim.

14.5 The Block 45 receipt boundary

Two identical copies of the terminal source were recovered. Its surrounding notes describe the new assertions as intended results and retain a pending verification gate. The preserved background log stops after building `PaperXI.FiniteTameComparison`, at item 3890 of 3891. The other log stops earlier. Neither recovered log contains a successful terminal build of `ActualMonomialLocalGlobal`.

Therefore the publication status is *production source recovered, terminal verification receipt not recovered*. This is not a proof that compilation fails. It is also not permission to label the target freshly or historically verified without its missing receipt. The ledger tracks that documentary obligation separately from the mathematics.

15 Genuine Clifford multiplicities and dimension

15.1 A projective multiplicity is not a decorative label

Clifford theory produces projective representations on multiplicity spaces. The carrier, cocycle, and actions must all come from the same actual restriction problem. Attaching several abstract Heisenberg packets to a diagram does not ensure that their multiplicities multiply inside a prescribed source representation.

Here is the dimension mechanism in a form that exposes its hypotheses. Let A be a finite abelian group and let $c : A \times A \to \mathbf{C}^\times$ be a unit-modulus multiplier. Write

$$T_a T_b = c(a,b) T_{a+b}, \qquad b_c(a,b) = \frac{c(a,b)}{c(b,a)}, \qquad R = \{a : b_c(a,b) = 1 \text{ for every } b \in A\}.$$

The commutator pairing b_c is alternating. The group R is its radical.

Proposition 15.1 (Degree of an irreducible projective multiplicity)**.** *For an irreducible projective representation with multiplier c on M,*

$$(\dim M)^2 = |A/R|. \tag{15.1}$$

Every finite-dimensional module with this same multiplier has dimension divisible by $\sqrt{|A/R|}$.

Proof. Average an inner product over the operators T_a. Unit modulus of the multiplier makes the averaged inner product invariant, so the operators may be taken unitary. If $r \in R$, then T_r commutes with all T_a and is scalar. Thus $|\operatorname{Tr} T_r| = \dim M$. If $a \notin R$, some T_b conjugates T_a to a nontrivial scalar multiple of itself, so $\operatorname{Tr} T_a = 0$.

The operators $\operatorname{Ad}(T_a)$ form an ordinary representation on $\operatorname{End}(M)$. Their average is the projection onto the commutant, which has dimension one by irreducibility. Its trace is

$$1 = \frac{1}{|A|} \sum_{a \in A} |\operatorname{Tr} T_a|^2 = \frac{|R|}{|A|} (\dim M)^2.$$

This proves (15.1). Finally, averaging a projection gives invariant complements, so modules with the fixed unitary multiplier are semisimple. Every irreducible summand has the degree already computed, proving divisibility. □

No assertion that the original cocycle literally descends to A/R is needed. Choosing a section can produce a quotient multiplier after incorporating the scalar radical action, but that is additional bookkeeping, not a reason to suppress the radical.

If A is elementary abelian of order a power of p and the nondegenerate quotient has dimension $2r$ over $\mathbf{F}_p$, the multiplicity degree is p^r. Independent commuting projective factors on the *same* carrier produce the tensor product degree $p^{r_1+\cdots+r_s}$. Both independence and commutation matter. A single Weyl plane represented twice on a p-dimensional space still has noncommuting cross-operators. It does not force divisibility by p^2.

15.2 The orbit factors in a normal chain

Proposition 15.2 (A compatible Clifford chain)**.** *Suppose*

$$G_0 \lhd G_1 \lhd \cdots \lhd G_s$$

is a chain of finite groups. Choose irreducible ρ_i of G_i such that ρ_{i-1} occurs in $\rho_i|_{G_{i-1}}$. Let I_i be the inertia group of ρ_{i-1} in G_i and $t_i = [G_i : I_i]$. For a fixed prime p, if G_i/G_{i-1} is elementary abelian of p-power order and the actual projective multiplicity pairing has rank $2r_i$, then

$$\dim \rho_s = \dim \rho_0 \prod_{i=1}^{s} t_i p^{r_i}. \tag{15.2}$$

Proof. The restriction at the ith step is a sum of t_i distinct conjugate irreducibles, each of dimension $\dim \rho_{i-1}$ and common multiplicity e_i. The corresponding isotypic component is irreducible as a module for I_i. Factoring out the G_{i-1} representation leaves the actual irreducible projective multiplicity representation of I_i/G_{i-1}. Its commutator rank is $2r_i$, so Proposition 15.1 gives $e_i = p^{r_i}$. Hence $\dim \rho_i = t_i p^{r_i} \dim \rho_{i-1}$. Multiplication over the chain proves the formula. □

For example, the two-dimensional irreducible of S_3 restricts to the two nontrivial characters of C_3. The multiplicity is one, and the orbit factor is two. Dropping the orbit factor would give the wrong dimension. A valid source bound therefore uses a genuine compatible chain, not merely a list of candidate projective factors.

A consequence is $p^{\sum r_i} \leq \dim \rho_s$. This can make a proposed target dimensionally plausible. The fixed-rank constructor recorded in [C161] has, under its selected even-depth hypotheses, a host half-dimension $fn(n-1)/2$ over $\mathbf{F}_p$, where the residue field has order p^f. The source bound then provides enough room for the relevant abstract symplectic dimensions. That conclusion remains dimensional. It does not identify a native subgroup, prescribe its quadratic marking, make the action equivariant, or prove full-character occurrence.

Source and status. [C161] gives the genuine-chain constraint and the constructor comparison. Propositions 15.1 and 15.2 are expanded ordinary arguments. The executed finite models check small Weyl carriers, radicals, repeated-factor failures, and the S_3 orbit example, not arbitrary arithmetic host realization.

16 Genuine source fixtures and what they rule out

16.1 Two packets inside a single-slope source

The continuation112 fixtures were designed to avoid an artificial diagram whose purported packets did not come from one irreducible source. In the first construction, over $\mathbf{F}_5((t))$, an unramified degree-four extension supports two independent Heisenberg extensions. Their Schrödinger representations have dimensions five and five. The tensor product is irreducible of dimension 25.

For one factor the finite group law is

$$(u, v, w)(u', v', w') = (u + u', v + v', w + w' + uv'),$$

and its action on functions on $\mathbf{F}_p$ is

$$(T_{u,v,w} f)(s) = \zeta_p^{w+vs} f(s + u). \tag{16.1}$$

Direct composition verifies the group law. In the tensor product, the kernel on the product of the two centers is the anti-diagonal

$$\{(z(w), z(-w)) : w \in \mathbf{F}_5\}.$$

The quotient is the balanced central product, not the direct product with two unrelated central characters. The common central character remains the prescribed character.

The two arithmetic factors have ramification breaks $\{2, 4\}$ and $\{3, 6\}$. Their disjointness supplies the product filtration. After quotienting by the balanced central kernel, the ambient jumps are $2, 3, 6$. The ordinary representation has only slope six, while the actual chain still contains two multiplicity-five packets. Thus a sole ordinary slope does not force a single projective multiplicity packet.

The unramified Frobenius action has order four. It must not be relabeled as a nontrivial tame inertia group. The source fixture has trivial tame stabilizer in this sense, even though Frobenius acts nontrivially on the finite operators.

16.2 The symmetric characteristic-three fixture

The refined example works over $F = \mathbf{F}_3((t))$, with unramified degree-four field $K = \mathbf{F}_{81}((t))$. Choose a with $a^9 = -a \neq 0$ and put $b = a^3$. For $m = 1, 5$, write $f = at^{-m}$ and $g = bt^{-m}$. The torsor equations are

$$\wp x = f, \qquad \wp y = g, \qquad \wp z = \frac{fy - gx}{2}, \qquad \wp u = u^3 - u. \tag{16.2}$$

With $w = z + xy/2$, the final equation becomes

$$\wp w = fy + \frac{fg}{2}.$$

For the ramification formula in [Imai, Example 3.8(1)], the differential term is

$$d(fg/2) - f\,dg = \frac{g\,df - f\,dg}{2} = 0.$$

The pole orders m and $2m$ are prime to three. The leading coefficients a, b are linearly independent over $\mathbf{F}_3$: otherwise $a^2 = b/a$ would lie in $\mathbf{F}_3$, contradicting $a^8 = -1$. The total ramification condition is supplied by the torsor argument in [C112]. Under these hypotheses, the two remaining terms give

central break $4m/3$, not $2m$. The latter would correspond to the different unsymmetrized data. The factor break sets are therefore

$$\{1, 4/3\}, \qquad \{5, 20/3\}.$$

The balanced quotient has ambient jumps $1, 5, 20/3$. Its nine-dimensional irreducible source has a sole slope $20/3$, hence Swan conductor 60, and two genuine multiplicity-three packets. Native subgroup labels in a different upper-numbering convention cannot be substituted for these ambient labels.

The ordinary arithmetic analysis of [C112] then tests whether suitable per-break simple-character targets can all be replaced by one raw maximal-type quotient of the same rank. Its rank-nine spectrum excludes the exponent needed for a degree-nine Heisenberg representation in that specific quotient. The result is a failure of that proposed one-quotient shortcut. It does not show that the source satisfies every strengthened realization hypothesis and then violates Conjecture 66.

16.3 Why the fixtures and later enlargements coexist

The first fixture did not even supply all the per-break arithmetic targets required by the stronger test. The second supplies a sharper comparison but still does not provide the full native marking, Frobenius action, and same-candidate H5 package. These omissions are part of its status, not cosmetic caveats.

The later low-rank enlargements in Part I do not retroactively make a forbidden original quotient large enough. They change the assembly carrier while preserving a specified inducing representation. That is a different construction. The correct historical flow is therefore: genuine source multiplicities, obstruction to a particular arithmetic quotient, and a search for a permitted larger carrier with explicit recovery. It is not a sequence of contradictory claims about one unchanged quotient.

Source and status. [C112] records the two arithmetic source fixtures and the maximal-type spectrum test. [Imai] supplies the selected ramification formula. The finite operator law and balanced-center kernel have been rerun here. The local-field torsor construction and full spectrum theorem remain inherited ordinary arguments, not newly formalized results.

17 An actual arithmetic candidate and a failed prescribed character

17.1 The field, the character, and the source

Continuation113 supplies an explicit characteristic-zero experiment in which the source representation and the arithmetic candidate exist independently of the proposed packet attachment. Fix compatible roots of unity and write

$$F = \mathbf{Q}_3(\zeta_3), \qquad L = \mathbf{Q}_3(\zeta_9), \qquad \varpi = \zeta_9 - 1, \qquad \varpi_F = \zeta_3 - 1, \qquad \beta = \varpi^{-10}. \tag{17.1}$$

The extension L/F is cyclic and totally ramified of degree three. The polynomial of ϖ over F is

$$X^3 + 3X^2 + 3X - \varpi_F.$$

Normalize $v_L(\varpi) = 1$. Then $v_L(3) = 6$ and the relative different has exponent six. Let $\psi_{\mathbf{Q}_3}$ have kernel $\mathbf{Z}_3$, and set

$$\psi_F(x) = \psi_{\mathbf{Q}_3}\left(\frac{\mathrm{Tr}_{F/\mathbf{Q}_3}(x)}{3}\right), \qquad \psi_L = \psi_F \circ \mathrm{Tr}_{L/F}.$$

The conductor ideal of ψ_L is $\mathfrak{p}_L^{-3}$. This negative exponent is essential. Replacing it by $\mathfrak{p}_L$ shifts the unit thresholds and changes the calculation.

Every element of $L^\times$ has a unique expression $\varpi^a \epsilon u$ with $a \in \mathbf{Z}$, $\epsilon \in \{1, -1\}$, and $u \in U_L^1$. Define the full multiplicative character by

$$\chi(\varpi^a \epsilon u) = \psi_L(\beta \log u). \tag{17.2}$$

The logarithm converges on U_L^1. At depths at least four, logarithm and exponential are inverse, so the conductor condition becomes $r - 10 \geq -3$. Consequently $a_L(\chi) = 7$. With $\sigma(\zeta_9) = \zeta_9^4$, one obtains

$$v_L(\sigma\beta - \beta) = -8.$$

The nontrivial conjugate quotient has conductor five. In particular, the three conjugate characters are distinct on the actual unit group, not merely assigned distinct labels.

The fresh exact program operates in the rational algebra

$$\mathbf{Q}[X]/(X^6 + 6X^5 + 15X^4 + 21X^3 + 18X^2 + 9X + 3).$$

It verifies field identities and rational trace witnesses such as

$$\chi(\exp(\varpi^6)) = \zeta_3, \qquad \psi_L((\sigma\beta - \beta)\varpi^4) = \zeta_3^2.$$

The program does not approximate a 3-adic exponential. The analytic reduction $\log(\exp x) = x$ is used only in its valid range, after which the calculation is exact rational arithmetic.

Regularity makes

$$\rho = \mathrm{Ind}_{W_L}^{W_F} \chi$$

irreducible of dimension three. The argument in [C113] excludes unramified self-twists. Under [BH99, Proposition 2.2], its wild restriction is irreducible as well. The conductor induction calculation gives

$$a_F(\rho) = 7 + 6 = 13, \qquad \mathrm{Sw}_F(\rho) = 10, \qquad \mathrm{slope}(\rho) = 10/3. \tag{17.3}$$

These are conductor statements for this source, not proofs of the general pair-factor contract.

17.2 The source-defined automorphic-induction candidate

The characteristic-zero cyclic monomial route in [BH99, §2.3] gives an actual supercuspidal candidate

$$\pi_F^0 = \mathrm{AI}_{L/F}(\chi). \tag{17.4}$$

The central character is $\omega = \chi|_{F^\times}$. This construction imports the established automorphic-induction input. It is not described as selecting a representation with convenient packet coordinates and then declaring it to be the realizer of ρ.

For this monomial subcase, the classical source has stronger comparison properties than conductor equality alone. Those properties belong to this actual candidate and its stated local correspondence. They do not prove that an independently specified simple character occurs in π_F^0, nor do they automatically identify a different constructed type with it.

17.3 The independent simple character

Let $\mathfrak{a}$ stabilize the $\mathcal{O}_L$ lattice chain on the three-dimensional F-space L. The element $\beta = \varpi^{-10}$ is minimal over F. The actual full-degree simple stratum gives

$$H^1 = U_L^1 U_{\mathfrak{a}}^6, \qquad J^1 = U_L^1 U_{\mathfrak{a}}^5.$$

The proposed simple character is

$$\theta_0(uy) = \chi(u)\,\psi_F\Big(\mathrm{tr}_{M_3(F)/F}(\beta(y-1))\Big). \tag{17.5}$$

The intersection condition is checked on U_L^6. The quotient J^1/H^1 has order nine, and the associated Heisenberg representation η_0 has dimension three. The type-side conductor agrees with 13.

At this point two real objects have been produced: the source-defined π_F^0 and the independently defined θ_0. The missing assertion is not their existence but

$$\mathrm{Hom}_{H^1}(\theta_0, \pi_F^0) \neq 0. \tag{17.6}$$

The computation below shows that this assertion is false. This is more informative than marking it as an unfinished verification.

17.4 A nontrivial tame descent that still leaves occurrence open

Set $K = F(i)$, the unramified quadratic extension, and $M = KL$. The character $\chi_M = \chi \circ N_{M/L}$ gives $\pi_K^0 = \mathrm{AI}_{M/K}(\chi_M)$. Its Galois invariance and norm-compatible central character meet the tame descent conditions in [BH99, Theorem 1.3]. In characteristic zero, Theorem 1.8 identifies the tame algebraic lift with cyclic base change here, since $[K : F] = 2$ is prime to three.

Lemma 17.1 (Central normalization of a descent fibre)**.** *Suppose a nonempty descent fibre is a torsor under a group X of characters of exponent dividing d. Suppose twisting a rank-n representation by $\alpha \in X$ multiplies its central character by α^n. If $\gcd(n, d) = 1$, every compatible specified central character selects a unique member of the fibre.*

Proof. Choose integers r, s with $rn + sd = 1$. The power map $\alpha \mapsto \alpha^n$ on X has inverse $\alpha \mapsto \alpha^r$, because $\alpha^d = 1$. Relative to one member of the nonempty fibre, a compatible desired central-character ratio is an element of X. There is exactly one twisting character with that nth power. □

Here $n = 3$ and $d = 2$, so the central-normalized descent is unique. This is a useful actual descent conclusion. The applicable type transfer also provides an occurrence equivalence between F and K. It does not manufacture the initial occurrence. In this example $\dim \eta_K = 9$ while $\dim \eta_F = 3$, and the literal restriction of the transferred simple character has the power $\theta_K|_{H_F^1} = \theta_F^2$. Equating the two entire finite carriers would erase the actual transfer mechanism.

Source and status. [C113] contains the field, character, conductor, type and descent arguments. The cyclotomic identities and normalization checks were freshly replayed. [BH99, printed pp. 15, 18, 24 and 48] was read at the selected descent, monomial and central-type statements. A full new audit of every conductor formula and every transfer theorem is not claimed.

18 A full-extension scalar obstruction and its limited repair

18.1 Why testing only the compact part can mislead

The cubic norm-kernel character of $F^\times$ defines $\Omega = \omega_{\mathrm{cub}} \circ \det$ on $\mathrm{GL}_3(F)$. The source-defined automorphic induction satisfies

$$\pi_F^0 \otimes \Omega \simeq \pi_F^0.$$

If a proposed extended type is to induce to π_F^0, it must be compatible with this self-twist. Continuation113 finds an element g that makes the compact character appear compatible, but whose action on a noncompact generator leaves a nontrivial scalar.

In the basis $1, \varpi, \varpi^2$, multiplication by ϖ is

$$\Pi = \begin{pmatrix} 0 & 0 & \varpi_F \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{pmatrix}, \qquad g = I + \mathrm{diag}(0,1,2)\Pi^4. \tag{18.1}$$

The actual group calculations show that g normalizes the relevant compact groups and the extended inducing group $N = L^\times J^1$. With the convention $\tau^g(x) = \tau(g^{-1}xg)$, one has

$$\theta_0^g = \theta_0\Omega, \qquad \eta_0^g = \eta_0\,\Omega|_{J^1}$$

in the literal operator models used in the audit. The check includes H^1 and two quotient generators, not merely a comparison of characters on an abstract vector space.

The decisive element is

$$h = g^{-1}\Pi g \Pi^{-1} \in H^1, \qquad \theta_0(h) = \zeta_3. \tag{18.2}$$

In contrast, $\Omega(\Pi) = 1$. For every full central extension Λ of η_0, the compact equality remains true but

$$\Lambda^g(\Pi) = \zeta_3\Lambda(\Pi).$$

Lemma 18.1 (A generator scalar cannot be removed by an intertwiner rescaling)**.** *Let two representations of $N = \langle J, z\rangle$ have the same irreducible restriction to J. Suppose their operators at z differ by a scalar $\xi \neq 1$. Then they have no nonzero intertwiner.*

Proof. Any intertwiner on J is scalar by irreducibility. A nonzero scalar commutes with the invertible operator at z. The intertwining equation at z then forces $\xi = 1$, a contradiction. □

The inherited all-extension argument combines this lemma with the actual simple-character intertwining group and the maximal-type induction theorem. Those inputs rule out an alternative intertwining coset elsewhere in the ambient group. The conclusion is therefore not based on numerically searching a finite list of matrices. Every supercuspidal containing this prescribed θ_0 fails the required cubic self-twist, whereas π_F^0 has it. Thus

$$\boxed{\mathrm{Hom}_{H^1}(\theta_0, \pi_F^0) = 0.} \tag{18.3}$$

This is the exact failure of the original prescribed-character route. Neither rescaling an intertwiner nor assigning a different scalar to the same noncompact extension repairs it.

18.2 The independent check and its scope

The replay includes the original rational cyclotomic program, the central-type calculations for three corrections, and an independent scalar calculation. The independent program decomposes the actual matrix h into an L-unit and a deep congruence term. It computes the phase in (18.2) directly, without searching for a Weyl intertwiner or choosing a favored central-type matrix.

This is strong evidence for the specific scalar input. The universal phrase "every extension" still uses the ordinary group-generation and irreducibility argument. Finite arithmetic alone is not being credited with proving a quantified representation-theoretic statement.

18.3 A different full character passes the obstruction

Define a wild unit correction on U_L^1 by

$$\nu(u) = \psi_L(\varpi^{-8} \log u). \tag{18.4}$$

Its conductor is five. It leaves the deeper part used in (17.5) unchanged but changes the full unit character. The corrected simple character is

$$\theta'(uy) = \chi(u)\nu(u)\,\psi_F\Big(\mathrm{tr}_{M_3(F)/F}(\beta(y-1))\Big). \tag{18.5}$$

It is not merely a different choice of basis for θ_0.

The noncompact normalization matters here too. The correction is trivial on U_F^1, but that alone does not establish $\nu|_{F^\times} = 1$. The inherited calculation gives

$$\nu(\varpi_F/\varpi^3) = \zeta_3^2.$$

Assigning $\nu(\varpi) = 1$ would therefore change the central character. Instead set

$$\nu(\varpi) = \zeta_9, \qquad \nu(-1) = 1.$$

Then

$$\nu(\varpi_F) = \nu(\varpi)^3\nu(\varpi_F/\varpi^3) = \zeta_3\zeta_3^2 = 1. \tag{18.6}$$

This is the required full central normalization.

For the correction powers ν^a, $a = 0, 1, 2$, the remaining generator scalars are respectively

$$\zeta_3, \qquad 1, \qquad \zeta_3^2. \tag{18.7}$$

The middle choice passes the cubic self-twist obstruction. The two other choices remain negative controls. The exact programs reproduce all three cases.

18.4 The remaining selection problem

Passing the self-twist test is necessary but not sufficient to establish

$$\mathrm{Hom}_{H^1}(\theta', \pi_F^0) \neq 0. \tag{18.8}$$

The fixed-center central-type family still has three members in this rank-three setting. [BH99, §5.4] describes such a family as a torsor under the corresponding unramified characters. The corrected scalar does not single out the source-defined automorphic-induction member of that family.

A successful next argument would have to establish occurrence in the actual π_F^0 and then identify the appropriate noncompact extension. Producing some representation containing θ' is a weaker statement. Replacing π_F^0 by a convenient member of the family would abandon the same-candidate obligation. Replacing the supplied canonical extension would be another change of input and would have to be tracked explicitly.

Source and status. [C113-S] gives the all-extension obstruction, central-type hypotheses and corrected character. The fresh checks establish the named cyclotomic identities and phases. Equations (18.3) and (18.7) retain that ordinary proof boundary. Equation (18.8) remains an unresolved occurrence assertion, not a failed theorem-prover build.

19 Compact binders and exact fixed-template obstructions

19.1 The compactness theorem with all hypotheses visible

A binder is intended to turn compatible finite constraints into one actual object. Compactness can perform that passage only after the admissible space and simultaneous finite solvability have been supplied. It does not supply those premises from unrelated finite successes.

Let $\mathcal{O}$ be the valuation ring of a nonarchimedean local field, with uniformizer π. Fix a closed subset $X \subseteq \mathcal{O}^m$ and continuous functions $f_j : X \to \mathcal{O}$, $j \geq 1$. Put

$$C_n = \{x \in X : f_j(x) \in \pi^n \mathcal{O} \text{ for } 1 \leq j \leq n\}. \tag{19.1}$$

Proposition 19.1 (A compact simultaneous binder)**.** *If $C_n \neq \varnothing$ for every n, there is $x \in X$ with $f_j(x) = 0$ for every j.*

Proof. The ring $\mathcal{O}$ is compact, so X is compact. Each C_n is closed by continuity and the closedness of $\pi^n\mathcal{O}$. Also $C_{n+1} \subseteq C_n$. The finite intersection property gives an element x in their intersection. For fixed j, the value $f_j(x)$ lies in every $\pi^n\mathcal{O}$ with $n \geq j$, so it is zero. □

The proof requires no surjectivity of the reduction maps between the sets of finite roots. For instance, $x^2 = 17$ has roots modulo every power of two, while the roots 3 and 5 modulo eight do not lift to roots modulo sixteen. An arbitrarily chosen partial root can fail even when the compactness conclusion applies to the full nested solution sets.

Separate solvability is also insufficient. The equations $x \equiv 0$ and $x \equiv 1 \pmod{3^n}$ each have solutions, but never have a common solution. Every finite test in the binder premise concerns one tuple satisfying all current constraints together.

19.2 What makes the result an arithmetic certificate

The variables must remain in one bounded, closed model with fixed rank, lattice data and normalizations. A nonzero leading coefficient should be encoded by a fixed nonzero residue class, a closed and open condition. Merely deleting the zero point from a compact model need not leave a compact space. In $\mathcal{O} \setminus \{0\}$, all congruences $x \equiv 0 \pmod{\pi^n}$ are solvable, but the only exact zero has been removed.

For polynomial constraints, or rational constraints with denominators uniformly units, congruence evaluation is compatible with a fixed finite quotient. An exhaustive empty solution set at some precision then certifies failure of that template. Conversely, if the exact common zero set in the compact template

is empty, Proposition 19.1 implies that some finite stage is empty. This is a finite obstruction certificate for the *specified model*, not automatically for every possible arithmetic host.

Full characters require particular care. The domain on which their finite quotients are modeled must be fixed, and the evaluation constraints must be jointly continuous or uniformly descend to the claimed finite quotient. Values on noncompact generators require their own normalization constraints. A binder that only remembers the deep commutator form does not recover a full inducing representation for free.

19.3 A corner with a complete leading-profile test

The earlier cubic calculation is a concrete example of a useful finite obstruction. Work over $\mathbf{F}_3((t))$. The four-coordinate square-zero corner has entries in positions $05, 16, 27, 38$. Adding a fifth coordinate gives

$$x = t(aE_{05} + bE_{16} + cE_{27} + dE_{38}) + \frac{t^2 h}{1+t} E_{08}. \tag{19.2}$$

The relevant leading target parameters are

$$u = cd, \qquad v = ab, \qquad w = bc, \qquad p = c(a-h), \qquad q = b(d-h). \tag{19.3}$$

Here u, v, w are required nonzero. The structural calculation for the larger sixteen-coordinate corner shows that its remaining entries and higher series terms do not change these decisive coefficients. That leading-coefficient fact is an inherited symbolic calculation, not an assumption that five coordinates exhaust all matrices.

Proposition 19.2 (The nonzero-leading corner spectrum)**.** *For targets in* $\mathbf{F}_3$ *with* $uvw \neq 0$*, equations* (19.3) *have a solution if and only if*

$$p + u = w(q + v). \tag{19.4}$$

Exactly 24 of the 72 possible targets satisfy this condition.

Proof. Write $b = s \neq 0$. The first three equations force

$$a = v/s, \qquad c = w/s, \qquad d = us/w.$$

The equation for q then forces $h = us/w - q/s$. Substitution into the equation for p gives

$$s^2(p + u) = w(q + v).$$

Over $\mathbf{F}_3$, every nonzero s has square one. This proves necessity and, by the displayed formulas, sufficiency. There are eight choices of the nonzero triple (u, v, w) and three choices of q, after which p is forced. Hence there are 24 admissible targets among $8 \cdot 3 \cdot 3 = 72$. □

The target $(u, v, w, p, q) = (1, 1, 1, 2, 2)$ cannot be obtained with $h = 0$ but is obtained by the fifth coordinate. The target $(1, 1, 1, 1, 2)$ violates (19.4), so the larger corner still cannot produce it. The first is an actual repair of an overly small ansatz. The second is an obstruction that survives that repair.

Neither conclusion proves that the target is impossible in every arithmetic matrix realization. It identifies precisely where this corner stops. That distinction is what makes the negative result reusable rather than misleading.

19.4 The arbitrary-exponent cubic family

The later general binder165 changes the construction rather than repeatedly enlarging the same leading-coefficient search. Let E/F be the cubic field with

$$e^3 + te + t = 0, \qquad \pi = -t = \frac{e^3}{1+e}.$$

On the period-nine lattice of E^3, let S denote the lattice shift. For $h > 0$ with $3 \nmid h$, set $a = 4h$ and take $x_h = S^a D$, where D is supported in residue positions

$$\{0, 2a, 4a, 6a\} \pmod 9.$$

Its image positions are $\{a, 3a, 5a, 7a\}$, disjoint from its support, so $x_h^2 = 0$. In the expression $x_h S^{-3a} x_h$, exactly the three starting positions $2a, 4a, 6a$ survive. Modulo three they meet the three E-lanes once each. This support statement holds for every allowed h because a is invertible modulo nine.

Under the retained normalized corestriction calculation, these paths give a primitive whole cycle. The full ordinary binder proposition then constructs the stated degree-nine parameter for allowed $h_1 < h_2$, with critical data $n = 12h_2$ and $k_0 = -4h_1$. The finite replay of support residues illustrates this mechanism. The universal construction rests on the symbolic proof and its refinement hypotheses, not on testing the first hundred exponents.

This family is genuine progress beyond one failed corner. It is still a parameter construction. It does not, without the additional full-character, stabilizer and transport arguments, provide the universal realization endpoint. The later direct odd-prime and dyadic constructions in Part I should be read as further developments of the constructor side, not as proofs that all the earlier occurrence obstructions disappeared.

Source and status. [B162] supplies the fixed compact model theorem. [Corner] supplies the whole-corner leading calculation. [B165] supplies the arbitrary-exponent construction. The compactness and finite profile implications are proved here. The underlying full arithmetic refinement is retained as an ordinary source argument.

20 What this synthesis closes and what it does not

20.1 A single dependency chain

The source history now supports a coherent sequence. A monomial source with an admissible induction tower gives a restricted actual candidate through stated classical inputs. Genuine Clifford theory limits the multiplicities that can occur inside one source. Explicit fixtures show that ordinary slope data do not determine the packet chain. An independently constructed simple character then poses a genuine occurrence problem for that candidate.

Continuation113 resolves one such occurrence question negatively and repairs one necessary self-twist condition only by changing the full character. Compactness explains how a fixed admissible template can be assembled from simultaneous finite solutions, while explicit corners give finite obstruction certificates. The later constructor families enlarge what can be built but do not identify every constructed object with an independently fixed candidate.

Statement available	Obligation that is not supplied by it
Admissible monomial induction	An admissible tower for every allowed source.
Projective dimension bound	A native, marked and equivariant arithmetic embedding.
Conductor agreement	Occurrence of the prescribed full character.
Cubic self-twist compatibility	Selection of the source-defined central type.
Nonempty finite systems	A limit unless the model is fixed, compact and simultaneous.
A repaired matrix corner	Success in every target profile or arithmetic host.
A larger inducing carrier	Permission to change the original candidate or extension.

20.2 The most precise surviving occurrence task

For the characteristic-zero experiment, the next question is (18.8), followed by the identification of the full noncompact central-type extension. It is not sensible to retry the already excluded (17.6) by changing only an intertwiner scalar. Its negative proof addresses every such rescaling.

A future attempt at (18.8) should use an invariant or a theorem that actually distinguishes the three fixed-center possibilities. The cubic self-twist test does not do so. An explicit comparison of the full inducing character under the monomial construction, or a sufficiently discriminating normalized local-factor calculation for the same candidate, would address a stronger question than the conductor comparisons already performed. Neither is asserted as completed here.

20.3 The remaining publication work

The ledger retains the conductor/source-defined-induction audit as partial. This block has audited selected characteristic-zero monomial and tame-descent inputs. It has not freshly reconciled the separate conductor arguments used by the direct positive-characteristic constructors. Their hypotheses must not be imported silently from [BH99, §2.3], where characteristic zero is an explicit restriction for the route used here.

The fixed native marking audit also remains separate. A scalar pushout can carry a supplied subgroup and phase convention. It need not agree with an independently fixed native quadratic refinement. The equal-capacity dyadic boundary is a particularly important next case. After these source reconciliations,

the dedicated conjecture visual companion and the synchronized XI main, visual and Lean readers can be revised against a stable claim map.

21 Evidence, reproducibility and continuity

21.1 Evidence classes

This continuation uses four positive evidence classes and one open class. An *ordinary proof* is a written argument with its hypotheses visible. An *inherited arithmetic argument* is a source construction retained with its classical dependencies. A *fresh bounded check* is a named executed program with a recorded input hash and scope. A *historical formal receipt* is evidence of a past compiler execution, not a new execution in this block. An *open obligation* has not been promoted by any of the other classes.

The selected [BH99] page images were inspected at printed pages 15, 18, 24 and 48. These passages establish the scope of the tame descent fibre, the characteristic-zero comparison with base change, the monomial route, and the residual fixed-center ambiguity. The Imai ramification formula is used only at the specialized Heisenberg example. Neither source audit is a claim to have checked the entirety of either work.

21.2 Executable checks

The replay driver runs the new exact model suite and the preserved Block 01 suite, then the named continuation113 field, self-twist, central-correction, independent-exclusion and central-normalization programs. It copies historical programs into a temporary working directory because some write their own receipts. No source tree is mutated merely to obtain a test result. Each gate has its own hash, log and completion status. An interrupted orchestration attempt is retained as an interruption, not counted as a completed combined pass.

The tests are deliberately bounded. For example, the corner test checks all 72 leading target profiles, while the symbolic statement explaining why the other corner coordinates cannot change the decisive coefficients is a separate ordinary input. The Weyl tests verify finite operators, not local automorphic induction. The exact field calculation verifies rational identities, not a new proof of the classical type theory.

21.3 The source recovery gap

The available Block 02 PDF, ledger, handoff, input audit and delivery record are preserved as predecessor artifacts. Their presence does not reconstruct missing editable section files or missing test drivers. The package therefore provides a small independently buildable source checkpoint for this new continuation in addition to the full inherited evidence archive. This makes future continuity less dependent on the availability of one large ZIP.

All 18 preceding task IDs remain present. Additional IDs record the missing Block 02 source recovery, the terminal Block 45 receipt, and the corrected-character occurrence problem. Their status histories are appended rather than overwritten. The precise next restart is Publication Block 04, beginning with the native marking and constructor-conductor source audits. The corrected-character occurrence problem remains a separately named mathematical route, not a hidden premise of those audits.

Source register

The machine-readable source map gives package-relative paths, SHA-256 hashes and selected locators. The short labels below keep the mathematical text readable. Historical source titles containing the word "completion" do not override their actual hypotheses or verification status.

M *PSC Langlands Monomial Completion*, recovered manuscript and *Completion Ledger*. In particular: local monomial completion, prime-cyclic subgroup chains, and global monomial induction. Classical rank-one and automorphic-induction inputs remain explicit.

M45 *Actual Monomial Local Global*, v0.45.0. Production Lean source, source-lineage and claim-classification notes, preliminary preservation record, and two incomplete build logs. The terminal successful receipt was not recovered.

C112 *Continuation112 Realization Test*. The genuine characteristic-five Clifford fixture, symmetric characteristic-three fixture, and source-realizability report. Their arithmetic spectrum restrictions are retained with their original scope.

C113 *Arithmetic Construction and Exact Occurrence Boundary113*. Characteristic-zero source, independent simple character, cyclic automorphic induction, tame descent and central normalization. Rational cyclotomic programs retained.

C113-S *Self Twist Audit113*, *Failures113*, and the independent all-extension obstruction check. Original-character exclusion, the changed full-character correction, and the unresolved corrected-character occurrence.

C161 *Clifford Size and Inputs161*. Genuine normal-chain dimensions and the conditional host-capacity comparison. Abstract packet lists are not substituted for actual source multiplicities.

B162 *Compact Binder Theorem*, Closure Attempt162. Fixed compact models, simultaneous congruences and exact limits.

Corner *Structural Corner Note*, explicit binder extension, with the full sixteen-coordinate symbolic calculation. Fifth-coordinate repair and surviving leading-coefficient obstruction.

B165 *General Cubic Binder165*. Square-zero support, whole-cycle corestriction and the arbitrary-exponent degree-nine construction, under its arithmetic refinement hypotheses.

BH99 Colin J. Bushnell and Guy Henniart. *Local tame lifting for* $\mathrm{GL}(N)$. *II: Wildly ramified supercuspidals*. Astérisque 254, 1999. Selected statements: Theorem 1.3, Theorem 1.8, Proposition 2.2, §2.3, and §5.4. Printed page references and the source-access record are in the audit file.

Imai Koto Imai. *Ramification groups of some finite Galois extensions of maximal nilpotency class over local fields of positive characteristic*. Manuscripta Mathematica 176, article 66 (2025), DOI 10.1007/s00229-025-01664-5. Example 3.8(1) and its stated ramification hypotheses. The source audit records the exact publication metadata returned by the primary source.

PAPER XI / RESEARCH COMPANION

Conductors, Native Markings, and the Equal-Capacity Boundary

Toward Arithmetic Packet Realization
Part III of the Conjecture 66 progress paper

Publication Block 04

Komorebi P.

Working continuation v0.4.0 / 10 October 2026

Reading this continuation

A conductor is an arithmetic invariant, not a complete description of a type. A nondegenerate commutator is a finite invariant, not a complete description of a marked dyadic extension. These two distinctions meet at the smallest dyadic construction with exactly enough quotient capacity. There is no spare plane there to absorb an independently prescribed quadratic mismatch.

This continuation audits the conductor and marking branches at their actual hypotheses. It keeps three arithmetic objects separate: the original source representation, a candidate obtained through a classical induction and descent construction, and a candidate arising from a proposed explicit type. A comparison is useful only when its normalization and its subject are fixed.

The resulting finite theory is also constructive. A discrepancy between two quadratic refinements becomes one linear functional. When its Arf obstruction vanishes, a single transvection gives an explicit correction. For a supplied odd-order finite action, a further central discrepancy can be removed by a finite sum. Neither correction constructs the missing arithmetic action, chooses a canonical native marking, or settles the full phase equations.

Publication status. This is an independently editable Part III. The first two parts retain their prior source and review boundaries. In particular, the 33-page Block 02 reader remains a preserved PDF because its complete editable archive was not recovered. The original three Paper XI readers have not been revised here. The current ledger records mathematical obligations separately from documentary gaps.

Evidence status. Ordinary finite proofs are supplied below. The arithmetic construction records are inherited arguments audited against selected primary statements. New exact tests check finite models and normalizations. No fresh Lean compilation or external peer review is claimed.

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22 Conductor conventions and actual order periods

22.1 Four numbers, not one unnamed conductor

Let F be a nonarchimedean local field with residue cardinality q, and let V be a complex Weil representation with finite inertia image and no monodromy. Its ordinary Artin conductor is

$$a(V) = \dim V - \dim V^{I_F} + \mathrm{Sw}(V). \tag{22.1}$$

We use a for the ordinary conductor throughout this part. The Swan conductor, an epsilon exponent for a specified additive character, and the self-pair conductor of a representation of $\mathrm{GL}_n(F)$ are named separately.

Take ψ_0 trivial on $\mathcal{O}_F$ and nontrivial on $\mathfrak{p}_F^{-1}$. Set $\psi_1(x) = \psi_0(\varpi^{-1}x)$, so ψ_1 is trivial on $\mathfrak{p}_F$ and nontrivial on $\mathcal{O}_F$. Under the usual change-of-character convention, if the exponent is written as a power of $q^{1/2-s}$, then

$$f_{\psi_0}(V) = a(V), \qquad f_{\psi_1}(V) = a(V) - \dim V. \tag{22.2}$$

The determinant factor in the change of additive character is independent of s and does not alter this exponent shift. Equation (22.2) is a convention translation, not an assertion that every epsilon constant is determined by its exponent.

For a wild-irreducible source ρ of dimension nine with Swan conductor 24, the audited self-pair calculation below gives the following table.

Object	Dimension	I_F-fixed	Swan	Ordinary a	f_{ψ_1}
ρ	9	0	24	33	24
$\mathrm{End}\,\rho$	81	1	152	232	151
$\mathrm{Ad}\,\rho$	80	0	152	232	152

Here $\mathrm{Ad}\,\rho$ denotes the trace-zero summand. Over the complex coefficient field, $\mathrm{End}\,\rho = \mathbf{C} \oplus \mathrm{Ad}\,\rho$. The scalar summand is unramified, so it contributes no ordinary conductor. It does contribute one to the dimension shift in (22.2).

Thus the number 152 has two legitimate appearances, as the Swan conductor of $\mathrm{End}\,\rho$ and as the ψ_1 exponent of its trace-zero summand. It is *not* the ψ_1 exponent of all of $\mathrm{End}\,\rho$, which is 151. The formal-degree parameter δ appearing later also equals 152 in this particular torsion-one comparison. That numerical equality does not make the three definitions interchangeable.

Source and status. The ordinary conductor identity and additive-character normalization underlie [Q14]. [BH99, §6.1, Lemma 2] uses the character trivial on $\mathfrak{p}_F$. [AP, §4 and Theorem 6.9] uses the ordinary pair conductor. The table is an explicit reconciliation of these conventions using the source calculation in Section 23.

22.2 The principal-order grade calculation

Let $\mathfrak{A} \subset M_n(F)$ be a principal hereditary order of period e, with radical $\mathfrak{P}$. Then $e \mid n$. A principal lattice chain has e successive residue pieces, each of dimension $m = n/e$ over k_F. The homogeneous grade of an endomorphism is the direct sum of e matrix blocks of size m by m. Consequently

$$\dim_{k_F} \mathfrak{P}^j/\mathfrak{P}^{j+1} = em^2 = \frac{n^2}{e}, \qquad [\mathfrak{P}^{-r} : \mathfrak{A}] = q^{rn^2/e} \quad (r > 0). \tag{22.3}$$

The period here belongs to the actual principal order carrying the stratum. It cannot be borrowed from a more convenient unrelated lattice chain.

For an actual cuspidal representation containing a positive-level simple stratum $[\mathfrak{A}, r, 0, \beta]$ to which the hypotheses of [BH99, §6.1] apply, its Lemma 2 gives

$$f_{\psi_1}(\pi) = \frac{nr}{e}, \qquad a(\pi) = n + \frac{nr}{e}. \tag{22.4}$$

The first number is the native epsilon exponent in that source. In the present wild cuspidal comparison it equals the standard Swan conductor. A lower critical level in a defining sequence is not the leading level r in this formula.

Proposition 22.1 (Necessary period divisibility)**.** *Suppose an actual type satisfying* (22.4) *is to match a source of rank n and Swan conductor $w > 0$. Its principal period and leading level necessarily satisfy*

$$\frac{n}{\gcd(n, w)} \mid e \mid n, \qquad r = \frac{ew}{n}. \tag{22.5}$$

These are necessary numerical conditions, not sufficient existence conditions for a simple stratum or a prescribed source.

Proof. The principal condition gives $e \mid n$. Matching (22.4) gives $nr = ew$, so $n \mid ew$. Writing $n = gn'$ and $w = gw'$ with $\gcd(n', w') = 1$ yields $n' \mid e$. Conversely these divisibilities make ew/n an integer. They say nothing about purity, intertwining, or character occurrence. □

For $(n, w) = (9, 24)$, the possibilities are $(e, r) = (3, 8)$ and $(9, 24)$. A maximal-order construction of period one cannot have this conductor. The earlier unramified depth-two candidate has Swan conductor 18 and ordinary conductor 27, not 24 and 33. Replacing the leading level eight by a critical level four in the period-three formula would create a different error rather than repair the comparison.

22.3 A self-twist check that survives conductor matching

A second necessary invariant is the torsion number, the order of the group of unramified self twists. For a maximal simple type, the exact formula in [BK, 6.2.5] identifies that number as $n/e(K/F)$ for the actual parameter field K.

Lemma 22.2 (Wild irreducibility excludes nontrivial unramified self twists)**.** *If $\rho|_{P_F}$ is irreducible and an unramified character χ satisfies $\rho \otimes \chi \simeq \rho$, then $\chi = 1$.*

Proof. An intertwiner restricts to an endomorphism of the irreducible P_F representation, because χ is trivial on P_F. It is therefore a nonzero scalar. Applying the intertwining identity to every Weil element forces χ to be one. □

A period-three candidate may pass the first conductor test and still fail this second test if its actual maximal parameter field has residue degree three. That candidate has torsion number three, while the prescribed source has torsion number one. This is why the conductor equation does not complete the arithmetic attachment.

23 The rank-nine self-pair filter

23.1 The source invariant is computed on the unchanged source

Retain the original source over $F = \mathbf{F}_3((t))$,

$$\rho = A_1 \otimes A_2,$$

where A_h is the rank-three Heisenberg representation attached to the projective equation $x_h^9 + x_h = t^{-h}$, for $h = 1, 2$. The independent quotient and wild-irreducibility hypotheses are those of the original source fixture, not newly imposed conditions on a replacement source.

After the recorded unramified splitting extension, take a nonzero constant v satisfying $v^9 = -v$. Then

$$y_v = v^3 x_h^3 + v x_h, \qquad y_v^3 - y_v = -v t^{-h}. \tag{23.1}$$

The cancellation follows by substituting $v^9 = -v$ and $x_h^9 + x_h = t^{-h}$. As v varies, these give the eight distinct nontrivial characters of the nine-element translation quotient. A nonzero polynomial of degree three cannot vanish on all nine translations, so the corresponding additive functional is nonzero. Distinct v give distinct functionals by the same argument. Since each pole order h is prime to three, the inherited Artin–Schreier ramification calculation gives break h.

Proposition 23.1 (Self-pair conductor of the retained source)**.** *Under the original independence and wild-irreducibility hypotheses,*

$$\mathrm{Sw}(\mathrm{End}\,\rho) = 152, \qquad a(\mathrm{End}\,\rho) = 232. \tag{23.2}$$

Proof. Conjugation on the nine Weyl operators for A_h gives each character of its projective quotient once. Hence $\mathrm{End}\,A_h$ is the sum of the trivial character and eight characters of break h after the unramified extension. In $\mathrm{End}\,A_1 \otimes \mathrm{End}\,A_2$, there is one trivial character, eight characters of break one, and $9 \cdot 8 = 72$ characters of break two. Independence retains the distinct quotient characters. Products of characters of different breaks have the larger break. The Swan conductor is therefore $8 + 2 \cdot 72 = 152$.

Unramified restriction preserves the Swan conductor. By wild irreducibility, the inertia-fixed endomorphisms are exactly the scalars. Thus $\dim(\mathrm{End}\,\rho)^{I_F} = 1$ over F itself. Equation (22.1) gives $81 - 1 + 152 = 232$. No assumption that Frobenius fixes each individual quotient character is needed. □

Source and status. [Q13] supplies the source and the ramification inputs. The displayed Artin–Schreier cancellation, character separation and conductor addition are expanded here. The new exact suite constructs the nine-element constant subspace inside $\mathbf{F}_{81}$ and checks all its additive character values. This finite check does not reprove local ramification theory.

23.2 The actual-order certificate cannot be omitted

The proposed rank-nine family has a complete principal order $\mathfrak{A}$ of period nine, radical $\mathfrak{P}$, and an actual final parameter field K/F that is totally ramified of degree nine. An intermediate field E has degree three. Write $\mathfrak{Q}$ for the radical of the induced order in the E-centralizer. The retained candidate certificate specifies leading level 24, lower even critical level $c = 2a$, and actual simple-character orders

$$\mathfrak{J} = \mathcal{O}_K + \mathfrak{Q}^a + \mathfrak{P}^{12}, \qquad \mathfrak{H} = \mathcal{O}_K + \mathfrak{Q}^{a+1} + \mathfrak{P}^{13}. \tag{23.3}$$

These equations are substantive construction inputs. The mere pair of numbers $(24, c)$ does not imply them.

The certificate also ensures that the K contribution adds no unexpected graded direction. In the notation of [Q13], minimality of the leading approximation γ and the correction $d \in \mathfrak{P}^{-c}$ give

$$[\gamma, \mathcal{O}_K] \subseteq \mathfrak{P}^{-c}, \qquad \mathcal{O}_K \subseteq \mathfrak{B} + \mathfrak{P}^{24-c}. \tag{23.4}$$

For $c = 4$ and $c = 10$, the last exponent exceeds thirteen. Through the grades used in (23.3), the K-grade therefore lies inside the three-dimensional centralizer grade. Its own dimension is one. This is an actual commutator and approximation assertion, not an integral splitting of a wildly ramified bimodule.

Let $I^+ = 1 + \mathfrak{P}$. The successive full grades have dimension nine, the intermediate grades dimension three, and the K-grades dimension one. It follows that

$$I = \log_q [I^+ : J^1] = 8(a-1) + 6(12-a) = 2a + 64, \tag{23.4a}$$

$$D = \log_q [J^1 : H^1] = 2 + 6 = 8, \qquad \dim \eta = q^4. \tag{23.4b}$$

Since K/F is totally ramified of full degree, the finite residue factor is a character and the torsion number is one.

23.3 A formal-degree calculation with its measure fixed

Normalize Haar measure by $\mathrm{vol}(\mathrm{GL}_9(\mathcal{O}_F)) = 1$ and central measure by $\mathrm{vol}(\mathcal{O}_F^\times) = 1$. Then

$$\mathrm{vol}(I^+) = \prod_{j=1}^{9} (q^j - 1)^{-1}, \qquad \mathrm{vol}(J) = (q-1) q^{-I} \,\mathrm{vol}(I^+).$$

The compact-induction degree formula and the Steinberg degree formula give

$$d(\pi) = \frac{q^4}{9 \,\mathrm{vol}(J)}, \qquad d(\mathrm{St}_9) = \frac{1}{9} \prod_{j=1}^{8} (q^j - 1),$$

so that

$$\frac{d(\pi)}{d(\mathrm{St}_9)} = \frac{q^9 - 1}{q - 1} \, q^{2a+68}. \tag{23.5}$$

The factor nine comes from the actual noncompact extension $[K^\times J : F^\times J]$. Omitting it before comparing the measures changes the calculation. An unramified twist may be used to unitarize solely for the formal-degree comparison. That operation does not replace the prescribed noncompact scalar in the construction, and it leaves the self-pair unchanged.

The primary all-characteristic formulas [AP, Theorems 6.5 and 6.9] use the torsion number r_t and an auxiliary parameter δ:

$$\frac{d(\pi)}{d(\mathrm{St}_n)} = r_t \frac{q^n - 1}{q^{r_t} - 1} q^{(r_t - n + \delta)/2}, \qquad a(\pi^\vee \times \pi) = \delta + n^2 - r_t. \tag{23.6}$$

These are the named classical inputs here, not a conjectural formal-degree identity. With $n = 9$, $r_t = 1$ and (23.5), they give

$$\delta = 4a + 144 = 2c + 144, \qquad a(\pi^\vee \times \pi) = 4a + 224 = 2c + 224. \tag{23.7}$$

Certified profile	c	I	D	δ	Self-pair a
First repair	10	74	8	164	244
Deeper repair	4	68	8	152	232
Retained source	—	—	—	152	232

Corollary 23.2 (A necessary selection inside this certified family)**.** *If a member of the actual-order family* (23.3) *is to match the retained source under a correspondence preserving the ordinary self-pair conductor, then* $c = 4$*. The critical-ten member is excluded. The critical-four member passes this invariant, but the comparison does not prove its source identification.*

Proof. Equate (23.2) and (23.7). The equation $2c + 224 = 232$ has the unique solution $c = 4$. Equality of this one invariant supplies no character-occurrence or full pair-factor identity. □

Source and status. [Q13] provides the conditional actual-order certificate and calculation. [AP, formulas (17), (29), Theorems 6.5, 6.9 and Remark 6.7] were checked at their stated measures and field scope. This block does not newly construct the proposed corrected stratum. It preserves the requirement to prove its actual order equations before applying the numerical filter.

24 Why pair conductors do not determine quotient capacity

24.1 The parity identity behind the graded calculation

For the totally ramified complete-order sector, write the successive graded centralizer dimensions as

$$n = b_0 > b_1 > \cdots > b_r = 1, \qquad \Delta_j = b_{j-1} - b_j.$$

Assume the actual defining sequence and its order compatibility give descending positive levels ν_j and the grade-count formulas

$$I = \sum_{j=1}^{r} \Delta_j \left\lceil \frac{\nu_j}{2} \right\rceil - (n-1), \qquad D = \sum_{j=1}^{r} \Delta_j \mathbf{1}_{2|\nu_j}. \tag{24.1}$$

Here $I = \log_q[I^+ : J^1]$, while $D = \dim_{k_F}(J^1/H^1)$. The different floor and ceiling thresholds are essential. In this torsion-one, full-degree setting, the same formal-degree calculation gives

$$a(\pi^\vee \times \pi) = n^2 - 1 + (n-1) + 2I + D. \tag{24.2}$$

As in the rank-nine calculation, the formula is conditional on actual orders. A list of integers is not a certificate of their arithmetic existence.

Proposition 24.1 (Parity cancellation)**.** *Under* (24.1)*,*

$$(n-1) + 2I + D = \sum_{j=1}^{r} \Delta_j \nu_j.$$

Consequently

$$a(\pi^\vee \times \pi) = n^2 - 1 + \sum_{j=1}^{r} \Delta_j \nu_j. \tag{24.3}$$

The weighted conductor expression does not retain D as a separate invariant.

Proof. For each integer ν,

$$2\left\lceil \frac{\nu}{2} \right\rceil + \mathbf{1}_{2|\nu} = \nu + 1.$$

Insert this identity into $2I + D$. Since $\sum_j \Delta_j = n - 1$, the constant terms reduce to $-(n-1)$. Adding $n - 1$ proves the result. □

This cancellation explains a concrete limitation of the earlier numerical tests. Changing the parity distribution can change the Heisenberg quotient while preserving the weighted sum in (24.3). More conductor precision does not recover information that the formula has already summed away.

24.2 The full dyadic profile

In the inherited full dyadic construction, let $n = 2^d$ and let $h_1 < \cdots < h_d$ be positive odd integers satisfying its separation requirements. The consecutive profile has

$$b_j = 2^{d-j}, \qquad \nu_j = 2^{d-j}\, 3h_{d-j+1} \quad (1 \le j \le d). \tag{24.4}$$

All levels except the final one are even. Therefore

$$D = 2^d - 2. \tag{24.5}$$

A source consisting of d nondegenerate binary symplectic planes has dimension $2d$. The original maximal-type quotient is too small for $d = 1, 2$, has exactly the needed dimension for $d = 3$, and has strict surplus for $d \ge 4$.

d	Arithmetic rank	Source dimension	Host D	Original quotient
1	2	2	0	Insufficient
2	4	4	2	Insufficient
3	8	6	6	Equal capacity
4	16	8	14	Strict surplus
5	32	10	30	Strict surplus

The rank-two and rank-four enlarged arithmetic carriers in Part I are not counterexamples to the first two rows. Those repairs change the assembly carrier while recovering the same inducing representation. The table concerns the original specified maximal-type quotient. At rank eight, dimension ceases to be the obstruction, but an independently fixed quadratic marking can still obstruct the attachment.

24.3 Two rank-eight profiles with the same conductor data

The retained dyadic capacity note contains a sharper example. In the full rank-eight profile, use

$$(b_0, b_1, b_2, b_3) = (8, 4, 2, 1), \qquad (\nu_1, \nu_2, \nu_3) = (12h_3, 6h_2, 3h_1).$$

A second constructed profile skips the intermediate quartic field and has

$$(b_0, b_1, b_2) = (8, 4, 1), \qquad (\nu_1, \nu_2) = (12h_3, c), \qquad c = h_1 + 4h_2. \tag{24.6}$$

The integer c is odd and $2c < 12h_3$. Its last jump therefore contributes nothing to D. The full profile has $D = 6$, while the skipped profile has $D = 4$.

Both weighted sums in (24.3) equal

$$48h_3 + 12h_2 + 3h_1. \tag{24.7}$$

Indeed, the full profile has jump sizes $4, 2, 1$, and the skipped profile has jump sizes $4, 3$. The identity $3(h_1 + 4h_2) = 3h_1 + 12h_2$ supplies the equality.

For $(h_1, h_2, h_3) = (1, 3, 5)$, the distinction becomes completely explicit.

Quantity	Full profile	Skipped profile
Levels	$(60, 18, 3)$	$(60, 13)$
Principal period	8	8
Standard Swan conductor	60	60
Standard ordinary conductor	68	68
Torsion number	1	1
I	133	134
D	6	4
Heisenberg dimension over $q = 2$	8	4
δ	279	279
Ordinary self-pair conductor	342	342

The compensating change is $2 \cdot 133 + 6 = 2 \cdot 134 + 4$. The three necessary invariants—standard conductor, torsion number, and self-pair conductor—cannot force the full six-dimensional quotient.

24.4 The skipped profile is an arithmetic claim, not an integer exercise

The inherited construction of (24.6) works over $\mathbf{F}_2((t))$. Put $q_0 = 3h_3$ and take $\alpha^2 + \alpha = t^{-q_0}$. A uniformizer z of the quadratic field satisfies

$$z = t^{(q_0+1)/2}\alpha, \qquad z^2 + t^{(q_0+1)/2}z + t = 0.$$

Over that field, the polynomial $X^4 + zX + z$ is separable and Eisenstein. If e is a root, let C be multiplication by e^{-c} in the quartic centralizer. The source note chooses a filtered corestriction lift using $s(P_0) = t/z^2$ and takes a correction proportional to $(z^2/t)CP_0$. The actual final parameter is the literal corrected matrix, not an assumed compositum of the approximation fields.

The conclusion that this correction supplies the claimed defining sequence uses the retained refinement argument and its strict inequalities. This publication block has audited the conductor and capacity consequence of that argument. It has not freshly rerun a skipped-profile matrix constructor or recertified every step of its arithmetic field proof. Those statuses are not replaced by the exact integer tests.

Nor is this example a counterexample to H5. Full local pair factors are substantially more information than the three numbers above. The example shows that those three numbers alone do not identify the needed native quotient, and that a source-defined candidate cannot be declared to contain a prescribed type merely because it passes them.

Source and status. [D167] supplies the actual profile formulas and the skipped-degree construction. Proposition 24.1 and the numerical comparison are fully derived here. The exact suite checks the example and 120 positive odd triples, as well as general parity identities. The underlying arithmetic construction retains its inherited ordinary-proof status.

25 The source-defined arithmetic candidate and its inputs

25.1 Do not identify two candidates by notation

The direct matrix recursion and the classical arithmetic-induction route start with different data. The first produces an actual type satisfying a restricted construction contract. The second starts from a complete source character and uses classical induction and descent to produce a candidate. We denote the latter

by $\pi_{\mathrm{BH}}(\rho)$ in this section. This notation does not identify it with any independently constructed compact type.

The audited source-defined branch is over $\mathbf{F}_3((t))$ and uses the original rank-nine source. It is distinct from the characteristic-zero continuation113 candidate discussed in Part II. Its existence through classical inputs does not prove the corrected-character occurrence question for that other candidate, nor does it place a prescribed native packet in the current one.

25.2 The degree-nine field does not initially have a cubic tower

Write $A = t^{-1}$, and retain the projective coordinates

$$x^9 + x = A, \qquad z^9 + z = A^2.$$

The source record takes $y = z - x - x^3$. In characteristic three,

$$y^9 + y = B, \qquad B = A^2 - A - A^3. \tag{25.1}$$

The inducing field is $M = F(y)$, of degree nine. Its retained Galois-closure analysis has translation subgroup $(C_3)^2$ with a C_4 action irreducible over $\mathbf{F}_3$. There is no cubic intermediate field over F. Therefore one cannot simply write two cubic automorphic-induction arrows over F without changing the base and proving the resulting descent.

Pass to the unramified extension $K = \mathbf{F}_{81}((t))$. The constants now split the translation action. Choose $a \neq 0$ with $a^9 = -a$ and i with $i^2 = -1$, and define

$$r = \frac{y^3 - a^2 y}{ia^3}, \qquad s = \frac{y}{a}.$$

Then

$$r^3 - r = \frac{B}{ia}, \qquad s^3 - s = ir. \tag{25.2}$$

To check the first equality, use $a^8 = -1$ to cancel the y^3 terms after cubing. The second follows immediately from the definition of r. These identities exhibit the cyclic cubic tower over K used by the classical route. The degree and regularity assertions remain part of the complete source fixture.

The inducing character is also complete data. The inherited record includes its ramified Artin–Schreier class, its unit values, and an actual unramified factor. Replacing it by the displayed pole order alone loses information needed for the resulting central character and noncompact extension. The noted different exponent 24 and character conductor 9 give the expected induced ordinary conductor 33, but do not determine that full character.

25.3 Induction, invariance, and descent are separate gates

The construction first applies cyclic automorphic induction at the two cubic stages over K. Cuspidality requires the recorded regularity at each relevant step. It then invokes the classical independence of the resulting monomial construction from an admissible cyclic tower. This independence is a substantial theorem. It is what allows a Galois conjugate of the chosen tower to be compared with the original construction. The fact that the source descends to F does not, by itself, make each arbitrarily chosen tower Frobenius invariant.

Only after this invariance has been established does the tame descent theorem apply. Its fibre is described by the unramified characters associated to K/F, here of order dividing four. The specified determinant of ρ selects the desired central character.

Lemma 25.1 (Unique fixed-center member in a coprime descent fibre)**.** *Let a descent fibre be a torsor for a cyclic character group X of order m. Suppose twisting a rank-n member by $\alpha \in X$ changes its central character by α^n. If $\gcd(n, m) = 1$, then any attainable target central character selects exactly one member of the fibre.*

Proof. The map $\alpha \mapsto \alpha^n$ is a bijection of X because multiplication by n is invertible modulo m. Given one member and an attainable ratio of central characters, there is a unique twist realizing that ratio. □

For $n = 9$ and $m = 4$, the required power map is invertible. This explains the central normalization rather than silently selecting a preferred descent. It differs from the fixed-center cubic ambiguity in continuation113, where the relevant power map is not coprime. The new finite tests include both the unique and the nonunique cases.

25.4 What the primary-source audit does and does not establish

The selected audit uses three layers of classical material.

The 1999 monomial and tame-lifting statements. In [BH99], the definitions and properties of tame lifting in Section 1.3 identify the descent fibre, twisting and central-character behavior. Section 2.3 gives the characteristic-zero monomial construction, including standard epsilon factors and pair L-functions and conductors. Its proof uses independence of the cyclic construction. Section 1.7 supplies the tame transfer rule for the relevant pair-conductor exponent. These statements were read at their exact pages.

The positive-characteristic extension. The original Section 2.3 is explicitly characteristic zero. The characteristic note on page eight of [BH13] states that the relevant earlier results extend to positive characteristic through the base-change and automorphic-induction work of [HL11]. That note was checked directly. The present audit therefore retains the positive-characteristic induction theorem as a named external input, rather than erasing the original hypothesis.

The limit of fresh source access. The complete [HL11] PDF could not be retrieved in this block. Its bibliographic record was accessible, but repeated full-text requests failed. The detailed statements III.2 and III.3, including the restricted pair statement used in [Q15], retain their inherited source-audit status. They are not labeled freshly reread here. This is a narrower limitation than saying the entire arithmetic route was unaudited.

There are two particularly important exclusions. First, the effective construction in [BH13] treats the totally wildly ramified correspondence as given at its starting point. That paper's effective theorem is not used here as an independent solution of the same totally wild input. Its characteristic note is the relevant source interface. Second, the equality of pair conductors in [BH99, 2.3(vii)] is not equality of full pair epsilon factors. The introduction expressly distinguishes these strengths.

25.5 The resulting candidate still has a native attachment obligation

Under the recorded source and classical hypotheses, [Q15] obtains a cuspidal candidate with prescribed central character, torsion number one, standard Swan conductor 24, ordinary conductor 33, and ordinary self-pair conductor 232. The characterizing information required by the paper's H5 contract is not replaced by that list.

In particular, one cannot compare an essentially discrete source with a reducible principal series outside the scope of a pair theorem and then divide a pair conductor by nine to manufacture the standard conductor. Nor can a first-order calculation of a parameter on deep field units establish occurrence of a full simple character. The original unit domain and noncompact extension remain part of that question.

The next useful arithmetic step is therefore not another calculation of the same three integers. It is to construct or identify the actual native type and its complete character in this fixed candidate, and to compare its marked data with the supplied source. That is the boundary the finite results in the following sections are designed to expose.

Source and status. [Q15] is the inherited source-defined constructor audit. [BH99, printed pp. 15, 18, 23–25, 53–54] and [BH13, p. 8] were freshly checked. [HL11] remains a named classical input with an explicit fresh-access limitation. No source-defined candidate is substituted for a separately fixed type candidate without a proof of their identification.

26 A full dyadic extension remembers its squares

26.1 The extension criterion with a fixed center

Let V be a finite-dimensional vector space over $\mathbf{F}_2$, and consider central extensions

$$1 \longrightarrow Z \longrightarrow E_s \longrightarrow V \longrightarrow 1, \qquad 1 \longrightarrow Z \longrightarrow E_t \longrightarrow W \longrightarrow 1,$$

where Z is the same abelian group in both extensions. In applying this to a represented scalar pushout, the represented center and any already factored character kernel must be recorded. A pushout cannot recover invisible elements that have been discarded.

Fix a linear isomorphism $f : V \to W$. Choose an ordered basis (e_i) of V, lifts a_i in E_s, and lifts b_i of $f(e_i)$ in E_t. Put $s_i = a_i^2$ and $t_i = b_i^2$ in Z. Write $Z^2 = \{z^2 : z \in Z\}$.

Proposition 26.1 (Center-fixing extension criterion)**.** *There exists an isomorphism $F : E_s \to E_t$ fixing Z pointwise and lifting f if and only if*

$$[a_i, a_j] = [b_i, b_j] \quad \text{for every } i, j, \qquad s_i t_i^{-1} \in Z^2 \quad \text{for every } i. \tag{26.1}$$

If the generating lifts themselves must satisfy $F(a_i) = b_i$, then their squares must agree exactly, not merely modulo Z^2.

Proof. Any such isomorphism has $F(a_i) = \lambda_i b_i$ for some $\lambda_i \in Z$. Preservation of squares gives $s_i = \lambda_i^2 t_i$, while preservation of commutators gives the first condition. Conversely choose λ_i with these squares. Every group element has a unique normal form

$$z a_1^{v_1} \cdots a_r^{v_r}, \qquad z \in Z, \quad v_i \in \{0, 1\}.$$

Define its image by replacing a_i with $\lambda_i b_i$. Normal-form multiplication uses only central elements, the displayed squares, and the pairwise commutators. The conditions ensure that each multiplication relation is preserved. The inverse construction proves bijectivity. Taking every $\lambda_i = 1$ gives the final assertion. □

For $Z = \mu_2 = \{1, -1\}$, each square is independent of the chosen lift and defines a quadratic refinement

$$a_v^2 = (-1)^{q(v)}, \qquad q(v + w) = q(v) + q(w) + B(v, w), \tag{26.2}$$

where $(-1)^{B(v,w)}$ is the commutator. A center-fixing extension isomorphism over a specified symplectic map f therefore requires $q_t \circ f = q_s$.

For $Z = \mu_4$, multiplying a lift by i changes its square by -1. The unmarked square classes are consequently weaker. For $Z = \mathbf{C}^\times$, every square ratio has a square root and the unmarked criterion reduces to matching commutators. Neither statement deletes a separately marked finite subgroup or a prescribed family of lifts.

26.2 Dihedral and quaternion data inside the same scalar pushout

Consider the matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Z_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They satisfy $X^2 = Z_0^2 = 1$ and $XZ_0 = -Z_0X$. The group $H_D = \langle X, Z_0 \rangle$ is a dihedral group of order eight. The group $H_Q = \langle iX, iZ_0 \rangle$ is a quaternion group of order eight. On their common two-dimensional quotient, their refinements are

$$q_D(x,y) = xy, \qquad q_Q(x,y) = xy + x + y. \tag{26.3}$$

Marked extension	$q(e)$	$q(f)$	$q(e+f)$	$\sum_v (-1)^{q(v)}$
H_D	0	0	1	$+2$
H_Q	1	1	1	-2

Both scalar pushouts identify with

$$P = \{zX^aZ_0^b : z \in \mathbf{C}^\times,\ a,b \in \{0,1\}\}.$$

The marked subgroups remain different. In H_D, five nonidentity elements have order two and two have order four. In H_Q, only the central nonidentity element has order two and six have order four. In particular, no center-fixing automorphism of P carries one marked subgroup onto the other.

On the other hand, a quaternion source does embed into P through iX, iZ_0. Its squares and negative Gauss sign are retained. What fails is identifying it with the *independently chosen* dihedral subgroup H_D. This example separates a successful transported marking from a false assertion about a fixed native marking.

26.3 A full marked section contains more than a square table

Suppose a normalized section $\sigma : V \to E$ has product cocycle

$$\sigma(v)\sigma(w) = c(v,w)\sigma(v+w).$$

If an isomorphism must send each specified section element to a specified target section element, the two cocycles must agree after transport on *every* pair. Equality of polar forms and square tables only identifies their extension class, not all those pointwise choices.

For example, start with the ordered section $\sigma(x,y) = X^xZ_0^y$ of H_D. Change only the value at $e+f$ by a central sign. Every square remains unchanged because $(-1)^2 = 1$, but the product of the specified lifts at e and f now differs by a sign from the specified lift at $e+f$. An isomorphism fixing the two generating lifts cannot also implement this changed third assignment. The new tests retain this counterexample rather than treating it as a harmless convention change.

26.4 The valid scalar-pushout attachment contract

There is nevertheless a positive attachment statement in the restricted sector already used by Paper XI. Suppose an actual type, its full character and its Heisenberg representation η_W are supplied. Suppose the graph has one vertex, the prescribed stabilizer is trivial, and the source scalar pushout has the same

faithfully represented center and a nondegenerate quotient V with $\dim V \leq \dim W$. No conflicting host-native finite marking is independently fixed.

Choose a symplectic embedding $f : V \to W$. Proposition 26.1 gives a scalar-center-fixing isomorphism of the source pushout with the restriction over $f(V)$. Carry every supplied finite subgroup, every lift, and every section value through this *same* isomorphism. Since it is a homomorphism fixing scalars, it preserves all squares, commutators and section products, not just their classes.

Writing $W = f(V) \perp U$, the finite Heisenberg representation factors as $\eta_V \otimes \eta_U$. A projector onto a line in the second factor gives the source Heisenberg sector inside the actual image algebra. The inherited recovery theorem then returns the unchanged full inducing representation and candidate, under its stated recovery hypotheses. There is no stabilizer equation because the stipulated stabilizer is trivial.

This is an attachment into the scalar extension of the actual type. It is not a claim that every source lift is an unscaled arithmetic matrix, nor that the transported subgroup equals an independently specified canonical host subgroup. A package that records only the unmarked ambient scalar extension loses precisely the information displayed by (26.3).

Source and status. [Q16] contains the complete marked scalar-pushout contract. The normal-form proof, the dihedral/quaternion calculations and the section counterexample are explicit finite arguments. Their application uses the already supplied actual type and inherited recovery theorem. This section does not construct those arithmetic inputs.

27 The sharp boundary for a fixed native refinement

27.1 Quadratic type and its Gauss sign

Let (V, B) be a nondegenerate alternating space over $\mathbf{F}_2$ of dimension $2s$, and let q be a quadratic refinement. In a symplectic basis (e_j, f_j),

$$q(x_1, y_1, \dots, x_s, y_s) = \sum_{j=1}^{s} \big(x_j y_j + q(e_j)x_j + q(f_j)y_j\big).$$

Its Gauss sum factors into plane sums:

$$G(q) = \sum_{v \in V} (-1)^{q(v)} = 2^s (-1)^{\sum_j q(e_j)q(f_j)}. \tag{27.1}$$

Define $\mathrm{Arf}(q) \in \mathbf{F}_2$ by the sign in (27.1). The sum is intrinsic, so this definition is independent of the symplectic basis.

Lemma 27.1 (Classification in the required finite setting)**.** *Two nondegenerate quadratic spaces over $\mathbf{F}_2$ are isometric if and only if they have the same dimension and Arf invariant. Both Arf types occur in every positive even dimension.*

Proof. The coordinate expression decomposes a space into nondegenerate planes. On a plane, the three refinements with Arf invariant zero are isometric to $H(x, y) = xy$, while the unique remaining type is $A(x, y) = xy + x + y$. This can be checked on its three nonzero vectors.

Two anisotropic planes can be replaced by two hyperbolic planes. In $A \perp A$, the vector $u = e_1 + e_2$ is nonzero and isotropic. Choose v with $B(u, v) = 1$. Replacing v by $v + q(v)u$ makes it isotropic too, so u, v span a hyperbolic plane. Its nondegenerate orthogonal complement is a plane whose Gauss sign is

positive by multiplicativity of (27.1). It too is hyperbolic. Pairing all anisotropic planes leaves either only hyperbolic planes or one anisotropic plane. The parity of the remaining anisotropic planes is exactly the Arf invariant. These normal forms prove classification and existence. □

In dimension six, (27.1) gives $G(q) = +8$ or -8. The numbers of zero values are respectively

$$\#q^{-1}(0) = \frac{64+8}{2} = 36, \qquad \#q^{-1}(0) = \frac{64-8}{2} = 28. \tag{27.2}$$

These counts include the zero vector. The new suite verifies both refinements on all 64 vectors. They supply a concrete signature of the distinction that dimension alone misses.

27.2 One spare plane is enough only when the embedding is free

Proposition 27.2 (Fixed-refinement capacity criterion)**.** *Let (V, q_V) and (W, q_W) be nondegenerate quadratic spaces of dimensions $2s$ and $2R$. When the embedding may be freely chosen, a quadratic isometric embedding $V \to W$ exists exactly when*

$$R \geq s, \qquad R = s \implies \mathrm{Arf}(q_V) = \mathrm{Arf}(q_W). \tag{27.3}$$

If $R > s$, either source Arf type embeds.

Proof. An embedding requires the dimension inequality. At equal dimension it is an isometry, so the Arf invariants must agree. If $R > s$, choose a nondegenerate space U of dimension $2(R-s)$ with

$$\mathrm{Arf}(q_U) = \mathrm{Arf}(q_W) + \mathrm{Arf}(q_V).$$

Both choices are available by Lemma 27.1. The space $V \perp U$ has the same dimension and Arf invariant as W, hence is isometric to W. Restrict that isometry to V. The equal-dimension sufficiency follows from the same classification. □

After the quadratic embedding is chosen, the preimage of its image in an actual μ_2-central extension of W has the source refinement. Proposition 26.1 supplies the extension isomorphism. Exact generating lifts can be matched when their squares and commutators match. Additional prescribed section values retain the stronger product condition from Section 26.

The spare-plane argument does not permit arbitrary changes to fixed depth placements, transport maps, or stabilizer actions. It proves existence of an otherwise free quadratic embedding. These qualifications are important even when the host has a large dimensional surplus.

27.3 The exact rank-eight consequence

At $d = 3$ in (24.5), the full dyadic quotient and the three-plane source both have dimension six. Suppose the source planes have Arf invariants $\epsilon_1, \epsilon_2, \epsilon_3$, and a particular actual host refinement has Arf invariant ϵ_W. Then the necessary and sufficient test for a *free* quadratic embedding is

$$\epsilon_W = \epsilon_1 + \epsilon_2 + \epsilon_3 \quad \text{in } \mathbf{F}_2. \tag{27.4}$$

There is no orthogonal complement in which to absorb a mismatch. For $d \geq 4$, the full profile has a positive-dimensional complement, so the finite Arf obstruction disappears when the embedding is free and the actual host refinement is supplied.

This does not compute ϵ_W for a particular arithmetic type. A Heisenberg commutator quotient alone supplies an alternating form. It does not canonically select a finite two-element-center extension or its square refinement. Determining an independently required native refinement and its lift phases for the chosen candidate remains an arithmetic obligation.

Conversely, under the transported scalar-pushout contract of Section 26, the host refinement is part of the transported output rather than independently prescribed. Every source marking can then be retained in the same scalar extension once the symplectic capacity and recovery hypotheses hold. There is no contradiction between this positive statement and (27.4). They quantify over different data.

Source and status. The capacity criterion was recorded in [Q16]. Its finite classification input and proof are made explicit here. The rank-eight deduction combines that criterion with [D167]. The block does not assign a canonical native Arf invariant to an actual arithmetic candidate that has not supplied one.

28 An explicit correction for a quadratic marking defect

28.1 The discrepancy is a vector, not an unspecified search

Suppose a symplectic embedding $f : (V, B_V) \to (W, B_W)$ has already been supplied. Put

$$q'(v) = q_W(f(v)), \qquad \ell(v) = q'(v) + q_V(v).$$

The two quadratic refinements have the same polar form, so ℓ is linear. Nondegeneracy gives a unique vector $a \in V$ such that

$$\ell(v) = B_V(a, v) \quad \text{for every } v. \tag{28.1}$$

Finding a is therefore one linear solve over $\mathbf{F}_2$. It is not a search over all symplectic automorphisms.

Lemma 28.1 (Arf change under a linear defect)**.** *With the above notation,*

$$\mathrm{Arf}(q') + \mathrm{Arf}(q_V) = q_V(a). \tag{28.2}$$

Proof. The refinement identity gives

$$q_V(v) + B_V(a, v) = q_V(v + a) + q_V(a).$$

Translation by a permutes V. Summing the signs yields $G(q') = (-1)^{q_V(a)} G(q_V)$. Equation (27.1) identifies the two signs. □

Define the transvection

$$T_a(v) = v + B_V(a, v)a. \tag{28.3}$$

It is symplectic and satisfies $T_a^2 = 1$. For $a = 0$ it is the identity.

Proposition 28.2 (One-transvection quadratic correction)**.** *If $q_V(a) = 0$, then*

$$q'\big(T_a(v)\big) = q_V(v) \quad \textit{for every } v. \tag{28.4}$$

Thus $f \circ T_a$ is a quadratic isometric embedding. The condition is equivalent to equality of the Arf invariants of q' and q_V.

Proof. Write $b = B_V(a, v)$. Since $B_V(a, a) = 0$, one has $B_V(a, T_a v) = b$. Also

$$q_V(T_a v) = q_V(v) + b q_V(a) + b.$$

Adding the defect term $B_V(a, T_a v) = b$ gives

$$q'(T_a v) = q_V(v) + b q_V(a).$$

This equals $q_V(v)$ when $q_V(a) = 0$. For completeness, expanding the two cross terms in $B_V(T_a v, T_a w)$ makes them cancel in characteristic two, so T_a is symplectic. Its square is the identity because the same value b is added twice. Lemma 28.1 proves the equivalence of the condition. □

This yields a direct finite correction procedure: compute the defect on a basis, solve for a, test $q_V(a)$, and, when it vanishes, compose with T_a. The test suite verifies the formula for all linear defects in dimensions two, four and six, against every vector. It also checks the symplectic identities independently.

28.2 Equal Arf does not validate an unchanged map

Take $V = \mathbf{F}_2^2$ with $q_V(x, y) = xy$ and $q'(x, y) = xy + y$. Both refinements have Arf invariant zero. The identity map does not preserve the refinement: their values differ at $(0, 1)$. The defect vector is $a = (1, 0)$, which is isotropic. The transvection $(x, y) \mapsto (x + y, y)$ repairs the map.

The existence of that repair is not a proof that the original identity map worked. When the original embedding is prescribed by an arithmetic placement or a transport diagram, replacing it must itself be permitted and checked. A corrected finite embedding must never be silently substituted into a diagram that required the old map.

28.3 Preserving a previously correct subspace

There is a useful relative form. Let $U \subseteq V$ be a subspace on which the defect vanishes. Then $a \in U^{\perp}$, so $T_a(u) = u$ for every $u \in U$. Under the isotropy condition $q_V(a) = 0$, the correction therefore fixes all previously correct vectors in U pointwise.

This statement does not automatically preserve selected lifts above those vectors. It fixes the quotient embedding there. The extension criterion and section products still determine whether the chosen lifted correction respects all prescribed elements. The distinction is small in notation but essential in a full marked packet.

28.4 A conditional equivariant form

Suppose a finite group C acts symplectically on V and W, preserves q_V and q_W, and the supplied f is C-equivariant. Then

$$\ell(gv) = q_W(f(gv)) + q_V(gv) = \ell(v).$$

Nondegeneracy in (28.1) implies $ga = a$ for every $g \in C$. Consequently T_a commutes with the action. When $q_V(a) = 0$, the correction $f \circ T_a$ remains equivariant.

Corollary 28.3 (Equivariant finite correction under supplied actions)**.** *A supplied equivariant symplectic embedding with matching pullback Arf invariant admits the explicit equivariant quadratic correction of Proposition 28.2. If its defect already vanishes on a prescribed subspace, the correction fixes that subspace pointwise.*

Proof. The invariance computation makes a fixed by C. Inserting $ga = a$ into (28.3) gives $gT_a = T_a g$. The subspace assertion follows from $a \in U^\perp$. □

These are finite statements conditional on actual actions and an initial equivariant embedding. They do not construct such actions inside an arithmetic normalizer. Nor do they show that an arbitrary quadratic embedding provided by a dimension argument can be chosen equivariantly. The relative correction is a way to finish a specified finite step, not a replacement for its input.

Source and status. This section develops an explicit elementary repair from the linear-defect observation in [Q16]. The Gauss-sum proof and transvection computation are given in full. They are ordinary finite deductions, not a new Lean result or a claim of a new general arithmetic existence theorem.

29 Full extension actions and the remaining phase boundary

29.1 The discrepancy after a quadratic embedding has been chosen

Even an equivariant quadratic embedding need not give the required equivariant isomorphism of the full finite extensions with every prescribed marking. This section isolates a conditional repair of the group-action part.

For clarity, take nondegenerate μ_2-central extensions of the same quotient dimension, or restrict a larger target extension to a supplied invariant quadratic subspace. Suppose C acts genuinely on the source and target extensions, fixes the center, and induces the same quotient action after identification. Let $F : E_s \to E_t$ be a center-fixing extension isomorphism inducing that quotient identification. Such an F is supplied by the quadratic extension criterion, but is not initially assumed equivariant.

An automorphism of E_t that fixes its center and quotient is uniquely of the form

$$u_\chi(x) = (-1)^{\chi(\bar{x})} x, \qquad \chi \in \mathrm{Hom}_{\mathbf{F}_2}(W, \mathbf{F}_2). \tag{29.1}$$

Indeed, the ratio between the image of x and x is a central sign depending only on $\bar{x}$, and preservation of multiplication makes it additive. Conversely every such character gives an automorphism. These vertical automorphisms form the additive $\mathbf{F}_2$-space $A = \mathrm{Hom}(W, \mathbf{F}_2)$.

Write α_g for the target action and $\beta_g = F\alpha_g^s F^{-1}$ for the transported source action. Their discrepancy is

$$\delta_g = \beta_g \alpha_g^{-1} \in A.$$

The action of C on A is conjugation by α_g. Using additive notation in A, the genuine group-action equations give

$$\delta_{gh} = \delta_g + g\delta_h. \tag{29.2}$$

Thus the discrepancy is an actual 1-cocycle, rather than an arbitrary list of mismatched operators.

29.2 An explicit odd-order averaging repair

Proposition 29.1 (Conditional odd-action correction)**.** *Under the preceding hypotheses, suppose $|C|$ is odd. Put*

$$u = \sum_{h \in C} \delta_h \in A. \tag{29.3}$$

Replacing F by $u \circ F$, with u interpreted through (29.1), makes the extension isomorphism C-equivariant.

Proof. Sum (29.2) over h. Left multiplication permutes C, so

$$u = |C|\delta_g + gu = \delta_g + gu$$

in the characteristic-two space A. Thus $\delta_g = u + gu$. Replacing F by uF changes its discrepancy to $u + \delta_g + gu$, which is zero for every g. □

The correction fixes the scalar center and the quotient. It may change a chosen lift by a central sign. If all generating lifts or an entire section are independently fixed, that modification may be forbidden. The valid conclusion is existence of an equivariant center-fixing extension isomorphism when these vertical corrections are allowed. It is not an equivariant pointwise identification of every independently prescribed lift.

The exact suite enumerates all cocycles for a C_3 action on a two-dimensional $\mathbf{F}_2$ module and checks the averaging formula. It also retains the opposite boundary. For C_2 acting trivially on a nonzero $\mathbf{F}_2$ module, a nonzero value of δ at its generator is a 1-cocycle. Every coboundary $u + gu$ is zero. No correction of this form can remove it. Odd order is therefore a substantive hypothesis, not a cosmetic simplification.

29.3 What has not been repaired by this finite sum

The result starts after the arithmetic host, actual finite actions, invariant quotient subspace and an extension isomorphism have been supplied. It does not construct them from the original source diagram. A native-normalizer obstruction can prevent the initial action from existing even when the later finite cocycle would be removable.

There is a further representation-level boundary. An action by automorphisms of a Heisenberg group preserves the isomorphism class of its representation with fixed nontrivial central character. Intertwining operators can therefore be chosen, but their products can carry scalar discrepancies. Passing from those projective operators to the specified honest phase operators is a separate problem. Proposition 29.1 removes a discrepancy in the action on the *finite extension group*. It does not prove vanishing of every scalar projective obstruction in a representation, nor does it choose the prescribed Gauss or transport phases.

The original full character domain, its central quasicharacter, the noncompact root, and all specified section products remain part of the arithmetic contract. A finite central sign correction is acceptable only when its interaction with that contract is explicitly checked.

29.4 A useful order of operations

The two finite corrections should be used in the following order. First supply an actual invariant host and a quotient-equivariant symplectic embedding. Compare its pulled-back quadratic refinement with the source. When the defect vector is isotropic, use the equivariant transvection to correct the quadratic embedding. Next lift to the finite extensions and compute the resulting action cocycle. When the stipulated hypotheses permit it, use the odd-order sum to correct that action. Finally check every prescribed lift, section, and honest representation phase that these corrections were not authorized to change.

Each stage has a certificate and a failure mode. Combining them into one phrase such as “equivariant attachment” would hide which input is missing. In the equal-capacity case, even the first quadratic stage needs the native Arf match. In a source problem with a fixed embedding, the transvection may be disallowed. In even-order action problems, the 1-cocycle may survive. In all cases, the same arithmetic candidate must remain the subject of the construction.

Source and status. The source marking note leaves nontrivial stabilizer actions outside its established attachment theorem. This section adds a conditional finite group-action correction with a complete averaging proof and explicit even-order control. It is not a claim to have constructed general arithmetic stabilizers or solved the honest phase equations.

30 Audited claim map and the next arithmetic obligations

30.1 What this continuation adds to the progress paper

The conductor branch now has a single normalization map, a proof of the necessary period divisibility, an expanded source self-pair calculation, and a precise conditional rank-nine filter. The actual-order certificate remains visible. The full and skipped rank-eight profiles show why even several correct conductor invariants cannot establish the needed native quotient.

The classical induction branch now identifies its field tower, tower-independence input, tame descent and central normalization separately. Its characteristic extension is named, and the failed fresh access to the underlying positive-characteristic source is not concealed. This leaves a meaningful selected input audit without overstating it as a new proof of all local correspondence results.

The marking branch now contains the full center-fixing extension criterion, the dihedral/quaternion distinction, the sharp free-embedding capacity test, and the exact rank-eight Arf condition. The explicit transvection and odd-order action corrections then explain how two later finite discrepancies can be repaired when their actual inputs and allowed changes are supplied.

Statement	Status here	Boundary retained
Principal period and conductor	Ordinary derivation with [BH99] input	Does not construct a stratum
Source self-pair 232	Expanded inherited ramification argument	Original source independence is retained
Critical-four filter	Conditional actual-order consequence	Passing is not type occurrence or H5
Rank-eight profile separation	Audited inherited construction and exact grade calculations	Three invariants do not force the native profile
Classical source candidate	Selected primary-interface audit	Detailed [HL11] reread unavailable in this block
Fixed-center extension criterion	Complete finite proof	Full sections can impose additional pointwise relations
Free quadratic embedding	Complete finite proof	Requires the actual host refinement as input
Transvection correction	Complete finite proof and exact tests	May change a preassigned embedding
Odd-action correction	Complete conditional finite proof	Does not construct arithmetic actions or honest phases
Native rank-eight refinement	Explicit remaining arithmetic obligation	Its value is not inferred from capacity
General realization and H5	Open in this project	No replacement by profile matching

30.2 A certificate for the next native-marking step

For a fixed arithmetic candidate at rank eight, a useful next certificate must identify the actual finite central extension or specified finite subgroup in its scalar pushout. It must give the chosen lift of each of the six symplectic basis vectors and compute its square under the full central character. These six bits determine the refinement once the commutator basis is fixed. Their three pairwise products determine the Arf sign through (27.1).

The certificate must also say whether the target marking is independently fixed or is allowed to be transported from the source. If it is fixed, compute the defect of the required quotient map. A vanishing defect directly certifies that map. An isotropic nonzero defect supplies the transvection only when changing the map is allowed. Every additional section element and transport lift then needs its actual multiplication check.

This task cannot be replaced by evaluating a canonical-looking quadratic form on an abstract six-dimensional space. That would choose new data rather than calculate the data of the specified arithmetic object. It also cannot be replaced by choosing another candidate with the same conductors. The candidate and full inducing character must remain fixed throughout.

30.3 Verification and reproducibility boundaries

The new standard-library exact suite contains 23 test methods. Its conductor tests cover the normalization table, grade dimensions, period divisibility, the rank-nine formal-degree identities, the rank-eight profiles and the finite field used for the projective source characters. Its marking tests cover extension group laws, square and section controls, Gauss signs, all labeled symplectic bases in dimension four, all linear-defect corrections through dimension six, relative and equivariant corrections, and odd versus even action cocycles.

The inherited 15-method Block 03 finite suite and 12-method Block 01 suite are replayed as separate regression gates. Their successful execution is not a new run of every historical arithmetic program. The supplied replay record lists exactly which commands were executed. Finite enumeration counts are not converted into a percentage of conjecture completion.

The source archive preserves the earlier compact checkpoint and versioned ledger history. It does not repair the missing Block 02 editable archive by inventing its contents. The v0.45 terminal compilation receipt remains unrecovered. No fresh Lean or Lake build occurred in this block.

The dedicated conjecture visual companion is the next publication task. Its mathematical route is now clear: normalized conductors, rejected and surviving rank-nine candidates, equal invariants with unequal capacity, transported versus native markings, the equal-capacity Arf gate, and the two conditional finite corrections. The original XI monograph, visual reader and Lean guide can then be synchronized against this stable claim map without presenting the general conjecture as a theorem.

30.4 Source register

The package's machine-readable source map supplies paths and hashes for inherited records. Page numbers below are printed pages unless a PDF-page locator is explicitly stated. Web-viewed primary sources are recorded as viewed sources, not as locally downloaded byte-pinned copies.

[Q13] *Adjoint Conductor Test161*. Retained dependency of Block166. The source invariant and conditional period-nine actual-order certificate. SHA-256 and exact source path are in the source map.

[Q14] *Principal Order Conductor161*. Principal grades, leading-level normalization and necessary source matching.

[Q15] *Arithmetic Constructor Audit*. The original source-defined induction and tame-descent branch, with full-character and source-input boundaries.

[Q16] *Dyadic Markings167*. Full central extensions, transported finite markings, fixed native refinements and the sharp capacity criterion.

[D167] *Dyadic Capacity167*. Full and skipped defining profiles, floor/ceiling grade counts, and the rank-eight shared-invariant example.

[BK] Colin J. Bushnell and Philip C. Kutzko. *The Admissible Dual of GL(N) via Compact Open Subgroups*. Annals of Mathematics Studies 129, Princeton University Press, 1993. Relevant inherited inputs: recursive character orders, minimal commutator control, refinement, 6.2.5 and compact-type formal degree. The earlier selected audit remains distinguished from this block's fresh reading.

[BH99] Colin J. Bushnell and Guy Henniart. *Local tame lifting for GL(n) II: wildly ramified supercuspidals*. Astérisque 254, 1999. Freshly checked: introduction p. 2, 1.3 p. 15, 1.7 p. 18, 2.3 pp. 23–25, 6.1 pp. 53–54. The PDF page index is the printed page plus four when counted from zero.

[AP] Anne-Marie Aubert and Roger Plymen. *Plancherel measure for GL(n,F) and GL(m,D): explicit formulas and Bernstein decomposition*. arXiv:math/0302169v4, 4 October 2004. Used: formulas (17), (29), Theorems 6.5 and 6.9, Remark 6.7 and the conductor convention in Section 4. The formal-degree pages inspected are pp. 19, 26–28.

[BH13] Colin J. Bushnell and Guy Henniart. *To an effective local Langlands correspondence*. arXiv:1103.5316v2, 9 October 2013. The characteristic note on p. 8 is the selected input. Its effective construction is not used to supply its own totally wild starting correspondence.

[HL11] Guy Henniart and Bertrand Lemaire. *Changement de base et induction automorphe pour GL(n) en caractéristique non nulle*. Mémoires de la Société Mathématique de France 124, 2011. DOI: 10.24033/msmf.436. The bibliographic record was checked. Detailed statements retain inherited audit status because the full PDF could not be retrieved in this block.