OPERATION KOMOREBI

REALIZATION CALCULUS

CORE SERIES XI

Tame and Wild Packet Geometry

For local Langlands

Traditional paper

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REALIZATION CALCULUS

Abstract. Let K be a nonarchimedean local field. We resolve several finite layers that occur inside the local Langlands correspondence for general linear groups. Frobenius semisimple Weil Deligne parameters first split into irreducible Weil atoms and finite monodromy chains. In the tame case, character projectors show that every irreducible atom is one transitive Frobenius orbit, and Mackey theory gives an exact greatest common divisor and least common multiple rule for tensor packets. Regular finite field characters produce primitive cuspidal projectors and finite Gauss phases. For wild data, projective intertwiners recover an alternating commutator form. Its radical free quotient determines a finite Heisenberg representation, and maximal isotropic averaging gives an exact selector. We prove a restricted one break bridge: an equivariant symplectic isometry between a declared Galois envelope and a type side quotient induces the Heisenberg comparison uniquely up to scalar, while one cyclic phase condition removes the remaining ambiguity. Odd residue characteristic is governed by finite Weil phases. Residue characteristic two requires a quadratic refinement and its Arf invariant. A matrix holonomy theorem then gives the exact gluing criterion on a finite transport graph. The general arithmetic packet realization is stated as Conjecture 66. Detailed ordinary proofs give restricted arithmetic realizations in all odd prime-power ranks and in all dyadic ranks for the stated increasing-break families. The complete comparison of pair local factors remains a conditional input for the same arithmetic candidate.

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Reading this edition

This monograph presents the finite packet theory and retains the detailed restricted arithmetic constructions. Two separate research companions explain the later progress toward Conjecture 66. They do not replace the monograph's proved statements or turn its conditional comparison into an existence theorem.

Five readers with different jobs

Reader	Route and purpose
Core monograph	Finite tame and wild theory, restricted arithmetic constructions, and the exact realization/comparison boundary.
Core visual guide	Geometric route through the core paper. The original arithmetic examples retain their stated scope.
Lean and evidence guide	Formal statements, literal excerpts and evidence lineage. The guide distinguishes the selected source census, compiler replay and arithmetic hypotheses.
Progress paper	Toward Arithmetic Packet Realization, 77 pages, v0.4.0. Connected proofs, failed routes, conductor conventions and native-marking conditions.
Research visual atlas	Arithmetic Packet Realization, 36 pages, v0.5.0. Thirty-two worked plates accompany the progress paper.

The endpoint is a quantified construction problem

Conjecture 66 asks for an actual arithmetic witness for each admitted source with its supplied H1–H4 datum. The comparison proposition then uses H5 for that same realized candidate. Historical obligation 66, historical Theorem 16.2, and all-records row 95 refer to this endpoint's lineage. Historical register row 66 is a different statement.

The research contract in Section 1 and atlas plate V01 make the quantifiers explicit. Supplying finite operators does not construct an arithmetic representation. Likewise, proving a factor identity for a different candidate does not identify the packet's realizer.

How to use the links

Keep all five PDFs together with their delivered filenames. The colored reading links resolve to actual pages of the companion files. Section numbers remain the primary mathematical locators. The included JSON concordance records the resolved page destinations for this edition.

Current scope and research routes

Question	Route beyond the core statement
Parameter, full character and recovery	Progress Sections 7–10 and atlas V06–V10. The full character, central character and noncompact completion remain fixed.
Monomial, Clifford and occurrence routes	Progress Part II. The original character route is excluded under its hypotheses. A different corrected character passes a scalar test, but occurrence in the same candidate is still open.
Conductor and capacity	Progress Sections 22–25 and atlas V11–V19. Swan, ordinary conductor and additive-character epsilon exponent are named separately. Equal numerical invariants need not give equal quotient capacity.
Native markings and allowed repairs	Progress Sections 26–29 and atlas V20–V29. A freely transported marking is different from a prescribed native marking. Finite corrections require their stated inputs and permission to change the relevant map.
Remaining arithmetic certificate	Progress Section 30 and atlas V30–V32. General realization, H5, corrected-character occurrence and the stronger native certificate remain separate tasks.

What this revision checks

This revision preserves the paper's structure and detailed construction appendices while correcting several local formulations. The breakwise phase product now uses its actual moving spaces and states the symplectic action explicitly. The twisted-algebra proof uses direct averaging. The type-side pairing fixes its prime-field character, and the tame comparison includes the rank-one conductor and trivial-character Gauss-sum boundaries. Interaction carriers are distinguished from minimal character orbits.

Appendix B restores the approximation index $B^{[j]}$ that was converted into a footnote. The earlier dimension-factor and 44 superscript repairs remain in place. All 15 original construction notes and the exact source diff accompany the release. These source restorations and local mathematical clarifications do not supply a general arithmetic witness.

The selected Henniart–Lemaire induction inputs [20] have now been checked against the full official text, including their characteristic, rank and normalization hypotheses. The new source review is bounded to the cited application; it does not establish H5 or corrected-character occurrence.

For the current compiler and axiom results, consult the Lean and evidence guide and the dated verification receipts. Historical evidence retains its original scope. Internal review is not external peer review. The complete Block 2 editable archive remains unrecovered, so its research-reader pages are preserved rather than claimed as a unified source rebuild.

1 Introduction

The local Langlands correspondence for $\mathrm{GL}_n(K)$ is an established theorem. It matches irreducible smooth representations of $\mathrm{GL}_n(K)$ with Frobenius semisimple Weil Deligne representations of dimension n. Its proof and normalization draw on global methods, harmonic analysis, types, local factors, and the geometry of Shimura varieties [19, 17, 8]. The present article asks a narrower structural question. Which parts of the local correspondence admit finite packet coordinates whose projectors, transfers, phases, and gluing laws can be proved directly?

The answer has a sharp division. The tame layers are explicit. A tame irreducible Weil representation is induced from a character of an unramified extension. Its inertia restriction is one Frobenius orbit. Regular characters of elliptic finite field tori give cuspidal blocks through the Deligne and Lusztig construction [13]. Their tensor products are controlled by a greatest common divisor and least common multiple decomposition. The corresponding cycle orientation and finite Gauss phases can be computed exactly. These features are closely related to the tame lifting and essentially tame correspondence of Bushnell and Henniart [4, 5, 6, 7, 9].

Wild data are different. The restriction to wild inertia need not split into cyclic characters. On the type side, simple character theory produces finite symplectic quotients and Heisenberg representations [11]. Canonical beta extensions fix a phase ambiguity in the extension to the normalizer [2, 14]. On the Galois side, higher ramification gives a canonical filtration and strong numerical constraints [10]. The missing construction is not another finite matrix calculation. It is a choice free passage from an arbitrary wild inertia atom to a finite symplectic envelope that agrees with the type side quotient.

We therefore separate three levels of assertion. First, the article proves finite algebraic theorems about tame orbits, Heisenberg groups, twisted group algebras, finite Weil phases, and transport graphs. Second, it proves a restricted one break bridge once a normal wild datum and an equivariant symplectic identification are declared. Third, it records the exact additional input needed to pass from that restricted bridge to an independent construction of all wild atoms.

The finite Heisenberg theory used here is classical. We use the Stone von Neumann theorem, finite Weil representations, and standard projective representation theory [15, 25, 26, 22, 16]. The contribution is the way these results are arranged into one comparison diagram and the matrix holonomy theorem that controls finite gluing. In residue characteristic two, the alternating form does not determine the phase. A quadratic refinement and its Arf invariant are required. The underlying quadratic form theory is standard [24].

Local factors provide the final test. Henniart showed that pair epsilon factors characterize the local correspondence in the required range [18]. The tame packet calculations below recover exact local polynomials and finite Gauss phases for the declared tame atoms. They do not supply the complete pair factor comparison for arbitrary wild atoms. This boundary is retained throughout the paper.

This reading edition separates the missing arithmetic realization from the conditional identification of one actual candidate. The endpoint formerly listed as proof obligation 66 (historical Theorem 16.2, row 95 of the all-records register) is now conjecture 66. The finite theorem statements and proofs include the geometric-Frobenius convention correction and the additional local corrections recorded in this edition's reading guide. The later restricted constructions are developed in section 19 and the detailed construction appendices. Their evidence is ordinary proof with retained internal reviews, rather than new Lean certificates. These reviews are not external peer review.

The accompanying release includes Markdown (`.md`) files containing detailed proofs, construction notes, proof audits, and research records. These documents distinguish established results, conditional results, and the remaining obligations associated with Conjecture 66.

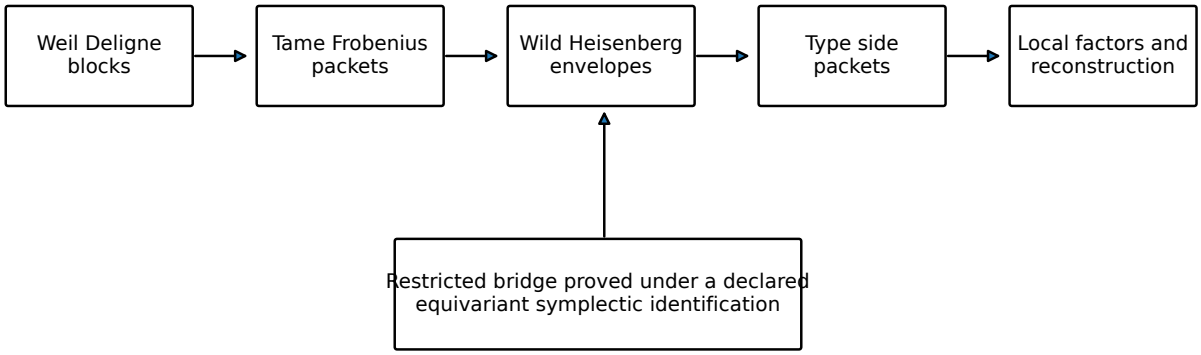


Figure 1. The finite theorem layers and the remaining wild boundary.

The article is organized as follows. Weil Deligne chains appear in section 2. Tame Frobenius packets, finite field cuspidal projectors, tensor refinement, and Gauss phases are developed in sections 3 to 6. Finite Heisenberg groups and projective commutator envelopes appear in sections 7 and 8. The restricted one break Galois envelope and its comparison with the type side are treated in sections 9 to 11. Odd and dyadic phase data are separated in sections 12 and 13. Matrix holonomy and breakwise reconstruction appear in sections 14 and 15. The local-factor boundary appears in section 16. Examples and scope follow in sections 17 and 18. Section 19 and the appendices develop the restricted arithmetic constructions. The separate progress paper reconciles subsequent routes, corrections and remaining obligations without replacing these source records.

2 Weil Deligne chains

Let W_K be the Weil group of K, let I_K be inertia, let P_K be wild inertia, and choose a geometric Frobenius lift Φ. A Weil Deligne representation is a pair (r, N) with

$$r(w)Nr(w)^{-1} = |w|_K N.$$

Standard Fact 2.1 (Indecomposable monodromy blocks)**.** Every Frobenius semisimple Weil Deligne representation is a direct sum of blocks

$$\mathrm{Sp}_m(\tau),$$

where τ is an irreducible Weil representation and the nilpotent operator is one Jordan chain of length m.

This standard decomposition reduces the construction of a local correspondence to two tasks. One must understand irreducible Weil atoms, then one must attach the correct generalized Steinberg chain. The second task is formal once the first is known.

Definition 2.2 (Monodromy packet chain)**.** Let $V_0, \dots, V_{m-1}$ be copies of the space of τ. Put

$$V = \bigoplus_{j=0}^{m-1} V_j.$$

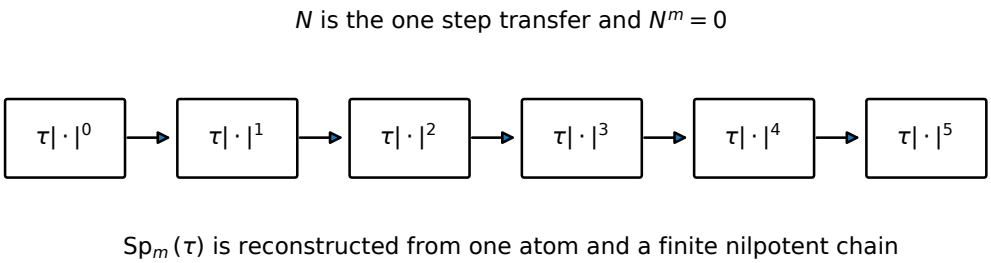


Figure 2. A Weil Deligne chain is reconstructed from one atom and finitely many nearest neighbor transfers.

A monodromy packet chain is a nilpotent operator N with

$$N(V_j) \subseteq V_{j-1}$$

for $j > 0$, with $N(V_0) = 0$, together with a Frobenius operator whose conjugation scales N by q^{-1}.

Proposition 2.3 (Exact block reconstruction)**.** *Let P_j be the projector onto V_j. Then*

$$N = \sum_{j=1}^{m-1} P_{j-1} N P_j, \qquad N^m = 0.$$

Every power N^r is the sum of the packet paths of length r along the chain.

Proof. The projectors resolve the identity. Since N maps only from V_j to V_{j-1}, every block except the displayed nearest neighbor blocks vanishes. A product of r nonzero blocks moves down exactly r levels. No path of length m exists, so $N^m = 0$. □

Proposition 2.4 (Scalar covariance model)**.** *Let*

$$F = \mathrm{diag}(1, q, q^2, \ldots, q^{m-1})$$

and let $N_{j-1,j} = 1$. Then

$$FNF^{-1} = q^{-1}N.$$

Proof. The only nonzero entry in column j and row $j-1$ is multiplied by $q^{j-1}q^{-j} = q^{-1}$ under conjugation. □

Theorem 2.5 (Formal atomic reconstruction)**.** *Assume that every irreducible Weil representation τ has been assigned a supercuspidal representation σ_τ, compatibly with isomorphism and unramified twist. Then the rules*

$$\mathrm{Sp}_m(\tau) \longmapsto \mathrm{St}_m(\sigma_\tau)$$

and normalized parabolic induction determine a candidate correspondence on all Frobenius semisimple Weil Deligne parameters.

Proof. Decompose the parameter into indecomposable monodromy blocks. Replace each Weil atom by its assigned supercuspidal atom and each monodromy chain by the corresponding generalized Steinberg block. Order the resulting essentially square integrable factors in Langlands order and take the unique

irreducible quotient. The multiset of indecomposable Weil Deligne blocks is intrinsic, so the result does not depend on the chosen display. □

Remark 2.6. Theorem 2.5 is a reconstruction theorem. It does not produce the atomic assignment $\tau \mapsto \sigma_\tau$, and it does not prove the required local factor identities.

3 Tame Frobenius packets

Assume that P_K acts trivially. After passing to a finite quotient, tame inertia is cyclic. Let t generate a quotient of order M, and put $\zeta_M = e^{2\pi i/M}$. For $j \in \mathbb{Z}/M\mathbb{Z}$, define

$$P_j = \frac{1}{M} \sum_{a=0}^{M-1} \zeta_M^{-ja} r(t)^a.$$

Proposition 3.1 (Tame character projectors)**.** *The operators P_j are pairwise orthogonal idempotents whose sum is the identity. Frobenius permutes them by*

$$\Phi P_j \Phi^{-1} = P_{qj}.$$

Proof. The first assertion is the Fourier idempotent calculation for a cyclic group. For the geometric Frobenius convention of section 2, the relation is $\Phi^{-1} t \Phi = t^q$. Equivalently, $\Phi t \Phi^{-1} = t^{q^{-1}}$, with inverse taken modulo M since $\gcd(q, M) = 1$. Conjugating the Fourier sum and changing its summation index therefore gives $\Phi P_j \Phi^{-1} = P_{qj}$. The inverse Frobenius convention reverses the label direction while preserving the same orbits. □

Theorem 3.2 (Tame orbit induction)**.** *Let τ be an irreducible complex representation of W_K with trivial wild inertia. Then the nonzero tame character packets form one transitive Frobenius orbit of length f. If K_f/K is the unramified extension of degree f, there is a character ξ of W_{K_f} such that*

$$\boxed{\tau \cong \operatorname{Ind}_{W_{K_f}}^{W_K} \xi.}$$

Proof. The nonzero tame packets are permuted by Frobenius. A union of Frobenius orbits is stable under W_K, so irreducibility forces one orbit. Choose one packet V_θ. Its stabilizer is $W_{K_f} = \langle I_K, \Phi^f \rangle$. A nonzero proper stable subspace of V_θ would generate a nonzero proper W_K stable subspace after summing its Frobenius translates. Thus V_θ is irreducible for the stabilizer. Inertia acts on it by the scalar character θ, and the remaining action is generated by Φ^f. The stabilizer action on this packet is therefore one dimensional. Induction reconstructs the complete orbit. □

Corollary 3.3 (Dimension and regularity)**.** *The dimension of τ equals the Frobenius orbit length of its tame inertia character. The inducing character is regular, meaning that no smaller positive power of Frobenius fixes it.*

Proof. The induced representation has one line for every Frobenius translate of the chosen character. A shorter orbit would produce a proper stabilizer and contradict the minimality of f. □

Proposition 3.4 (Dual and self dual tame packets)**.** *For a tame atom*

$$\tau_\xi = \operatorname{Ind}_{W_{K_f}}^{W_K} \xi,$$

we have

$$\tau_\xi^\vee \cong \operatorname{Ind}_{W_{K_f}}^{W_K} \xi^{-1}.$$

The atom is self dual exactly when

$$\xi^{-1} = \xi^{\Phi^j}$$

for some $j \in \mathbb{Z}/f\mathbb{Z}$.

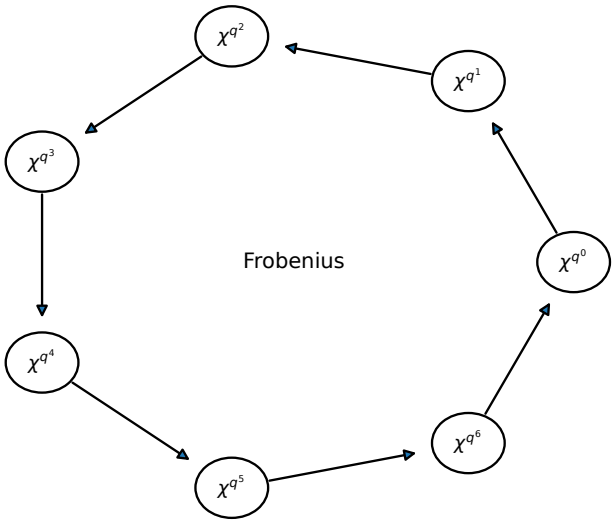


Figure 3. A tame irreducible atom is one transitive Frobenius orbit.

Proof. Duality commutes with finite index induction, giving the first statement. Two regular induced tame atoms are isomorphic exactly when their inducing characters lie in the same Frobenius orbit. Applying this to ξ and ξ^{-1} gives the criterion. □

Proposition 3.5 (Unramified twist)**.** *Let ν be an unramified character of W_K. Then*

$$\tau_\xi \otimes \nu \cong \mathrm{Ind}_{W_{K_f}}^{W_K} \left(\xi \, \nu|_{W_{K_f}} \right).$$

The tame inertia orbit is unchanged. If the original Frobenius cycle has closing scalar a, the twisted cycle has closing scalar

$$a\nu(\Phi)^f.$$

Proof. The tensor identity follows from the projection formula for induction. Since ν is trivial on inertia, it does not change any tame character packet. It multiplies every Frobenius step by $\nu(\Phi)$, so a closed cycle of length f gains the factor $\nu(\Phi)^f$. □

3.1 Regular orbit counts

Characters of $\mathbb{F}_{q^n}^\times$ are indexed by exponents modulo $q^n - 1$, and Frobenius acts by multiplication by q.

Proposition 3.6 (Möbius orbit count)**.** *The number of regular Frobenius orbits of length n is*

$$\boxed{c_n(q) = \frac{1}{n} \sum_{d \mid n} \mu(d) \, (q^{n/d} - 1).}$$

Proof. For every divisor $r \mid n$, the characters fixed by Frob^r are exactly those descending to $\mathbb{F}_{q^r}^\times$, and there are $q^r - 1$ of them. Möbius inversion gives the number of characters of exact orbit length n. Division by n gives the number of orbits. □

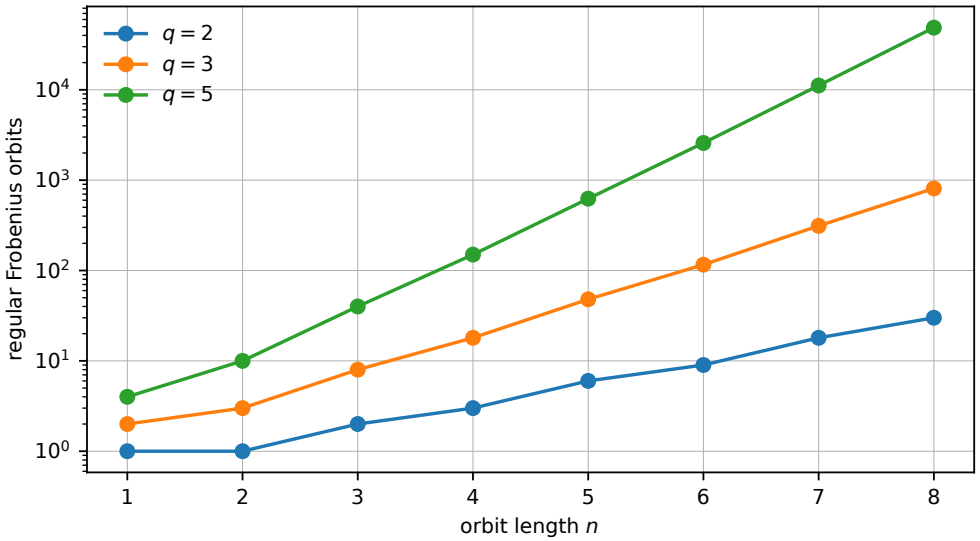


Figure 4. Regular Frobenius orbit counts for three residue field sizes.

Example 3.7 (Degree four over $\mathbb{F}_2$)**.** The regular exponent orbits modulo fifteen are

$$(1,2,4,8), \qquad (3,6,12,9), \qquad (7,14,13,11).$$

The formula gives

$$\frac{1}{4}\big((2^4-1)-(2^2-1)\big)=3.$$

4 Finite field cuspidal projectors

Let

$$G_n=\mathrm{GL}_n(\mathbb{F}_q), \qquad T_n\cong\mathbb{F}_{q^n}^{\times}$$

with T_n an elliptic maximal torus.

Standard Fact 4.1 (Regular elliptic torus characters)**.** A regular character $\theta : T_n \to \mathbb{C}^{\times}$ determines an irreducible cuspidal representation κ_θ of G_n. Frobenius conjugate characters determine isomorphic representations, and every irreducible cuspidal representation of G_n occurs in this way.

This is the finite field input from Deligne and Lusztig theory [13]. We use standard finite group character theory for the central idempotents [28, 21]. Once κ_θ is known, its block is selected by a central idempotent.

Proposition 4.2 (Primitive cuspidal idempotent)**.** *Let χ_θ be the irreducible character of κ_θ, and let $d_\theta=\chi_\theta(1)$. Then*

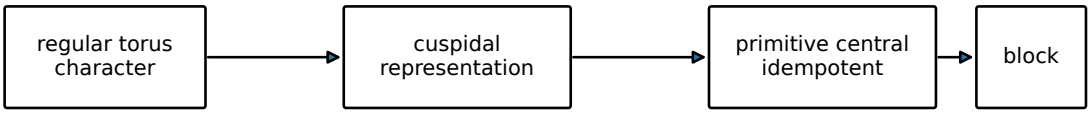
$$e_\theta=\frac{d_\theta}{|G_n|}\sum_{g\in G_n}\chi_\theta(g^{-1})g$$

is the primitive central idempotent selecting the κ_θ block in $\mathbb{C}[G_n]$.

Proof. On an irreducible representation of character ψ, the central element acts by a scalar. Taking traces shows that the scalar is

$$\frac{d_\theta}{|G_n|\,\psi(1)}\sum_{g\in G_n}\chi_\theta(g^{-1})\psi(g).$$

Character orthogonality makes this scalar one for $\psi=\chi_\theta$ and zero otherwise. The element is therefore central, idempotent, and primitive. □



Parabolic unipotent averages vanish on the cuspidal block.

Figure 5. A regular torus character determines a cuspidal block and its primitive central projector.

Proposition 4.3 (Parabolic annihilation)**.** *Let $P = LU$ be a proper parabolic subgroup of G_n, and put*

$$e_U = \frac{1}{|U|} \sum_{u \in U} u.$$

If κ_θ is cuspidal, then

$$\boxed{e_U e_\theta = e_\theta e_U = 0.}$$

Proof. The operator e_U projects onto the subspace of U fixed vectors. A cuspidal representation has no nonzero fixed vector under the unipotent radical of a proper parabolic. Thus e_U acts by zero on the κ_θ block. Since e_θ is central, both products vanish. □

4.1 Depth zero reconstruction and cycle orientation

The depth zero construction inflates a finite cuspidal representation to $\mathrm{GL}_n(\mathcal{O}_K)$, extends it to the standard compact modulo center subgroup, and then compactly induces it to $\mathrm{GL}_n(K)$. This is classical type theory [11, 8].

The induced tame Weil atom has an elementary cycle invariant. In an orbit basis $e_0, \dots, e_{n-1}$, write

$$\tau(\Phi) e_j = e_{j+1}$$

for $j < n - 1$, and

$$\tau(\Phi) e_{n-1} = a e_0.$$

Proposition 4.4 (Frobenius cycle polynomial)**.** *The Frobenius transfer matrix has characteristic polynomial*

$$X^n - a$$

and determinant

$$\boxed{\det \tau(\Phi) = (-1)^{n-1} a.}$$

Proof. The matrix is the companion matrix of $X^n - a$. Its only nonzero determinant term is the closed n cycle. That cycle has sign $(-1)^{n-1}$, and the closing edge contributes a. □

Corollary 4.5 (Unramified orientation character)**.** *The sign $(-1)^{n-1}$ is the orientation of the closed Frobenius packet cycle. It is unchanged by rescaling the individual orbit basis vectors.*

Proof. A basis rescaling conjugates the transfer matrix by a diagonal matrix, so the determinant and characteristic polynomial remain unchanged. □

5 Tensor refinement of tame atoms

Let

$$\tau_\xi = \operatorname{Ind}_{W_{K_n}}^{W_K} \xi, \qquad \tau_\eta = \operatorname{Ind}_{W_{K_m}}^{W_K} \eta,$$

where K_n/K and K_m/K are unramified. Put

$$d = \gcd(n, m), \qquad L = \operatorname{lcm}(n, m).$$

Theorem 5.1 (Greatest common divisor and least common multiple packets)**.** *There is a decomposition*

$$\boxed{\tau_\xi \otimes \tau_\eta \cong \bigoplus_{i=0}^{d-1} \operatorname{Ind}_{W_{K_L}}^{W_K} \left(\xi_{L,n} \eta_{L,m}^{\Phi^i} \right),}$$

where $\xi_{L,n}$ and $\eta_{L,m}$ are the norm lifts to K_L.

Proof. Apply Mackey decomposition [23]. The double cosets

$$W_{K_n} \backslash W_K / W_{K_m}$$

are the orbits of simultaneous translation on $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$. There are d orbits, each of length L. The stabilizer of a point in each orbit is W_{K_L}. The second character is shifted by the chosen Frobenius representative. Summing the induced terms over the double cosets gives the decomposition. □

Corollary 5.2 (Packet count and dimension)**.** *The tensor product has d induced interaction packets, each with carrier dimension and Frobenius coset-cycle length L. The interaction character may have a shorter inertia orbit, in which case a packet may split further into irreducible atoms. The total dimension is preserved because*

$$dL = nm.$$

Proof. The double coset calculation in theorem 5.1 gives exactly d summands, and every stabilizer is W_{K_L}, so every induced packet has dimension L. The identity $dL = nm$ follows from the definitions of greatest common divisor and least common multiple. □

5.1 Exponent coordinates

Represent the residue characters by exponents

$$a \pmod{q^n - 1}, \qquad b \pmod{q^m - 1}.$$

Proposition 5.3 (Refined exponent formula)**.** *The ith interaction packet on $\mathbb{F}_{q^L}^\times$ has exponent*

$$\boxed{c_i = a\frac{q^L - 1}{q^n - 1} + bq^i \frac{q^L - 1}{q^m - 1} \pmod{q^L - 1}.}$$

It is trivial on tame inertia exactly when $c_i = 0$.

Proof. Norm lifting a character exponent from $\mathbb{F}_{q^n}^\times$ to $\mathbb{F}_{q^L}^\times$ multiplies the exponent by $(q^L - 1)/(q^n - 1)$. The Frobenius shift multiplies the second exponent by q^i. Multiplication of characters adds exponents. Triviality is equivalent to the resulting exponent being zero modulo $q^L - 1$. □

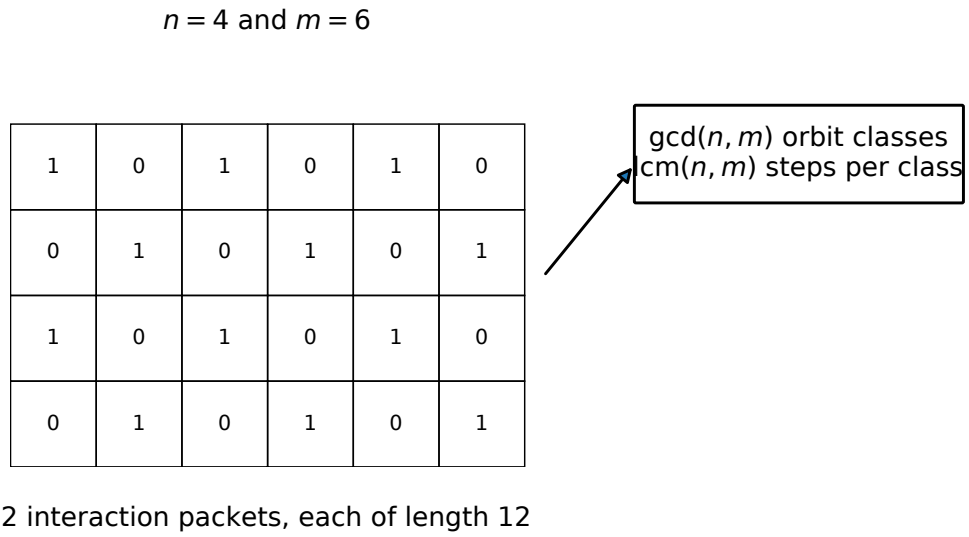


Figure 6. Tensor refinement separates into greatest common divisor many packets, each with least common multiple carrier dimension.

5.2 Tame local polynomials

Let V be one interaction packet. If its inertia character is nontrivial, then $V^{I_K} = 0$. If the inertia character is trivial, Frobenius acts on the invariant packet by a companion cycle.

Proposition 5.4 (Local polynomial of one tame packet)**.** *Suppose an interaction packet has orbit length r and closing scalar a. Its local polynomial is*

$$\det\left(I - u\Phi \mid V^{I_K}\right) = \begin{cases} 1 - au^r, & \text{if the inertia character is trivial,} \\ 1, & \text{otherwise.} \end{cases}$$

Proof. For nontrivial inertia, the invariant space is zero. In the trivial case, Frobenius on the invariant packet is the companion matrix of $X^r - a$. Substituting $X = u^{-1}$ and multiplying by u^r gives $1 - au^r$. □

Corollary 5.5 (Tame tensor local factor)**.** *The local factor of the tame tensor product is the product of*

$$(1 - a_i u^{r_i})^{-1}$$

over precisely those refined packets whose inertia exponent vanishes.

Proof. The invariant space is the direct sum of the invariant refined packets, and determinant is multiplicative on direct sums. □

Proposition 5.6 (Trivial tensor packet criterion)**.** *For every refined interaction packet, let r_i be its cycle length and a_i its closing scalar. The dimension of the W_K fixed space in $\tau_\xi \otimes \tau_\eta$ equals the number of packets satisfying both*

$$c_i = 0 \qquad \text{and} \qquad a_i = 1.$$

Proof. A nonzero fixed vector must lie in an inertia trivial packet, so $c_i = 0$. On that packet, Frobenius is a companion cycle with characteristic polynomial $X^{r_i} - a_i$. Its fixed space is the eigenspace for eigenvalue one. This eigenspace is one dimensional exactly when $a_i = 1$, and it is zero otherwise. Summing over the refined packets proves the formula. □

Corollary 5.7 (Tame Hom test)**.** *For irreducible tame atoms,*

$$\dim \mathrm{Hom}_{W_K}(\tau_\xi^\vee, \tau_\eta)$$

is zero or one, and it is one exactly when the refined tensor packet test in proposition 5.6 finds one trivial packet.

Proof. The Hom space is the fixed space of $\tau_\xi \otimes \tau_\eta$. Schur's lemma gives dimension zero or one, and proposition 5.6 gives the packet criterion. □

5.3 Tame conductor

Proposition 5.8 (Conductor of a nontrivial tame atom)**.** *Let τ be an irreducible tame Weil representation of dimension $n > 1$. Then*

$$\mathrm{Swan}(\tau) = 0, \qquad \dim \tau^{I_K} = 0, \qquad \mathrm{Art}(\tau) = n.$$

Proof. Wild inertia acts trivially, so the Swan conductor is zero. An inertia fixed vector would generate a nonzero Frobenius stable sum of trivial character packets. Irreducibility would then make inertia act trivially on all of τ. An irreducible representation of the quotient generated by Frobenius would be one dimensional, contrary to $n > 1$. The Artin conductor is therefore

$$\dim \tau - \dim \tau^{I_K} + \mathrm{Swan}(\tau) = n.$$

□

6 Finite Gauss phases

Let $k = \mathbb{F}_q$, let $\psi : k \to \mathbb{C}^\times$ be a nontrivial additive character, and let $\chi : k^\times \to \mathbb{C}^\times$ be a multiplicative character. Define

$$G(\chi, \psi) = \sum_{x \in k^\times} \chi(x)\psi(x).$$

The standard theory of Gauss and Jacobi sums will be used in the form recorded in [12, 1].

Proposition 6.1 (Gauss magnitude)**.** *If χ is nontrivial, then*

$$\boxed{|G(\chi, \psi)| = q^{1/2}.}$$

Proof. Compute

$$|G(\chi, \psi)|^2 = \sum_{x,y \in k^\times} \chi(xy^{-1})\psi(x - y).$$

Write $x = ty$. The inner sum over y equals $q - 1$ when $t = 1$ and equals -1 otherwise. Since $\sum_{t \in k^\times} \chi(t) = 0$, the result is q. □

Definition 6.2 (Normalized Gauss phase)**.** For nontrivial χ, define

$$\gamma(\chi, \psi) = q^{-1/2} G(\chi, \psi).$$

Then $|\gamma(\chi, \psi)| = 1$.

Proposition 6.3 (Duality of Gauss phases)**.** *For nontrivial χ,*

$$\boxed{\gamma(\chi, \psi)\gamma(\chi^{-1}, \psi) = \chi(-1).}$$

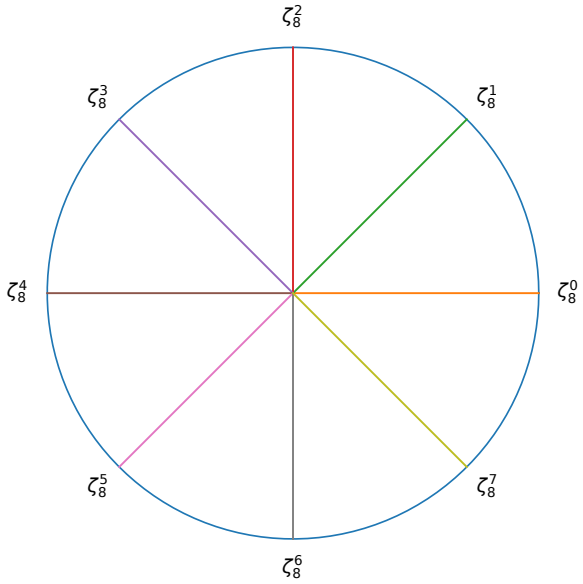


Figure 7. Normalized finite Gauss sums lie on the unit circle.

Proof. The standard Gauss identity gives

$$G(\chi,\psi)G(\chi^{-1},\psi) = \chi(-1)q.$$

Divide by q. □

Standard Fact 6.4 (Hasse and Davenport lifting)**.** Let k_r/k be the extension of degree r. Lift χ by the norm and ψ by the trace. Then the lifted Gauss sum satisfies the Hasse and Davenport product relation, with a sign determined by the chosen additive character normalization.

Remark 6.5. Finite Gauss phases reproduce the finite field part of tame and depth zero normalization. They do not by themselves give the full epsilon factor of a general wild pair.

7 Finite Heisenberg packets

Let p be odd. Let X be an r dimensional vector space over $\mathbb{F}_p$, and put

$$V = X \oplus X^*.$$

Define the alternating form

$$b\big((x,\xi),(y,\eta)\big) = \eta(x) - \xi(y).$$

Let $Z = \mathbb{F}_p$, and define the Heisenberg group

$$\mathfrak{H}(V) = V \times Z$$

with multiplication

$$(v,z)(w,t) = \left(v+w, z+t+\frac{1}{2}b(v,w)\right).$$

Fix the additive character

$$\psi(z) = \mathrm{e}^{2\pi i z/p}.$$

Definition 7.1 (Schrödinger operators)**.** On $\mathcal{H} = \mathbb{C}^X$, define

$$\big(\rho(x,\xi,z)f\big)(u) = \psi\left(z + \xi(u) + \frac{1}{2}\xi(x)\right) f(u+x).$$

Theorem 7.2 (Finite Schrödinger representation)**.** *The map $\rho : \mathfrak{H}(V) \to \mathrm{GL}(\mathcal{H})$ is a representation. It is irreducible, its center acts by ψ, and*

$$\dim \mathcal{H} = p^r = \sqrt{|V|}.$$

Proof. A direct calculation gives

$$\rho(v,z)\rho(w,t) = \rho\left(v+w, z+t+\frac{1}{2}b(v,w)\right).$$

Translations by X and multiplications by characters from X^* generate the full matrix algebra on $\mathbb{C}^X$. Indeed, the character multipliers give all diagonal projectors by finite Fourier inversion, and translations move each coordinate line to every other coordinate line. The commutant is therefore scalar, so the representation is irreducible. The dimension is $|X| = p^r$. □

Proposition 7.3 (Character support)**.** *The character of the Schrödinger representation satisfies*

$$\boxed{\operatorname{Tr}\rho(v,z) = \begin{cases} p^r\psi(z), & v = 0, \\ 0, & v \neq 0. \end{cases}}$$

Proof. If the translation component x is nonzero, the operator permutes the standard basis without a fixed point, so its trace is zero. If $x = 0$ but $\xi \neq 0$, the trace is

$$\psi(z)\sum_{u\in X}\psi(\xi(u)) = 0.$$

When $x = \xi = 0$, the operator is the scalar $\psi(z)$. □

Standard Fact 7.4 (Finite Stone von Neumann theorem)**.** For a finite Heisenberg group with nondegenerate commutator form and a fixed nontrivial central character, there is one irreducible representation up to isomorphism. Its dimension is the square root of the order of the symplectic quotient.

The standard fact follows from the twisted group algebra argument in section 8. It is also the finite form used in the classical Weil representation literature [15, 26].

7.1 Maximal isotropic selectors

Let $L \subseteq V$ be maximal isotropic. Choose a subgroup lift $\widetilde{L} \subseteq \mathfrak{H}(V)$ on which the central coordinate is zero.

Proposition 7.5 (Isotropic averaging projector)**.** *The operator*

$$P_L = \frac{1}{|L|}\sum_{\ell\in\widetilde{L}}\rho(\ell)$$

is an orthogonal projector. In the Schrödinger model with $L = X$, it has rank one and selects the constant functions.

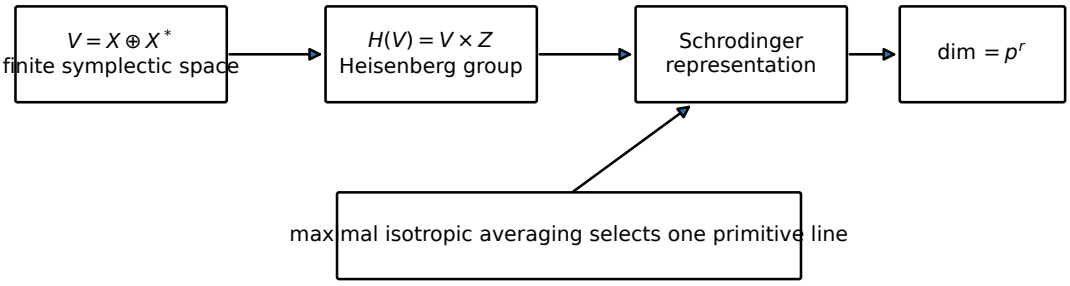


Figure 8. The finite Heisenberg packet and its maximal isotropic selector.

Proof. The lifted isotropic subgroup is abelian, so group averaging is an idempotent. Every operator in the average is unitary, and the subgroup is closed under inversion, so the average is self adjoint. For $L = X$, the average is the translation average

$$(P_L f)(u) = \frac{1}{|X|} \sum_{x \in X} f(u + x),$$

which is the constant function with value equal to the mean of f. Its range is one dimensional. □

Corollary 7.6 (Primitive line selector)**.** *Every maximal isotropic subgroup selects a one dimensional fixed line in the irreducible Heisenberg representation.*

Proof. The projector rank equals its trace. The identity element contributes p^r, while every nonidentity element of the lifted isotropic subgroup is noncentral and has trace zero by proposition 7.3. Since $|L| = p^r$, we get

$$\operatorname{Tr} P_L = \frac{p^r}{p^r} = 1.$$

Thus the projector has rank one. □

8 Twisted group algebras and commutator envelopes

Let A be a finite abelian group, and let

$$c : A \times A \to \mathbb{C}^\times$$

be a normalized two cocycle. The twisted group algebra $\mathbb{C}^c[A]$ has basis $(u_a)_{a \in A}$ and multiplication

$$u_a u_b = c(a, b) u_{a+b}.$$

Define the commutator bicharacter

$$b_c(a, b) = c(a, b) c(b, a)^{-1}.$$

Proposition 8.1 (Scalar choice invariance)**.** *Replacing u_a by $\lambda_a u_a$, with $\lambda_a \in \mathbb{C}^\times$, changes c by a coboundary and leaves b_c unchanged.*

Proof. The new cocycle is

$$c'(a,b) = \lambda_a \lambda_b \lambda_{a+b}^{-1} c(a,b).$$

The scalar coboundary factor is symmetric in a and b, so it cancels from $c'(a,b)c'(b,a)^{-1}$. □

Proposition 8.2 (Alternating commutator form)**.** *The map b_c is an alternating bicharacter. Its radical*

$$R = \{a \in A : b_c(a,x) = 1 \text{ for every } x \in A\}$$

is a subgroup, and b_c descends to a nondegenerate alternating bicharacter on A/R.

Proof. Associativity of the twisted product gives multiplicativity of scalar commutators in each variable. We have $b_c(a,a) = 1$ and $b_c(b,a) = b_c(a,b)^{-1}$. The radical is the kernel of the homomorphism from A to its character group defined by $a \mapsto b_c(a,\cdot)$. The induced pairing on the quotient is nondegenerate by construction. □

Theorem 8.3 (Nondegenerate twisted group algebra)**.** *Assume that b_c is nondegenerate. Then $|A|$ is a square, and*

$$\boxed{\mathbb{C}^c[A] \cong M_{\sqrt{|A|}}(\mathbb{C}).}$$

Consequently, $\mathbb{C}^c[A]$ has one irreducible module up to isomorphism, of dimension $\sqrt{|A|}$.

Proof. Let

$$z = \sum_{a \in A} z_a u_a$$

be central. Commuting with u_x gives

$$z_a b_c(x,a) = z_a$$

for every x. Nondegeneracy forces $z_a = 0$ for every nonzero a. Thus the center is $\mathbb{C}$. To prove semisimplicity directly, choose a linear projection P onto a submodule of a finite dimensional $\mathbb{C}^c[A]$-module and form

$$\overline{P} = \frac{1}{|A|} \sum_{a \in A} u_a P u_a^{-1}.$$

Every summand maps into the submodule and is the identity there. Conjugation by any u_x permutes the summands: the cocycle scalars cancel between each operator and its inverse. Hence $\overline{P}$ is an equivariant projection, and its kernel is an invariant complement. The algebra is therefore semisimple. Its center $\mathbb{C}$ permits only one simple factor. A finite dimensional central simple algebra over $\mathbb{C}$ is a full matrix algebra. Its dimension is $|A|$, so the matrix size is $\sqrt{|A|}$. □

Corollary 8.4 (Radical free envelope)**.** *For an arbitrary scalar projective action of A, the quotient*

$$\boxed{A^{\mathrm{symp}} = A/\operatorname{rad} b_c}$$

is the maximal quotient detected by noncommuting projective intertwiners. If A is an elementary abelian p group, then A^{symp} is a symplectic $\mathbb{F}_p$ space.

Proof. Every radical element commutes with all projective generators, so it contributes only central data. Quotienting by the radical leaves the nondegenerate commutator form. In the elementary abelian case, multiplicative values in the group of pth roots can be written as an alternating bilinear form over $\mathbb{F}_p$. □

Remark 8.5. The envelope depends only on the projective representation. It does not by itself prove that a given wild inertia representation carries a canonical finite quotient with the required projective action.

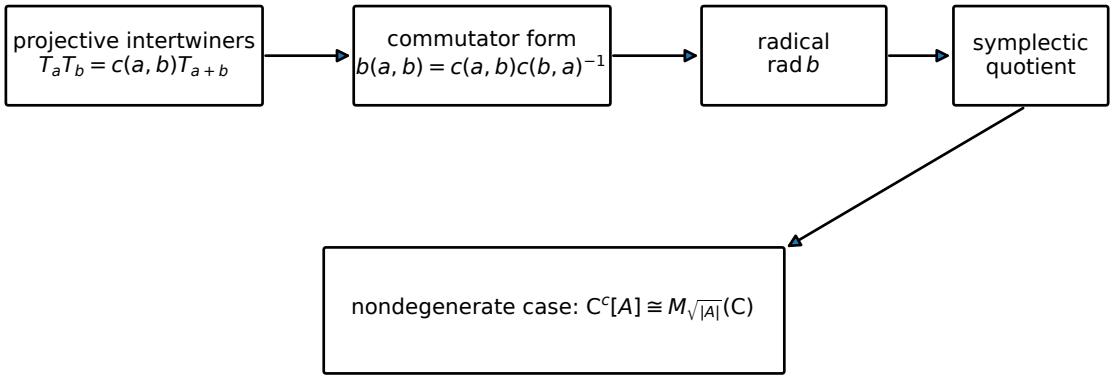


Figure 9. Projective intertwiners determine a radical free symplectic envelope.

9 A restricted one break Galois envelope

We now isolate a finite Clifford setting in which the quotient projective action is defined on a multiplicity space. This avoids assigning a quotient action directly to arbitrary representatives of G/N.

Definition 9.1 (One break Clifford datum)**.** A one break Clifford datum consists of finite groups and representations

$$N \triangleleft G, \qquad A = G/N, \qquad \Pi : G \to \mathrm{GL}(\mathcal{X}), \qquad \rho : N \to \mathrm{GL}(W),$$

with the following properties.

(1) The quotient A is a finite elementary abelian p group.
(2) The representations Π and ρ are irreducible.
(3) The isomorphism class of ρ is fixed by G.
(4) The restriction of Π to N is ρ isotypic.
(5) The quotient records one declared ramification break.

The multiplicity space is

$$M_\rho = \mathrm{Hom}_N(W, \mathcal{X}).$$

The isotypic condition gives an N equivariant evaluation isomorphism

$$W \otimes M_\rho \longrightarrow \mathcal{X}.$$

Choose a section $s : A \to G$. For every $a \in A$, choose an intertwiner I_a from ρ to its conjugate by $s(a)$. Under the evaluation isomorphism, there is a unique operator T_a on M_ρ such that

$$\Pi(s(a)) = I_a \otimes T_a.$$

The operators T_a form a projective representation of A.

Proposition 9.2 (Clifford multiplicity action)**.** *The projective equivalence class of the action $(T_a)_{a\in A}$ is independent of the section, the intertwiners, and the chosen isotypic evaluation isomorphism. The projective action is irreducible.*

Proof. Changing an intertwiner I_a by a scalar changes T_a by the inverse scalar. Changing the isotypic evaluation isomorphism conjugates all T_a by one element of $\mathrm{GL}(M_\rho)$. Changing the section multiplies $s(a)$ by an element of N. That factor is absorbed by the ρ action on W, leaving a scalar change on the multiplicity operator by Schur's lemma. Thus all choices give projectively equivalent actions.

If a nonzero proper subspace of M_ρ were stable under all projective operators T_a, then its tensor product with W would give a nonzero proper G stable subspace of $\mathcal{X}$. This contradicts irreducibility of Π. □

Definition 9.3 (Clifford cocycle class)**.** Choose scalars $c(a,b)$ such that

$$T_aT_b = c(a,b)T_{a+b}.$$

The cohomology class

$$[c_\rho] \in H^2(A,\mathbb{C}^\times)$$

is the Clifford cocycle class of the datum.

Proposition 9.4 (Intrinsic commutator pairing)**.** *The pairing*

$$b_\rho(a,b) = c(a,b)c(b,a)^{-1}$$

is independent of every choice in definition 9.1. It is an alternating bicharacter

$$b_\rho : A \times A \to \mu_p.$$

Proof. The projective equivalence class is intrinsic by proposition 9.2. Conjugation does not change scalar commutators, and scalar changes alter the cocycle by a coboundary. The commutator pairing is therefore unchanged by proposition 8.1. Its alternating and bilinear properties follow from proposition 8.2. Since every element of A has order p, every value has order dividing p. □

Definition 9.5 (One break symplectic envelope)**.** The one break symplectic envelope is

$$\boxed{V_\rho = A/\operatorname{rad} b_\rho.}$$

Theorem 9.6 (Nondegenerate one break packet)**.** *If b_ρ is nondegenerate, then*

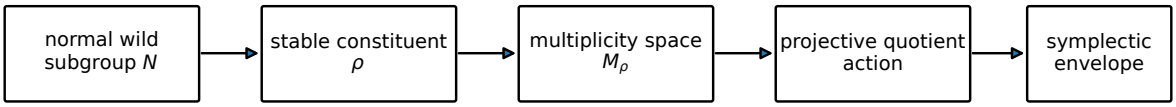
$$\mathbb{C}^{c_\rho}[A] \cong \mathrm{End}(M_\rho), \qquad \dim M_\rho = \sqrt{|A|}.$$

Proof. The projective action gives a nonzero representation of the twisted group algebra on M_ρ. The algebra is a full matrix algebra by theorem 8.3. Its unique irreducible module has dimension $\sqrt{|A|}$. The projective multiplicity action is irreducible by proposition 9.2, so it is that module. □

Proposition 9.7 (Isomorphism invariance)**.** *The cocycle class and the equivariant isometry class of V_ρ depend only on the isomorphism class of the one break Clifford datum.*

Proof. An isomorphism of data transports the restriction, the multiplicity space, and its projective quotient action. It therefore transports the cocycle class, the commutator pairing, its radical, and the quotient. □

Proposition 9.8 (Tame stabilizer action)**.** *Suppose a finite group C acts by automorphisms of the one break Clifford datum. Then C acts on V_ρ by symplectic automorphisms.*



The construction is intrinsic for the declared one break Clifford datum, up to equivariant isometry.

Figure 10. The restricted one break construction is made on the Clifford multiplicity space.

Proof. The automorphism action transports the intrinsic projective multiplicity representation. It therefore preserves the scalar commutator pairing and its radical. The induced action on the quotient is symplectic. □

Corollary 9.9 (Heisenberg representation attached to one break)**.** *Fix a nontrivial central character. The envelope V_ρ determines one irreducible finite Heisenberg representation up to isomorphism.*

Proof. Apply the finite Stone von Neumann theorem to the nondegenerate symplectic quotient. □

Remark 9.10. The group cohomology class records the extension obstruction for the multiplicity action. General background on projective representations and group cohomology can be found in [22, 3]. The theorem remains restricted because the finite quotient and the one break isotypic datum are assumed rather than constructed from arbitrary wild inertia.

10 The type side symplectic quotient

Let θ be a maximal simple character for $\mathrm{GL}_n(K)$, with compact open subgroups

$$H^1_\theta \subseteq J^1_\theta \subseteq J_\theta.$$

The quotient

$$V_\theta = J^1_\theta / H^1_\theta$$

is regarded as a finite vector space over $\mathbb{F}_p$, where p is the residue characteristic. Fix a nontrivial additive character $\psi_0 : \mathbb{F}_p \to \mu_p \subset \mathbb{C}^\times$, where μ_p is the group of p-th roots of unity. The simple-character commutator takes values in μ_p. Since ψ_0 is injective, the equation

$$\psi_0\big(h_\theta(\bar{x}, \bar{y})\big) = \theta([x, y])$$

uniquely defines the $\mathbb{F}_p$-valued commutator pairing h_θ. We use the same identification of μ_p with the additive group of $\mathbb{F}_p$ on the Galois side when comparing the two symplectic spaces.

Standard Fact 10.1 (Type side Heisenberg quotient)**.** The pairing h_θ is nondegenerate and alternating. There is one irreducible representation η_θ of J^1_θ containing θ, and its dimension is

$$\sqrt{\left|J^1_\theta / H^1_\theta\right|}.$$

This is part of the simple character and Heisenberg representation theory used in the construction of types [11]. The extension of η_θ from J^1_θ to J_θ is not unique before a normalization is chosen.

Standard Fact 10.2 (Canonical beta extension)**.** The canonical beta extension is obtained from a primary extension together with finite symplectic and tame orientation corrections. The construction is functorial in the declared simple character data and removes the remaining extension ambiguity.

The normalization and its relation to finite Weil representations are developed in [2, 14]. The present article uses this theorem as type side input. It does not rederive simple character theory.

Proposition 10.3 (Type side projective commutator)**.** *The projective action of J_θ / J^1_θ on the Heisenberg representation preserves the commutator form on V_θ. Any two choices of projective lifts give the same alternating form.*

Proof. Conjugation by J_θ preserves θ, so it preserves the commutator pairing defined from $\theta([x,y])$. Two choices of projective lifts differ by scalars. The scalar commutator is therefore unchanged by proposition 8.1. □

11 The restricted Galois to type bridge

The full bridge would construct an intrinsic Galois envelope for every wild atom and identify it canonically with V_θ. We prove the comparison after the required symplectic identification has been supplied.

Definition 11.1 (Bridge datum)**.** A restricted bridge datum consists of

(1) a one break Galois envelope (V_ρ, b_ρ)
(2) a type side quotient (V_θ, h_θ)
(3) an equivariant symplectic isometry
$$\iota : V_\rho \longrightarrow V_\theta$$
for a finite cyclic stabilizer $C = \langle c \rangle$
(4) the same nontrivial central character on the two Heisenberg groups
(5) when $p = 2$, compatible quadratic refinements and an isomorphism of the resulting Heisenberg central extensions lifting ι.

Theorem 11.2 (Restricted Heisenberg bridge)**.** *A restricted bridge datum determines an isomorphism between the two Heisenberg representations, unique up to a nonzero scalar. More precisely, there is an invertible map*

$$U : \mathcal{H}_\rho \longrightarrow \mathcal{H}_\theta$$

such that

$$U\omega_\rho(h) = \omega_\theta(\iota_H(h))U$$

for every Heisenberg element h, and every other such map is a scalar multiple of U.

Proof. When p is odd, the symplectic isometry and the common central character induce the standard isomorphism ι_H of the two Heisenberg groups. When $p = 2$, the required lift of ι is part of the bridge datum because the alternating form alone does not determine the central extension. Pulling the

type side representation back through ι_H gives an irreducible Heisenberg representation with the same central character as the Galois side representation. The finite Stone von Neumann theorem gives an isomorphism. Schur's lemma gives uniqueness up to scalar. □

The stabilizer actions need not agree after choosing U. Their discrepancy is scalar because both actions implement the same automorphisms of the irreducible Heisenberg representation.

Proposition 11.3 (Cyclic discrepancy character)**.** *Assume that the two projective stabilizer actions have been linearized to honest representations of the cyclic group C. Then there is a character*

$$\delta : C \to \mathbb{C}^\times$$

such that

$$U\Omega_\rho(g) = \delta(g)\Omega_\theta(g)U$$

for every $g \in C$.

Proof. For each g, the operator

$$\Omega_\theta(g)^{-1}U\Omega_\rho(g)U^{-1}$$

commutes with the type side Heisenberg representation. It is therefore scalar. Since both stabilizer actions are honest representations, multiplication of group elements shows that the scalars form a character. □

Lemma 11.4 (Linearization over a cyclic stabilizer)**.** *Every finite dimensional projective representation of a finite cyclic group over $\mathbb{C}$ is projectively equivalent to an honest representation.*

Proof. Let $C = \langle c \rangle$ have order m, and let T be a lift of the projective generator. Since $c^m = 1$, the operator T^m is scalar. Choose $\lambda \in \mathbb{C}^\times$ with $\lambda^m T^m = I$. Replacing T by λT gives an honest cyclic representation. □

Theorem 11.5 (One phase closes the cyclic bridge)**.** *In a restricted bridge datum, choose linearizations of the cyclic stabilizer actions. If the two actions have the same scalar phase on the generator c, then the Heisenberg intertwiner can be chosen to intertwine the full semidirect product action.*

Proof. By proposition 11.3, the discrepancy is a character δ. Equality of the generator phase gives $\delta(c) = 1$. Since c generates C, the character is trivial. The chosen map U then intertwines every stabilizer element as well as the Heisenberg group. □

Corollary 11.6 (Bridge uniqueness)**.** *When the semidirect product representation is irreducible, the completed bridge is unique up to scalar.*

Proof. Any two completed bridges differ by an endomorphism commuting with the semidirect product representation. Schur's lemma applies. □

Remark 11.7. The theorem does not construct ι. It proves that, once an equivariant symplectic isometry exists, the remaining finite comparison is one dimensional for a cyclic stabilizer.

12 Odd residue characteristic phases

Let p be odd, let (V, b) be a finite symplectic $\mathbb{F}_p$ space of dimension $2r$, and let

$$g \in \mathrm{Sp}(V)$$

have no fixed vector. Fix one of the standard finite Weil normalizations used in the literature [15, 25, 16].

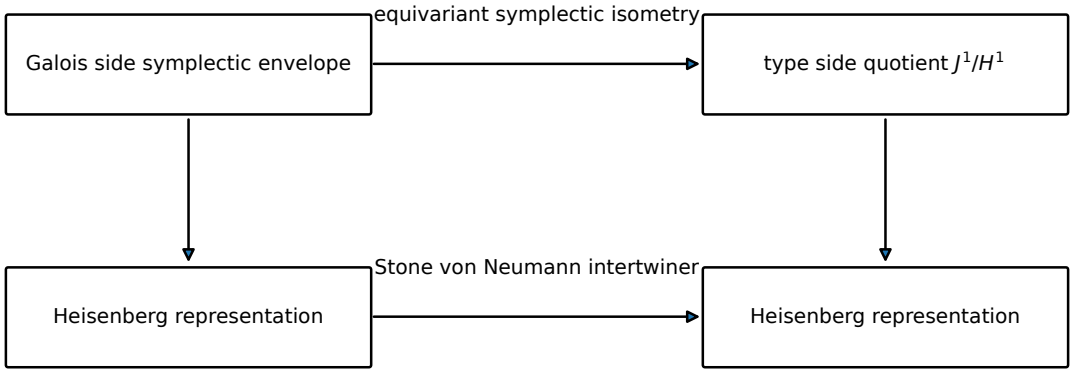


Figure 11. An equivariant symplectic isometry reduces the finite comparison to one scalar cyclic phase.

Standard Fact 12.1 (Fixed point free Weil trace)**.** Under the declared normalization,

$$\boxed{\operatorname{Tr}\omega(g) = \left(\frac{-1}{p}\right)^{r}\left(\frac{\det(g-I)}{p}\right).}$$

Here $(\frac{\cdot}{p})$ denotes the quadratic character of $\mathbb{F}_p^{\times}$.

Definition 12.2 (Odd finite Weil phase)**.** For fixed point free g, define

$$\Phi_p(g,V) = \left(\frac{-1}{p}\right)^{r}\left(\frac{\det(g-I)}{p}\right).$$

Proposition 12.3 (Symplectic invariance)**.** *If*

$$\iota : (V,b) \longrightarrow (V',b')$$

is a symplectic isometry and $g' = \iota g \iota^{-1}$, then

$$\Phi_p(g,V) = \Phi_p(g',V').$$

Proof. The dimensions agree, and conjugate matrices have the same determinant of $g-I$. □

Proposition 12.4 (Orthogonal multiplicativity)**.** *Suppose*

$$V = V_1 \perp \cdots \perp V_s$$

and g preserves every summand. If no $g|_{V_j}$ has a fixed vector, then

$$\boxed{\Phi_p(g,V) = \prod_{j=1}^{s} \Phi_p(g|_{V_j}, V_j).}$$

Proof. The half dimensions add, and

$$\det(g - I \mid V) = \prod_j \det(g - I \mid V_j).$$

The quadratic character is multiplicative. □

Corollary 12.5 (Characteristic polynomial certificate)**.** *Let*

$$P_g(X) = \det(XI - g \mid V).$$

Then

$$\boxed{\Phi_p(g, V) = \left(\frac{-1}{p}\right)^r \left(\frac{P_g(1)}{p}\right).}$$

Proof. Since $\dim V$ is even,

$$\det(g - I) = \det(I - g) = P_g(1).$$

□

Proposition 12.6 (Fixed and moving decomposition)**.** *Assume that $g \in \mathrm{Sp}(V)$ has order prime to p. Then*

$$V = \ker(g - I) \perp \mathrm{im}(g - I).$$

The restriction of the symplectic form to $\mathrm{im}(g - I)$ is nondegenerate, and $g - I$ is invertible on that moving summand.

Proof. The order hypothesis makes g semisimple, so

$$V = \ker(g - I) \oplus \mathrm{im}(g - I).$$

If $x \in \ker(g - I)$ and $y = (g - I)z$, then

$$b(x, y) = b(x, gz) - b(x, z) = b(g^{-1}x, z) - b(x, z) = 0.$$

Thus the two summands are orthogonal. Nondegeneracy on each summand follows from nondegeneracy on V. The last statement is immediate from the direct sum decomposition. □

Definition 12.7 (Moving phase)**.** Under the hypotheses of proposition 12.6, define

$$\Phi_p^{\mathrm{mov}}(g, V) = \Phi_p\left(g|_{\mathrm{im}(g-I)}, \mathrm{im}(g - I)\right).$$

Proposition 12.8 (Hyperbolic phase formula)**.** *Let U be a finite dimensional $\mathbb{F}_p$ space, let $a \in \mathrm{GL}(U)$ have no eigenvalue one, and let g act on*

$$H(U) = U \oplus U^*$$

by

$$g = (a, (a^{-1})^*).$$

Then

$$\boxed{\Phi_p(g, H(U)) = \left(\frac{\det a}{p}\right).}$$

Proof. Put $d = \dim U$. We have

$$\det\big((a^{-1})^* - I\big) = (-1)^d (\det a)^{-1} \det(a - I).$$

Hence

$$\det(g - I) = (-1)^d (\det a)^{-1} \det(a - I)^2.$$

Multiplying its quadratic character by the normalization factor $(\frac{-1}{p})^d$ cancels the sign and the square. Since a quadratic character equals its inverse, the result is $(\frac{\det a}{p})$. □

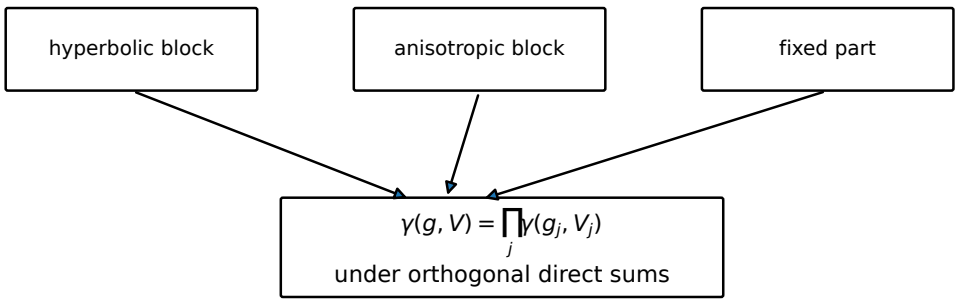


Figure 12. Odd finite Weil phases multiply across orthogonal atomic blocks.

Corollary 12.9 (Phase only bridge test)**.** *Two odd characteristic moving packets have the same normalized phase whenever their moving half dimensions agree and the determinants of $g - I$ lie in the same square class.*

Proof. Apply the definition of Φ_p. □

Remark 12.10. When g has fixed vectors, one first separates the fixed part and applies the formula to the moving quotient. Additional normalization data may enter the full trace on the fixed part.

13 Dyadic quadratic refinements

Let (V, b) be a symplectic vector space over $\mathbb{F}_2$. The alternating form alone is not enough to determine the finite phase.

Definition 13.1 (Quadratic refinement)**.** A quadratic refinement of b is a map

$$q : V \to \mathbb{F}_2$$

satisfying

$$q(x + y) - q(x) - q(y) = b(x, y).$$

Proposition 13.2 (Affine space of refinements)**.** *If one quadratic refinement exists, then the set of all quadratic refinements is an affine space over V^*.*

Proof. The difference of two refinements has zero polar form, so it is additive and hence linear over $\mathbb{F}_2$. Conversely, adding a linear form does not change the polar form. □

Definition 13.3 (Arf invariant)**.** Choose a symplectic basis $(e_i, f_i)_{i=1}^r$. Define

$$\mathrm{Arf}(q) = \sum_{i=1}^{r} q(e_i)q(f_i) \in \mathbb{F}_2.$$

Standard Fact 13.4 (Arf invariance and classification)**.** The value in definition 13.3 is independent of the symplectic basis. Two nondegenerate quadratic refinements of one finite symplectic space are isometric exactly when their Arf invariants agree.

This is the classical classification of nonsingular quadratic forms over $\mathbb{F}_2$ [24].

Theorem 13.5 (Dyadic Gauss sign)**.** *Let* $\dim V = 2r$*, and let* q *be nondegenerate. Then*

$$\boxed{\sum_{x \in V} (-1)^{q(x)} = (-1)^{\mathrm{Arf}(q)} 2^r.}$$

Proof. Orthogonally decompose (V, q) into two dimensional nonsingular planes. On a plane with Arf invariant zero, the four values of $(-1)^q$ sum to 2. On a plane with Arf invariant one, they sum to -2. The total sum is the product of the plane sums. The parity of the number of anisotropic planes is the Arf invariant. □

Corollary 13.6 (Orthogonal dyadic multiplicativity)**.** *For an orthogonal sum*

$$(V, q) = (V_1, q_1) \perp (V_2, q_2),$$

we have

$$\mathrm{Arf}(q) = \mathrm{Arf}(q_1) + \mathrm{Arf}(q_2)$$

and the normalized Gauss sign is the product of the two normalized signs.

Proof. Take symplectic bases of the two summands and concatenate them. The Gauss sum factors as a product over the Cartesian decomposition. □

Proposition 13.7 (The form alone does not determine the phase)**.** *On every nonzero symplectic* $\mathbb{F}_2$ *space, there are quadratic refinements of both Arf invariants. Their polar forms agree, but their normalized Gauss signs are opposite.*

Proof. On a symplectic plane, the standard hyperbolic refinement has Arf invariant zero. Adding a suitable nonzero linear form produces a refinement with Arf invariant one. Orthogonally adjoining a fixed refinement on the complement gives the two choices in every positive even dimension. The signs differ by theorem 13.5. □

Proposition 13.8 (Translation of quadratic refinements)**.** *For* $v \in V$*, put*

$$q_v(x) = q(x) + b(v, x).$$

Then

$$\boxed{\mathrm{Arf}(q_v) = \mathrm{Arf}(q) + q(v).}$$

Proof. Using the quadratic identity,

$$q_v(x) = q(x + v) + q(v).$$

Therefore

$$\sum_x (-1)^{q_v(x)} = (-1)^{q(v)} \sum_x (-1)^{q(x)}.$$

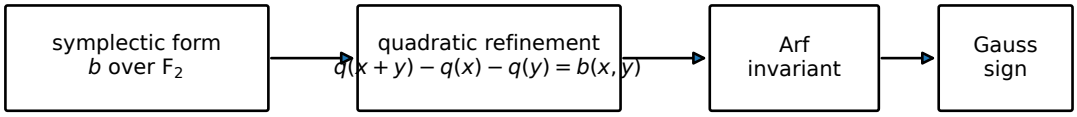
Compare the signs in theorem 13.5. □

Theorem 13.9 (Number of dyadic refinements)**.** *Let* $\dim V = 2r$*. The number of quadratic refinements with Arf invariant zero is*

$$\boxed{2^{r-1}(2^r + 1),}$$

and the number with Arf invariant one is

$$\boxed{2^{r-1}(2^r - 1).}$$



$$\sum_x (-1)^{q(x)} = (-1)^{\mathrm{Arf}(q)} 2^r$$

The alternating form alone does not determine the dyadic phase.

Figure 13. At residue characteristic two, the quadratic refinement and its Arf invariant carry phase information not contained in the alternating form.

Proof. Every refinement is q_v for a unique $v \in V$, because b identifies V with V^*. By proposition 13.8, the Arf invariant changes according to $q(v)$. Let Z and O be the numbers of zeros and ones of q. Then

$$Z + O = 2^{2r}$$

and theorem 13.5 gives

$$Z - O = (-1)^{\mathrm{Arf}(q)} 2^r.$$

Solving gives the two displayed numbers, with their roles exchanged when the starting refinement has Arf invariant one. Since translation runs through every refinement, the final counts are independent of the starting choice. □

14 Matrix holonomy and finite gluing

Let Γ be a connected finite graph. At every vertex v, place finite dimensional complex vector spaces V_v and W_v of the same dimension. For every oriented edge $e : u \to v$, let

$$A_e : V_u \longrightarrow V_v, \qquad B_e : W_u \longrightarrow W_v$$

be invertible. The reversed edge carries the inverse map.

Definition 14.1 (Path transport)**.** For an oriented path

$$\gamma = (v_0 \to v_1 \to \cdots \to v_m),$$

define

$$A_\gamma = A_{v_m v_{m-1}} \cdots A_{v_1 v_0}, \qquad B_\gamma = B_{v_m v_{m-1}} \cdots B_{v_1 v_0}.$$

For a loop based at r, these are the based holonomies.

Theorem 14.2 (Matrix holonomy criterion)**.** *Fix a root r and an isomorphism*

$$U_r : V_r \longrightarrow W_r.$$

The following conditions are equivalent.

(1) There are isomorphisms $U_v : V_v \to W_v$, with the declared U_r, such that

$$U_v A_e = B_e U_u$$

for every edge $e : u \to v$.

(2) For one fundamental cycle basis based at r,

$$U_r A_\gamma = B_\gamma U_r.$$

(3) The same identity holds for every based loop γ.

Proof. If the vertex maps exist, multiply the edge equations around a loop. Every intermediate vertex map cancels, giving condition three. Condition three implies condition two.

Conversely, choose a spanning tree. Starting from U_r, define U_v by transport along the unique tree path from r to v. The edge equations hold on every tree edge by construction. Every non tree edge closes one fundamental cycle. The edge equation on that edge is equivalent to the root holonomy identity for its fundamental cycle. Condition two therefore gives all remaining edge equations. □

Corollary 14.3 (Unbased gauge classification)**.** *There exists some family of vertex isomorphisms if and only if the two holonomy representations*

$$\pi_1(\Gamma, r) \longrightarrow \mathrm{GL}(V_r), \qquad \pi_1(\Gamma, r) \longrightarrow \mathrm{GL}(W_r)$$

are conjugate.

Proof. Apply theorem 14.2 and allow U_r to vary. □

Corollary 14.4 (Scalar and projective gluing)**.** *If every based loop acts by a scalar, finite gluing is equivalent to equality of the scalar holonomies on a fundamental cycle basis. The number of independent tests is*

$$\boxed{|E| - |V| + 1.}$$

Proof. A connected graph has a spanning tree with $|V| - 1$ edges. Every remaining edge gives one fundamental cycle. In the scalar case, conjugacy has no effect. □

Proposition 14.5 (Projective cocycle form)**.** *Suppose the edge maps are defined only up to scalar, in the setting where all based holonomies of both systems are scalar. Choose representatives. The ratio of the two scalar holonomies on each loop is unchanged by vertex rescaling. It defines a character of the first homology group when the scalar transport is abelian.*

Proof. A vertex rescaling appears once with positive exponent and once with negative exponent around a closed path, so it cancels. Multiplication of loops multiplies the scalar ratios. In the abelian case, the ratio factors through the abelianization of the fundamental group, which is the first homology group. □

Proposition 14.6 (Exact edge residual from cycle defect)**.** *Fix a spanning tree and construct the vertex maps by tree transport from U_r. Let $e : u \to v$ be a non tree edge. Let p_u and p_v be the tree paths from the root to u and v. Define the fundamental cycle defect*

$$D_e = U_r A_{p_v}^{-1} A_e A_{p_u} - B_{p_v}^{-1} B_e B_{p_u} U_r.$$

Then the edge residual is exactly

$$\boxed{U_v A_e - B_e U_u = B_{p_v} D_e A_{p_u}^{-1}.}$$

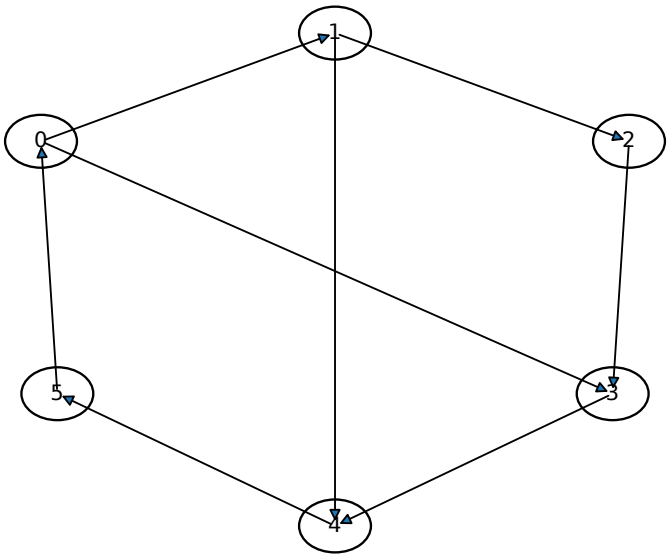


Figure 14. A spanning tree fixes the vertex gauges. The non tree edges test the remaining cycle holonomies.

Proof. Substitute

$$U_u = B_{p_u} U_r A_{p_u}^{-1}, \qquad U_v = B_{p_v} U_r A_{p_v}^{-1}$$

into the left side and factor B_{p_v} on the left and $A_{p_u}^{-1}$ on the right. □

Corollary 14.7 (Quantitative finite gluing)**.** *With the notation of proposition 14.6,*

$$\|U_v A_e - B_e U_u\| \le \left\|B_{p_v}\right\| \|D_e\| \left\|A_{p_u}^{-1}\right\|.$$

Thus small fundamental cycle defects give small non tree edge residuals whenever the tree transports are uniformly conditioned.

Proof. Take operator norms in the exact identity. □

Remark 14.8. The theorem is an exact finite gluing result. It assumes that the packet spaces and edge transports have already been constructed on both sides.

15 Breakwise assembly

Wild inertia carries an upper ramification filtration. In favorable settings, a wild atom separates into finitely many break pieces. We state the finite algebra needed once such pieces are intrinsic.

Definition 15.1 (Orthogonal break decomposition)**.** An orthogonal break decomposition is a symplectic direct sum

$$V = \bigperp_{s \in S} V_s$$

that is preserved by the declared tame stabilizer. Each V_s is the finite symplectic packet assigned to one break.

Proposition 15.2 (Heisenberg tensor factorization)**.** *Fix compatible central characters. The irreducible Heisenberg representation of an orthogonal sum is the tensor product of the irreducible Heisenberg representations of its summands:*

$$\boxed{\omega_V \cong \bigotimes_{s\in S} \omega_{V_s}.}$$

Proof. The Heisenberg group of an orthogonal sum is the central product of the Heisenberg groups of the summands. The tensor product representation has the declared central character and has dimension

$$\prod_s \sqrt{|V_s|} = \sqrt{|V|}.$$

It is irreducible by the finite Stone von Neumann theorem, so it is the required representation. □

Corollary 15.3 (Breakwise phase product)**.** *In odd residue characteristic, let $g \in \mathrm{Sp}(V)$ preserve every break packet and suppose its restriction to $M = \mathrm{im}(g - I)$ is fixed point free. Put $g_s = g|_{V_s}$ and $M_s = \mathrm{im}(g_s - I)$. The moving restrictions are symplectic, and*

$$\boxed{\Phi_p(g|_M, M) = \prod_{s\in S} \Phi_p(g_s|_{M_s}, M_s).}$$

In residue characteristic two, the analogous normalized Gauss sign is the product of the signs determined by the breakwise quadratic refinements.

Proof. The invariant orthogonal decomposition gives $M = \perp_{s\in S} M_s$. Symplectic invariance gives $M^\perp = \ker(g - I)$. Fixed point freeness on M therefore makes the restricted form nondegenerate on M and on each M_s. Each restriction of $g_s - I$ to M_s is invertible, so proposition 12.4 applies to the moving spaces. A zero moving space contributes one. In characteristic two, apply corollary 13.6. □

Theorem 15.4 (Breakwise restricted bridge)**.** *Suppose every break has a restricted bridge datum, the symplectic isometries are compatible with the stabilizer action, and the scalar phases agree on a generating set of the stabilizer. Then the tensor product of the breakwise Heisenberg intertwiners gives a bridge for the full finite envelope. Any remaining gluing obstruction is exactly the matrix holonomy obstruction of theorem 14.2.*

Proof. Tensor the breakwise Heisenberg intertwiners from theorem 11.2. Phase agreement makes the tensor map equivariant on the chosen stabilizer generators. The breakwise direct sum and tensor factorization make the local comparisons compatible. If the packets are arranged over a transport graph, the only remaining condition is consistency of the local maps around cycles, which is exactly the holonomy criterion. □

16 Local factor tests and the remaining bridge

Local class field theory identifies one dimensional Weil characters with characters of $K^\times$ [27, 29]. The tame induction theorem extends this explicit picture to higher dimensional tame atoms. The local polynomial and conductor calculations in propositions 5.4 and 5.8 therefore give explicit finite invariants for the declared tame packets. Finite Gauss sums give the residue field phase. The full local factor, however, also includes ramification depth, Haar measure normalization, the additive character conductor, and wild interaction data.

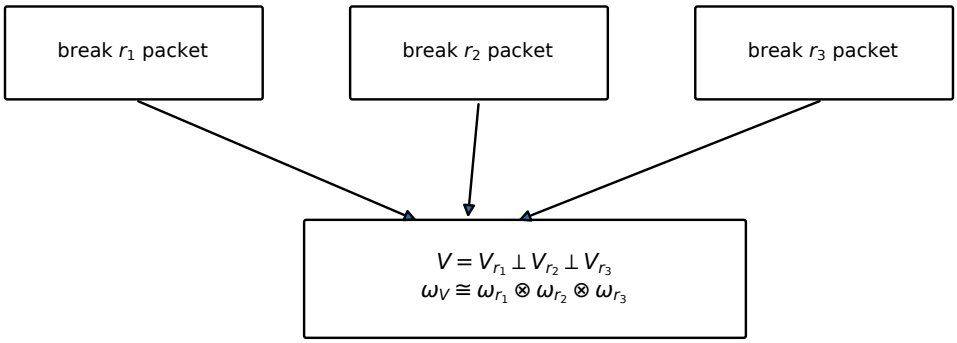


Figure 15. Orthogonal break packets produce tensor factors of the finite Heisenberg representation.

Proposition 16.1 (Finite tame comparison package)**.** *For a tame induced atom, the following data are determined by its packet coordinates:*

(1) the dimension and inertia orbit
(2) the Frobenius cycle polynomial
(3) the Artin conductor
(4) the local polynomial
(5) the tensor refinement into interaction packets
(6) the finite Gauss sum of each declared residue character, with unit-modulus phase normalization when that character is nontrivial.

Proof. The dimension and orbit follow from theorem 3.2. The cycle polynomial is proposition 4.4. For rank greater than one, the conductor is proposition 5.8. In rank one, the Swan conductor is zero, and the Artin conductor is zero for trivial inertia and one otherwise, by the formula $\mathrm{Art} = \dim V - \dim V^{I_K} + \mathrm{Sw}$. The local polynomial is proposition 5.4. Tensor refinement is theorem 5.1. The Gauss sum is its defining finite sum; for a nontrivial multiplicative character its unit phase is definition 6.2. For the trivial character and a nontrivial additive character ψ, the sum is $G(1, \psi) = -1$, because the sum of ψ over the whole field is zero. □

16.1 Arithmetic realization and the conditional comparison

Fix the paper's reciprocity, additive-character, local-factor, and phase normalizations. Let $\mathcal{A}$ be the declared class of irreducible wild Weil atoms under consideration. For $\rho \in \mathcal{A}$, the supplied assembled finite datum d_ρ consists of the following inputs, historically denoted H1–H4:

(1) an intrinsic finite break decomposition on the Galois side;
(2) the declared equivariant symplectic bridges to the supplied breakwise type-side quotients;
(3) compatible odd or dyadic central-extension and phase data on every break, including every prescribed finite marking;

(4) a compatible transport system with vanishing matrix holonomy obstruction.

These are finite comparison inputs. Supplying separate breakwise target types does not itself supply one common arithmetic candidate. If a common candidate is already supplied, that is a stronger restricted hypothesis.

An *arithmetic realization witness* $\mathcal{R}(d_\rho, \pi)$ consists of actual arithmetic inducing data for an admissible supercuspidal π, together with explicit comparison and recovery maps carrying the prescribed assembled finite packet. The maps preserve its declared stabilizer action, all character values explicitly included in the supplied datum, central-extension and phase data, and compatible transports. Recovery returns that same π, with its fixed central character and noncompact completion. An enlarged actual inducing carrier is permitted; the whole packet is not required to be the original maximal-simple-type quotient. Any stronger native-marking or numerical-layer requirement is part of the declared datum and must be verified separately.

Conjecture 66 (Arithmetic packet realization)**.** For every $\rho \in \mathcal{A}$ with its supplied H1–H4 assembled finite datum d_ρ, there exists an actual admissible supercuspidal representation π of $\mathrm{GL}_{\dim \rho}(K)$ and an arithmetic realization witness $\mathcal{R}(d_\rho, \pi)$.

This is the realization problem isolated from historical Theorem 16.2 and original proof obligation 66 (row 95 of the historical all-records register). It asks for one witness for each admitted source, rather than for success of every proposed host or every noncompact completion. No proof of the unrestricted statement is claimed.

Proposition 16.3 (Comparison of one realized candidate)**.** *Let $\rho \in \mathcal{A}$, and let π be one actual admissible supercuspidal representation of $\mathrm{GL}_{\dim \rho}(K)$ with $\mathcal{R}(d_\rho, \pi)$. Suppose H5 holds for this same π: its required pair local factors equal those prescribed for ρ on the stipulated characterizing family of twists, with the fixed normalizations. Assume the stated local-factor characterization applies to that family and class. Then π is isomorphic to the representation assigned to ρ by the established local Langlands correspondence.*

Proof. H5 identifies the tested local factors for this actual candidate. The applicable local-factor characterization identifies its isomorphism class with the established correspondence [18]. The realization witness ensures that it is this same candidate which carries the prescribed assembled packet. No existence conclusion is inferred from the finite gluing conditions alone. □

Remark 16.4. H5 remains a conditional premise about the same fixed candidate. It is not a formal consequence of the finite symplectic bridge, conductor profiles, or a choice of noncompact root. This proposition adds no unconditional pair-factor obligation to the construction conjecture.

16.2 Exact open boundary

The following issues delimit the unrestricted program. The first four construct and normalize the finite inputs for broader source classes. Conjecture 66 asks for arithmetic realization once H1–H4 are supplied. The fifth is the independent route to replacing the retained conditional H5 input.

(1) Construct a choice free finite symplectic envelope from an arbitrary irreducible wild inertia atom, with its ramification filtration and tame stabilizer action.
(2) Prove a canonical equivariant symplectic isometry from that envelope to the type side quotient J^1_θ / H^1_θ.
(3) Recover the canonical beta extension phase in every break, including the required dyadic quadratic refinement.
(4) Prove matrix holonomy compatibility for the resulting packet transport system.

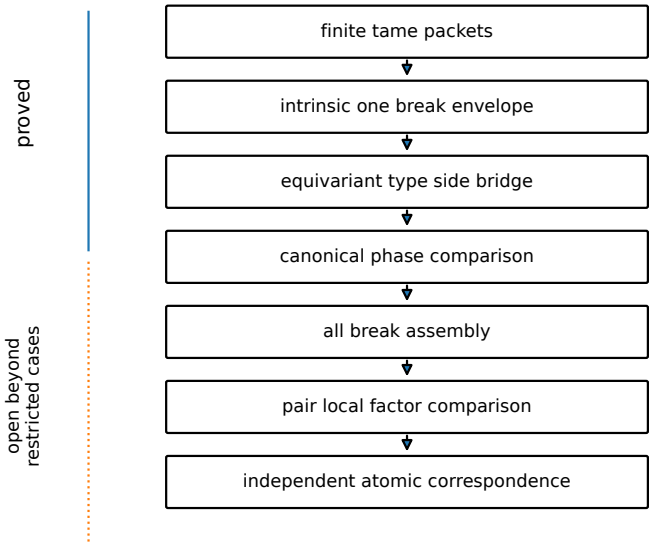


Figure 16. The proved finite layers and the remaining independent local correspondence steps.

(5) Compare the full family of pair L, epsilon, and gamma factors without invoking the established correspondence.

The first item is a construction problem. The second is a comparison problem. The third is a normalization problem. The fourth is a descent problem. The fifth concerns proving the local factor premise needed for identification. Conjecture 66 does not assert this fifth step. None is replaced by a larger finite computation.

17 Examples and finite certificates

The examples in this section illustrate the exact finite formulas. They are not used as proofs of the general theorems.

Example 17.1 (A degree four tame orbit)**.** Over $\mathbb{F}_2$, the exponent orbit

$$1 \longmapsto 2 \longmapsto 4 \longmapsto 8 \longmapsto 1$$

has length four. It therefore gives a tame atom induced from the unramified extension of degree four. The Frobenius transfer has one four cycle, and the number of regular degree four orbits is three by proposition 3.6.

Example 17.2 (Tensor packets of lengths four and six)**.** For $n = 4$ and $m = 6$,

$$\gcd(4, 6) = 2, \qquad \operatorname{lcm}(4, 6) = 12.$$

The tensor product splits into two interaction packets of depth twelve. Their total dimension is $2 \cdot 12 = 24$, equal to the product $4 \cdot 6$.

Example 17.3 (Rank one Heisenberg group)**.** Take $V = \mathbb{F}_3^2$. The Heisenberg group has order $3^3 = 27$, and its irreducible representation with a fixed nontrivial central character has dimension three. The trace vanishes on every noncentral element. Averaging the translation subgroup gives a rank one projector.

Example 17.4 (Nondegenerate twisted algebra)**.** Let $A = \mathbb{F}_p^2$ with its standard alternating form. A cocycle with that commutator form gives

$$\mathbb{C}^c[A] \cong M_p(\mathbb{C}).$$

The unique irreducible projective representation has dimension p.

Example 17.5 (Dyadic phase ambiguity)**.** On a symplectic plane over $\mathbb{F}_2$, choose refinements q_0 and q_1 with Arf invariants zero and one. Their polar forms are the same, while

$$\sum_x (-1)^{q_0(x)} = 2, \qquad \sum_x (-1)^{q_1(x)} = -2.$$

Thus the alternating packet alone cannot select the dyadic sign.

Example 17.6 (One cycle gluing obstruction)**.** Let a connected graph have one non tree edge. Tree transport determines every vertex map from one root map. The final edge equation holds exactly when the two holonomies around the unique fundamental cycle intertwine. Changing the scalar on that edge changes the cycle holonomy and produces a genuine obstruction.

18 Scope of the finite reconstruction

The article proves the following statements under their displayed hypotheses.

(1) Tame irreducible Weil atoms are transitive Frobenius packets induced from unramified extensions.
(2) Regular tame packets have the stated Möbius count.
(3) Finite cuspidal blocks have exact primitive central projectors.
(4) Tame tensor products satisfy the greatest common divisor and least common multiple packet formula.
(5) Tame local polynomials, conductors, and finite Gauss phases are explicit.
(6) Nondegenerate scalar projective commutators give full matrix twisted group algebras and unique finite Heisenberg representations.
(7) A declared one break Clifford datum has a choice independent radical free symplectic envelope.
(8) An equivariant symplectic isometry gives a Heisenberg bridge unique up to scalar.
(9) One generator phase removes the scalar discrepancy for a cyclic stabilizer.
(10) Odd finite Weil phases and dyadic Arf signs are multiplicative across orthogonal break packets.
(11) Matrix holonomy on a fundamental cycle basis is the exact finite gluing obstruction.

The article does not independently prove the local Langlands correspondence, simple character theory, the canonical beta extension, or universal wild companion existence. It does not construct the Galois to type symplectic isometry for an arbitrary wild atom. It does not compare all pair local factors in the wild case.

19 Restricted arithmetic realization theorems

The later constructions provide actual arithmetic candidates and exact finite packet recovery in substantial restricted families. They do not establish conjecture 66 for every admitted source. Their detailed ordinary proofs are typeset in the appendices, and the original reviewed Markdown files accompany this edition. In these appendices F denotes the base local field, while $K = F[\beta]$ denotes a parameter field; these are local notation conventions. All references to "claim 66" in the retained notes mean the general endpoint now isolated as Conjecture 66.

19.1 The odd-prime trace-zero construction

Theorem 19.1 (Restricted odd-prime arithmetic realization). *Let p be an odd prime, $F = \mathbb{F}_p((t))$, fix $d \geq 1$, and let*

$$0 < h_1 < \cdots < h_d, \qquad p \nmid h_i, \qquad q_i = (p+1)h_i, \qquad N = p^d.$$

There is an explicit maximal simple stratum in $M_N(F)$ with totally ramified full-degree parameter field, a literal defining sequence of degrees $p, p^2, \ldots, p^d$, and levels $n_j = p^{d-j}q_{d-j+1}$. Every native parameter and consecutive correction has trace zero over its recursive base field. The stratum admits a full simple character trivial on U_F^1. Its fixed canonical beta extension can be completed to an actual irreducible admissible supercuspidal π with any prescribed smooth central quasicharacter trivial on U_F^1, when its completion choices are free or already compatible.

Assume actually supplied labelled full-character extensions with whole two-dimensional symplectic quotients whose balanced complex-scalar central product has d planes, one vertex, and trivial stabilizer. Then the same π admits a full scalar-pushout realization of that packet and exact recovery of its unchanged inducing representation. The prescribed central character must lie in the covered tame class. No literal embedding of the old matrix groups, canonical native marking, or prescribed numerical host-depth placement is inferred.

Proof. The explicit Artin–Schreier recursion and its normalized filtered lift construct the maximal parameter and literal defining sequence; see appendix B. Its trace transport identity makes every native trace zero, rather than relying on amplification by p. Native-rank full-character construction, full beta-determinant transfer, trace-zero correction, and maximal-field character extension give the scalar triviality and central completion; see appendix C. The actual leading symplectic layer has dimension $(p-1)p^{d-1} \geq 2d$ and survives every later approximation (appendix D). A labelled symplectic embedding gives a center-fixing full scalar-pushout isomorphism. Finite Stone–von Neumann decomposition separates the source factor from its orthogonal complement. Maximal-isotropic averaging on the complementary Heisenberg factor selects a one-dimensional representation line and hence a source-sized sector. The retained induction/idempotent recovery returns the same Λ and π, as proved in appendix A. □

For $p > 3$, a native minimal rank-p type quotient has dimension $p-1$, so it is not automatically a plane. The theorem requires genuinely supplied whole-plane data. Independence of source tensor factors is also an input. An arbitrary single-factor wild determinant need not be tame.

Under additional actual-inertia hypotheses, genuine scalar phases on the entire source carrier are realized by actual central matrices of a finite cover and descend to the stipulated operators. A direct matrix homomorphism from the stabilizer itself need not exist; the exact statement and descent proof are in appendix E.

19.2 The weighted dyadic construction and enlarged carriers

Theorem 19.2 (Restricted dyadic marked assembly realization). *Let $F = \mathbb{F}_2((t))$, fix an additive character of conductor $t\mathcal{O}_F$, fix $d \geq 1$, and let $h_1 < \cdots < h_d$ be positive odd integers. Put $q_i = 3h_i$, $N = 2^d$, and $n_j = 2^{d-j}q_{d-j+1}$. The explicit recursive matrix generates a separable totally ramified degree-N maximal parameter field, normalizes the complete lattice chain, and has the literal defining sequence of degrees $2, 4, \ldots, 2^d$. It admits a full simple character trivial on U_F^1 and an actual supercuspidal completion with any prescribed central quasicharacter trivial on U_F^1, when the noncompact completion is free or already compatible.*

For actually supplied full marked whole-plane scalar-pushout data with one vertex, trivial stabilizer, and prescribed source central character trivial on U_F^1, an arithmetic assembly carrier exists at every length and recovers the same candidate: inside its original type for $d \geq 3$, and in explicit enlarged inducing carriers for $d = 1, 2$. Every supplied full character value and transported whole finite subgroup, lift, square, and phase marking is retained

at the represented marking level. Any central-character kernel already factored out remains recorded separately; scalar pushout does not recover its invisible elements. This does not identify the transported marking with an independently prescribed canonical native marking.

Proof. The Artin–Schreier filtered lift has native trace one. Consequently each native correction δ satisfies

$$v_F(\operatorname{tr}\delta) + s \geq 1 \quad (s = 1),$$

which makes its additive scalar correction trivial on U_F^1. The actual field and defining-sequence proof is appendix H. Full beta-determinant transfer and character extension at each new maximal native field prove the full-character claim, including the odd-critical endpoint $2(\lfloor r/2 \rfloor + 1) - r = 1$; see appendix I. The exact original quotient has dimension $2^d - 2$, which accommodates the required $2d$ source dimensions precisely for $d \geq 3$ (appendix J).

At rank two, with $q = 2m + 1$, enlarge $J = U_K^1 U_A^{m+1}$ to $P = U_K^1 U_A^m$. The actual character ratio $\lambda = \theta^g/\theta$ for $g \notin J$ is nontrivial and sign-valued. With $H_0 = \ker\lambda$, the quotient is $P/H_0 \simeq \mathbb{F}_2^2$, and $\operatorname{Ind}_J^P \theta$ is an irreducible two-dimensional Heisenberg carrier (appendix L). At rank four, the actual outer quadratic centralizer gives $P = JU_B^m$, $[P : J] = 2$, and $P/H_0 \simeq \mathbb{F}_2^4$, with carrier $\operatorname{Ind}_J^P \eta$ of dimension four (appendix M). The finite scalar center can be larger than μ_2; no literal extraspecial quotient is claimed.

At both ranks $K^\times$ normalizes P. Thus $\mathcal{N} = K^\times J$ is normal of index two in $\mathcal{M} = K^\times P$, and

$$\Sigma = \operatorname{Ind}_{\mathcal{N}}^{\mathcal{M}} \Lambda, \qquad \pi \simeq \operatorname{cInd}_{\mathcal{M}}^{\mathrm{GL}_N(F)} \Sigma.$$

Induction in stages preserves the full original Λ, its central character, and its noncompact root; see appendices N and O. The full marked scalar-pushout comparison applies to these actual enlarged pairs as well as to the original higher-rank types (appendix K). The full synthesis and scope are in appendix G. □

The original low-rank maximal-type capacity obstruction remains true. The new pair is a common arithmetic assembly carrier; it is not renamed as the original maximal-simple-character pair. For an independently fixed native quadratic marking, a preassigned embedding must satisfy $q_W(f(v)) = q_{\mathrm{src}}(v)$. At equal capacity, a free embedding requires equal Arf invariant. Spare capacity for $d \geq 4$ admits either source Arf type when the embedding is free. Retaining only the central $\{\pm 1\}$ would erase the distinction between whole marked finite extensions such as D_8 and Q_8.

19.3 Scope, numerical invariants, and evidence

For the same candidates the constructions give

	p odd	$p = 2$
$\dim_{\mathbb{F}_p}(J^1/H^1)$	$p^d - 1$	$2^d - 2$
$\operatorname{Swan}(\pi)$	$p^{d-1}(p+1)h_d$	$3 \cdot 2^{d-1} h_d$

and torsion number one. Here Swan denotes the Swan conductor, and a below denotes the ordinary pair conductor. An epsilon exponent for a specified additive character is a different quantity. The separate progress paper, Sections 22–24, reconciles these conventions and the actual-order hypotheses. Their self-pair conductor formulas are

$$a(\pi^\vee \times \pi) = p^{2d} - 1 + \sum_{i=1}^{d} (p^{2i} - p^{2i-2}) h_i \quad (p \text{ odd}),$$

$$a(\pi^\vee \times \pi) = N^2 - 1 + \sum_{i=1}^{d} 3 \cdot 4^{i-1} h_i \qquad (p = 2).$$

These identities do not prove H5, identify an endo-class, or construct a canonical source-to-host marking. At rank eight, the retained skipped-degree host has the same displayed numerical profiles as the full tower and a smaller Heisenberg quotient. The explicit construction, rather than numerical matching alone, supplies the arithmetic object.

The proofs use the stated Bushnell–Kutzko simple-type theorems [11] and the retained conductor inputs. Exact source locations and hypothesis checks are recorded in the component proofs and accompanying audits. The finite rank-nine and rank-eight runs check particular matrices, traces, normalized lifts, and filtration grades. They are bounded checks, not arbitrary-length proofs. No new Lean build of these arithmetic constructions is claimed.

Nontrivial prescribed symplectic stabilizer actions, independently fixed canonical native markings, arbitrary source/transport diagrams, and any required numerical layer placement remain outside the restricted results. The current totally ramified host cannot implement nontrivial prime-to-p symplectic actions by the stronger character-preserving inner-normalizer mechanism; this obstruction is stated precisely in appendix F. That stronger mechanism is not silently imposed on the original conjecture. H5 remains conditional for the same fixed candidate.

The accompanying archive retains the mathematical reviews, computational evidence, and source records underlying the displayed scope.

Detailed ordinary proofs and construction records

The following appendices typeset the complete selected current construction proofs from Blocks 166 and 167. Their source notation, numbered source subsections, exact hypotheses, and mathematical scope are retained. The synthesis appendices state the final combined results; component notes also document narrower lemmas, alternative routes, and failed interpretations. In particular, the separate classical arithmetic-induction application in the leading-capacity note is not an input to the direct constructor.

Historical research identifiers, filenames, and references to earlier blocks are preserved to make the arguments traceable to the accompanying Markdown files. Earlier "pending" review language is superseded by the retained internal review records included with those source files, at their recorded dates and scopes. No external peer review is claimed. Negative capacity conclusions concern the original maximal-type carrier; the later low-rank enlargement does not repeal them. These appendices are ordinary mathematical proofs. Neither their typesetting nor the inherited formalization certificates constitute a fresh Lean proof of these arithmetic results or of Conjecture 66.

A Odd-prime synthesis and complete scope

Date: 2026-10-05. Parent: `Paper_XI_Base_Realization165.zip`.

This is a synthesis of the independently reviewed proofs in this continuation. The result is an ordinary mathematical proof, using the stated classical simple-type theorems. It is not a Lean certificate. Universal claim 66 remains open. Original H5 remains the original conditional premise about the same fixed candidate; no unconditional pair-factor comparison is asserted here.

A.1 1. The arithmetic construction now proved

Let p be odd, $F = \mathbb{F}_p((t))$, and

$$0 < h_1 < \cdots < h_d, \qquad p \nmid h_i, \qquad q_i = (p+1)h_i, \qquad N = p^d.$$

For every such tuple there is an explicit matrix $\beta \in M_N(F)$ whose field $K = F[\beta]$ is totally ramified of degree N, normalizes the complete lattice chain, and gives a maximal simple stratum. It has a literal defining sequence of degrees $p, p^2, \ldots, p^d$, with leading level n_1 and successive critical levels $n_2, \ldots, n_d$ given by

$$n_j = p^{d-j} q_{d-j+1} \quad (1 \le j \le d).$$

Every native parameter in the construction, every amplified approximation, and every consecutive correction has matrix trace zero over its base field.

For this actual stratum there exists a full simple character θ with $\theta|_{U_F^1} = 1$. Retaining its canonical beta extension, the residue character and noncompact extension can be selected so that the resulting irreducible admissible supercuspidal π has any prescribed smooth central quasicharacter ω trivial on U_F^1. These completion choices must be free or already compatible; incompatible immutably supplied choices are not replaced.

The same candidate has

$$\begin{aligned} \operatorname{rank}(\pi) &= N, & t(\pi) &= 1, \\ \operatorname{Sw}(\pi) &= p^{d-1}(p+1)h_d, & \dim_{\mathbb{F}_p}(J^1/H^1) &= N-1, \\ \dim \eta &= p^{(N-1)/2}, \end{aligned}$$

$$a(\pi^\vee \times \pi) = p^{2d} - 1 + \sum_{i=1}^{d} (p^{2i} - p^{2i-2}) h_i.$$

These numerical identities do not identify a Weil representation, an endo-class, or the pair epsilon factors required by H5.

A.1.1 Construction and proof

At a recursive base (L, t_L), introduce

$$E = L(\alpha), \qquad \alpha^p - \alpha = t_L^{-q_d}.$$

This is a separable totally ramified extension of degree p, with $v_E(\alpha) = -q_d$ and $\operatorname{Tr}_{E/L} \alpha = 0$. Choose $1 \le b < p$ with $q_d b \equiv -1 \pmod p$, set $a = (q_d b + 1)/p$, and put $z = t_L^a \alpha^b$. Then z is an explicit uniformizer. Write its Eisenstein polynomial as $f(X) = X^p + \cdots + a_0$.

Let P_0 extract the constant coefficient in the basis $1, z, \ldots, z^{p-1}$. The normalized corestriction $s = \kappa \operatorname{pr}$, where $\kappa = -f'(z)/z^{p-1}$, satisfies

$$u_0 = s(P_0) = a_0/z^p \in \mathcal{O}_E^\times, \qquad \operatorname{Lift}_E(C) = u_0^{-1} C P_0, \qquad s(\operatorname{Lift}_E(C)) = C.$$

The lift preserves the exact filtration valuation. Its crucial additional identity is

$$\mathrm{tr}_L(\mathrm{Lift}_E(C)) = \mathrm{Tr}_{E/L}\big(\kappa^{-1}\,\mathrm{tr}_E(C)\big).$$

Define

$$B(L, t_L; h_1, \ldots, h_d) = \alpha I_{p^{d-1}} + \mathrm{Lift}_E\big(B(E, z; h_1, \ldots, h_{d-1})\big),$$

with the degree-p multiplication matrix as the base case. Induction proves trace zero at every native rank. This is stronger than the automatic zero trace obtained by multiplying any matrix multiplicity by p in characteristic p.

For the inductive field step $d \geq 2$, put $n = p^{d-1}q_d$ and $R = p^{d-2}q_{d-1}$. The derived matrix is the recursively constructed maximal-field parameter, with $k_0 = -q_1$. The inequalities $n > pR > 2R$ imply $-2R > -n - q_1$. Bushnell–Kutzko Remark 2.2.7 therefore applies to the actual displayed matrix. It supplies its maximal field, normalization, and critical exponent. Valuation preservation and literal truncation of the recursion supply the whole defining sequence. The full details, quotient count, and conductor dependencies are in `tower/TRACE_ZERO_AS_TOWER166.md` and `tower/RECURSIVE_TOWER166.md`.

The full-character argument uses a specific order of operations. First construct a full character at native rank p^j, trivial on U_F^1. Transfer its full beta-determinant unit character to multiplicity p, using BK 3.6.13(ii) and 3.6.14. On a scalar $u \in U_F^1$, this gives $\theta_j(u^p) = 1$. Next add the actual trace-zero correction, using BK 3.3.18. It remains trivial on the scalar intersection of its domain. Finally extend at the next **maximal native parameter field**: the forced deep character and the trivial character of U_F^1 agree on their intersection, so they extend through a finite abelian unit quotient. BK 3.2.5 constructs the full simple character.

This procedure never substitutes an unjustified extension through a nonmaximal centralizer. It proves full-character compatibility, including shallow units, rather than merely cancellation of a leading trace. The residue factor fixes ω on $\mathbb{F}_p^\times$; an N-th-root choice for the noncompact generator fixes $\omega(t)$. Every choice is recorded before H5. See `central/TRACE_ZERO_CHARACTER_INDUCTION166.md` for the complete induction.

A.2 2. The finite packet that this same candidate realizes

Assume the one-vertex scalar-pushout contract: there are d labelled, actually supplied full character extensions with whole symplectic quotients V_i of dimension two over $\mathbb{F}_p$. Their balanced $\mathbb{C}^\times$-central product has quotient $V_{\mathrm{src}} = \bigoplus_i V_i$ and Heisenberg carrier dimension $p^d = N$. Initially take the declared stabilizer $C = \{1\}$.

For $p = 3$, this applies directly to the retained cubic family with whole rank-three break-type quotients. For $p > 3$, it applies only when the supplied plane data genuinely satisfy the stated original bridge contract. A native minimal rank-p type quotient has dimension $p - 1$, so it must not automatically be called a plane.

The actual candidate from Section 1 has a nondegenerate leading subspace

$$S \subset J^1/H^1, \qquad \dim_{\mathbb{F}_p} S = (p-1)p^{d-1} \geq 2d.$$

Here S is the image of the concrete congruence subgroup at half the leading even level. BK's high-depth equality preserves this subgroup and its commutator pairing through every later approximation; see `capacity/LEADING_LAYER_CAPACITY166.md`, Section 1. This part of that note applies to the direct constructor and does not require its separate classical arithmetic-induction application.

Choose a labelled symplectic embedding $V_{\mathrm{src}} \to S$. For each original full character form

$$\mathcal{P}_i = (\mathbb{C}^\times \times J_i^1)/\{(\theta_i(h)^{-1}, h) : h \in H_i^1\}.$$

The balanced product and the corresponding host restriction have the same alternating commutator. The center-fixing extension isomorphism, obtained with matched Darboux lifts and complex roots, therefore preserves every supplied value $\theta_i(h)$, including the shallow values.

Write $W = V_{\mathrm{src}} \perp V_{\mathrm{src}}^{\perp}$. Finite Stone–von Neumann gives $\eta_W \simeq \eta_{\mathrm{src}} \otimes \eta_{\perp}$. Choose an isotropic eigenspace giving a line in the complementary factor. Its actual Heisenberg image-algebra projector e has rank N. The retained induction/idempotent theorem recovers the **same** inducing representation Λ from $e\Lambda$, and then the same π by compact induction. No candidate is changed at recovery.

This attachment preserves the full source characters as scalar-pushout data. It does not identify the old matrix groups with literal subgroups of the new type, retain the old lower summand conjugacy classes, or place each source label at its old numerical host depth. Those stronger requirements have not been proved.

When the retained independent extraspecial tensor source has $d \geq 2$, its full determinant is trivial on wild inertia: each factor determinant has order dividing p, and tensor determinants raise it to p^{d-1}. Thus its central character belongs to the class in Section 1. For $d = 1$, wild triviality of the source determinant must be checked separately. Independence of the tensor factors is an input, not a consequence of distinct h_i.

A.3 3. A proved extension to genuine scalar inertia phases

There is also a positive result beyond $C = \{1\}$. Suppose the finite prime-to-p stabilizer acts trivially on the source scalar extension, and its prescribed honest operators on the **entire** rank-N Weil carrier are $\chi(c)I_N$, each lying in the actual inertia image. Assume the candidate's central character is the determinant of that same source under the fixed rank-one reciprocity convention.

Then a finite cover $\widetilde{C} \twoheadrightarrow C$ acts by actual central matrices in the unchanged candidate. Its represented operators descend to exactly χ on C. Hence the attachment and recovery above are equivariant for that action. A direct matrix homomorphism from C itself is not asserted and can fail.

Indeed, for residue cardinality q, let $B = \omega(\mathrm{Teich}(k^{\times}))$. The actual inertia hypothesis gives $\chi(C)^N \subset B$. Since N is a power of p and $\chi(C)$ has order prime to p, this implies $\chi(C) \subset B$. Choose k_0 inverse to N modulo $q-1$, and set

$$\delta(u) = \omega(\mathrm{Teich}(u))^{k_0}, \qquad \widetilde{C} = \{(c,u) : \chi(c) = \delta(u)\}.$$

The matrix $\mathrm{Teich}(u^{k_0})I_N$ acts by $\chi(c)$. The kernel acts trivially on the entire sector induction and ambient compact induction, because these matrices are central. This preserves the fixed Λ, its canonical beta-extension factor, and all completion choices. It does not assert that the canonical beta extension alone has scalar character χ.

The full theorem and the nonsplitting example are in `stabilizer/SCALAR_NATIVE_ATTACHMENT166.md`, with an independent review in `audit/SCALAR_NATIVE_REVIEW166.md`. An arbitrary abstract phase or a phase on a proper multiplicity space does not satisfy its hypotheses merely because it is scalar.

A.4 4. What was repaired, and what remains

The old fixed-field central argument genuinely failed. At $p = 3$, $(h_1, h_2) = (1, 5)$, its naturally transferred full character takes the value $\zeta_3 \neq 1$ on $1 + t^2$, although the desired wild center is trivial. A shallow character choice cannot repair a forced value. The two-stage changing-field repair is retained in `centra l/CENTRAL_REPAIR166.md`; the trace-zero tower above resolves this central issue uniformly in every length.

The construction does not use an inverse established correspondence or the classical arithmetic-induction/descent candidate to select π. It does use the named BK construction and

type theorems, the retained type/formal-degree conductor formulas, and exact sector recovery. The capacity note separately retains an alternative consequence for a candidate supplied by the substantial classical arithmetic constructor; that is a different route and is not an input to Section 1.

The unresolved universal obligations are precise:

1. **Nontrivial prescribed symplectic stabilizer actions.** For this totally ramified full-degree host, the stronger inner-normalizer interpretation forces prime-to-p actions on the quotient to be trivial. It therefore cannot implement a genuinely nontrivial supplied symplectic action. The raw original contract does not automatically require that stronger interpretation; it still needs an actual valid arithmetic comparison, not relabelling of external operators.
2. **Arbitrary admissible source packets and transport diagrams.** The proved attachment has the whole-plane, one-vertex, finite scalar-pushout hypotheses above. It does not establish all source classes, arbitrary graphs, or prescribed numerical layer placement.
3. **The dyadic construction with all markings.** A separate field-level recursion and some capacity statements exist at $p = 2$. The odd-prime trace-zero/full-character theorem here does not settle that case.

Original H5 stays exactly where it was: if supplied for this fixed candidate and the stipulated tests, the retained conditional endpoint applies. No claim that the displayed profiles imply H5, or that one of the noncompact root choices must satisfy H5, is made. The unresolved construction issues above are not renamed epsilon-factor obligations.

A.5 5. Evidence and reproducibility

The arbitrary-length theorem is proved symbolically and independently reviewed. The exact $p = 3, d = 2, (h_1, h_2) = (1, 2)$ computation in `tower/as_rank9check/` verifies the displayed matrix, trace identities, corestriction, and levels; it is a finite check, not the universal proof. The separate rank-27 run in `tower/run_rank27/` checks the earlier non-trace-zero recursion and must not be described as an AS central completion example. No Lean build was run.

`STATUS166.json` records the closed and open obligations. The component audits identify the exact reviewed files and hashes. The dependency file pins prior checkpoints and the authorized primary-source copy without redistributing that publication.

B The trace-zero Artin–Schreier tower

Date: 2026-10-05. Ordinary proof, submitted for independent audit.

B.1 Exact statement

Let p be odd, F=F_p((t)), and 0<h_1<…<h_d with p not dividing any h_i. Put q_i=(p+1)h_i. The constructor below gives an actual matrix B_d over F of size p^d with exactly the field, simplicity, approximation-level, odd-prime quotient, and conductor conclusions of `RECURSIVE_TOWER166.md`. In addition, its matrix trace is exactly zero. Every native shorter constructor and every amplified arithmetic approximation also has matrix trace zero. Therefore every difference between consecutive arithmetic approximations has matrix trace zero.

The trace statement is an identity over the current coefficient field, not merely a congruence, a field trace inferred from characteristic zero, or a division by the matrix size. All coefficient fields have characteristic p, so division by p^j would be invalid.

B.2 1. An explicit separable minimal extension with a traceless parameter

At an arbitrary recursive step let L have residue field F_p and uniformizer t_L. Given q>0 prime to p, set

```
E=L(alpha),  alpha^p-alpha=t_L^(-q).
```

The right side is not an Artin–Schreier coboundary in L: the negative valuation of x^p-x, if x has negative valuation, is divisible by p. Thus the polynomial is irreducible of degree p. The extension is separable, and the valuation equation forces ramification index p and

```
v_E(alpha)=-q.
```

It is therefore totally ramified and alpha is minimal over L. For odd p its field trace is

```
Tr_(E/L)(alpha)=sum_(c in F_p)(alpha+c)=0.
```

Choose the unique b in {1,…,p-1} with qb congruent to -1 modulo p, put a=(qb+1)/p, and define

```
z=t_L^a alpha^b.
```

Then v_E(z)=pa-qb=1, so z is a uniformizer and E=L(z). Its monic minimal polynomial $f(X) = X^p + a_{p-1}X^{p-1} + \cdots + a_0$ is Eisenstein. It is an explicit polynomial: compute the product over the p conjugates

```
f(X)=product_(c in F_p) (X-t_L^a(alpha+c)^b)
```

and reduce coefficients using $\alpha^p - \alpha = t_L^{-q}$. The invariant coefficients belong to L. This is a finite algebraic instruction, with no choice of an unknown extension or matrix.

B.3 2. Normalized corestriction and an explicit filtered right inverse

Use the basis 1,z,…,z^(p-1) of E over L. If T_ij is its coefficient matrix unit and pr is the trace corestriction, then

```
pr(T_ij)=z^i z_j^*,
z_j^*=sum_(k=j+1)^p a_k z^(k-j-1) / f'(z),  a_p=1.
```

Set

```
kappa=-f'(z)/z^(p-1),  s=kappa pr.
```

The displayed dual-basis formula gives

```
s(T_ij)=-z^(i-j) + terms of strictly larger E-valuation.
```

Indeed v_E(a_k)>=p for 1<=k<p, so the contribution indexed by a_k has valuation at least k greater than the first contribution. Applying s entrywise on E^m gives a B-bimodule corestriction from End_L(E^m) to B=End_E(E^m). With the complete interleaved period-pm L-chain and period-m E-chain, it preserves grades and satisfies $s(P^r) = Q^r$.

Let P0 extract the constant coefficient on every E-coordinate. The constant dual coefficient satisfies

```
z_0^*=-a_0/(z f'(z)),
u_0=s(P0)=kappa z_0^*=a_0/z^p in O_E^times.
```

For any C in M_m(E), define the actual L-linear matrix

```
Lift_E(C)=u_0^(-1) C P0.
```

Then

```
s(Lift_E(C))=C.
```

This lift preserves the exact filtration valuation. To see this, first multiply C by the unit u_0^(-1), which does not change its valuation in the E-chain. Expand each entry in the basis 1,z,…,z^(p-1). The only nonzero columns of Lift are its constant-coefficient columns. The L-grade of a coefficient c_i z^i E_uv is

```
pm*v_L(c_i)+mi+u-v,
```

equal to its E-grade, and distinct i have valuations distinct modulo p. Thus the minimum cannot cancel. This also proves the stated surjectivity $s(P^r) = Q^r$ without a compactness or lifting-existence assumption.

B.4 3. Exact trace identity

For y in E, extraction of its constant coefficient is

```
coeff_0(y)=Tr_(E/L)(z_0^* y).
```

The ordinary L-matrix trace of the rank-one operator yP0 is coeff_0(y). Consequently, for every C in M_m(E),

```
tr_L(Lift_E(C))
   =Tr_(E/L)(z_0^* u_0^(-1) tr_E(C))
   =Tr_(E/L)(kappa^(-1) tr_E(C)).            (T)
```

Here tr_E(C) is the ordinary m-by-m matrix trace over E. In particular, if tr_E(C)=0, then the lift has ordinary L-matrix trace zero exactly.

B.5 4. The recursive actual matrix

Define B(L,t_L;h_1,…,h_d) by induction. For d=1 use the multiplication matrix of the alpha from Section1 with q=q_1. For d>=2 use Section1 with q=q_d, put m=p^(d-1), recursively construct

```
C=B(E,z;h_1,...,h_(d-1)),
```

and set

```
B(L,t_L;h_1,...,h_d)=alpha I_m+Lift_E(C).      (ASR)
```

Everything in (ASR) is explicit in finite field operations and the displayed degree-p polynomial extensions. It differs from the old conjugacy-preserving two-summand construction. Preserving old lower summand conjugacy classes is not claimed.

Inductively tr_E(C)=0. Equation (T) and the Artin–Schreier trace give

```
tr_L(B)=m Tr_(E/L)(alpha)+tr_L(Lift_E(C))=0.
```

For the induction base the trace is Tr_(E/L)(alpha)=0. Thus this argument proves tracelessness in the current base field for every native rank, not merely tracelessness after an amplification whose size is divisible by p.

B.6 5. Actual fields and profiles

Set gamma=alpha I_m and b=Lift_E(C). The minimal outer stratum has

```
n=p^(d-1)q_d,  k_0(gamma)=-n.
```

The inner actual maximal field supplied by induction has

```
R=p^(d-2)q_(d-1),  v(C)=-R,  k_0(C)=-q_1.
```

We have s(b)=C, v(b)=-R, and n>pR>2R because p is odd and the h_i increase. Hence

```
-2R > -n-q_1 = k_0(gamma)+k_0(C).
```

BK Remark2.2.7 applies to this pure derived maximal field. It gives the field generated by the actual displayed gamma+b, with degree and ramification p^d, residue degree one, normalization of the complete chain, and k_0=-q_1. Its valuation remains -n. Thus [A,n,0,B] is simple.

This repeats the precise argument of `RECURSIVE_TOWER166.md` with a new outer minimal extension and normalized right inverse. No step of that argument required the old polynomial e^p+t e+t.

B.7 6. Native approximations, amplification, and traceless corrections

For the full rank p^d matrix put, for 1<=j<=d,

```
B^[j]=B(F,t;h_(d-j+1),...,h_d) tensor I_(p^(d-j)).
```

These are literal truncations of (ASR). At the outer level the same largest q_d selects the same Artin–Schreier extension and uniformizer in all truncations. At the next level the same q_(d-1) does the same, and so on. Thus no hidden change of bases or incompatible parameter fields enters the truncations. Lift_E commutes with amplification by an identity because P0 acts separately on each coordinate.

Every native shorter B has trace zero by Section4. Its amplification therefore has trace zero. In particular

```
tr_F(B^[j])=0,
tr_F(B^[j+1]-B^[j])=0.                          (Z)
```

The same is true at native rank p^(j+1): the correction between B(F,t;h_(d-j),…,h_d) and the identity-p amplification of B(F,t;h_(d-j+1),…,h_d) has trace zero. This native assertion is the one relevant when constructing characters before each transfer.

The field degrees and critical levels of $B^{[j]}$ are exactly

```
degree p^j,  n_j=p^(d-j)q_(d-j+1).
```

Valuation-preservation of every Lift gives

```
v_A(B^[j+1]-B^[j])=-n_(j+1).
```

Thus the approximation and defining-sequence proof of the first note applies unchanged. The odd-prime J/H dimensions, grade-index formula, Swan conductor and self-pair conductor depend on these same levels and centralizer degrees, so all their formulas carry over unchanged.

B.8 7. Precisely what the trace supplies

For any displayed correction delta and any scalar u in L^times for which its additive correction character is defined,

```
psi_delta(u I)=psi_L((u-1)tr_L(delta))=1.
```

This proves triviality of the actual additive correction on the scalar subgroup of its domain. It does not, by itself, construct a simple character with prescribed values on larger scalar groups. Compatible extension, transfer, and intersection arguments for the full simple character are separate obligations.

The note also does not identify the constructed candidates with any Weil source, choose them by inverse LLC, realize arbitrary stabilizer operators, or prove same-candidate H5. Its new output is an explicit arithmetic defining sequence whose trace vanishes at every native and amplified stage, suitable as input to that separate character argument.

B.9 Primary dependency

Bushnell–Kutzko, The admissible dual of GL(N) via compact open subgroups, Remark2.2.7 (printed p55), with the defining-sequence and ring recursion already pinned in `RECURSIVE_TOWER166.md`. Retained source SHA256: `d69db0827256707a34cd2562af68b9de11fc889ab3f43535ec47b620d58ca9c8`. The trace-dual and Artin–Schreier calculations above are given in full.

C Native-rank full characters and tame central completion

Parent checkpoint: Paper_XI_Base_Realization165. New arithmetic input: `attachment166/tower/TRACE_ZERO_AS_TOWER166.md`, together with its `RECURSIVE_TOWER166.md` defining-sequence proof. Input hashes are recorded in `TRACE_ZERO_CHARACTER_PINS166.json`.

Evidence: ordinary proof using the explicitly constructed arithmetic parameters and the named Bushnell–Kutzko character and transfer theorems. No arithmetic-induction/descent constructor or inverse correspondence is used. No Lean verification is claimed.

C.1 1. Statement

Let p be odd, F=F_p((t)), and

```
0<h_1<...<h_d,       p not dividing any h_i,
q_i=(p+1)h_i,        N=p^d.
```

Let beta_d be the actual trace-zero Artin–Schreier matrix constructor in the named tower note. There exists an actual full simple character theta for its maximal simple stratum such that

```
theta restricted to U_F^1 is the trivial character.            (1)
```

Every step constructing theta is specified below: a finite abelian character extension, the standard simple-character transfer, or the explicit additive correction from the displayed defining sequence. The field, order, and parameter are the actual tower ones, not an unspecified arithmetic realization.

The actual canonical beta extension can then be completed to a maximal extended type and supercuspidal pi with any prescribed smooth central quasicharacter omega_F trivial on U_F^1. Its rank, Swan conductor, torsion number and self-pair conductor remain those proved for the same tower. The residue and noncompact extension choices must be free or already compatible with omega_F.

For p=3, this gives full scalar-pushout attachment and same-candidate sector recovery for any number d of the supplied cubic source planes, with declared stabilizer C={1} and the one-vertex contract. All original full source character values are retained in the attachment. A general nontrivial stabilizer, transport diagram, or H5 conclusion is not part of this statement.

C.2 2. The exact arithmetic inputs

For 1<=j<=d, put

```
b_j=B(F,t;h_(d-j+1),...,h_d) in M_(p^j)(F),
```

where B is the trace-zero constructor. The tower proof gives:

1. K_j=F[b_j] is a totally ramified maximal field of degree p^j, normalizing the native complete period-p^j chain A_j.
2. For $1 \le j < d$, in rank p^(j+1), gamma_j=b_j tensor I_p is a simple approximation to b_(j+1), at the critical level r_j=q_(d-j)>0 of b_(j+1). They are equivalent as strata at that level.
3. For $1 \le j < d$, delta_j=b_(j+1)-gamma_j has grade -r_j and F-matrix trace zero.
4. At native rank p, b_1 is multiplication by alpha in the explicit Artin–Schreier field
   `E=F(alpha),  alpha^p-alpha=t^(-q_d).`
   Its actual field trace is zero, since p is odd.

For completeness, the tower's exact trace identity is

```
tr_L Lift_E(C)=Tr_(E/L)(kappa^(-1) tr_E(C)),
```

where s=kappa*pr is its normalized trace corestriction and Lift_E(C)=s(P0)^(-1) C P0. The recursive inner matrix has trace zero, so this proves zero trace at every native rank. It is not merely the automatic zero obtained by amplifying any matrix p times in characteristic p. The literal compatibility of the truncations is proved in Section6 of the tower note.

C.3 3. A finite character-extension fact

Let K/F be a finite totally ramified local extension. Suppose chi is a smooth character of U_K^a, and chi is trivial on U_K^a intersect U_F^1. Then chi extends to a smooth character of U_K^1 trivial on U_F^1.

Indeed, on U_K^a U_F^1 define chi_0(ku)=chi(k). Agreement on the intersection makes this a character. Choose M>=a such that chi is trivial on U_K^M. The character chi_0 descends to a subgroup of the finite abelian group U_K^1/U_K^M. It extends to that whole group by successively adjoining generators and selecting complex roots. Pulling it back proves the assertion. This also describes an actual finite procedure for making the extension choices.

This fact will be used only when the parameter field is maximal in the current native matrix rank. Direct use in a nonmaximal centralizer would require checking its determinant image and would not follow from the scalar intersection alone.

C.4 4. Base character at native rank p

Let E=K_1, and let n_1=q_d be the native level of alpha. Since n_1 is even, set a=n_1/2+1. On U_E^a the forced field-unit character is

```
phi(v)=psi_F(Tr_(E/F)(alpha(v-1))).
```

It is trivial on U_E^a intersect U_F^1 because Tr_(E/F)(alpha)=0. Section3 gives a full character chi_E of U_E^1 extending phi and trivial on U_F^1. No additive-character normalization is needed for this implication: the trace is exactly zero.

For the native minimal stratum, $H^1 = U_E^1$ U_(A_1)^(a). Define theta_1 by chi_E on U_E^1 and by psi_alpha on U_(A_1)^a. They agree on the intersection, so BK3.2.1 supplies the actual full simple character. It satisfies theta_1|U_F^1=1.

C.5 5. Transfer the full field character before adding the next layer

Suppose theta_j has been constructed on the native rank-p^j stratum of b_j, with theta_j|U_F^1=1. Its restriction to U_(K_j)^1 is the full beta-determinant character chi_j, since its native centralizer is K_j itself.

Use the standard simple-character transfer to the multiplicity-p realization gamma_j=b_j tensor I_p in rank p^(j+1). Denote the transferred full character by Theta_j. The exact primary result is BK3.6.13(ii), with existence in Theorem3.6.14: transfer preserves the full beta-determinant character on U_(K_j)^1. Consequently its restriction to the centralizer order is chi_j composed with det_(K_j).

For every literal scalar u in U_F^1 this gives

```
Theta_j(u I)=chi_j(u^p)=theta_j(u^p)=1.            (2)
```

The determinant exponent p is essential. We do not replace (2) by an unsupported pullback along a scalar matrix embedding. This transfer also deals with nonminimal b_j: beta-determinant preservation is the general theorem, not just the minimal-character formula.

C.6 6. Add the actual correction and extend at maximal rank

Let r=r_j=q_(d-j), m=floor(r/2)=r/2, beta=b_(j+1), gamma=gamma_j, and delta=beta-gamma. The actual defining sequence gives

```
D=H^(m+1)(beta)=H^(m+1)(gamma).
```

By BK3.3.18, the character

```
vartheta=Theta_j restricted to D times psi_delta
```

is an actual simple character at level m for beta. For every scalar u in D intersect U_F^1, equation (2) and the exact trace identity give

```
vartheta(u)=Theta_j(u)
               psi_F((u-1)tr_F(delta))=1.        (3)
```

The new parameter field K=K_(j+1) is maximal and totally ramified, and the native chain has K-period one. Thus

```
U_K^1 intersect D=U_K^(m+1),
U_F^1 intersect D=U_F^ceil((m+1)/p^(j+1)).
```

In particular (3) supplies the exact compatibility condition for extending vartheta|U_K^(m+1) while imposing triviality on all U_F^1. Apply Section3 to obtain such a character chi_K of U_K^1.

BK3.2.5, including its explicit extension formula, now constructs

```
theta_(j+1)(kd)=chi_K(k) vartheta(d),
k in U_K^1, d in D.
```

The proposition supplies normalization of vartheta by the field units, well-definedness, and the full simple-character condition. This theta_(j+1) has the claimed trivial wild center. Induction proves (1) at the actual final rank p^d.

The proof deliberately performs every arbitrary character extension at maximal native rank. Extending directly along amplified intermediate centralizers M_(p^(d-j))(K_j) would introduce determinant-image compatibility conditions. Merely knowing that their ambient matrix trace is zero would not discharge those conditions. Transfer first, followed by maximal-rank extension, removes that ambiguity rigorously.

C.7 7. Complete this same type with the prescribed tame center

Keep the actual canonical beta extension kappa of the resulting theta. The final K/F is totally ramified of degree N, with residue rank one. The compact type group satisfies

```
J/J^1 = k_K^times = F_p^times.
```

On U_F^1 the beta extension acts by theta, hence trivially. For a prescribed omega_F trivial on U_F^1, choose the residue character

```
sigma=omega_F|F_p^times divided by omega_kappa|F_p^times.
```

Its inflation gives lambda=kappa tensor sigma with the right center on O_F^times. Complete lambda to an actual extension Lambda_0 on the inducing group K^times J using the retained maximal-type completion theorem. If its center is omega_0, choose s in C^times with

```
s^N=omega_F(t)/omega_0(t).
```

Multiply Lambda_0 by the character of K^times J trivial on J and taking a K-uniformizer to s. The quotient (K^times J)/J is Z, and t has image N; this proves exact agreement of the resulting central character with omega_F on all F^times. Canonical kappa is unchanged.

The resulting pi=c-Ind Lambda is irreducible, admissible and supercuspidal. All these choices are made before H5. If a residue character or noncompact extension has been immutably supplied, it must already satisfy this compatibility; an incompatible datum is not silently replaced.

By the exact same parameter and defining-sequence calculations,

```
rank(pi)=N=p^d,
torsion(pi)=1,
Swan(pi)=p^(d-1)(p+1)h_d,
a(pi^vee x pi)=p^(2d)-1
                + sum_i (p^(2i)-p^(2i-2))h_i.
```

No determinant twist changes the depth. These are the proved arithmetic tower profile formulas, not a proof of H5.

C.8 8. Full packet attachment in arbitrary cubic length

For p=3, retain d original supplied full characters theta_i and their actual character groups H_i^1 normal in J_i^1, with whole symplectic quotients V_i that are the d labelled planes. Assume the declared stabilizer is C={1} and the diagram is one vertex. The source balanced scalar extension has quotient V_src of dimension 2d and Heisenberg carrier dimension 3^d, the original rank.

The actual final host has total quotient dimension N-1. Its leading nondegenerate symplectic layer has dimension

```
(p-1)p^(d-1)=2*3^(d-1) >= 2d.
```

This follows either from the explicit defining-sequence quotient calculation or the independently proved leading-layer lemma. Choose a labelled symplectic embedding of the supplied V_src in that layer. Write W=V_src perpendicular-sum V_perp. Choose a maximal isotropic subspace L in V_perp. Its associated eigenspace in the actual Heisenberg carrier has dimension 3^d, and L-perp/L is V_src.

Form the actual reduced scalar extension using the full host character and its extension to the inverse image of L. Form each original full pushout

```
P_i=(C^times x J_i^1)/
          {(theta_i(h)^(-1),h): h in H_i^1}.
```

The center-fixing symplectic-extension isomorphism, constructed by matched Darboux lifts and complex roots, maps their balanced product to that actual reduced scalar extension. In particular each original h in H_i^1 maps to the scalar theta_i(h). Every prescribed shallow and deep source character value is retained. Source character attachment does not claim that the original field units are the same literal scalar matrices in the new host.

Finite Stone–von Neumann supplies the full carrier intertwiner. The retained sector induction and recovery idempotent then recover this same Lambda and this same pi from the source-sized sector. The canonical beta extension, residue factor, central character, and noncompact scalar are those actually selected in Section7. No source-independent alternate candidate is substituted in recovery.

For the retained independent tensor family with d>=2, each factor's wild determinant has order dividing p, because the abelianization of its extraspecial order-p^3 image is elementary abelian. The tensor determinant raises each factor determinant to p^(d-1), so the full source determinant is trivial on wild

inertia. Thus the source's prescribed central quasicharacter lies in the class covered by Section7. At d=1, matching this central conclusion requires the source determinant itself to be trivial on wild inertia; it is not inferred merely by taking a tensor power of length one.

For p>3, the same arithmetic host and character theorem hold, and the same attachment works for any actually supplied valid packet whose whole quotient and markings meet the stated symplectic contract. A proposed source plane is not automatically a whole rank-p type quotient: that latter quotient has dimension p-1. No admission of all such plane sources into original H2 is inferred.

C.9 9. Boundary and primary-source check

This provides the previously missing full-character and actual wild-center construction for the explicit odd-prime arithmetic tower in every prime-power rank. The proof uses no classical automorphic-induction/descent candidate. It retains the supplied finite characters through the declared scalar-pushout attachments.

It does not preserve old summand matrix conjugacy classes, produce a prescribed nontrivial stabilizer action, preserve arbitrary numerical depth labels, handle arbitrary transport graphs, settle the dyadic case, or prove original same-candidate H5. It therefore does not close unrestricted claim66.

The retained primary BK text, SHA256 `d69db0827256707a34cd2562af68b9de11fc889ab3f43535ec47b620d58ca9c8`, was directly checked at:

- Definition3.2.1 and Proposition3.2.5, printed98–100: minimal character condition and explicit full-character extension.
- Corollary3.3.18, printed112–113: the actual additive correction takes the approximation character to the next simple character.
- Condition3.6.13(ii) and Theorem3.6.14, printed136–137: transfer preserves the full beta-determinant character, including at m0=0.

The actual field and defining-sequence assertions are the separately proved and independently reviewed trace-zero tower inputs, with BK2.2.7 and the retained simple-stratum theorems. No newly downloaded source or new novelty claim is involved.

D The surviving leading symplectic layer

Status: ordinary proof from the retained Bushnell–Kutzko theorems and, for the source-defined candidate, the substantial published Bushnell–Henniart arithmetic induction/descent constructor. No Lean build or bounded computation is used. The completed independent review is retained in the accompanying audit files.

D.1 The useful simplification

Full critical-profile rigidity is unnecessary for the trivial-stabilizer finite attachment contract. An even leading level produces a symplectic subspace that survives every later simple approximation. For the independent rank-p Heisenberg tensor sources, this single subspace already has enough dimension to hold every source plane, however many there are.

The argument below concerns the actual type in the source-defined arithmetic candidate. It does not identify that type with a separately guessed binder parameter.

D.2 1. Leading-layer theorem

Let F have residue field k of cardinality q and residue characteristic p. Let pi be an actual irreducible supercuspidal representation of GL_n(F) with torsion number one, where n=p^r. Suppose

```
a(pi)=n+N,     N>0 even,     v_p(N)=r-1.
```

Then an actual maximal simple type in pi, with Heisenberg quotient $W = J^1/H^1$, contains a nondegenerate symplectic F_p-subspace S of dimension

```
[k:F_p] (p-1)p^(r-1).
```

For k=F_p this is exactly (p−1)p^(r−1). The subspace is the image of the concrete congruence group U^(N/2)(A) in J^1/H^1. It is present regardless of the later critical exponents, their parities, or the number of intermediate approximation fields.

D.2.1 1.1. Actual parameter field and leading level

By BK maximal-type existence, pi contains a maximal simple type. Its positive level follows from a(pi)>n. Write its stratum as [A,N_A,0,beta], with parameter field K=F[beta]. BK 6.2.5 gives

```
t(pi)=n/e(K/F).
```

Thus e(K/F)=n. Since [K:F] divides n, K/F is totally ramified of degree n and has residue degree one. The principal-order assertion after BK 6.2.5 gives e(A)=n. Accordingly A is the complete principal hereditary order. The established conductor formula

```
a(pi)=n+n N_A/e(A)
```

now implies N_A=N. This is the same source-checked formula used in Block165, originating in BH1999 Section 6.1 Lemma 2 and retained in `PRINCIPAL_ORDER_CONDUCTOR161.md`.

Put P=rad(A), $U^a(A) = 1 + P^a$ for a>=1, and m=N/2.

D.2.2 1.2. The terminal minimal approximation has degree p

Apply the defining-sequence construction BK 2.4.1–2 to beta. Write gamma for its terminal minimal approximation. Every transition has a critical level c<N, and gamma has the same A-level N as beta.

All approximation residue degrees divide f(K/F)=1, and all approximation ramification degrees divide n. Therefore E=F[gamma] is totally ramified of degree d dividing n.

The scalar case is impossible: a scalar of F has A-valuation divisible by n, whereas v_p(N)=r−1. Thus gamma is noncentral and minimal in the ordinary field sense. Its field valuation is

```
−v_E(gamma)=Nd/n.
```

Minimality, by BK 1.4.14–15, requires gcd(Nd/n,d)=1. Since d is a p-power and v_p(N)=r−1, integrality first gives p|d, while coprimality forces d=p. Hence E/F is totally ramified of degree p.

Set B=End_E(F^n), B_order=A∩B, and Q=rad(B_order). The same complete lattice chain has period n/p over E. Each P-grade has k-dimension n and each Q-grade has k-dimension n/p. In particular,

```
dim_k P^m/(Q^m+P^(m+1))=n−n/p.
```

No assertion that E is literally a subfield of the final field K is needed. It is the actual minimal approximation field in the defining sequence.

D.2.3 1.3. The high subgroup survives every approximation

The minimal definitions BK 3.1.7–8 give

```
J_order(gamma)=B_order+P^m,
H_order(gamma)=B_order+P^(m+1).
```

At a nonminimal transition beta_i→beta_(i+1), with critical level c_i<N, the recursive J-order definition uses the threshold ceil(c_i/2). Since ceil(c_i/2)<=m, the inclusion P^m⊂J_order(beta_(i+1)) implies P^m⊂J_order(beta_i). Starting at gamma, induction proves

```
U^m(A)⊂J^1(beta).
```

For H, BK 3.1.9(ii) states that $H^a(\beta_i) = H^a(\beta_{i+1})$ as soon as

```
a>=floor(c_i/2)+1.
```

Every c_i is at most 2m−1, so this threshold is at most m. Consequently

```
H^m(beta)=H^m(gamma).
```

Here H^m means $H^1 \cap U^m(A)$. Therefore the image S of U^m(A) in W is canonically

```
S = U^m(A)/H^m(beta)
  = U^m(A)/H^m(gamma)
  ≅ P^m/(Q^m+P^(m+1)).
```

The last group identification uses m>=1, so multiplication is additive modulo P^(m+1). Thus |S|=q^(n−n/p).

D.2.4 1.4. Its actual commutator form is nondegenerate

Choose the actual simple character theta contained in the type of pi. By the standard deep-restriction property of simple characters, theta restricts to psi_beta on U^(m+1)(A). This follows directly from BK 3.2.1–4 by induction along the defining sequence.

For u,v in U^m(A), their group commutator is in $U^{2m}(A) = U^N(A)$, which lies in U^(m+1)(A). The terminal approximation satisfies

```
beta−gamma∈P^(1−N),
```

because every critical level in the sequence is less than N. Hence

```
(beta−gamma)([u,v]−1)∈P.
```

With the standard level-one additive character psi_A=psi_F∘tr_A/F, psi_A is trivial on P. We therefore have the exact equality

```
theta([u,v]) = psi_beta([u,v]) = psi_gamma([u,v]).
```

The right side is the Heisenberg commutator pairing for the minimal simple stratum of gamma. Its quotient $J^1(\gamma)/H^1(\gamma)$ is precisely $U^m(A)/H^m(\gamma)$, because $J^1(\gamma) = U^1(B_{\text{order}})U^m(A)$, with $U^1(B_{\text{order}}) \subset H^1(\gamma)$. BK's nondegeneracy theorem for simple-character commutators therefore says that the form on S is nondegenerate.

For general k, this is a nondegenerate alternating F_p-form of dimension $[k : \mathbb{F}_p](n - n/p)$. For $k = \mathbb{F}_p$, it has dimension $n - n/p$. This proves the theorem.

The argument uses no self-pair conductor and no determination of the later critical levels. Those invariants remain useful for finer arithmetic identification, but they are not needed for this finite-sector capacity claim.

D.3 2. Arbitrary-length odd-prime source family

Fix an odd prime p and F=F_p((t)). Let the complete marked Weil source rho be an independent tensor product of r rank-p wild Heisenberg factors at distinct positive levels

```
0<h_1<...<h_r,     p does not divide h_i.
```

Here independence means the stipulated balanced-central-product description of the wild image is valid, and rho|P_F is irreducible. We do not infer independence from the inequalities alone. Suppose the retained Tsushima source calculation supplies the usual highest slope

```
slope(rho)=(p+1)h_r/p.
```

Then rank(rho)=n=p^r and

```
Swan(rho)=N=(p+1)p^(r−1)h_r.
```

Let Pi_BH(rho) be the published source-defined arithmetic induction/descent candidate, with its full prescribed central character, as audited in Block165. The construction is applied to the complete actual rho, including its scalar normalization. Its codomain is the torsion-one supercuspidal class, and its standard epsilon-function property gives a(Pi_BH(rho))=n+N. These are uses of the substantial classical constructor, not consequences of finite Heisenberg algebra alone and not a selection of an inverse LLC image.

For odd p, N is even and v_p(N)=r−1. The leading-layer theorem therefore puts a symplectic space of dimension

```
D=(p−1)p^(r−1)
```

inside the actual Heisenberg quotient of a type in this same Pi_BH(rho).

The source packet is the orthogonal sum of r planes, so its symplectic dimension is 2r. The elementary inequality

```
(p−1)p^(r−1)>=2r
```

holds for every odd p and r>=1: it suffices that 3^(r−1)>=r, which follows by induction. Thus the actual candidate's leading symplectic layer alone can receive the entire source packet.

D.3.1 2.1. Full characters and recovery for declared C={1}

Work under the same finite scalar-pushout contract as `SAME_CANDIDATE_ATTACHMENT165.md`: declared stabilizer C={1}, one vertex, and full source character extensions retained, with no requirement of literal old matrix-group inclusion or equality of old numerical depths with host depths.

Let P_src be the balanced C×-central product of the r supplied full source character extensions. Its quotient V_src has dimension 2r over F_p and its central-character-one Heisenberg representation has dimension p^r=n. Choose a symplectic embedding V_src↠S⊂W.

The C×-pushout of the actual type character, restricted to this subspace, has the same alternating commutator as P_src. The retained extension theorem supplies a center-fixing isomorphism from P_src to that restriction. Consequently every source h∈H_i^1 is carried to the scalar theta_i(h), preserving the full character value. No shallow source values are discarded.

Let V be the image and write W=V⊥V_perp. Stone–von Neumann gives

```
eta_W ≅ eta_src⊗eta_perp.
```

Choose a line in eta_perp and the corresponding rank-n projector e in the actual finite Heisenberg image algebra. Then e eta_W≅eta_src. Applying the retained Block159 induction/idempotent theorem to the unchanged actual extended type Lambda in Pi_BH(rho) recovers Lambda and then this exact Pi_BH(rho). All source normalization and the actual noncompact scalar remain fixed.

This proves same-candidate finite attachment for arbitrary r in this C={1} odd-prime family. Original H5, when supplied for this same candidate, remains exactly the original conditional comparison premise. We do not prove it here.

D.3.2 2.2. First new rank: GL_27

For p=3,r=3, the source has rank27 and V_src has dimension6. Put h=h_3 with 3∤h. The actual candidate has

```
N=36h,       m=18h,
dim_F3 S=27−9=18.
```

Thus its high layer contains the entire three-plane source packet. At least a 3^9-dimensional Heisenberg contribution is present, while the required source sector has dimension27=3^3. The six surplus symplectic pairs can be projected onto a line in their representation, together with any additional lower layers. This proves the required capacity and sector realization for the new rank27 family without solving its two possible lower critical levels.

In particular, for (h_1,h_2,h_3)=(1,2,4), N=144 and the same high subspace has dimension18. These numbers are a specialization of the universal proof, not a numerical experiment.

D.4 3. Why trying to force every critical level was unnecessary

At rank27, a hypothetical complete degree pattern 27↠9↠3 has leading level N=36h_3 and lower critical levels a<b<N. The standard full-profile conductor count would give

```
Swan(End rho)=18N+6b+2a.
```

Comparing with the stipulated source formula 8(h_1+9h_2+81h_3) only gives

```
a+3b=4h_1+36h_2.
```

That single linear relation does not determine both a and b, and by itself does not force each parity. Nor does it establish that all three degrees occur rather than skipping degree9. This observation is only a warning about the insufficient equations; it is not presented as an arithmetic counterexample or as existence of every numerical profile satisfying the equation.

The leading-layer theorem bypasses these unresolved profile choices. It supplies a fixed nondegenerate high subspace in every actual candidate with the two required invariants.

D.5 4. Dyadic capacity boundary

The leading-layer theorem itself is valid at p=2. If an actual torsion-one candidate of rank2^r has

```
N=3·2^(r−1)h,     h odd,     r>=2,
```

then its high subspace has dimension 2^(r−1) over F_2. This alone is at least 2r for r>=4, with equality at r=4. It does not supply enough capacity by itself for r=2 or3. For r=1, N is odd and this theorem does not apply.

This is a capacity statement only. Applying it to a dyadic source requires the actual source family, its arithmetic candidate and whatever quadratic/phase markings the original contract requires. No universal dyadic attachment theorem is claimed from dimension alone.

D.6 5. Boundaries that remain

1. The full character attachment here is through center-fixing scalar-pushout maps. Literal embeddings of prescribed old arithmetic matrix groups are not proved.
2. No nontrivial prescribed stabilizer action is attached. Capacity does not imply equivariant placement.
3. All planes may be placed in the leading host layer. Preservation of supplied numerical ramification depths, or a specified map into individual host depth layers, would be an additional requirement and is not claimed.
4. General packet transport graphs are not handled here. The statement is for C={1} and one vertex.
5. The arithmetic induction/descent theorem is an explicit major external input. This is a new capacity/attachment consequence for its actual output, not a new elementary reconstruction of that classical constructor.
6. H5 remains conditional and refers to that unchanged actual candidate. No additional pair-factor premise is introduced.
7. No Lean verification is claimed.

D.7 6. Primary locators and retained dependencies

The authorized primary BK text was read at these exact results: 1.4.14–15 (minimality), 2.4.1–2 (defining sequence and degree divisibility), 3.1.7–9 (recursive orders and high-depth equality), 3.2.1–4 (deep restriction of simple characters), 3.4.1 (nondegenerate Heisenberg commutator), and 6.2.5 with the following paragraph (torsion number and principal period).

The standard conductor and published arithmetic-constructor properties are reused from the source-pinned Block165 audit and Block161 conductor proof. The finite scalar-pushout and sector-recovery arguments are exactly those retained and restated in `SAME_CANDIDATE_ATTACHMENT165.md`. No entire copyrighted source is copied into this continuation.

E Actual scalar inertia phases and finite-cover descent

Evidence: ordinary proof using rank-one reciprocity, the given central character, and elementary finite group arguments. The result constructs actual central matrix operators through a finite cover and proves their descent. It does not construct a new arithmetic candidate or alter a canonical beta extension.

E.1 1. Exact statement

Let F be a nonarchimedean local field with residue field k of cardinality q and residue characteristic p. Put Q=q-1 and n=p^r, allowing r=0. Let rho:W_F -> GL_n(C) be a smooth Weil representation with finite inertia image. Use the fixed reciprocity convention identifying its determinant with a character omega:F× -> C×.

Let C be a finite group of order prime to p, and suppose its prescribed operators on the **full source carrier** are

```
Omega_src(c)=chi(c)I_n,
```

where chi is a character and each Omega_src(c) belongs to the actual matrix image rho(I_F). A finite cyclic scalar tame image is the intended example. This hypothesis refers to actual source operators, not freely chosen phases on an unrelated multiplicity-space factor.

Let Pi be any already constructed irreducible smooth GL_n(F) representation with central character omega. Suppose it has its supplied actual inducing representation Lambda on N, with F× contained in N, and an attached finite sector E=eLambda realizing the prescribed source Heisenberg packet. Assume the prescribed C action on the source scalar extension is trivial, and that the honest scalar phase on the packet is the same chi above.

Theorem. There is a finite group C_tilde surjecting onto C and a genuine homomorphism

```
g:C_tilde -> F× I_n contained in N
```

whose action on Lambda and on Pi descends to the prescribed honest C action chi(c)I. Every finite sector is stable under this action. Any center-preserving Heisenberg attachment and its intertwiner are C-equivariant. The retained sector-induction recovery maps recover the same Lambda and Pi equivariantly. All actual inducing data, including the fixed canonical beta extension, remain unchanged.

The arithmetic action is supplied by actual central matrices, with an explicit finite-cover descent. The theorem does not assert that C itself embeds as a matrix subgroup. Such an embedding can fail even for a genuine scalar tame source, as Section4 shows.

E.2 2. Construction and proof

Let D=k×, and let Teich:D -> O_F× be the canonical multiplicative lift. For p=2 this is still the prime-to-p root-of-unity subgroup. When q=2, D is trivial and the argument below forces chi trivial as well.

Local reciprocity identifies the determinant image of inertia with

```
det rho(I_F)=omega(O_F×).
```

The unit group decomposes as

```
O_F× = Teich(D) times U_F¹.
```

The smooth character image omega(U_F¹) is a finite p-group, while omega(Teich(D)) has order prime to p. Therefore the prime-to-p elements of omega(O_F×) are exactly omega(Teich(D)).

For every c in C, the actual-source hypothesis implies

```
chi(c)^n=det(Omega_src(c)) in det rho(I_F).
```

This value has order prime to p, and hence lies in omega(Teich(D)). Since n is a p-power and chi(C) has order prime to p, the map z -> z^n permutes chi(C). Consequently

```
chi(C) is contained in omega(Teich(D)).                (1)
```

In particular every source phase has order dividing Q. This conclusion does not require selecting a splitting of the tame quotient or choosing Frobenius representatives.

Choose an integer k_0 with n k_0 congruent to1 modulo Q. When Q=1 take k_0=1. Define

```
delta(u)=omega(Teich(u))^(k_0),       u in D.
```

Because exponentiation by k_0 is an automorphism on every group of roots of unity of order dividing Q, delta(D)=omega(Teich(D)). Define the finite fiber product

```
C_tilde={(c,u) in C times D : chi(c)=delta(u)}.          (2)
```

It is a subgroup, and projection pr:C_tilde -> C is surjective by (1). Define the actual central matrix homomorphism

```
g(c,u)=Teich(u^(k_0)) I_n.                              (3)
```

Its action on Lambda and Pi is the central-character scalar

```
Lambda(g(c,u))=omega(Teich(u^(k_0)))I
              =delta(u)I=chi(c)I.                      (4)
```

If (1,u) belongs to ker pr, (4) is the identity. Hence the represented action factors uniquely through C. It is exactly the prescribed source action. Although selecting a particular matrix representative for c may require a choice of u, the represented operator is independent of that choice. Changing k_0 by a multiple of Q leaves (3) unchanged.

All matrices (3) are central. Their conjugation on actual groups, characters, and scalar pushouts is therefore the identity, and their Lambda operators are scalar. Thus they preserve every e and every sector E. The source extension action is also the identity by hypothesis, so any attachment of those scalar extensions is equivariant. Both honest carrier actions are chi(c)I, so every Heisenberg intertwiner intertwines C without a new phase choice.

For the recovery step, put P={a in N:Lambda(a)eLambda(a)^(-1)=e}. Then F× is contained in P. The retained maps

```
Phi:Ind_P^N E -> Lambda,
j:Lambda -> Ind_P^N E
```

are N-equivariant. They therefore commute with all actual matrices (3), and ker pr also acts trivially on Ind_P^N E: its matrices are central elements of P acting trivially on E, hence trivially on every induced translate. Thus the equality descends through pr to C. The recovering idempotent, and then compact induction to GL_n(F), preserve this equivariance. They recover the same fixed Lambda and Pi. QED.

E.3 3. What this closes and what it does not

For a source packet whose marked tensor carrier is the entire rank-n source, this proves the stabilizer step whenever its genuine tame operators are scalar and an actual host with the correct central character and full-character attachment is already present. There is no remaining scalar-phase equation to solve in that situation: the determinant and central character supply it.

If only the source symplectic C action is known to be trivial, the preceding `STABILIZER_CONTRACT166.md` proves triviality of its full scalar-extension action for prime-to-p C. Schur then gives a scalar packet character. To apply the present theorem one must still identify that character with the action of actual inertia on the full source. That identification holds when the supplied honest packet carrier is the whole source carrier. It is not automatic for a proper multiplicity-space factor, whose omitted factor can contribute to the determinant.

For several factors the total packet character is the product of the supplied factor characters. The theorem matches that product on the whole attached packet. It does not represent independent factor actions by distinct central matrices unless those additional marked actions are separately specified and verified.

The result uses the fixed full inducing representation Lambda. If Lambda|J=kappa_can tensor sigma, then kappa_can and sigma are both unchanged. Equation(4) concerns Lambda, and does not assert that kappa_can alone has the same scalar character. A contract prescribing an action specifically on the canonical beta extension requires that distinction to remain explicit.

No nontrivial source symplectic action is realized here. No new base candidate, new H5 comparison, dyadic quadratic refinement, or general rank theorem is claimed. Existing quadratic refinements are preserved because the actual central conjugation action is the identity.

E.4 4. Why a direct map C into residue units is too strong

Take q=5, n=5^r, and let omega on k×=F5× be the quadratic character. Let C=C2 and let chi be its nontrivial character. Such scalar tame behavior is obtained by twisting a tame-trivial rank-5^r source by the quadratic tame character, whenever such a source is supplied; the retained odd-prime two-factor family supplies rank25 examples.

A group homomorphism a:C2 -> F5× must send its generator to1 or -1. Both have quadratic character1. Consequently

```
omega(a(c)) = chi(c)^n
```

is impossible at the nontrivial c, since n is odd. The proposed direct section is not forced by genuine scalar tame source data. Reciprocity supplies a quotient of residue units; that quotient need not split.

The construction above still works. Here k_0=1 modulo4, C_tilde is the cyclic group of order4, and its central matrices act by the quadratic character. The kernel of C_tilde -> C2 acts trivially on the fixed candidate, so the actual operator action descends correctly. This example explains why finite-cover descent is necessary rather than a cosmetic reformulation.

E.5 Dependencies and review status

External input: rank-one local class field theory, with the same fixed normalization used for omega=det rho; the existing actual candidate's central character identity; the retained sector recovery theorem. The finite-cover construction and its descent are proved above and use no local Langlands candidate selection. No computation is needed. This version has undergone author self-review; a parent audit is requested. No Lean run is claimed.

F The precise stabilizer-action contract and normalizer obstruction

Evidence: direct audit of the original statement, a new elementary reduction, and an explicit actual-source/actual-type H1–H2 witness. Earlier artifacts are read-only. No Lean closure, class-wide arithmetic realization, or H5 witness is claimed.

Primary audit verdict: incomplete, with the smallest missing implication being the identification of the declared finite action with the action intended on a newly constructed common arithmetic candidate. The original text does not define that identification by a common matrix normalizer.

F.1 1. Exact contract, and what is already an input

The controlling text is `closure_attempt162/inputs/original_reading.txt`:

- Proposition9.8, starting line4834: a finite group acting by automorphisms of the one-break Clifford datum acts symplectically on its intrinsic envelope.
- Definition11.1, lines4911–4923: a bridge supplies an equivariant isometry to an actual type quotient for a finite cyclic stabilizer C, with common central character and additional dyadic lift data.
- Proposition11.3 and Theorem11.5: once the two honest cyclic actions are chosen, their discrepancy is a character, killed by generator-phase equality.
- Theorem15.4 and Theorem16.2: breakwise comparisons and transports are supplied and tensored. They do not construct a common arithmetic host from separate actual type targets.

Thus H2 and H3 already settle the finite action comparison **on each supplied break target**. They do not leave a second free stabilizer action to invent on that same target. Their transport into a new common arithmetic type is another assertion. Neither a map C into its matrix normalizer nor arbitrary transported operators are identified with that assertion by the original text.

The cyclic group in Definition11.1 is not explicitly required there to have order prime to p. The reduction below assumes that order condition; it applies to the declared tame prime-to-p case, and cannot silently be applied to an arbitrary finite-order Frobenius operator.

F.2 2. Full scalar-extension rigidity for a native tame action

Let an actual maximal simple type in GL_n(F) have a totally ramified parameter field K/F of degree n=p^r and residue matrix rank one. Let

$$W=J^1/H^1,\quad P=(C\times J^1)/\{(\theta(h)^{-1},h):h\in H^1\},$$

and let eta be its irreducible Heisenberg representation. Let C be finite of order prime to p. Suppose the C action on P fixes the scalar center and is genuinely induced by character-preserving matrix normalizers of this type. It is enough that it is an honest action whose action on W comes from those normalizers. Then:

1. C acts trivially on W.
2. C acts trivially on the full scalar extension P.
3. Every honest C action Omega on eta implementing this action is Omega(c)=chi(c)I for one character chi of C.

Proof. The retained normalizer theorem in Block162 gives

$$N=K\times J^1,\qquad N/(F\times J^1)=Z/nZ.$$

The action on W factors through this cyclic p-group, so a homomorphic image of the prime-to-p group C is trivial.

For any central extension P of the elementary abelian p-group W by C×, the kernel of

```
Aut_(C×)(P) -> Aut(W)
```

is naturally Hom(W,C×). Indeed, an automorphism phi fixing the quotient and the center has phi(x)=a(x_bar)x. The scalar a is independent of the lift x and multiplication shows that a is a character. Conversely every such character defines an automorphism. Since W has exponent p, this kernel is a finite elementary abelian p-group. The action of C lies in this kernel and therefore is trivial. Finally an implementing operator commutes with the irreducible P action, so is scalar by Schur. Honesty makes its scalar a character. QED.

This strengthens the earlier W-level obstruction: passing to the full scalar pushout cannot hide a nontrivial native prime-to-p action above an identity quotient action.

F.3 3. Exact attachment criterion in this native setting

Suppose a marked source scalar extension P_src has a symplectic embedding into W, and an actual finite sector of eta realizes that extension. Fix its center-preserving attachment and a Heisenberg intertwiner T. If the host has the native tame action of Section2, the attachment intertwines the full honest C actions if and only if

```
the source action on P_src is trivial, and
chi_src = chi_host.
```

Necessity follows from equivariance and Section2. For sufficiency, trivial extension actions make both honest carrier actions scalar by Schur, and equal characters commute through every Heisenberg intertwiner. Any sector is stable under the scalar host action.

If the source C action on its symplectic quotient is trivial, its action on the full scalar extension is already trivial by the Hom-kernel argument. For several break factors, the source character is the product of their characters. Thus the criterion is a single scalar-character equality, rather than a further system of symplectic placement equations.

This applies to the original C={1} fixture immediately. For nontrivial C, H3 equates source and **supplied break-target** phases. It does not equate their product with a separately built common host's phase without a dictionary. We do not change a fixed canonical beta extension to enforce this equality.

F.4 4. H2 and matching carrier dimensions do not force trivial C

Here is an actual H1–H2 witness under the literal original allowance of separate type targets. It is not a claimed H5-admitted input, and does not assert equal ambient matrix ranks for source and break target.

Take F=Fp((t)), p odd. The retained primary source by Tsushima, Definition3.1, Lemmas2.6,2.8,3.9,3.12,3.14 and Corollary3.15, with R(X)=X^p and m=1, constructs a genuine rank-p Weil representation irreducible on wild inertia. Its wild image is extraspecial of order p^3. The source envelope is a symplectic plane, and its actual tame image C=mu_(p+1) acts by multiplication on the roots of $X^{(p^2)}+X$. This action is nontrivial and has order p+1. The source's finite Heisenberg carrier has dimension p.

For the actual type target, take the unramified quadratic extension E/F and the maximal order A=End_OF(OE). Let beta=t^(-2)a, where a is a unit generating the residue extension. This is a minimal simple stratum of even level two. The standard minimal simple-character formulas give

```
J¹=U_A¹,  H¹=U_E¹ U_A²,
W_theta=End_Fp(Fp²)/Fp²,     dim_Fp W_theta=2,
dim eta_theta=p.
```

These are actual groups of a simple character, not an abstract replacement. The type residue units act by conjugation. Write sigma(x)=x^p and decompose

```
End_Fp(Fp²)=Fp² direct-sum Fp² sigma.
```

On the quotient represented by b sigma, conjugation by u in Fp²× is

```
b -> u^(1-p)b.
```

Its image is the norm-one torus, of order p+1. Consequently the actual projective residue action of Fp²×/Fp× on the two-dimensional quotient is nontrivial. It has the same symplectic C-module as the source: identify the cyclic groups using the displayed ratio map. The two invariant alternating forms differ by a nonzero Fp scalar, and multiplication by lambda in Fp²× rescales an alternating form by Norm(lambda). Surjectivity of the norm gives an equivariant symplectic isometry.

This supplies the actual source, actual type quotient, matching degree-p Heisenberg carriers, and nontrivial equivariant H2 action. For p=3, C has order4 and the action is an irreducible rotation of the F3 plane.

The target here is in GL_2(F), while the source has rank p. This is permitted by the literal breakwise definitions, which contain no relation between the individual type targets' ambient ranks. Block161 already recorded that exact omission. If a stricter intended contract requires every break target in a specified GL_(p^r)(F), this example does not settle that stricter question. We have not supplied H5 for a common candidate or certified fixed canonical beta-extension phases for this example. It refutes only the proposed inference that actual H2 plus equality of finite carrier dimensions forces C trivial.

F.5 5. Result for the next construction attempt

There are three distinct cases:

Given contract	Exact remaining issue
Original C={1} fixture	No stabilizer issue remains.
Native prime-to-p C action in a degree-p^r totally ramified host	Source extension action must be trivial; then compare one scalar character.
Nontrivial source symplectic C action under the original finite contract	A common-host arithmetic action dictionary is needed. Requiring an inner normalizer makes this impossible for the preceding host; transporting external operators alone was already done in161.

Thus "solve arbitrary nontrivial stabilizer attachment by adjusting the current totally ramified binder" is not a viable native-inner target. That is a precise failure of that stronger construction route, not a refutation of16.2.

F.6 Evidence and self-review

Dependencies: original Sections9–11,15–16; Block162's source-backed normalizer theorem; Block165's base realization contract and same-candidate attachment; the retained published Tsushima text; the retained minimal-stratum group formulas. The new Hom-kernel lemma and attachment criterion are ordinary elementary proofs. The witness uses the displayed arithmetic constructor inputs, not a

numerical experiment. No independent review of this new note is claimed in this file, and no new Lean compilation was attempted.

Self-review explicitly checked: the prime-to-p qualification, honest action versus projective implementers, center-fixed scalar extension automorphisms, separate target rank versus original rank, and the absence of an H5 witness.

G Dyadic synthesis and all-length assembly scope

Date: 2026-10-05. Base: the final reviewed Block166, pinned in `TARGET167.md`. Evidence: ordinary mathematical proof and independent component reviews. No Lean verification or universal claim66 closure is asserted.

G.1 1. Combined arithmetic theorem

Let $F = \mathbb{F}_2((t))$, fix an additive character ψ_F of conductor $t\mathcal{O}_F$, and take increasing odd positive integers $h_1 < \cdots < h_d$. Put

$$q_i = 3h_i, \qquad N = 2^d, \qquad n_j = 2^{d-j} q_{d-j+1}.$$

There is an explicit $\beta \in M_N(F)$ generating a separable totally ramified degree-N field that normalizes the complete lattice chain. It gives a maximal simple stratum with leading level n_1, final critical exponent $-q_1$, and successive approximation-field degrees $2, 4, \dots, 2^d$, with critical levels $n_2, \dots, n_d$.

The actual stratum admits a full simple character θ trivial on U_F^1. Its canonical beta extension can be completed to an actual irreducible admissible supercuspidal π of $\mathrm{GL}_N(F)$ with any prescribed smooth central quasicharacter ω trivial on U_F^1, provided the noncompact completion is free or already compatible. Over this residue field, such ω is unramified.

For this same constructed candidate,

$$t(\pi) = 1, \qquad \mathrm{Sw}(\pi) = 3\,2^{d-1} h_d,$$
$$\dim_{\mathbb{F}_2} W = N - 2, \qquad \dim \eta = 2^{N/2-1}, \qquad W = J^1/H^1,$$
$$a(\pi^\vee \times \pi) = N^2 - 1 + \sum_{i=1}^{d} 3\,4^{i-1} h_i.$$

Matching these profiles does not identify the source, its endo-class, or the pair factors in original H5.

G.2 2. The explicit construction and its depth invariant

At a recursive coefficient field L of characteristic two, with residue field $\mathbb{F}_2$ and uniformizer t_L, put

$$E = L(\alpha), \qquad \alpha^2 - \alpha = t_L^{-q}, \qquad a = (q+1)/2, \qquad z = t_L^a \alpha.$$

The extension is separable and totally ramified of degree two; z is a uniformizer and

$$z^2 + t_L^a z + t_L = 0, \qquad \mathrm{Tr}_{E/L}(\alpha) = 1.$$

For constant-coefficient projection P_0 in the basis $1, z$, the normalized trace corestriction and exact filtered right inverse are

$$\kappa = \alpha^{-1}, \qquad s = \kappa\,\mathrm{pr}, \qquad s(P_0) = t_L/z^2, \qquad \mathrm{Lift}_E(C) = (z^2/t_L) C P_0.$$

They satisfy $s(\mathrm{Lift}_E(C)) = C$, preserve the exact filtration valuation, and give the identity

$$\mathrm{tr}_L \mathrm{Lift}_E(C) = \mathrm{Tr}_{E/L}\big(\alpha\,\mathrm{tr}_E C\big).$$

The actual recursive matrix is

$$B(L, t_L; h_1, \dots, h_d) = \alpha I_{2^{d-1}} + \mathrm{Lift}_E\big(B(E, z; h_1, \dots, h_{d-1})\big),$$

with multiplication by α as the rank-two base case.

Every native constructor has trace one. At every next native rank, the preceding parameter amplified by two has trace zero, so the native correction also has trace one. At full ambient rank, the earlier amplified approximations instead have trace zero and the final one has trace one. These are different statements; no amplified trace is used as a substitute for the native trace needed in character construction.

At an inductive field step $d \geq 2$, set $n = 2^{d-1}q_d$, $R = 2^{d-2}q_{d-1}$. Then $n > 2R$ and $-2R > -n - q_1$. BK Remark 2.2.7 gives the maximal field generated by the actual displayed matrix. Literal truncation and the filtered lift give the defining sequence. Nonzero native trace also proves separability: the irreducible characteristic polynomial has a nonzero coefficient of X^{N-1}, so its derivative is nonzero. See `tower/DYADIC_AS_TOWER167.md` for all field and filtration details.

The replacement for trace zero is the depth inequality

$$\boxed{v_F(\operatorname{tr}\delta) + s \geq 1.}$$

It makes the additive correction trivial on scalar U_F^s, because $(u-1)\operatorname{tr}\delta \in t\mathcal{O}_F$. Our native corrections have trace one, so the inequality holds at $s = 1$. This is the exact source-to-target transfer of the weighted-visibility principle; the Euler coefficient theorem is motivation, not an input to the arithmetic proof. In characteristic two, the weights are uniformizer depths, rather than integer powers of two in the coefficient field.

G.3 3. Full characters, including the odd critical endpoint

At the minimal rank-two base, extend the forced character on $U_E^{(q+1)/2}$ to U_E^1, imposing triviality on U_F^1. The prescriptions agree on their intersection because the parameter trace is one and $\psi_F(u-1) = 1$. This is an actual finite abelian character extension and gives the full minimal simple character.

For a later native step, first transfer the full beta-determinant unit character to multiplicity two. BK 3.6.13(ii) and 3.6.14 give, on scalar units,

$$\Theta_j(uI) = \theta_j(u^2) = 1.$$

For odd critical level r, use $m = \lfloor r/2 \rfloor$ and $D = H^{m+1}(\beta) = H^{m+1}(\gamma)$. The endpoint is valid because $2(m+1) - r = 1$: the additive-character cross term lies in the radical on which the matrix additive character is trivial. BK 3.3.18 supplies $\vartheta = \Theta_j|_D\, \psi_{\beta-\gamma}$. Integral trace makes it trivial on $D \cap U_F^1$.

The next parameter field is maximal. Extend the forced character on its deep units together with the trivial character of U_F^1, then apply BK 3.2.5 to obtain the next full character. All new extensions occur at maximal native rank. No unverified determinant-image extension in a nonmaximal centralizer is used.

For the final totally ramified field, $J/J^1 = \mathbb{F}_2^\times = 1$, so $J = J^1$ and the canonical beta extension is η. Extend to $K^\times J$, and choose the noncompact scalar to match $\omega(t)$. The needed N-th root exists in $\mathbb{C}^\times$. The compact type stays fixed. All choices precede H5. See `central/DYADIC_FULL_CHARACTER167.md` for the complete proof.

G.4 4. Exact packet capacity and marked attachment

The dyadic floors cannot be replaced by the odd-prime dimension formula. At stage j, the centralizer grade has dimension 2^{d-j}. Its contribution to J^1/H^1 is

$$\left(2^{d-j+1} - 2^{d-j}\right)\mathbf{1}_{n_j \text{ even}}.$$

Every n_j for $j < d$ is even; $n_d = q_1$ is odd and contributes zero. The sum is exactly $N-2$. BK nondegeneracy makes this a symplectic space. A whole source packet of d planes fits if and only if $N - 2 \geq 2d$, equivalently $d \geq 3$.

Ambient rank	Host quotient dimension	Host Heisenberg carrier	Required source carrier	Outcome
2	0	1	2	Insufficient
4	2	2	4	Insufficient

Ambient rank	Host quotient dimension	Host Heisenberg carrier	Required source carrier	Outcome
8	6	8	8	Exact capacity
16	14	128	16	Spare capacity

Now assume $d \geq 3$, the one-vertex contract, declared $C = \{1\}$, and actually supplied full character extensions with labelled whole-plane quotients. Retain their complete dyadic finite central-extension markings, including their specified central embeddings, lifts, square values, refinements and phases. Any central-character kernel already factored out must be recorded; a scalar pushout cannot recover invisible kernel elements.

Under the scalar-pushout realization contract, with no independently fixed conflicting host marking, choose a labelled symplectic embedding into W. Matching commutators gives a center-fixing isomorphism between the source $\mathbb{C}^\times$ extension and the corresponding host restriction. **Retain the image of the whole marked finite subgroup and every lift.** For this isomorphism F_0,

$$F_0(a)^2 = F_0(a^2).$$

Thus its square bits, refinement and Gauss sign survive, along with all supplied full character values. They are not inferred from the unmarked pushout. Stone–von Neumann gives the source factor inside η; a complementary line gives an actual rank-N image-algebra projector. The retained induction/recovery idempotent recovers the same Λ and the same π.

This gives an actual represented marked finite subgroup in the scalar extension of the actual type. It permits complex scalars times type operators, as the declared contract does. It does not assert literal inclusion of old matrix groups, numerical depth placement, or identification with a separately fixed canonical native finite subgroup.

If a genuine host-native quadratic refinement and its finite μ_2-central extension are independently supplied, quadratic embedding has a sharper condition. With otherwise free embeddings, at rank eight the equal-dimensional source and host must have equal Arf invariants. For $d \geq 4$, the host has at least one spare symplectic plane, so either source Arf type embeds. If the original embedding is preassigned, it must instead satisfy the pointwise equation $q_W(f(v)) = q_{\mathrm{src}}(v)$. These conclusions require the actual host marking as input; dimension does not manufacture it.

The full finite proofs are in `markings/DYADIC_MARKINGS167.md`. In particular, the D8 and Q8 unmarked complex pushouts coincide while their whole marked finite μ_2-central extensions differ. Merely recording the central $\{\pm 1\}$ subgroup would lose that distinction.

G.5 5. Failed repairs and boundaries

The rank-two and rank-four carrier failures cannot be repaired by changing the maximal simple parameter while keeping the specified Swan exponent and torsion number. At rank two, odd Swan forces a minimal odd-level stratum and $J^1 = H^1$. At rank four, Swan $6h_2$ with h_2 odd forces the degree pattern $4 \to 2$; the maximal quadratic derived field forces an odd lower critical level. Every such maximal-type carrier has dimension two, below the required four. Scalar twists preserve this capacity.

This is an obstruction to the whole-source sector inside that maximal-type carrier. It is not a counterexample to the original theorem or a proof that a different arithmetic carrier in the same representation cannot work. Section 7 constructs such carriers and proves their exact recovery.

At rank eight, numerical profiles alone also fail to determine capacity. `capacity/DYADIC_CAPACITY167.md`, Section 6, constructs an actual skipped-degree $8 \to 2$ host with the same Swan, torsion and self-pair

conductor but quotient dimension four rather than six. Our successful result uses the explicit full tower, rather than inferring its profile from invariants.

Thus the completed dyadic scope is:

- Actual fields, full characters and central completion in every rank 2^d.
- Full marked scalar-pushout packet realization for the stated family when $d \geq 3$, with $C = 1$, one vertex, and transported markings retained.
- A conditional quadratic embedding into an independently supplied native marked host, with the exact Arf or pointwise condition above.

Section 7 constructs different actual arithmetic carriers at the two low ranks and recovers the same candidate from them. The maximal-type capacity failure above remains true. Original H5 stays the original same-candidate conditional premise; these conductor identities and noncompact root choices do not prove it.

G.6 6. Evidence

The arbitrary-length field and full-character arguments are ordinary proofs. The exact rank-eight run for $h = (1, 3, 5)$ checks actual matrix identities, normalized lifts, leading and adjacent-correction grades $-60, -18, -3$, native traces one and ambient traces $0, 0, 1$. It outputs a matrix with 25 nonzero Laurent-polynomial entries. The corresponding critical exponents are proved symbolically. The run does not prove simplicity, full-character existence, or universal marked realization. Those conclusions use the separate proofs and their stated scope. No Lean build was run.

The base synthesis before the low-rank continuation is preserved in `history/PROOF167_BEFORE_ENLARGEMENT.md`, together with its review. Its previously open enlarged-carrier route is executed next.

G.7 7. The low-rank repair: enlarge the actual inducing carrier

The maximal-type carrier is not the only arithmetic carrier available in the constructed representation. The following constructions keep the original β, θ, Λ and π fixed.

G.7.1 Rank two

Write the odd minimal level as $q = 2m + 1$, with $m \geq 1$, and let $J = H^1 = J^1 = U_K^1 U_A^{m+1}$. Define

$$P = U_K^1 U_A^m.$$

This is an actual compact open group, with $J \triangleleft P$, $[P : J] = 2$ and $P \cap (K^\times J) = J$. For $g \in P \setminus J$, the actual intertwining theorem makes $\lambda = \theta^g/\theta$ nontrivial. The commutator $[P, J] \subset U_A^{m+1}$ lies in the forced additive-character domain, so λ has values $\{\pm 1\}$. Put $H_0 = \ker \lambda$. Then

$$P/H_0 \simeq \mathbb{F}_2^2, \qquad E = \operatorname{Ind}_J^P \theta \text{ is irreducible of dimension 2.}$$

The actual represented generators are a diagonal sign and an off-diagonal matrix that anticommute. Thus their commutator form is nondegenerate, and $\theta|_{H_0}$ is the full character of an actual new arithmetic Heisenberg pair. No abstract operators create that group action.

The identity compact-induction double coset realizes E inside the same π, as functions supported on $(K^\times J)P$. The complete group and recovery proof is `lowrank/RANK2_ENLARGED_CARRIER167.md`.

G.7.2 Rank four

Use the actual rank-four outer approximation field E_0/F of degree two and its centralizer $B = \mathrm{End}_{E_0}(F^4)$. The odd lower critical level is $r = 3h_1 = 2m + 1$; the half leading level is $a = 3h_2 > r$. Define

$$P = J\,U_B^m,$$

where $J = J^1$ is the actual compact type group. The filtered centralizer estimate for the actual final field, without assuming $E_0 \subset K$, proves that P is a compact group with J normal of index two. For a concrete new coset representative u, $\lambda = \theta^u/\theta$ on the actual $H = H^1$ is nontrivial with values $\{\pm 1\}$. Put $H_0 = \ker\lambda \subset H$. The calculation proves

$$P/H_0 \simeq \mathbb{F}_2^4, \qquad E = \mathrm{Ind}_J^P \eta \text{ is irreducible of dimension 4.}$$

The new represented commutators are scalar, its quotient has exponent two, and nondegeneracy follows from its exact scalar preimage and irreducibility. The finite scalar center may be larger than μ_2; a literal extraspecial quotient is not asserted. The full character values are retained by the actual finite Heisenberg extension. Again E occurs inside the same π through the identity double coset. See `lowrank/RANK4_ENLARGED_CARRIER167.md`.

G.7.3 Exact recovery, with the noncompact normalization retained

At both low ranks, $K^\times$ normalizes the new P. In rank two this follows directly from normalization of A and its radical. In rank four, a K-uniformizer is approximated by the derived chain shift R times a unit of B, with relative error in P_A^{n-r}. Its conjugation error on U_B^m lies in $U_A^{n-r+m} \subset J$, which proves normalization. The exact argument is `lowrank/RANK4_INDUCTION_BY_STAGES167.md`.

Consequently $\mathcal{M} = K^\times P$ is an actual open group compact modulo the center, $\mathcal{N} = K^\times J$ is normal of index two in $\mathcal{M}$, and

$$\Sigma = \mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}} \Lambda, \qquad \Sigma|_P \simeq E, \qquad \boxed{\pi \simeq \mathfrak{cInd}_{\mathcal{M}}^G \Sigma.}$$

Restriction to P is irreducible, so Σ is irreducible. Induction in stages proves the last equality for the original Λ, including its fixed central character and noncompact root. This is a genuine enlarged inducing representation; it changes neither the candidate nor its original canonical beta-extension data.

There is also a recovery from compact data explicitly recording that root. With $A_0 = \Lambda(\varpi_K)$ and $X = \mathfrak{cInd}_J^G \eta$, define $(Tf)(g) = A_0^{-1} f(\varpi_K g)$. Then $X/(T-1)X \simeq \pi$. After imposing the fixed central character, $\mathcal{N}/(F^\times J)$ is cyclic of order N, and the finite average $N^{-1}\sum_{i=0}^{N-1} T^i$ selects this same extension. These operators act on complex representation spaces, so their scalar division by N is valid. This is a different recovery from the Block159 projector onto a sector of η. The exact kernel and finite projector are proved in `lowrank/ENLARGED_CARRIER_CONTRACT167.md`.

G.7.4 What the low-rank repair admits

Apply the marked scalar-pushout theorem of Section 4 to the **actual new** pair (H_0, P) and its carrier E. It has exactly the capacity of the whole source at each low rank. All full source character values and the transported whole finite subgroup, lifts, squares and phase markings can be attached and retained. The displayed Σ and induction in stages recover the same candidate.

The old H2 break targets remain their supplied actual simple-character quotients. The enlarged pair is their common arithmetic assembly carrier; it is not renamed as the original maximal-simple H^1, J^1 pair. If a stronger reading requires the whole packet itself to be the quotient of a maximal simple character in the same candidate, the earlier dimension obstruction still applies. Likewise, agreement with an

independently fixed native quadratic marking must be checked separately. The contract note proves this distinction explicitly.

G.8 8. Final scope and next construction question

The weighted dyadic mechanism now gives actual arithmetic fields, full characters and tame central completion in every rank 2^d of the stated increasing odd-break family. Under the one-vertex, $C = 1$, full marked scalar-pushout **assembly** contract, the packet carrier can be realized at every length: inside the original type for $d \geq 3$, and in the explicit enlarged inducing carrier at $d = 1, 2$. The source's prescribed central character must belong to the covered tame class. In particular no arbitrary single-factor wild determinant is assumed to be trivial.

Remaining universal obligations are construction/comparison of independently prescribed canonical native markings, nontrivial symplectic stabilizer actions, general source/transport diagrams and any required numerical layer placement. Original H5 remains conditional for this same fixed candidate. No equality of profiles or choice among noncompact roots is promoted to H5.

The next concrete target is equivariance of the arithmetic attachment for a nontrivial prescribed symplectic action. Another matrix search inside the old low-rank maximal-type carrier would repeat its already proved obstruction. `STATUS167.json` and `ROUTES167.md` record the achieved scopes and open routes.

H The explicit dyadic arithmetic tower

Date: 2026-10-05. Ordinary proof. No character, quotient-dimension, source attachment, pair-factor, or universal Paper XI closure is asserted here.

H.1 1. Statement

Let F=F_2((t)), let 0<h_1<…<h_d be odd positive integers, and set

```
q_i=3h_i,  N=2^d.
```

The explicit recursion below constructs B_d in M_N(F) such that

```
F[B_d] is a totally ramified maximal field normalizing
   the complete period-N lattice chain,
v_A(B_d)=-2^(d-1)q_d,
k_0(B_d,A)=-q_1,
tr_F(B_d)=1.
```

There are explicit native constructors of ranks 2^j for the largest j exponents. Their identity-amplifications give the actual defining sequence with field degrees 2,4,…,2^d and critical levels

```
n_j=2^(d-j)q_(d-j+1),  1<=j<=d.
```

Every parameter in this ambient defining sequence has the same leading level n_1. The quantities n_2,…,n_d are its successive critical approximation levels, not new leading levels of the ambient matrices.

Every native constructor has trace 1. Every correction from an identity-2 amplification of the preceding native constructor to the next native constructor also has trace 1. In full ambient rank 2^d, the earlier amplified approximations instead have trace 0, because their multiplicities are even. This distinction is essential.

H.2 2. The minimal degree-two building block

Let L be a complete discretely valued field of characteristic 2 with residue field F_2, and choose a uniformizer t_L. For odd q>0 put a=(q+1)/2 and define

```
E=L(alpha),  alpha^2-alpha=t_L^(-q),
z=t_L^a alpha.
```

The Artin–Schreier polynomial is irreducible: an element x^2-x of negative L-valuation has even negative valuation. Its derivative is 1, and the valuation equation forces ramification index 2. Thus E/L is separable and totally ramified, v_E(alpha)=-q, and alpha is minimal.

The element z has E-valuation 2a-q=1 and satisfies

```
f(z)=z^2+t_L^a z+t_L=0.
```

Thus 1,z is a uniformizer power basis, and

```
alpha=z/t_L^a,  Tr_(E/L)(alpha)=1.
```

H.3 3. An explicit normalized corestriction and lift

On E use the coefficient basis 1,z. Its trace-dual elements are

```
z_0^*=(z+t_L^a)/t_L^a,
z_1^*=1/t_L^a.
```

Let pr(T_ij)=z^i z_j^* be the trace corestriction. Put

```
kappa=t_L^a/z=alpha^(-1),
s=kappa pr,
u_0=s(P0)=t_L/z^2.
```

Here P0 is constant-coefficient projection, applied separately to each coordinate when acting on E^m. The identity for u_0 follows from f(z)=0. It is a unit: v_E(t_L)=v_E(z^2)=2. More explicitly,

```
s(T_i0)=z^i(1+t_L^a/z),
s(T_i1)=z^(i-1).
```

Since v_E(t_L^a/z)=2a-1=q>0, these preserve the expected leading grades. On End_L(E^m), take the complete interleaved chain in the basis w_(mi+b)=z^i v_b. Its centralizer is M_m(E) with its complete E-chain. The formulas give s(P^r) contained in Q^r for all integers r.

For C in M_m(E), define the explicit L-linear operator

```
Lift_E(C)=u_0^(-1) C P0=(z^2/t_L) C P0.
```

B-bimodularity gives s(Lift_E(C))=C. The lift preserves exact valuation: u_0^(-1) is an E-unit, and expansion in 1,z gives coefficient valuations distinct modulo 2. Each coefficient entry has the same grade in the interleaved L-chain and the E-chain. In particular $s(P^r) = Q^r$.

H.4 4. Exact trace transport

For every C in M_m(E), ordinary matrix traces satisfy

```
tr_L(Lift_E(C))
   =Tr_(E/L)(z_0^* u_0^(-1) tr_E(C))
   =Tr_(E/L)(kappa^(-1) tr_E(C))
   =Tr_(E/L)(alpha tr_E(C)).                (T2)
```

The first equality follows by taking the L-trace of each rank-one operator yP0, which is the constant coefficient of y. This coefficient is Tr_(E/L)(z_0^* y).

Thus the trace of a lift is 1 whenever the inner ordinary E-matrix trace is 1. It is not asserted that arbitrary trace values are unchanged.

H.5 5. Recursive actual matrices and fields

Define B(L,t_L;h_1,…,h_d) as follows. For d=1 use the multiplication matrix of alpha from Section2 with q=q_1. For d>=2 apply Section2 with q=q_d, set m=2^(d-1), recursively construct

```
C=B(E,z;h_1,...,h_(d-1)),
```

and put

```
B(L,t_L;h_1,...,h_d)=alpha I_m+(z^2/t_L) C P0.       (D2)
```

This is a finite algebraic recipe. It uses specified separable quadratic extensions and no unknown support, genericity choice, or correspondence inverse.

The degree and simplicity proof is by induction. The base alpha is minimal and generates E. At a later step let gamma=alpha I_m and b=Lift_E(C). The outer minimal stratum has

```
n=2^(d-1)q_d,  k_0(gamma,A)=-n.
```

The inner induction hypothesis gives an actual maximal field E[C] normalizing the complete E-chain with

```
R=2^(d-2)q_(d-1),  v(C)=-R,  k_0(C)=-q_1.
```

Because h_d>h_(d-1), we have n>2R. Therefore

```
-2R > -n-q_1 = k_0(gamma,A)+k_0(C).
```

The derived stratum with parameter s(b)=C is pure, and its field is maximal. Bushnell–Kutzko Remark2.2.7 consequently gives the field generated by the actual gamma+b, not merely an equivalent replacement. It has ramification and degree 2^d, residue degree one, normalizes the complete chain, and has k_0=max(-n,-q_1)=-q_1. Since v(b)>v(gamma), its level is n and [A,n,0,gamma+b] is simple.

The derived model E[C] is not identified with L[gamma+b]. In particular no literal embedding of E into the latter field is claimed.

H.6 6. Trace 1 at every native stage

The base matrix has trace Tr_(E/L)(alpha)=1. Inductively tr_E(C)=1, so (T2) gives tr_L(Lift_E(C))=1. For d>=2 the L-trace of alpha I_m is

```
m Tr_(E/L)(alpha)=0,
```

because m is even. Hence tr_L(B)=1 at every native rank.

For any j>=1, let A_j denote the native rank-2^j constructor using a fixed list of the largest j exponents. The next native rank-2^(j+1) constructor A_(j+1) uses the same largest exponents and adds the next smaller one. With their compatible interleaved bases, put

```
Delta_(j+1)=A_(j+1)-A_j tensor I_2.
```

Then

```
tr(Delta_(j+1))=1-2*1=1                    in characteristic 2.
```

This native trace identity is stronger and more relevant than a claim obtained merely by amplifying to a rank divisible by two.

H.7 7. Full arithmetic defining sequence

For the final rank 2^d define

```
B^[j]=B(F,t;h_(d-j+1),...,h_d) tensor I_(2^(d-j)).
```

All truncations use the same top q_d to select the first extension and uniformizer, the same next q_(d-1) to select the next extension, and so on. The coefficient projection and lift commute with amplification. Thus these are literal truncations of (D2).

Their parameter fields have degree 2^j, and their critical levels in the ambient complete chain are n_j=2^(d-j)q_(d-j+1). Valuation-preserving lifts give

```
v_A(B^[j+1]-B^[j])=-n_(j+1).
```

Later differences have strictly greater valuation, since n_j>2n_(j+1). This proves that the displayed parameters form an actual defining sequence of simple approximations.

Their ambient traces are

`tr_F(B^[j])=0 for j<d,  tr_F(B^[d])=1.`

Consequently ambient corrections before the last step have trace 0, and the last correction has trace 1. Every one of these trace values is integral. On a scalar principal unit u, the additive correction character has value psi_F((u-1)tr(delta)); this is 1 for an additive character trivial on the maximal ideal. No assertion about full simple character extension follows here without its separate intersection and transfer argument.

H.8 8. Concrete rank-eight instance and limits

For h=(1,3,5), the construction gives actual approximation-field degrees 2,4,8 and critical levels 60,18,3. The supplementary exact program constructs its Laurent-polynomial matrices, verifies the normalized lift identities, and checks the native/ambient trace distinctions and matrix grades. It does not prove field existence or simplicity by bounded computation: those are consequences of Section5.

The odd-prime Heisenberg quotient and conductor counting from Block166 cannot simply be copied: q_1 is odd here. This note makes no dyadic quotient, full-character, stabilizer, phase, source-attachment, or H5 claim. It also changes lower parameter conjugacy classes, as in the corresponding odd-prime Artin–Schreier construction.

H.9 Dependencies and provenance

Primary theorem: Bushnell–Kutzko, The admissible dual of GL(N) via compact open subgroups, Remark2.2.7 (printed p55), together with the amplification normalization and defining-sequence conventions already checked in Block166.

Retained primary source text SHA256: `d69db0827256707a34cd2562af68b9de11fc889ab3f43535ec47b620d58ca9c8`.

Block166 inputs are pinned separately in `INPUT_PINS167.json` and are not modified by this continuation. The Artin–Schreier, trace-dual, and trace calculations needed specifically for p=2 are proved above.

I Dyadic full characters and the odd critical endpoint

Checkpoint: Paper_XI_Dyadic167. Parent: final Paper_XI_Attachment166. Author scope: this file only; no Block166 file is modified.

Evidence: ordinary mathematical proof using the actual Artin–Schreier matrix tower and the retained Bushnell–Kutzko character and transfer theorems. No Lean verification is claimed. This note proves no dyadic quotient-capacity, quadratic-phase, packet-attachment, or H5 result.

I.1 1. Exact theorem

Let F=F_2((t)), let psi_F be an additive character of conductor t O_F (trivial there and nontrivial on O_F), and let

```
0<h_1<...<h_d,       each h_i odd,
q_i=3h_i,           N=2^d.
```

Use the dyadic Artin–Schreier recursion specified in Section2. It gives the actual maximal simple parameter beta_d with its native arithmetic defining sequence. There is an actual full simple character theta of its maximal simple stratum such that

```
theta restricted to U_F^1 is trivial.                          (1)
```

The construction retains the actual canonical beta extension and can be completed with any prescribed central quasicharacter omega_F trivial on U_F^1. Because $k_F^\times = \mathbb{F}_2^\times$ is trivial, such an omega_F is unramified. Thus this covers every tamely ramified central quasicharacter in this particular residue-field setting.

The corresponding compact induction is an actual irreducible admissible supercuspidal of rank N, by the retained maximal-simple-type theorem. No claim is made here about its Heisenberg quotient size or about realization of a supplied dyadic finite packet.

The trace-zero argument from166 does not apply verbatim: at p=2 the primitive Artin–Schreier parameter has trace one. The correct invariant is **integral trace**, not necessarily zero trace. Since psi_F kills t O_F, integral correction trace is sufficient on scalar principal units. The recursion below preserves the stronger exact value one.

I.2 2. Actual dyadic recursion and exact native traces

At a recursive step let L be a characteristic-two local field with residue field F_2, uniformizer t_L, and a positive odd q. Define

```
E=L(alpha),       alpha^2-alpha=t_L^(-q).
```

The right side is not an Artin–Schreier coboundary: a negative valuation of x^2-x is even, whereas -q is odd. The degree-two polynomial is therefore irreducible and separable. Its valuation equation gives total ramification and v_E(alpha)=-q. Put

```
a=(q+1)/2,       z=t_L^a alpha.
```

Then z is an actual uniformizer and its Eisenstein polynomial is

```
f(Z)=Z^2+t_L^a Z+t_L.                                          (2)
```

Its two conjugate alpha values are alpha and alpha+1, so

```
Tr_(E/L)(alpha)=1.                                             (3)
```

Use the coefficient basis 1,z, the trace corestriction pr, and the same normalized corestriction convention as166:

```
kappa=-f'(z)/z = t_L^a/z = alpha^(-1),
s=kappa pr.
```

Minus and plus agree in characteristic two. If P0 extracts the constant coefficient on every E-coordinate, then

```
u_0=s(P0)=t_L/z^2 is a unit,
Lift_E(C)=u_0^(-1) C P0,
s(Lift_E(C))=C.
```

The trace-dual calculation in166 gives, as an identity of L-matrix traces for every matrix C over E,

```
tr_L Lift_E(C)
   =Tr_(E/L)(kappa^(-1) tr_E(C))
   =Tr_(E/L)(alpha tr_E(C)).                              (4)
```

The lift preserves the complete-chain filtration valuation, by expansion in 1,z. Thus it is the actual filtered right inverse used in the field constructor, not an arbitrary inverse on vector spaces.

Define B(L,t_L;h_1,…,h_s) recursively. At s=1 use the L-matrix of multiplication by alpha with q=q_1. At s>=2 use (2) with q=q_s, m=2^(s-1), and

```
C=B(E,z;h_1,...,h_(s-1)),
B(L,t_L;h_1,...,h_s)=alpha I_m+Lift_E(C).                  (5)
```

The base trace is one by (3). If tr_E(C)=1, then (4) gives lift trace one. Since m is even, the trace of alpha I_m as an L-matrix is m Tr_(E/L)(alpha)=0. Consequently

```
tr_L B(L,t_L;h_1,...,h_s)=1                               (6)
```

for every native recursion, by induction. There is no division by the rank and no confusion between a native trace and its amplified trace.

The actual field proof from166 applies with this degree-two extension and normalized lift. At a nontrivial step its hypotheses are n=2^(s-1)q_s, R=2^(s-2)q_(s-1), s(Lift(C))=C, and

```
n>2R,
-2R>-n-q_1.
```

The first strict inequality follows from h_s>h_(s-1). These are the conditions used in BK Remark2.2.7. The displayed beta itself generates the totally ramified degree-2^s field normalizing the complete chain, and its final critical exponent is -q_1. This statement does not identify the final field with its auxiliary derived field.

For the final length d write the native shorter matrices as

```
b_j=B(F,t;h_(d-j+1),...,h_d),       1<=j<=d.
```

They are the compatible native truncations of (5), because the outer extension, uniformizer, and lift depend on the same retained largest h at each stage. For $1 \leq j < d$, in rank 2^(j+1), put

```
gamma_j=b_j tensor I_2,
delta_j=b_(j+1)-gamma_j.
```

The defining-sequence calculation of166 gives gamma_j as a simple approximation at the critical level r_j=q_(d-j) of b_(j+1), and delta_j has grade -r_j. Equation (6) gives the exact scalar identity

```
tr_F(gamma_j)=0,
tr_F(delta_j)=1.                                        (7)
```

Thus for every scalar u in U_F^1 on which its additive correction is defined,

```
psi_(delta_j)(u I)=psi_F((u-1)tr_F(delta_j))
                  =psi_F(u-1)=1.                        (8)
```

It is (8), not a false trace-zero assertion, that drives the full character construction.

I.3 3. The odd-critical-level rounding

The critical levels r=q_i are odd. Write r=2s+1, and set

```
m=floor(r/2)=s,        m+1=(r+1)/2.
```

The relevant BK statements use the floor, with no assumption that r or the residue characteristic is odd or even:

- BK3.1.9(ii) identifies H^k(beta) and H^k(gamma) for k>=floor(r/2)+1.
- BK3.3.18 gives the correction bijection at floor(r/2)<=m<=r-1.
- BK3.2.5 extends the resulting character at floor(r/2) to a full character, its fibers being the indicated determinant characters.

At our exact endpoint m=s, the correction character is legitimate. For x,y in the radical power P^(s+1),

```
delta*x*y belongs to P^(2(s+1)-r)=P^1.
```

The level-one additive matrix character is trivial on P^1, so the multiplicative cross term contributes one. Therefore psi_delta is a character on U^(s+1), and hence on the recursive subgroup used below. This checks the borderline odd-level rounding directly. Replacing m by r/2 as if it were an integer would be incorrect.

The same check applies at the native minimal base level q, which is odd: its forced additive character begins at floor(q/2)+1=(q+1)/2, and 2*(q+1)/2-q=1.

I.4 4. Finite character extension at maximal rank

We reuse the elementary extension procedure with its compatibility condition explicit. If K/F is a finite totally ramified field, and chi is a smooth character of U_K^a trivial on U_K^a intersect U_F^1, define a character on U_K^a U_F^1 by

```
chi_0(ku)=chi(k).
```

It is well defined by the intersection condition. Choose M>=a with chi|U_K^M=1. Then chi_0 is a character of a subgroup of the finite abelian group U_K^1/U_K^M. Extend it by adjoining generators and choosing the required complex roots, and pull it back. The resulting chi_K extends chi and is trivial on U_F^1.

All freely chosen character extensions below take place at maximal native rank, where the centralizer is K itself. We do not apply this scalar-intersection argument directly inside a nonmaximal matrix centralizer, which would require a separate determinant-image check.

I.5 5. Base full character with trace one

At native rank two, let E=F[alpha], with q=q_d and a=floor(q/2)+1. The forced character on U_E^a is

```
phi(v)=psi_F(Tr_(E/F)(alpha(v-1))).
```

For scalar u in U_E^a intersect U_F^1, equation (3) gives

```
phi(u)=psi_F((u-1)Tr_(E/F)(alpha))=psi_F(u-1)=1.
```

Section4 constructs chi_E on U_E^1 extending phi and trivial on U_F^1. Define theta_1 on $H^1 = U_E^1$ U_(A_1)^a by chi_E on the first factor and psi_alpha on the second. The restrictions agree on the intersection. BK3.2.1 gives the actual full simple character, with theta_1 | U_F^1=1.

This uses the true trace value one and only the stated conductor condition on psi_F. It does not require an artificial zero-trace primitive parameter in a quadratic extension.

I.6 6. Native-rank induction via full beta-determinant transfer

Assume theta_j is a full simple character for native b_j in rank 2^j, with theta_j | U_F^1=1. Let K_j=F[b_j]. Since K_j is maximal, the character chi_j=theta_j | U_(K_j)^1 is its full beta-determinant character.

Transfer theta_j to the multiplicity-two realization gamma_j=b_j tensor I_2 in rank 2^(j+1), using BK3.6.14. Denote the full transferred character by Theta_j. BK3.6.13(ii) preserves the entire beta-determinant character on U_(K_j)^1 at level m0=0. Hence for every literal scalar u in U_F^1,

```
Theta_j(u I)=chi_j(det_(K_j)(u I_2))
           =chi_j(u^2)=theta_j(u^2)=1.                    (9)
```

This statement includes nonminimal b_j. The determinant exponent two is essential and has not been suppressed.

Now set beta=b_(j+1), gamma=gamma_j, delta=delta_j, r=r_j, m=floor(r/2), and

```
D=H^(m+1)(beta)=H^(m+1)(gamma).
```

The equality and endpoint m were verified in Section3. BK3.3.18 constructs the actual recursive simple character

```
vartheta=(Theta_j restricted to D)*psi_delta.
```

For u in D intersect U_F^1, equations (8) and (9) prove

```
vartheta(u)=1.                                            (10)
```

The new field K=F[beta] is maximal and totally ramified in native rank 2^(j+1). Its chain has K-period one, so

```
D intersect U_K^1=U_K^(m+1),
D intersect U_F^1=U_F^ceil((m+1)/2^(j+1)).
```

Thus (10) is exactly the compatibility required by Section4. Extend vartheta | U_K^(m+1) to chi_K on U_K^1 trivial on U_F^1. BK3.2.5 constructs the full simple character by

```
theta_(j+1)(kd)=chi_K(k)*vartheta(d),
k in U_K^1, d in D.
```

That proposition supplies normalization and well-definedness; its proof is a character-extension argument and imposes no exclusion of residue characteristic two. The new theta_(j+1) is trivial on U_F^1. Induction proves (1) in every native rank 2^d.

More generally the character argument needs only integral base field trace and integral traces for the native corrections. The dyadic tower establishes their exact value one, while the odd-prime tower of166 establishes zero.

I.7 7. Canonical beta extension and prescribed tame center

Let K/F be the final totally ramified degree-N parameter field and let eta be the Heisenberg representation for the constructed theta. The compact type group is

```
J=O_K^times J^1.
```

Here $\mathcal{O}_K^\times = U_K^1$ because the residue field is F_2. Since U_K^1 is contained in J^1, we have

```
J=J^1,       J/J^1=F_2^times={1}.                    (11)
```

Thus the actual canonical beta extension on J equals eta; there is no residue-character choice and no change of that extension. It acts trivially on the central U_F^1 by (1). Equation (11) does not identify any externally prescribed dyadic quadratic phases.

Let omega_F be any smooth quasicharacter trivial on U_F^1. Since $\mathcal{O}_F^\times = U_F^1$, it is determined by its value omega_F(t). The actual maximal type extends to K^times J. Concretely, (K^times J)/J is infinite cyclic, generated by a K-uniformizer. An intertwiner for its conjugation action on eta gives an extension Lambda_0. Denote its central character by omega_0.

Choose a complex scalar s with

```
s^N=omega_F(t)/omega_0(t).
```

Multiply Lambda_0 by the character of K^times J trivial on J and taking the K-uniformizer to s. Because t has valuation N in K, this changes its scalar by s^N and gives the requested central quasicharacter on all F^times. The value at every element of J, including the canonical beta extension, is unchanged. This does not assume that a chosen K-uniformizer raised to N equals t.

The retained maximal-simple-type theorem now gives the actual irreducible admissible supercuspidal pi=c-Ind_(K^times J)^GL_N(F) Lambda. The noncompact scalar is fixed before any H5 comparison. An incompatible immutably prescribed extension is not overwritten; as in166, its value must be free or already compatible.

I.8 8. What this closes and what it does not

This closes the full-character and actual tame-central-character construction for the explicit dyadic arithmetic tower. The failure of trace zero at p=2 is repaired by the correct integral trace invariant. Both the odd-level floor and the multiplicity-two determinant transfer are accounted for explicitly.

This note does not compute J^1/H^1 or prove that it contains a particular dyadic source packet. It does not establish a quadratic refinement, canonical phase matching, nontrivial stabilizer action, general transport graph, source admission, or H5. In particular, an actual supercuspidal host with the required central character is not asserted to be a completed realization of the original dyadic packet without those separate proofs.

I.9 9. Exact source checks and input pins

The following final166 files were read and reused without editing:

- `attachment166/central/TRACE_ZERO_CHARACTER_INDUCTION166.md` SHA256 `99cb6e80c419e0bf7325e2b2b3fbaba1391903448a3b6a0c4cd9f3944ed406a1`.
- `attachment166/tower/TRACE_ZERO_AS_TOWER166.md` SHA256 `49a18fb733f0209b3e38118f5132839a060961edd0bdc0211e1f7b441a4e5653`.
- `attachment166/tower/RECURSIVE_TOWER166.md` SHA256 `6f39a077df18eb03f9fab2b6e426529dbe14cf4f7e1cfce95ddb0fcd187d7b33`.

- `attachment166/audit/TOWER_CENTRAL_REVIEW166.md` SHA256 `4bc71dd050ddb321a253105c8a4ee7a86185af255532d5605caa7a9de383e26b`.

Primary retained source: Bushnell–Kutzko, *The admissible dual of GL(N) via compact open subgroups*, text SHA256 `d69db0827256707a34cd2562af68b9de11fc889ab3f43535ec47b620d58ca9c8`. The current dyadic check reread definitions3.1.7–8 and Proposition3.1.9(ii) (printed90–91), Definition3.2.1 and Proposition3.2.5 (printed98–100), and Corollary3.3.18 (printed112–113). Their level thresholds are floor(r/2)+1 or floor(r/2), as used above, and their statements do not exclude p=2. The transfer's preservation of the full beta-determinant character is BK3.6.13(ii), with existence Theorem3.6.14 (printed136–137), already directly checked in166 and reread for this application.

The character-extension and correction proofs use neither division by two nor an odd-order averaging argument. The exact dyadic trace calculation and the borderline correction-character check are given in Sections2–3. No new publication lookup, external arithmetic-induction/descent constructor, or inverse LLC is used.

J Exact dyadic capacity and numerical-profile obstruction

Status: ordinary proof from the supplied actual defining sequence and the retained primary Bushnell–Kutzko results. No Lean or numerical computation is used. This note calculates the quotient of an actual simple type. It does not prove universal H5 or identify an explicitly constructed host with the source-defined arithmetic candidate.

J.1 1. Exact hypotheses and result

Let F=F_2((t)). Fix d>=1 and increasing positive odd integers

```
h_1<...<h_d,       q_i=3h_i,       n=2^d.
```

Suppose the actual arithmetic constructor supplies a full-degree totally ramified parameter beta on the complete period-n chain, with actual approximating fields of degrees

```
2,4,...,2^d
```

and ambient critical levels

```
nu_j=2^(d−j) q_(d−j+1),       1<=j<=d.
```

These are the supplied dyadic specialization of the recursive tower's field/profile conclusions. The calculation applies equally to the original polynomial tower of Block166 and to an Artin–Schreier variant once that variant's field/profile conclusions are established. No trace-zero assertion is needed here.

The exact actual Heisenberg quotient is

```
dim_F2(J^1/H^1)=2^d−2,
dim eta=2^((2^d−2)/2).
```

Consequently its symplectic dimension is at least the 2d dimensions of the whole d-plane source packet exactly when d>=3. In rank8, d=3, the quotient has dimension6 and eta has dimension8: the source carrier fits exactly, with no complementary Heisenberg factor needed.

d	Ambient rank	Actual quotient dimension	eta dimension	Source carrier dimension	Capacity
1	2	0	1	2	Fails
2	4	2	2	4	Fails
3	8	6	8	8	Exact
4	16	14	128	16	Sufficient
general	2^d	2^d−2	$2^{2^{d-1}-1}$	2^d	Sufficient iff d>=3

This improves the leading-layer-only sufficient threshold d>=4 to the exact whole-profile threshold d>=3. It also exposes a genuine low-rank limitation of this proposed whole-carrier realization.

J.2 2. Floors and ceilings in the actual BK orders

Write P=rad(A), and let B_j be the A-centralizer of the degree-2^j approximation. Its complete-chain grade has F_2-dimension

```
b_j=2^(d−j),       b_0=n.
```

Put

```
a_j=ceil(nu_j/2),       t_j=floor(nu_j/2)+1.
```

The exact BK definitions, 3.1.7–8, use a_j as the J cutoff and t_j as the H cutoff. Thus a_j=t_j for an odd critical level, while t_j=a_j+1 for an even critical level.

The increasing h_i give

```
nu_j>2 nu_(j+1),       1<=j<d.
```

This strict separation is useful: the centralizer approximation estimate from BK 3.1.5 puts each newly introduced centralizer into the preceding one modulo a term above the preceding half-level. In the notation of these stages, the relevant error level is nu_j−nu_(j+1)>nu_j/2. Therefore the successive grade dimensions really are b_j; there is no unproved assumption that the approximation fields are literally nested inside the final field.

Applying the ring recursion gives the J-grade staircase:

- below a_d the grade dimension is b_d=1;
- on a_(j+1)<=k<a_j it is b_j;
- from a_1 upward it is b_0=n.

The H staircase uses the corresponding t_j. Hence the exact contribution of transition j to J^1/H^1 is

```
(b_(j−1)−b_j) · 1_(nu_j even).
```

One can also obtain this by subtracting the exact logarithmic indices of H and J in 1+P. It is an equality of the group index $|J^1/H^1|$, not a guess from the visible leading matrix support. BK 3.4.1 makes the resulting commutator pairing nondegenerate.

For j<d the level nu_j has a factor of2 and is even. The final level nu_d=q_1 is odd. Therefore

```
dim_F2 J^1/H^1
  =sum_(j=1)^(d−1) (2^(d−j+1)−2^(d−j))
  =2^d−2.
```

The omitted final increment would have dimension1. It cannot be treated as a symplectic plane or silently restored: at that odd level the two defining lattice cutoffs are exactly equal. Its disappearance is also consistent with the even dimension required of a nondegenerate alternating space.

The source packet requires dimension2d. At d=1,2, 2^d−2<2d. At d=3 equality holds. If the inequality holds for d>=3, then

```
2^(d+1)−2−2(d+1) = (2^d−2−2d)+(2^d−2)>0,
```

so it continues to hold for every larger d. This proves the exact threshold.

J.3 3. Conductors still match despite the lost final contribution

This section reuses the same retained formal-degree and self-pair conductor identities already audited in Blocks161 and165. It is not needed to prove the dimension formula, but shows why numerical conductor checks do not expose the low-rank capacity failure.

Put Delta_j=b_(j−1)−b_j. The exact grade count gives

```
I=log_2[(1+P):J^1]
  =sum_j Delta_j ceil(nu_j/2)−(n−1).
```

For this profile only the final level is odd and Delta_d=1. Therefore

```
2I=sum_j Delta_j nu_j+1−2(n−1).
```

The Heisenberg dimension exponent is D/2, where D=n−2. The retained formal-degree identity for a full-degree, totally ramified, torsion-one maximal type gives

```
a(Pi^vee×Pi)=n^2−1+(n−1)+2I+D.
```

Substitution yields exact cancellation of the odd-cutoff correction:

```
a(Pi^vee×Pi)=n^2−1+sum_j Delta_j nu_j
           =2^(2d)−1+sum_(i=1)^d 3·4^(i−1) h_i.
```

The ordinary conductor is n+nu_1, so Swan(Pi)=2^(d−1)·3h_d, and the torsion number is one. These are the usual three numerical invariants of the stipulated independent dyadic Heisenberg tensor source.

Thus all three numerical invariants can match while eta is too small for the whole source carrier at ranks2 and4. This is a capacity failure, not a conductor mismatch. It does not by itself refute any original H1–H5 statement.

J.4 4. Rank2 failure is forced by Swan and torsion alone

Let Pi be any actual supercuspidal representation of GL2(F) with torsion number one and odd Swan exponent N=3h_1. BK 6.2.5 forces its maximal parameter field K/F to be totally ramified of degree2 and its hereditary order to have complete period2. The ordinary conductor formula then fixes its native level as N.

Every parameter beta of a maximal simple type in Pi has v_K(beta)=−N, which is odd. Since the residue extension is trivial, BK 1.4.14–15 say beta is minimal over F. The two order definitions are therefore

```
J_order=O_K+P^ceil(N/2),
H_order=O_K+P^(floor(N/2)+1).
```

The exponents coincide because N is odd. Hence $J^1 = H^1$, eta is one-dimensional, and no two-dimensional symplectic source plane can embed into this quotient.

This proves failure for every maximal simple type of every candidate with these two invariants. It is stronger than failure of one explicit matrix formula. It remains a statement about attachment inside the actual maximal-simple-type Heisenberg carrier, not a proof that no other arithmetic realization relation could work.

J.5 5. Rank4 failure is also forced by Swan and torsion

Let Pi be any actual supercuspidal representation of GL4(F) with torsion number one and Swan exponent N=6h_2, where h_2 is odd. Its maximal parameter field is totally ramified of degree4, its order has period4, and its native level is N.

Here v_2(N)=1. The defining-sequence/minimality argument of Block166 forces the terminal minimal approximation field E/F to have degree2. There is no possible intermediate degree between2 and4. The scalar terminal case is excluded by 4 not dividing N, and a minimal full-degree4 parameter is excluded because gcd(N,4)=2. Thus the actual degree pattern is forced to be 4↠2.

Write c=−k_0(beta,A), so 1<=c<N. Select the critical approximation furnished by the construction in BK 2.4.1 and the remark preceding 2.4.7. Because the original field is maximal, this approximation can be selected with a simple maximal derived stratum in B=End_E(F^4)≅M2(E). Equivalently one can replace its derived stratum by an equivalent simple representative and use BK 2.2.2–3. The derived field has

degree2 and ramification index2 over E. A simple stratum [B_order,c,c−1,alpha] is minimal. Since the complete B-chain has period2 and the derived field is totally ramified, BK 1.4.14 forces

```
gcd(c,2)=1.
```

Therefore c is odd. It cannot be made even while preserving this degree pattern and a genuine maximal derived field.

Let j=(c+1)/2. The actual orders are

```
J_order=O_K+Q^j+P^(N/2),
H_order=O_K+Q^j+P^(N/2+1).
```

The final low transition contributes nothing, while the high transition has dimension4−2=2. Thus every such Pi has

```
dim_F2 J^1/H^1=2,       dim eta=2.
```

The desired two-plane source carrier has dimension4, so it cannot be a sector of this eta. Requiring the self-pair conductor additionally fixes c=3h_1 through

```
a(Pi^vee×Pi)=15+2N+c,
```

but is unnecessary for the capacity failure.

J.6 6. Rank8: the three invariants do not force the good profile

The full 2→4→8 tower has quotient dimension6 and enough capacity. Nevertheless Swan, torsion number and self-pair conductor alone do not force that degree pattern. Here is an actual skipped-degree arithmetic construction with the same three numerical invariants and a smaller quotient.

Fix increasing odd h_1<h_2<h_3 and put

```
q=3h_3,       N=4q=12h_3,       c=h_1+4h_2.
```

Then c is odd and 0<c<N; indeed c<5h_3, so even 2c<N.

Construct the separable totally ramified quadratic extension

```
E=F(alpha),       alpha^2+alpha=t^(−q).
```

Put a=(q+1)/2 and z=t^a alpha. Then z is a uniformizer and

```
z^2+t^a z+t=0.
```

The element alpha has E-valuation −q and is minimal over F. On E^4, gamma=alpha I_4 is minimal of ambient level N in the complete period-eight order A.

Now construct the separable Eisenstein quartic extension over E

```
L=E(e),       e^4+z e+z=0,
```

and let C be multiplication by e^(−c) on L as a four-dimensional E-space. Since c is odd, e^(−c) is minimal, generates L/E, and defines a simple maximal derived stratum [B_order,c,c−1,C].

Let P0 extract the coefficient of1 on each E-coordinate in the basis 1,z. The normalized outer corestriction from the Artin–Schreier right-inverse calculation has

```
s(P0)=t/z^2.
```

Consequently the explicit F-linear lift

```
b=(z^2/t) C P0
```

satisfies s(b)=C and has valuation −c. Define the actual rank-eight matrix

```
beta=alpha I_4+(z^2/t) C P0.
```

BK 2.2.3 applies because gamma is simple at level c<N and the derived stratum is simple maximal. It proves that F[beta]/F is totally ramified of degree8, normalizes the complete chain, has leading level N, and has critical exponent −c. Its defining degree pattern is 8→2, skipping degree4.

Since N is even and c is odd, the actual Heisenberg quotient has dimension

```
(8−4)·1+(4−1)·0=4,
```

so eta has dimension4. Yet its invariants are

```
Swan=N=12h_3,
torsion=1,
a(Pi^vee×Pi)=63+4N+3c
            =63+48h_3+12h_2+3h_1,
```

exactly the same as the full three-stage profile. Any maximal simple character and its standard type completion give an actual supercuspidal Pi with these values. The required full three-plane source carrier would have dimension8, larger than this eta.

This is an arithmetic counterexample to the proposed implication from those three numerical invariants to adequate rank-eight Heisenberg capacity. It is not an H5 counterexample, does not identify the skipped host with the source-defined classical candidate, and does not claim that it preserves the supplied full source characters.

J.7 7. Which repairs have actually been tested

J.7.1 Replacing the displayed type or parameter at fixed Swan and torsion

At ranks2 and4 this fails by Sections4–5 for every maximal simple type satisfying those invariants. A new matrix search or a different critical coefficient cannot manufacture the missing quotient dimensions.

At rank8 the full tower supplies the good profile, but the three invariants alone do not force it inside a separately selected candidate. Section6 proves the failure of that proposed inference with an explicit actual host. A finer endo-class or ramification comparison would be needed for that same-candidate assertion.

J.7.2 Skipping approximation degrees or changing critical parity

At rank2 there is no degree to skip. At rank4 the level excludes both a scalar minimal approximation and a full-degree minimal approximation. The remaining quadratic derived step has odd critical exponent by minimality. Thus neither skipping nor declaring that exponent even is a valid repair at fixed invariants.

At rank8 a degree skip is genuinely possible, but it decreases capacity in the explicit example above.

J.7.3 Scalar twists, shallow or deep

A scalar twist of the existing actual inducing representation does not increase its dimension. The exact Block164 pushout transport identifies the same quotient J/H and leaves its commutator pairing unchanged. Thus it cannot turn an insufficient eta into a whole source carrier.

For an actual parameter, replacing beta by beta+aI with a∈F leaves its field, centralizer and commutator map unchanged. A shallow scalar addition does not change the leading level. A dominating scalar addition can change that level, but it contributes no new noncentral Heisenberg directions: in the defining sequence the added scalar stage has equal H and J orders. This agrees with the exact representation-level twist transport. Enlarging a displayed numerical level is therefore not a capacity repair.

This paragraph does not assert that every arbitrary scalar character twist is represented by one globally specified additive parameter. The already proved group/representation transport suffices for its negative capacity conclusion.

J.7.4 Adding residue multiplicity or increasing ambient rank

The torsion-one condition forces a full-degree totally ramified parameter field and residue multiplicity one. A host with an unramified or residual multiplicity factor does not preserve that invariant. Increasing the ambient rank changes the required source rank. Neither maneuver closes the current fixed-rank construction.

J.7.5 Using a larger arithmetic carrier inside the same representation

A larger compact subgroup representation, a genuinely different arithmetic quotient, or an induction-stage carrier might have enough dimension. Such an enlargement is not ruled out here. It would need an actual finite Heisenberg action carrying the prescribed full source data and a proved recovery in the unchanged candidate. Artificially adding vector-space copies or transporting abstract operators to an induced space does not provide that arithmetic attachment or prove its kernel compatibility. No such repair is constructed in this note.

J.8 8. Quadratic refinements and marked centers

For d>=3, the dimension result gives a symplectic embedding of the source quotient into the actual host quotient. With C trivial, a C×-central pushout can then be matched by its alternating commutator, and Stone–von Neumann supplies the corresponding source sector. This is the same finite scalar-pushout attachment contract used previously.

At p=2, that statement must not be promoted to an identification of arbitrary preselected native quadratic refinements. The C× pushout permits rescaling lifts by i and can identify the pushouts of the dihedral and quaternion extraspecial planes, even though their native square functions differ. A chosen source embedding can preserve that source's full finite central extension and its character values. It does not thereby identify a separately marked target mu_2 extension or its Arf invariant.

In particular, symplectic-isotropic reduction respects a prescribed quadratic form only when the reduced subspace is totally singular for that form. The retained `DYADIC_AND_WEAK_ATTACHMENT159.md` gives the exact quadratic/Arf obstruction. Additional source or target marked centers, chosen lifts, quadratic refinements and nontrivial stabilizer actions therefore require their own comparison. They are neither erased from the task nor supplied by a dimension inequality.

J.9 9. Scope and source provenance

The strongest proved conclusion is exact capacity for the specified actual full defining profile, together with invariant low-rank failures and an explicit rank-eight counterexample to numerical profile forcing. No universal impossibility for original Theorem16.2 is asserted. In particular this note supplies no

complete original H1–H5 counterexample and does not decide whether that endpoint requires this precise whole-maximal-type-carrier realization.

Primary source directly reread: Bushnell–Kutzko, *The admissible dual of GL(N) via compact open subgroups*, 1.4.14–15; 2.2.2–3; 2.4.1–2 and the maximal-field remark immediately preceding 2.4.7; 3.1.5, 3.1.7–9; 3.2.1–4; 3.4.1; 6.2.5 and its following paragraph.

Authorized retained text:

```
closure_recovery/extracted/Paper_XI_Continuation158_General_Sectors_and_Arit
hmetic_Constraints/scratch158/information158/input_cache/BK_source.txt
SHA256 d69db0827256707a34cd2562af68b9de11fc889ab3f43535ec47b620d58ca9c8
```

Internal inputs reused at their stated scope:

- `attachment166/tower/RECURSIVE_TOWER166.md`: actual defining degrees, levels, filtered right inverse and separated grade count; its field constructor permits p=2.
- `attachment166/tower/TRACE_ZERO_AS_TOWER166.md`, Sections1–2: normalized corestriction and explicit right inverse. Their algebraic formulas apply at p=2; the odd-prime trace-zero assertion is not used.
- `attachment166/capacity/LEADING_LAYER_CAPACITY166.md`: the leading subspace argument and the minimal terminal degree calculation.
- `base_realization165/polarization/PROFILE_RIGIDITY165.md` and `closure_recovery/full161/scratch161/placement161/ADJOINT_CONDUCTOR_TEST161.md`: retained formal-degree/self-pair conductor identities and normalization.
- `input_pair164/TWIST_TRANSPORT.md`: exact character-twist/pushout transport.
- `closure_recovery/extracted/Paper_XI_Continuation159_Type_Completion_and_Sector_Recovery/scratch159/coverage159/DYADIC_AND_WEAK_ATTACHMENT159.md`: distinction between symplectic and marked quadratic reduction.

No full copyrighted source is redistributed here. No new external correspondence theorem or H5 premise is introduced.

K Whole finite extensions and dyadic marking criteria

Evidence: direct reading of the original dyadic contract and ordinary finite group proofs. This note does not prove a dyadic arithmetic field construction, its simple-character existence, nontrivial stabilizer attachment, H5, or a Lean closure. It identifies exactly what a supplied actual host and sector must preserve, and proves the marked finite attachment step for C={1}.

K.1 1. The original datum includes more than the alternating form

The original text `closure_attempt162/inputs/original_reading.txt` states in Definition11.1, starting line4911, that at p=2 a restricted bridge additionally supplies compatible quadratic refinements and an isomorphism of the resulting Heisenberg central extensions lifting the given symplectic isometry. Sections13.1–13.9 define the refinements and their Arf/Gauss signs.

The retained visual appendix is particularly explicit, at lines6892–6947: with center {I,-I} fixed, squares of lifts carry the quadratic bit. Multiplying a lift by i changes the particular finite central-extension realization. The appendix says that allowing complex scalar rescaling **without recording this distinction** erases data that the bridge is required to retain.

The finite record must therefore retain at least:

- the labelled symplectic quotient and its pairing;
- the supplied quadratic refinement q;
- the specified finite central extension, its central embedding/character, and any marked lifts or specified extension isomorphism;
- the resulting normalized Gauss sign, and all declared phase operators;
- the complete original character values retained by the scalar pushout.

The last item alone does not retain the preceding finite-subgroup/lift marking unless that marking is explicitly carried along. The raw text does not require literal embeddings of the old arithmetic matrix groups into a new type. Nor does it define a unique quadratic refinement extracted automatically from an unmarked C× scalar extension of every arithmetic type.

K.2 2. General finite central-extension criterion

Let V be an elementary abelian 2-group with ordered basis v_1,…,v_m. Let E_s and E_t be central extensions of V by the same abelian group Z. Fix the identity on Z and a quotient isomorphism f. Choose lifts a_i of v_i in E_s and b_i of f(v_i) in E_t. Write

```
a_i²=s_i,  b_i²=t_i,  [a_i,a_j]=c_ij, [b_i,b_j]=d_ij.
```

Every element has a unique normal form

```
z a_1^(epsilon_1) ... a_m^(epsilon_m), epsilon_i in {0,1},
```

and analogously on the target. The multiplication is completely determined by the displayed squares and commutators.

Criterion. A center-fixing isomorphism lifting f exists precisely when

```
c_ij=d_ij for every i,j,
s_i/t_i belongs to Z² for every i.                    (1)
```

Indeed, every such isomorphism must take a_i to lambda_i b_i for some lambda_i in Z, forcing lambda_i²=s_i/t_i. Conversely these equations make the assignment a_i -> lambda_i b_i preserve all relations, and the normal forms prove it is bijective.

If the **particular lifts** a_i and b_i must correspond exactly, the condition is s_i=t_i, with lambda_i=1. If a full chosen section is marked, rather than only generating lifts, its multiplication coefficients must also correspond:

```
a(v)a(w)=c_s(v,w)a(v+w),
b(fv)b(fw)=c_t(fv,fw)b(f(v+w)),
```

and exact section preservation requires c_s(v,w)=c_t(fv,fw) for every pair. Transporting the complete section under an existing group isomorphism retains these identities automatically. Preserving just its squares need not retain an additional independently specified section.

K.2.1 2.1. Fixed two-element center

For Z={+1,-1}, Z²={1}. The lift square is independent of the lift within its fiber and defines

```
a(v)²=(-1)^(q_s(v)).
```

The identity (ab)²=a²b²[b,a] gives the quadratic refinement law. Criterion(1) becomes the exact requirement

```
q_t(f(v))=q_s(v) for every v.                    (2)
```

For an unspecified symplectic isomorphism between equal-dimensional nondegenerate spaces, such an f exists exactly when their Arf invariants agree. For a preassigned f, equality of Arf invariants alone is insufficient: (2) must hold for that f.

K.2.2 2.2. Four-element and complex centers

For Z=mu_4, multiplication of a lift by i changes its square by -1. The unmarked extension criterion preserves square classes modulo Z²=mu_2. In particular, squares +1 and -1 do not distinguish unmarked mu_4 extensions. A separately marked finite mu_2-central extension subgroup (such as the full D8 or Q8), or prescribed lifts, still distinguishes them.

For Z=C× every square ratio has a square root. Thus matching commutators is sufficient for an unmarked center-fixing extension isomorphism. This fact does not delete a marked finite subgroup from the objects being compared.

K.3 3. Exact D8/Q8 check

In dimension two over C put

```
X=[[0,1],[1,0]],       Z=[[1,0],[0,-1]].
```

Then X²=Z²=I, XZ=-ZX, and [X,Z]=-I. The group generated by X and Z has order8 and is D8. Its refinement on V=F2² is

```
q_D(x,y)=xy,       Arf(q_D)=0,
```

with square table (+1,+1,+1,-1) at (0,e,f,e+f), and Gauss sum +2.

The generators iX,iZ also anticommute, but each squares to -I and their product also squares to -I. They generate Q8, with

```
q_Q(x,y)=xy+x+y,   Arf(q_Q)=1,
```

square table (+1,-1,-1,-1), and Gauss sum -2.

Both C× pushouts identify with the same ambient matrix group

```
P={z X^a Z^b : z in C×, a,b in {0,1}}.
```

Nevertheless the marked subgroups

```
H_D=<X,Z>,          H_Q=<iX,iZ>
```

are different. No isomorphism fixing the two-element center can map H_D onto H_Q: it would preserve q and hence the Arf invariant, which differs. There is also no center-fixing automorphism of P carrying the marked H_D onto H_Q. Their unmarked pushouts are isomorphic, but the marked objects (P,H_D) and (P,H_Q) are not.

Conversely a Q8 source embeds perfectly into the same P by sending its generators to iX and iZ. This preserves every source square and its negative Gauss sign. It does not identify those generators with the independently marked D8 lifts X,Z. This is the distinction needed for the arithmetic sector.

K.4 4. Marked scalar-pushout attachment with C={1}

Assume an actual type, its full character theta, and its Heisenberg quotient W have already been constructed. Let P_W be its actual C× scalar pushout, acting on its actual Heisenberg representation eta_W. Let V_src be a finite nondegenerate symplectic F2-space of dimension2s, with its complete supplied dyadic marked extension data. For several breaks use their labelled orthogonal sum, balanced central product, and supplied product refinement.

Assume the following precise finite contract:

1. The source full-character pushout P_src has quotient V_src and the same represented scalar center as P_W.
2. The supplied finite central extensions and their marked lifts are retained as subgroups/elements of P_src, or as already specified maps into P_src. Any central character kernel already factored out must be recorded; a scalar pushout cannot recover invisible elements of that kernel.
3. The host has enough symplectic capacity, dim W>=2s, and the new attachment into this host has no independently fixed conflicting quadratic marking.
4. The declared stabilizer is C={1}; the graph has one vertex.

Then a complete marked attachment exists in the **same actual candidate**.

Proof. Choose a symplectic embedding f:V_src -> W and let V be its image. The restriction P_V of P_W and P_src have the same commutator. Criterion(1) with center C× produces an isomorphism

```
F:P_src -> P_V
```

fixing every scalar and lifting f. Carry every supplied finite subgroup H_src to the actual subgroup F(H_src) of P_V, and every marked lift a to F(a). This operation does not erase square data:

```
F(a)²=F(a²),        [F(a),F(b)]=F([a,b]).              (3)
```

Since F fixes the scalar center, $a^2=(-1)^{q(v)}$ implies $F(a)^2=(-1)^{q(v)}$. The transported refinement is therefore exactly q, not a new choice with the same polar form. All marked section products, specified extension maps, and finite group relations are preserved by the same F. Consequently the Arf invariant and normalized Gauss sign are unchanged.

Every original h in a full character domain still maps to theta_i(h), because F fixes the entire scalar center. Thus the dyadic marking is retained together with the full original character values.

Write W=V perpendicular-sum U. Finite Stone–von Neumann gives

```
eta_W isomorphic to eta_src tensor eta_U.
```

Choose a line in eta_U and its projector e. It lies in the actual Heisenberg image algebra, and the marked source subgroup acts on e eta_W by precisely its supplied irreducible Heisenberg representation. Its lift squares remain the operators in (3). The existing sector recovery theorem then recovers the unchanged inducing representation Lambda, and the same arithmetic candidate. There is no C-equivariance equation to solve because C={1}. QED.

This realizes an actual finite marked subgroup in the scalar extension of the actual type, with actual represented operators that are complex scalars times type operators. It does not assert that every marked source element is the image of an unscaled element of the old or new arithmetic matrix group. That stronger literal-embedding requirement is absent from the present contract.

It also does not assert that the newly marked subgroup equals a separately specified canonical native subgroup of the host. The marked subgroup is part of the result and must be retained in the artifact. Omitting it and recording only the unmarked P_W would reproduce the loss displayed by D8/Q8.

K.5 5. If the host refinement is independently fixed

Suppose instead that the actual host arrives with a particular nonsingular quadratic refinement q_W and its actual finite mu_2 central extension H_W. An attachment into that marked extension requires a quadratic isometric embedding, not merely a symplectic embedding. There is a sharp capacity test.

Theorem. Let dim V_src=2s and dim W=2R. With embeddings otherwise free, a quadratic isometric embedding

```
(V_src,q_src) -> (W,q_W)
```

exists if and only if R>=s and, in the equal-capacity case R=s,

```
Arf(q_src)=Arf(q_W).                                    (4)
```

If R>s, either source Arf type embeds.

Proof. Necessity of dimension is immediate. An equal-dimensional embedding is an isometry, forcing (4). If R>s, choose a nondegenerate quadratic space U of dimension2(R-s) with

```
Arf(q_U)=Arf(q_W)+Arf(q_src).
```

Both Arf types exist in every positive even dimension, by original Proposition13.7. The orthogonal sum V_src perpendicular-sum U then has the same dimension and Arf invariant as W. Original Standard Fact13.4 gives an isometry to W; restriction to V_src gives the desired embedding. QED.

After this embedding is chosen, the preimage of its image in H_W has exactly q_src. The fixed-two-element-center criterion of Section2 gives the extension isomorphism. Generating lifts with the specified squares can be matched. Additional fully prescribed section elements still require their exact product relations, as described in Section2.

Thus **one spare symplectic plane absorbs the Arf mismatch**, without changing the host's own refinement. This conclusion remains conditional on the host's actual marked refinement/finite extension being supplied or constructed. It does not manufacture a canonical arithmetic quadratic datum from its dimension.

For a preassigned embedding f, the missing equation is instead

```
q_W(f(v))=q_src(v) for all v.                            (5)
```

Its discrepancy q_W composed with f minus q_src is a linear functional. Strict capacity allows a different quadratic embedding when it is free; it does not authorize replacing a fixed original map, fixed depth placement, or fixed transport marking.

K.6 6. Exact outcome for the dyadic construction

Under the retained one-vertex, C={1}, full scalar-pushout realization contract, an actual host with sufficient symplectic capacity does support every supplied finite dyadic marking, provided the transported finite subgroup, lifts, refinement, and phase are explicitly kept. The proof is Section4, rather than an assertion that commutators determine the dyadic sign.

If a particular host-native quadratic extension is fixed in advance, use Section5. Strict capacity suffices for its quadratic embedding; equal capacity requires the Arf equation, and a preassigned embedding requires (5). If only the type quotient is known and a specifically canonical host quadratic datum is requested, constructing and comparing that datum remains an arithmetic obligation. The scalar-pushout proof must not silently claim to have done it.

No part of this note handles nontrivial prescribed C actions. In that case quadratic embedding, full extension action, lift markings, and chosen honest phase operators must all be equivariant; dimension and Arf equality alone do not settle those equations.

K.7 Evidence and author self-review

Read directly: original Definition11.1, Sections13 and15, and visual appendix pagesV–22/V–23 in the retained raw file; final Block166 `PROOF166.md`, Section2, and `LEADING_LAYER_CAPACITY166.md`, Sections2.1 and4. The extension criterion and D8/Q8 calculation are complete finite proofs. The quadratic embedding theorem uses the Arf classification already explicitly supplied by the original paper. The sector conclusion reuses the retained exact recovery theorem and does not assert new arithmetic existence.

Self-review checked that complex lift rescaling is never treated as an automorphism changing squares within a fixed marked extension, that the marked subgroup remains in the output, that preassigned maps differ from freely chosen embeddings, and that fixed native refinements require their own actual input. No independent review of this note or new Lean execution is claimed here.

L The actual enlarged rank-two inducing carrier

Status: ordinary proof. The construction below supplies an actual arithmetic finite Heisenberg pair and exact recovery of an already fixed representation. It does not change the previously audited zero-dimensional BK quotient at rank two. No Lean verification or H5 comparison is claimed.

L.1 1. Exact setting and result

Let F=F_2((t)) and G=GL2(F). Let [A,q,0,beta] be a minimal simple stratum with K=F[beta] totally ramified of degree2 and A its complete period-two hereditary order. Assume

```
q=2m+1>=3,        m>=1.
```

Write R=rad(A), $U^a_A = 1 + R^a$ for a>=1, and $U^a_K = 1 + \mathfrak{p}^a_K$. Let theta be a full simple character. Since q is odd and the residue field is F2, the actual maximal simple type group and character groups coincide:

```
J=J^1=H^1=U_K^1 U_A^(m+1).
```

Fix the full actual extension Lambda of theta to

```
N=K×J,
```

and the already fixed irreducible supercuspidal candidate

```
Pi=c-Ind_N^G Lambda.
```

Lambda is one-dimensional here. Its value on a uniformizer of K, including the noncompact scalar normalization, is part of the fixed data.

Define the larger compact open arithmetic subgroup

```
P=U_K^1 U_A^m.
```

Then the following statements hold.

1. J is normal of index2 in P, and P∩N=J.
2. E=Ind_J^P theta is an actual irreducible two-dimensional P-representation.
3. There is an explicitly defined normal subgroup H_new⊂J of index2 such that P/H_new≅F2² and theta_new=theta|H_new has nondegenerate commutator pairing on that quotient. E is its Heisenberg representation.
4. E embeds into the restriction of the same Pi, as the subspace of inducing functions supported on NP.
5. With X=c-Ind_J^G theta, a uniformizer varpi_K and s=Lambda(varpi_K), there is an exact quotient
   ```
   Pi ≅ X/(T-s)X,
   (Tf)(g)=f(varpi_K g).
   ```
   Induction in stages identifies X with c-Ind_P^G E. This gives an exact same-candidate recovery using the retained noncompact scalar, not merely a nonzero cyclic-generation assertion.

The new Heisenberg pair is an actual pair of compact subgroups of GL2(F), but is not asserted to be the BK H^1/J^1 pair of a maximal simple type. Whether the original H2 realization relation allows this enlargement is a separate obligation.

L.2 2. The compact enlargement is a group, with normal subgroup of index two

K× normalizes A and every power of its radical, so U_K^1 normalizes U_A^m. Therefore P is a subgroup and is compact open. It is contained in U_A^1 because m>=1 and $U_K^1 \subset U_A^1$.

The standard radical calculation gives

```
[U_A^a,U_A^b]⊂U_A^(a+b),        a,b>=1.
```

Indeed, for x∈R^a and y∈R^b, the difference (1+x)(1+y)−(1+y)(1+x) is in R^(a+b), and multiplication by unit inverses preserves that ideal.

Consequently

```
[U_A^m,U_K^1]⊂U_A^(m+1),
[U_A^m,U_A^(m+1)]⊂U_A^(2m+1)⊂U_A^(m+1).
```

These facts, and normalization by U_K^1, show that J is normal in P. They also show

```
[P,J]⊂U_A^(m+1).                                    (1)
```

To calculate the index, use

```
U_A^m∩J=U_K^m U_A^(m+1).
```

For if ku∈U_A^m with k∈U_K^1 and u∈U_A^(m+1), then $k \in K^\times \cap U_A^m = U_K^m$. The reverse inclusion is immediate. The complete period-two A-grade R^m / R^{m+1} has F2-dimension2, and its field grade $\mathfrak{p}_K^m / \mathfrak{p}_K^{m+1}$ has dimension1. Thus

```
[P:J]=[U_A^m:U_K^m U_A^(m+1)]=2.                    (2)
```

Finally, if kj∈P with k∈K× and j∈J, then $k \in Pj^{-1} \subset U_A^1$. Hence k∈U_K^1 and kj∈J. Therefore

```
P∩N=J.                                              (3)
```

These arguments use the actual group intersections. They do not replace P or J by a vector-space surrogate.

L.3 3. Irreducibility and the conjugate-character ratio

For the minimal simple character, the primary BK intertwining theorem gives

```
I_G(theta)=J K×J=N.
```

Here K× normalizes J and theta. In particular, for g∈P\J, (3) says g does not intertwine theta. Because J is normal in P, this means that the two characters

```
theta,
theta^g(j)=theta(gjg^-1)
```

are distinct. The index-two induction E=Ind_J^P theta is therefore irreducible. This also follows directly from the two distinct J-weight lines, which are interchanged by g.

Define the actual character of J

```
lambda(j)=theta^g(j)/theta(j).
```

It is nontrivial. By (1),

```
lambda(j)=theta(gjg^-1j^-1),
```

and the argument of theta lies in U_A^(m+1). The full simple character has exact restriction

```
theta(1+x)=psi_F(tr_F(beta x)),       x∈R^(m+1).
```

Since F has characteristic2, its additive characters take values in {1,−1}. Consequently

```
lambda(J)={1,−1}.                                    (4)
```

No assertion about shallow values of theta is used here. Theta may have a larger finite image on J. Only its ratio with its actual conjugate is proved to have order2.

As $g^2 \in J$, conjugation by g^2 fixes the character theta. Thus $\lambda^g = \lambda^{-1} = \lambda$. It follows that

```
H_new=ker(lambda)
```

is normal in P. It has index2 in J and index4 in P.

L.4 4. The actual finite Heisenberg pair

Use the function model of E with f(jp)=theta(j)f(p) and right translation by P. Coordinates are f(1),f(g). Then for $j \in J$,

```
E(j)=diag(theta(j),theta^g(j))
    =theta(j) diag(1,lambda(j)),
```

while

```
E(g)=[[0,1],[theta(g²),0]].                          (5)
```

Therefore $E(g)^2=\text{theta}(g^2)I$. Choose $j_0 \in J$ with $\text{lambda}(j_0)=-1$, and put

```
Z=theta(j_0)^−1 E(j_0)=diag(1,−1),
X=E(g).
```

The exact matrix identity is

```
XZX^−1=−Z.                                           (6)
```

The projective image of J has order2, generated by Z, while X is off-diagonal and contributes an independent projective involution. Thus the projective image of P is exactly C2×C2, with nontrivial commutator (6). Its projective kernel is precisely H_new. In particular

```
P/H_new≅F2².
```

On H_new, E is the scalar theta_new=theta|H_new. This character is P-invariant: conjugation by J fixes theta and conjugation by g changes theta by lambda, which is trivial on H_new. The resulting alternating form

```
(pH_new,p'H_new)↦theta_new([p,p'])
```

is nondegenerate, since its two displayed generators have commutator −1. Therefore (H_new,P,theta_new) is an actual arithmetic finite Heisenberg pair, and its actual irreducible carrier E has dimension2.

All representations here have finite image on the compact group P: theta has open kernel, the induction has finite dimension, and an open normal core of that kernel acts trivially. Thus no infinite projective-image issue is being suppressed.

This proof does not require rescaling the actual matrix g inside GL2(F). Its represented square $\text{theta}(g^2)$ can be an arbitrary retained scalar. Rescaling X is only allowed later in a C× pushout if that is the attachment contract.

L.5 5. Explicit embedding into the same fixed candidate

Realize Pi as functions F:G→C with

```
F(nx)=Lambda(n)F(x),
```

compact support modulo N, and right translation by G. For f∈E define i(f) by

```
i(f)(np)=Lambda(n)f(p),       n∈N, p∈P,
```

and zero off NP. This is well-defined because N∩P=J and Lambda|J=theta. The set NP is a finite union of N-cosets, by [P:J]=2, so the support is compact modulo N. The map i is P-equivariant and injective.

Its image is exactly the two-dimensional P-stable subspace of Pi consisting of inducing functions supported on NP. Thus the enlarged carrier is already inside the given Pi. No new candidate is chosen, and all noncompact values of Lambda remain fixed.

Since Pi is irreducible and i(E) is nonzero, its G-span is Pi. More precisely, there is the canonical G-equivariant quotient

```
Q_E:c-Ind_P^G E→Pi,
Q_E(F)=sum_(Px∈P\G) Pi(x^−1)i(F(x)).                (7)
```

The sum is finite and independent of representatives. If x is replaced by px, then F(px)=E(p)F(x) and P-equivariance of i cancels the additional Pi(p^−1). Formula (7) is nonzero and hence onto. The next section identifies its kernel exactly.

L.6 6. Exact recovery by the noncompact Hecke eigenvalue

Let X=c-Ind_J^G theta, in the same left-covariant/right-translation convention. Induction in stages gives an explicit isomorphism

```
c-Ind_P^G E → X,
F ↦ [x↦F(x)(1)].                                   (8)
```

The inverse sends f∈X to the E-valued function F(x)(p)=f(px). Both covariance statements follow directly from the definitions, and the finite index [P:J] ensures the support conditions agree.

Fix a uniformizer varpi=varpi_K. Because K× normalizes theta, the operator

```
(Tf)(x)=f(varpi x)                                 (9)
```

is a well-defined invertible G-equivariant operator on X. Put s=Lambda(varpi), which is nonzero. Since k_K=F2 and J contains U_K^1,

```
N/J≅K×/U_K^1≅Z,
```

with generator varpi. Define

```
Q:X→Pi,
(Qf)(x)=sum_(ell∈Z) s^(−ell) f(varpi^ell x).       (10)
```

For each x the sum is finite, since f has compact support modulo the open subgroup J. Equation (10) satisfies

```
(Qf)(varpi x)=s(Qf)(x),
(Qf)(jx)=theta(j)(Qf)(x),       j∈J,
```

and therefore the required N-covariance. Its support is compact modulo N. It commutes with G-right translation and satisfies

```
QT=sQ.                                                    (11)
```

It is onto: a function in Pi has finite support modulo N, and on each occupied N-coset its value can be lifted to a function supported on one J-coset. Extend those finitely many choices using theta.

To compute the kernel, decompose X as the algebraic direct sum of its subspaces supported on the left N-cosets in G. Each such summand is the space of finitely supported sequences on N/J≅Z, with the fixed theta covariance. The operator T is a bilateral shift. On that summand, (10) is evaluation of its Laurent polynomial at the nonzero scalar s, up to the harmless convention of replacing the Laurent variable by its inverse. The kernel is therefore the principal ideal generated by T−s. The elementary identity

```
ker(C[z,z^−1]→C, z↦s)=(z−s)
```

proves, summand by summand, that

```
ker Q=(T−s)X.
```

All vectors occupy finitely many such summands, so no completion or closure is needed. Thus

```
Pi ≅ X/(T−s)X ≅ (c-Ind_P^G E)/(T−s)(c-Ind_P^G E),    (12)
```

where T is transported through (8).

Under (8), formula (10) agrees with (7): on the standard copy of E supported on P, it gives exactly i, and G-equivariance then gives equality everywhere. Thus (12) identifies the kernel of the canonical recovery map, not merely the kernel of an unrelated surjection.

The same description can be phrased as evaluation of the type Hecke algebra at the character determined by Lambda. Its intertwining support is N, each supported intertwining space is one-dimensional, and the retained extension Lambda normalizes the Laurent generator. The explicit formulas above avoid any ambiguity about convolution orientation.

The scalar s is essential. E alone, being compact data, does not determine it. Changing s can change the unramified normalization of the candidate. The construction retains the original Lambda and s, so it recovers the exact original Pi rather than an unspecified member of its inertial class.

L.7 7. Full scalar-pushout attachment and remaining marking obligations

Form the C× pushout of the actual pair (H_new,P,theta_new). It is a Heisenberg extension of the actual symplectic plane P/H_new. Any supplied source plane with the same chosen alternating form has a center-preserving pushout isomorphism into it. Every source full-character value is then preserved in the center, as in the retained full-character attachment theorem. Stone–von Neumann identifies its source carrier with the actual two-dimensional E.

Together with the embedding i and exact quotient (12), this provides an actual arithmetic carrier and recovery inside the same fixed candidate. It closes the carrier-capacity failure of the old rank-two eta by changing the compact arithmetic pair, not by increasing the old eta or discarding its dimension calculation.

At p=2, a C× pushout does not automatically identify a separately marked native mu_2 extension or quadratic refinement. The source finite central extension can be retained through its chosen embedding, while a required comparison with the target's preselected square function remains an additional question. No arbitrary native Arf invariant is asserted here.

Nontrivial stabilizer operators, transport graphs, and original H2's requirement that a target be a particular BK type-side quotient are likewise separate. In particular, the theorem proves that the

new pair is an actual arithmetic finite Heisenberg pair, but does not rename it as the unchanged maximal-simple-type pair. No new hypothesis is silently added to original Theorem16.2.

L.8 8. Scope, smallest boundary, and sources

The inequality q>=3 matters. If q=1, then m=0 and the displayed enlargement is not the group used above. Over residue field F2 the complete Iwahori unit group equals U_A^1, and a naive level-zero replacement does not give the asserted index-two enlargement. No q=1 claim is made. The source levels q=3h with positive odd h satisfy the required q>=3.

The proof uses the actual one-dimensional Lambda because the maximal parameter field has full degree and residue field F2. It does not apply the same scalar argument automatically to a higher-rank inducing representation.

Primary BK passages: 3.1.7–8 (odd minimal H/J equality); 3.2.1–2 (deep restriction and K× normalization); 3.3.2 and 5.1.1/5.1.8 (intertwining); and 6.2.1–5 (the fixed maximal-type extension and compact induction). The subgroup, matrix, Mackey-support, and Laurent-evaluation calculations are proved explicitly above. No external correspondence theorem is used in this enlargement.

Retained primary text:

```
closure_recovery/extracted/Paper_XI_Continuation158_General_Sectors_and_Arit
hmetic_Constraints/scratch158/information158/input_cache/BK_source.txt
SHA256 d69db0827256707a34cd2562af68b9de11fc889ab3f43535ec47b620d58ca9c8
```

The previously reviewed negative statement concerned eta of the original BK J^1/H^1. That quotient remains zero. The new positive statement concerns a different, explicitly constructed arithmetic pair inside the same ambient representation, with exact noncompact-normalization-aware recovery.

M The actual enlarged rank-four inducing carrier

Date: 2026-10-05. Ordinary proof submitted for independent review.

M.1 Result and boundary

For the actual rank-four Artin–Schreier candidate of Block167, one can enlarge the compact group J1 to an actual compact subgroup P with

```
J1 normal in P,  [P:J1]=2,
E=Ind_(J1)^P(eta) irreducible of dimension 4.
```

The representation E occurs inside the restriction of the same actual supercuspidal candidate pi. There is an explicit normal subgroup H0 of P with [P:H0]=16 such that P/H0 is elementary abelian and the commutator form supplied by the actual character on H0 is nondegenerate. Thus the carrier has an actual four-dimensional binary symplectic quotient. No abstract transfer of matrix operators is used to create this action.

The finite image of P has scalar center theta(H0), which can be larger than {+1,-1}. It is literally an extraspecial group if this scalar center has order two. That additional assertion is not assumed or proved here. The construction supplies an actual finite Heisenberg central extension with nondegenerate projective quotient F_2^4. Matching an independently prescribed quadratic phase is also a separate assertion.

M.2 1. Actual arithmetic groups and notation

Let F=F_2((t)), and take the audited rank-four constructor with odd 0<h_1<h_2. Write

```
q_i=3h_i,  n=2q_2,  a=n/2=q_2,
r=q_1=2m+1,  m=(r-1)/2>=1.
```

We have a>r, so n>2r. Let A be the complete period-four hereditary order, with radical P_A. Let gamma be the actual degree-two outer minimal approximation, E=F[gamma], B=End_E(F^4)=M_2(E), B0=A cap B, and Q=rad(B0). The actual final parameter is beta=gamma+delta, and

```
s(delta)=c,
```

where c is the minimal degree-two Artin–Schreier parameter over E, of B0-level r. The final field K=F[beta] is totally ramified of degree four; no embedding E subset K is asserted.

Use $U_A^j = 1 + P_A^j$ and $U_B^j = 1 + Q^j$. BK3.1.7–9 give the actual groups

```
H=H1(beta)=U_K^1 U_B^(m+1) U_A^(a+1),
J=J1(beta)=U_K^1 U_B^(m+1) U_A^a.
```

In particular J/H has dimension two over F_2. Let theta be the actual full simple character constructed for beta, and let eta be its unique Heisenberg representation on J, of dimension two. The argument below works with this supplied actual theta; it does not replace its values.

Choose the outer minimal simple character theta_gamma from its defining sequence. On H cap U_A^(m+1), the recursion says

```
theta=theta_gamma psi_delta.
```

Moreover H is contained in H1(gamma)=U_B^1 U_A^(a+1). The character theta_gamma is invariant under U_B^1: on U_B^1 it factors through the E-determinant, and on U_A^(a+1) it is the gamma-additive character.

M.3 2. The actual final field has the needed graded line

For every integer j>=1 and k in K cap P_A^j, the minimal-approximation commutator estimate gives

```
k=b+e,  b in Q^j,  e in P_A^(j+n-r).              (1)
```

Indeed [gamma,k]=-[delta,k] belongs to P_A^(j-r), and the minimal corestriction exact sequence for gamma solves its commutator with an error of grade j+n-r. The remaining difference centralizes gamma and therefore belongs to B. This is the usual minimal case of BK1.4.10.

Applying s to [beta,k]=0 and using (1) gives

```
[c,b] in Q^(j+n-2r).
```

Hence the leading B-grade of b commutes with the leading grade of c. Let R be the complete period-two E-chain shift, with R^2=z I, where z is the chosen uniformizer of E. In the constructor's basis

```
c congruent to R^(-r) modulo Q^(1-r).
```

Because r is odd, the centralizing subspace on every B-grade j is the one-dimensional line spanned by R^j. On the other hand K is a totally ramified maximal field normalizing the complete A-chain, so (K cap P_A^j)/(K cap P_A^(j+1)) has dimension one over F_2. Estimate (1) is injective on the leading grade, so its image is exactly that centralizing line.

This proves the required graded assertion for the actual K, without replacing it by the derived model E[c] or assuming nested fields.

M.4 3. Normality and index of the enlarged group

Set

```
P=J U_B^m.
```

We first prove this is a group. The group U_B^m normalizes U_B^(m+1) and every U_A^j. For k in U_K^1, (1) approximates k by an element of U_B^1 with error in U_A^(n-r+1). Therefore

```
[U_B^m,U_K^1]
   subset U_B^(m+1) U_A^(m+n-r+1)
   subset H,
```

where the last inclusion follows from a>r=2m+1. Thus U_B^m normalizes both H and J, and P is an actual compact subgroup with J normal in P.

The quotient U_B^m / U_B^{m+1} has dimension two. By Section2 the image of J cap U_B^m on this grade is exactly its centralizing line, of dimension one. For the converse inclusion needed here, the B-approximation of a K-element of grade m differs from it by grade m+n-r>a; that error lies in U_A^a subset J. Thus the whole centralizing line really occurs in the intersection, not only in an abstract associated graded space. Consequently

```
[P:J]=[U_B^m:J cap U_B^m]=2.                    (2)
```

For a concrete representative choose D=diag(1,0) in B and put

```
x=R^m D,  u=1+x.
```

Its leading grade is outside the centralizing line, so u represents the nontrivial coset of P/J. Since $u^2 = 1 + x^2$ in characteristic two, we have u^2 in U_B^(2m) subset U_B^(m+1) subset H.

M.5 4. The actual character has an orbit of length two

Define the character of H

```
lambda(h)=theta(u h u^(-1))/theta(h).
```

For h in H the preceding normality argument gives [u,h] in H cap U_A^(m+1). Since theta_gamma is invariant under u,

```
lambda(h)=psi_delta([u,h]) in {+1,-1}.          (3)
```

Thus lambda has order dividing two, and its values are actual values of the supplied simple character. It is J-invariant, since both theta and its u-conjugate are J-invariant.

It is nontrivial. Take y=R^(m+1)D and h=1+y in U_B^(m+1). The leading commutator in (3), after corestriction, has value

```
psi_E(tr_(M_2(E)/E)(c[x,y]))=-1.
```

Here the commutator remainder has B-grade at least 3m+1, which becomes positive after multiplication by c of grade -2m-1. On the leading grades, writing X=R^m diag(A,B) and Y=R^(m+1)diag(C,D), one obtains

```
tr(R^(-2m-1)[X,Y])=(A+B)(C+D) in F_2.
```

Both sums equal one for the chosen D=diag(1,0). The normalized psi_E is nontrivial on the residue field and trivial on its maximal ideal. For this AS normalization one can check the sign directly from psi_E(1)=psi_F(Tr_(E/F)(alpha))=psi_F(1)=-1.

Hence theta^u differs from theta, eta^u differs from eta, and Clifford theory applied to (2) gives

```
E_carrier=Ind_J^P(eta) irreducible, dim(E_carrier)=4.       (4)
```

M.6 5. An actual nondegenerate symplectic quotient of dimension four

Put H0=ker(lambda) inside H, and theta0=theta restricted to H0. In the two-coset induction model, h in H acts by

```
diag(theta(h) I_2, theta^u(h) I_2).
```

Thus it acts scalarly precisely when h belongs to H0. We check that H0 is the entire scalar preimage and that its quotient is symplectic.

First, lambda is invariant under u. Indeed u^2 belongs to J and theta is J-invariant, so $\lambda^u = \lambda^{-1} = \lambda$ by (3). Consequently H0 is normal in P, and theta0 is P-invariant.

Second, lambda is trivial on U_A^(a+1). There the correction formula is valid, gamma commutes with u, and the delta contribution to a commutator has grade at least a+1+m-r=a-m>0. Therefore it lies in the kernel of the additive character.

Write J=H U_A^a. Since lambda is a character and J-invariant, it kills [H,H] and [H,U_A^a]. It also kills $[U_A^a, U_A^a]$, which lies in U_A^(2a). Hence lambda([J,J])=1. It kills every square in J as well: squares of H have lambda-value one, squares of U_A^a lie in U_A^(2a), and the mixed commutator contributions have just been handled.

Finally lambda([u,H])=1 by u-invariance, and [u,U_A^a] belongs to U_A^(a+m) subset U_A^(a+1), so lambda([u,J])=1. Also lambda(u^2)=1 because u commutes with u^2. It follows that

```
[P,P] subset H0,  g^2 in H0 for every g in P,
[P:H0]=[P:J][J:H][H:H0]=2*4*2=16.
```

Therefore P/H0 is an actual elementary abelian group of dimension four. If an element of J acts scalarly on the carrier, its first eta-block is scalar, so it belongs to H by the original nondegenerate Heisenberg quotient J/H. It then belongs to H0. An element outside J exchanges the two coset summands and cannot act scalarly. Thus H0 is precisely the scalar preimage.

The pairing

```
(gH0,kH0) |-> theta0([g,k])
```

is alternating and well defined. Since the projective quotient has exponent two, its values lie in {+1,-1}. It is nondegenerate: an element in its radical commutes with the entire irreducible carrier image and therefore acts scalarly by Schur's lemma, forcing it into H0.

This is the actual four-dimensional binary Heisenberg quotient sought for capacity. Its operators are supplied by elements of P.

M.7 6. The carrier occurs in the same supercuspidal candidate

Let pi=c-Ind_N^(GL_4(F))(Lambda) be the existing actual candidate, with N=K^times J. The residue field is F_2, so the compact extended type restricts to eta on J. Since P is contained in U_A^1,

```
N cap P=J.
```

Indeed an element k j in P with k in K^times has k in U_A^1, hence k in U_K^1 subset J. The identity double-coset term in the restriction of compact induction to P is therefore

```
Ind_J^P(eta)=E_carrier.
```

It is realized concretely by the functions supported on N P. Thus the four-dimensional carrier occurs in the same pi, with its already fixed full type character and noncompact extension scalar. No new arithmetic candidate, beta extension, or inverse correspondence is used.

M.8 7. Exact limits: finite image versus literal extraspecial group

The actual finite image is P/ker(theta0), because H0 is the scalar preimage. Its center is the scalar cyclic 2-group theta0(H0), and its commutator subgroup has order two. Its quotient by its center is F_2^4.

If theta0(H0) has order two, this finite image is extraspecial of order 32. If its scalar center is larger, quotienting arbitrary central elements need not preserve its commutator: in a cyclic 2-group the order-two subgroup can lie inside the squares. Thus a literal extraspecial quotient is not justified solely by the calculations above.

The stronger statement proved is the one with full actual character: an actual compact subgroup, actual induced carrier, actual scalar subgroup and nondegenerate actual commutator pairing. Passing to the usual full scalar pushout retains all theta0-values. It does not prove that an independently supplied quadratic refinement, stabilizer action, or source phase agrees with this carrier. Those requirements must be checked separately. No H5 or universal claim66 closure is asserted.

M.9 Provenance

Uses the audited dyadic parameter/character construction, BK1.4.10, BK3.1.7–9, and the defining-sequence character formula. The source normalization and all field/chain data are the actual ones in `dyadic167/tower/DYADIC_AS_TOWER167.md`. This note changes neither that proof nor its recorded exact computation. Clifford induction and the identity compact-induction double-coset term are used in their ordinary finite-index forms, with hypotheses proved above.

N Rank-four normalization and induction in stages

Date: 2026-10-05. Corollary to `RANK4_ENLARGED_CARRIER167.md`, whose reviewed bytes are kept unchanged. Ordinary proof submitted for independent audit.

Use that note's actual groups and notation:

```
n=2a,  r=2m+1,  a>r,
H subset J=J1(beta) normal in P=J U_B^m,
[P:J]=2,  N=K^times J,
pi=c-Ind_N^G(Lambda),  Lambda|J=eta.
```

The field K normalizes P. Consequently the following are actual groups and actual smooth representations:

```
M=K^times P,
Sigma=Ind_N^M(Lambda),
dim Sigma=4,
Sigma|P=Ind_J^P(eta),
pi is isomorphic to c-Ind_M^G(Sigma).
```

The representation Sigma is irreducible, and M is compact modulo the center of G. Thus the enlarged carrier has an exact arithmetic inducing datum for the same candidate, not merely an occurrence in its restriction.

N.1 1. Normalization by the actual parameter field

The units O_K^times equal U_K^1 because the residue field is F_2. They belong to J and hence normalize P. It remains to check a uniformizer k of K. This element has A-grade one and normalizes A.

The actual-field approximation estimate proved in Section2 of the carrier note gives

```
k=b+e,  b in Q^1,  e in P_A^(n-r+1).
```

The leading grade of b is the nonzero centralizing line of the primitive derived parameter. Over F_2 it is exactly the leading grade of R, the period-two E-chain shift. Therefore

```
b=R v,  v in 1+Q^1.
```

In particular b is invertible and normalizes both the complete E-chain and the complete interleaved F-chain. We can write

```
k=b(1+epsilon),  epsilon in P_A^(n-r).
```

For u in U_B^m, the usual filtration commutator bound gives

```
(1+epsilon)u(1+epsilon)^(-1)u^(-1)
   in U_A^(n-r+m).
```

This group is contained in U_A^a subset J, because

```
n-r+m=2a-m-1>a.
```

Conjugation by b preserves both relevant filtrations. Since b normalizes U_B^m, it follows that

```
k U_B^m k^(-1) subset U_B^m J=P.
```

Also kJk^(-1)=J, as K^times normalizes its actual simple-character groups. Thus kPk^(-1) is contained in P and contains J. Its index over J is two, equal to [P:J], so it equals P. This proves normalization by k, its inverse, and hence all K^times.

No equality or inclusion between the fields E and K was used. The element b belongs to the actual outer centralizer B and only approximates k to the stated precision.

N.2 2. The two inducing groups have index two

Since K^times normalizes P, the product M=K^times P is a group. Both P/J and the conjugation action of K^times on it are defined; P/J has order two, whose automorphism group is trivial. Therefore

```
[K^times,P] subset J.
```

Together with J normal in P and K^times normalizing J, this proves N=K^times J normal in M. The intersection K^times cap P is U_K^1, which is contained in J, since P subset U_A^1. Consequently

```
[M:N]=[P:J]=2.
```

The quotient $K^\times/F^\times$ is compact because K/F is a finite totally ramified extension, and P is compact. Hence M is compact modulo F^times. It is open in G because it contains the open subgroup J.

N.3 3. The enlarged inducing representation is irreducible

Let u in P represent its nontrivial coset modulo J. The carrier note proved that theta^u differs from theta, and therefore eta^u differs from eta. It follows that Lambda^u differs from Lambda, since their restrictions to J already differ. Applying finite-index Clifford theory to N normal in M gives

```
Sigma=Ind_N^M(Lambda) irreducible,
dim Sigma=2 dim Lambda=4.
```

There is one N–P double coset in M, and N cap P=J. Thus restriction to P gives the actual four-dimensional Heisenberg carrier:

```
Sigma|P is isomorphic to Ind_J^P(eta).
```

Its central character on F^times is unchanged from Lambda, because F^times is central and lies in N.

N.4 4. Exact recovery by induction in stages

Induction in stages for these open subgroups gives

```
c-Ind_M^G(Sigma)
   is isomorphic to c-Ind_M^G(Ind_N^M Lambda)
   is isomorphic to c-Ind_N^G(Lambda)
   =pi.
```

The second induction is finite-index, so ordinary and compact induction from N to M agree. This is an exact equality of the resulting isomorphism classes and applies to the same fixed Lambda, including its noncompact scalar choice. It introduces no new candidate and no H5 premise.

N.5 Scope

This establishes the arithmetic inducing datum and exact recovery for the enlarged carrier. It leaves unchanged the distinction between the original two-dimensional BK quotient J/H and the

new four-dimensional actual quotient P/H0. It also leaves unchanged the source-phase and literal extraspecial-center limitations recorded in the carrier note.

Dependency pin: `RANK4_ENLARGED_CARRIER167.md` SHA256 `1517e26138776fe0f32d74a174d309ec5dd6a0f572a979473d8af3d20b7a4155`.

O Exact enlarged-carrier recovery and original-contract comparison

Evidence: a bounded contract audit and elementary conditional recovery lemmas. This note does not prove that the proposed compact enlargement is a group, that its induced representation is Heisenberg, or that the source markings attach to it. Those are the arithmetic inputs to be verified by the separate rank-two and rank-four constructions.

Verdict: an enlarged compact carrier is not, merely by being Heisenberg or an equivalent type, a new instance of the original H2 quotient. It can serve as a common assembly carrier for the already supplied H2 targets if its actual occurrence, marked attachment, and same-candidate recovery are proved and retained. The recovery mechanism differs from the finite sector idempotent used when the packet lies inside the original inducing carrier.

O.1 1. What original H2 literally requires

Original Section10 fixes an actual maximal simple character theta, its actual groups H_theta¹ subset J_theta¹ subset J_theta, and

```
V_theta=J_theta¹/H_theta¹.
```

Its Heisenberg representation is the particular eta_theta containing theta. Definition11.1 supplies an equivariant symplectic isometry from a source envelope to that type-side quotient, together with the central character, and at p=2 the compatible quadratic refinements and central-extension lift. Standard Fact10.2 separately supplies the canonical beta extension of this simple-character datum.

Consequently the following implications are invalid without more proof:

```
Xi is a finite Heisenberg representation
   therefore Xi=eta_theta' for a maximal simple character theta';

(P,Xi) is equivalent to a known arithmetic type
   therefore it has that type's J¹/H¹ or canonical beta extension.
```

An equivalence of generated representation categories does not identify the particular finite group quotient in Definition11.1. If a fixed conductor and torsion profile forces every maximal simple-character carrier in the candidate to have dimension one or two, constructing a larger Xi in that candidate does not refute the dimension theorem. Xi is another arithmetic carrier.

There is nevertheless an important distinction between **replacing an H2 target** and **assembling already supplied H2 targets**. The accepted sector construction itself uses a finite sector that need not be a whole maximal simple-character carrier. Its old actual break targets remain unchanged and are attached to that sector. An enlarged P-carrier can play the analogous assembly role, provided the complete dictionary and recovery below are proved. Calling it merely an arbitrary vector space would understate that result; calling it a new canonical maximal simple type would overstate it.

O.2 2. Precise conditional setup

Let G=GL_n(F), Z=F×I_n, and let the already constructed candidate be

```
pi=c-Ind_N^G Lambda,
```

with its fixed actual extended type and fixed central character omega. Let J be a compact open normal subgroup of N such that

```
eta=Lambda|J is irreducible,       N/J is infinite cyclic.
```

In the proposed F2-residue, full-degree totally ramified setting, J is the actual compact type group J^1=J, not a new group called J merely to fit this notation. Let P be an actual compact open subgroup of G satisfying

```
J subset P,          N intersect P=J.
```

Define the genuine finite-dimensional compact induction

```
Xi=Ind_J^P eta.                                    (1)
```

Neither irreducibility nor Heisenberg structure of Xi is assumed to follow from its dimension. They must be proved separately when needed.

If the known eta intertwining satisfies I_G(eta)=N, then Xi is irreducible: the finite Mackey endomorphism formula gives

```
End_P(Ind_J^P eta)=End_J(eta)=C,
```

because no nontrivial double coset in P lies in I_G(eta). Finite-dimensional smooth representations of compact P factor through a finite group and are semisimple, so scalar endomorphisms imply irreducibility. This criterion requires its actual group and intertwining hypotheses; dimension alone is not a substitute.

O.3 3. Xi occurs in the same pi, and can be an equivalent compact type

The subspace of the actual compact-induction model of pi consisting of functions supported on NP is stable under P. Mackey restriction identifies that subspace with

```
Ind_(N intersect P)^P(Lambda|_(N intersect P))=Xi.     (2)
```

Thus Xi is an actual compact P-subrepresentation of the same pi, with no change of arithmetic candidate or noncompact scalar.

Induction in stages also gives the exact G-isomorphism

```
c-Ind_P^G Xi isomorphic to c-Ind_J^G eta.            (3)
```

If (J,eta) is already a type for a stated Bernstein block and Xi is irreducible, then (P,Xi) detects the same irreducible representations:

```
Hom_P(Xi,V) isomorphic to Hom_J(eta,V).
```

This is the precise equivalent-type statement that follows from (1). It does not assert that Xi is the eta of another maximal simple character, or attach a canonical beta extension to Xi. The known canonical extension in Lambda remains the one originally supplied.

O.4 4. Exact recovery before fixing the center: a quotient

Choose u in N mapping to a generator of N/J, and retain the actual matrix Lambda(u). Use the model

```
X=c-Ind_J^G eta,
f(jg)=eta(j)f(g),
```

with G acting by right translation. Define the invertible G-equivariant shift operator

```
(Tf)(g)=Lambda(u)^(-1)f(ug).                       (4)
```

Normality of J and the actual intertwining relation for Lambda(u) ensure that Tf has the same J-covariance. Its normalization explicitly retains the fixed noncompact extension; changing Lambda(u) changes the relation below.

Define

```
(Qf)(g)=sum_(k in Z) Lambda(u)^(-k)f(u^k g).          (5)
```

The sum is finite on each argument and has the required compact support modulo N. It is N-covariant and therefore lies in pi. It is surjective and satisfies Q(Tf)=Q(f). Decomposing support into N/J-orbits reduces its kernel to the elementary augmentation kernel on finitely supported sequences on Z. That kernel consists exactly of consecutive differences. Hence

```
X/(T-1)X isomorphic to pi.                           (6)
```

Through (3), this is a recovery of pi from the enlarged compact carrier plus the original noncompact relation. It does not select the correct relation from the finite Heisenberg packet alone.

In general this quotient cannot be replaced by a G-equivariant split inclusion before central normalization. A nonzero vector of X has finite support modulo compact J. The valuation of its determinant support is a finite subset of Z. A scalar uniformizer z=tI_n shifts these valuations by n; consequently X has no nonzero eigenvector for z. Every vector of pi is such an eigenvector, with eigenvalue omega(t). Therefore no nonzero G-map pi -> X exists. In particular (6) does not split.

This is why the Block159 sector summand projector cannot simply be quoted: there the finite sector was inside Lambda and its stabilizer contained the noncompact center. Here Xi is larger than Lambda and lies in pi|P, with P compact. These are different induction problems.

O.5 5. After retaining the center: a finite Hecke idempotent

There is also an exact summand version once the already fixed central character is imposed. Put J_Z=ZJ and P_Z=ZP. Assume, as in the proposed totally ramified full-degree setting, that

```
N/J_Z is finite cyclic of order ell
```

(here ell=n). Let Lambda_0=Lambda|J_Z. Because Xi occurs in the same pi, its action on Z intersect P agrees with omega. It therefore extends to P_Z by letting Z act through omega; call this Xi_omega. Induction in stages gives

```
Y=c-Ind_(P_Z)^G Xi_omega
  isomorphic to c-Ind_(J_Z)^G Lambda_0.              (7)
```

Using the same normalized shift (4), now on Y, one has T^ell=1. Indeed u^ell belongs to J_Z and the J_Z covariance cancels Lambda(u)^ell. The finite operator

```
e_Lambda=(1/ell) sum_(j=0)^(ell-1) T^j               (8)
```

is a G-equivariant idempotent whose image is exactly the fixed pi. To see this directly, N-covariant functions embed in J_Z-covariant functions because N/J_Z is finite. On that embedded pi the finite version of (5) equals ell times the identity. Its average is precisely (8), and its image consists exactly of the N-covariant functions.

Equivalently, Ind_(J_Z)^N Lambda_0 is Lambda tensored with the regular representation of N/J_Z. Formula(8) selects the Lambda summand. It uses the already fixed Lambda(u), including its root choice, so it preserves the same candidate. Other root choices select the other extensions in this finite induction. They are not interchangeable input data.

This is a finite Hecke/induction projector after central-character normalization. It is not the original finite image-algebra projector onto a sector of eta. A final artifact must retain (7), (8), and the normalization Lambda(u), rather than describing recovery merely as "the carrier is an equivalent type."

O.6 6. The exact remaining admissibility checks

For either proposed low-rank enlargement, the new proof must supply all of the following concrete inputs:

1. **Actual compact group:** prove that the proposed P is a subgroup, is compact open, contains the actual J, and satisfies P intersect N=J. Compute [P:J] and the resulting Xi dimension.
2. **Actual carrier:** verify eta=Lambda|J, its actual intertwining if the irreducibility criterion is used, and the Mackey realization (2).
3. **Actual finite Heisenberg structure:** exhibit the scalar preimage in P and the resulting elementary abelian projective quotient. Compute its represented commutator and prove nondegeneracy. An irreducible carrier of dimension two or four need not automatically have this structure.
4. **Original attachments:** attach the unchanged supplied actual H2 break targets, retaining their full source character values and dyadic finite extension/lift/square markings. The precise transported-versus-native distinction remains the one proved in `DYADIC_MARKINGS167.md`.
5. **Recovery data:** retain the actual Lambda, central character and noncompact generator normalization, and prove the isomorphisms and quotient/idempotent above for those exact groups.

When these are supplied, the result is stronger than occurrence of an unrelated finite vector space: it is a marked arithmetic carrier in the same candidate, equivalent through explicit induction and recovering that candidate exactly. For C={1} and one vertex, it can repair the failed low-rank **assembly-carrier** realization under the retained scalar-pushout contract. The original H2 targets remain their original simple-character quotients; they are not renamed Xi.

If the intended target instead requires the source packet itself to equal J_theta'[1]/H_theta'[1] for a maximal simple character in this same candidate, the enlargement does not solve it. The retained low-rank dimension obstruction still applies. A separate theorem identifying Xi with such an eta would be needed, and under the fixed obstructed profile it cannot be inferred from type equivalence.

No H5 condition is added or discharged. If H5 is subsequently supplied, it must concern this fixed recovered candidate. Nontrivial stabilizer actions, canonical native quadratic comparison, and arbitrary transport graphs remain outside this bounded argument.

O.7 Evidence and self-review

Read directly: original Section10 and Definition11.1 in the raw manuscript; the Block159 `NONINVARIANT_SECTOR_RECONSTRUCTION159.md`; current `PROOF167.md` and `TARGET167.md`; the retained dyadic marking theorem. The Mackey, induction, shift-quotient, and central-normalized projector arguments above are proved conditionally for the displayed groups, rather than assumed for the proposed congruence formula. No actual low-rank P computation or new Lean execution is claimed in this contract note.

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