The abc Conjecture

Radical support, prime contact, and the global bridge

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READER EDITION 1.0 / 8 OCTOBER 2026

A full reader edition of the conjecture and Operation Komorebi's exact reductions. The local structures are proved. The arithmetic estimate linking them to a uniform radical bound remains open.

EXACT IDENTITIES / FINITE EVIDENCE / OPEN CONJECTURE

How to read this piece

The equation $a + b = c$ joins two kinds of structure. Multiplication describes prime factors and their repeated powers. Addition forces those three factorizations to fit one exact relation. The *abc* conjecture says that this compatibility limits how large the sum can be compared with the distinct primes that appear.

This edition develops the conjecture and the existing Operation Komorebi research in one mathematical order. The opening sections explain the radical, the quantifiers, the polynomial analogue and the consequences. The middle sections give the prime contact calculus, including its exact divisibility, lattice, signal, filtered and logarithmic constructions. The final sections develop the September jet determinant block and identify the global theorem it still needs.

Reading path	Sections to emphasize
First encounter	Foundations, examples, the polynomial theorem, consequences, and the final global bridge
Proof spine	Prime packets through coherence, then the jet filter, Hankel determinant, local valuation and truncation proofs
Research boundary	The first-order barrier, exact synthesis costs, integral projectors, coherence endpoint, corrected defect and failed rational truncation
Visual companion	The separate guide follows the same objects with exact figures, short reading instructions and worked examples

The statement labels have precise roles. A theorem has a proof under its stated hypotheses. A finite calculation has a declared tested domain. A conditional theorem proves an implication from a hypothesis that remains unproved. A structural limitation closes a specified method or target, rather than all possible approaches.

Present boundary. This work does not prove the general *abc* conjecture. The strongest new determinant statements are formal, real analytic and p-adic local theorems. A nonzero global arithmetic realization with a uniform radical-scale height bound has not been constructed.

The research has also made a substantive correction. If $Q = abc/\operatorname{rad}(abc)$, then the defect relevant to the conjecture is $Q/(ab) = c/\operatorname{rad}(abc)$. The target $Q = R^{o(1)}$ is false. Its missing factor ab is restored explicitly in every final certificate.

The elementary and finite proofs below are self-contained where practical. References identify external theories and the authenticated literature used to place this work. No claim of historical priority follows from a newly organized derivation.

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1 Radical support and the exact conjecture

1.1 The multiplicative quantity that ignores repetition

Every positive integer has a unique prime factorization

$$n = \prod_{p \mid n} p^{e_p}, \qquad e_p \geq 1.$$

Its radical is

$$\operatorname{rad}(n) = \prod_{p \mid n} p, \qquad \operatorname{rad}(1) = 1.$$

The radical retains which primes occur and discards how many times they occur. Thus $\operatorname{rad}(72) = \operatorname{rad}(2^3 \cdot 3^2) = 6$, whereas $72/6 = 12$ records the repeated-power contribution. If n is squarefree, then $\operatorname{rad}(n) = n$.

For a positive primitive additive triple

$$a + b = c, \qquad \gcd(a, b) = 1,$$

all three pairs are coprime. For example, a divisor of a and c also divides $c - a = b$, hence is one. Therefore the prime supports are disjoint and

$$R = \operatorname{rad}(abc) = R_a R_b R_c, \qquad R_u = \operatorname{rad}(u).$$

Put $M_u = u/R_u$. The exact decomposition is

$$abc = R M_a M_b M_c.$$

The conjecture compares c with R, rather than with abc itself. Since abc already contains c, an estimate against the full product would lose the intended content.

Conjecture 1.1 (Masser–Oesterlé *abc*)**.** *For every real $\varepsilon > 0$ there is a constant $K_\varepsilon > 0$, depending only on ε, such that every positive integer triple with $a + b = c$ and $\gcd(a, b) = 1$ satisfies*

$$\boxed{c \leq K_\varepsilon \operatorname{rad}(abc)^{1+\varepsilon}.}$$

The order of the quantifiers is the statement. First choose ε. Then choose a constant that works for every primitive triple. The constant may become large as ε approaches zero. It may not depend on the particular triple.

This formulation is the one stated in Stewart and Yu's primary treatment of the conjecture [3]. Replacing $\leq$ by $<$ does not change the all-ε conjecture after enlarging the constant. Requiring $a \leq b$ also leaves it unchanged because the two inputs can be exchanged.

1.2 Equivalent output-sector formulations

The form best adapted to this research separates the repeated depth of the output from the input prime support.

Proposition 1.2 (Exact normalization)**.** *For every primitive positive triple,*

$$\frac{c}{R} = \frac{M_c}{R_a R_b}, \qquad Q := \frac{abc}{R}, \qquad \frac{Q}{ab} = \frac{c}{R}.$$

Consequently Conjecture 1.1 is exactly equivalent to each all-ε assertion

$$M_c \leq K_\varepsilon R_a R_b R^\varepsilon, \qquad Q \leq K_\varepsilon ab R^\varepsilon.$$

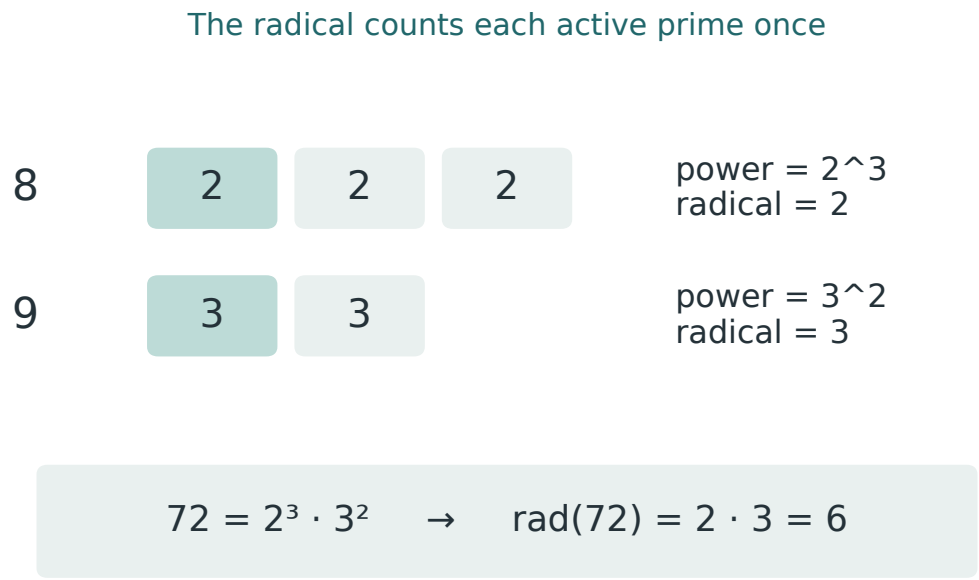


Figure 1: Prime support and repeated depth are different quantities. The radical keeps one copy of each active prime. The powerful part keeps the remaining copies.

Proof. Use $R = R_a R_b R_c$ and $c = R_c M_c$, then cancel the indicated factors. Multiplying any displayed inequality by the positive cancelled factors recovers the original conjecture, with the same constant and exponent. □

A local construction may measure M_c naturally. A Wronskian may instead be divisible by Q. Both are legitimate, provided the final global height includes the matching normalization.

1.3 Quality measures the exponent in one example

Define the quality of a primitive triple by

$$q(a,b,c) = \frac{\log c}{\log R}.$$

The denominator is positive because $c \geq 2$ and $R \geq 2$. The logarithm base does not matter. The identity $c = R^q$ is exact, although decimal qualities are approximations.

A triple has $q > 1$ exactly when $c > R$. It then has more output size than the global distinct-prime product. Such a triple is often called an *abc* hit. A hit is an interesting example, not a counterexample to the conjecture. The constant K_ε is part of the conjecture.

a	b	c	R	Factorization of the repeated terms
1	8	9	6	$8 = 2^3, 9 = 3^2$
3	125	128	30	$125 = 5^3, 128 = 2^7$
1	4374	4375	210	$4374 = 2 \cdot 3^7, 4375 = 5^4 \cdot 7$
2	$109 \cdot 3^{10}$	23^5	15042	$23^5 = 6436343$

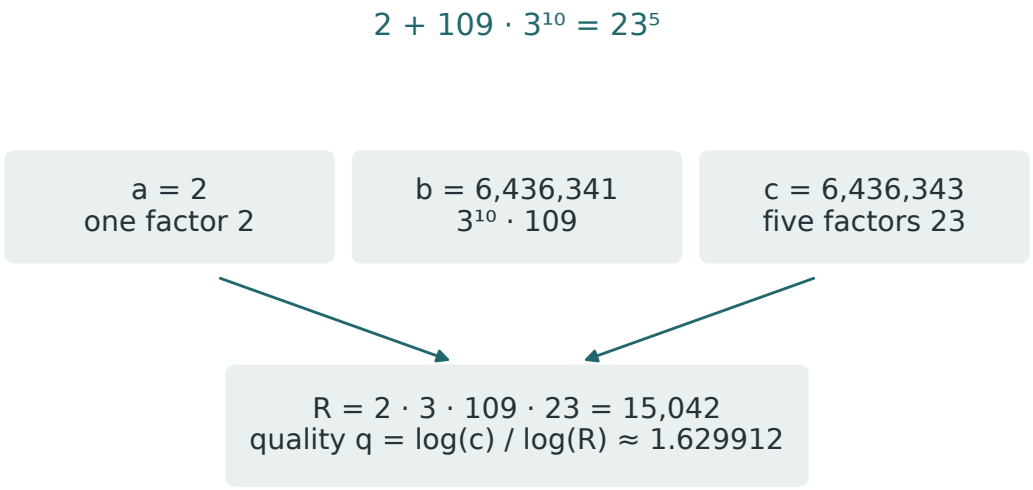


Figure 2: Exact examples compare the output with the radical. Decimal quality is a diagnostic derived from the exact integer data. A finite collection of high qualities does not decide the all-triple conjecture.

The last equality is $2 + 109 \cdot 3^{10} = 23^5$. Every factor in the table matters. Its radical is $2 \cdot 3 \cdot 109 \cdot 23 = 15042$, giving quality about 1.629912. The third triple has quality about 1.567887. These are illustrative exact examples, without any claim that either is the largest known quality.

Proposition 1.3 (Finiteness formulation)**.** *Conjecture 1.1 is equivalent to the assertion that, for every $\varepsilon > 0$, there are only finitely many primitive positive triples with $q(a, b, c) > 1 + \varepsilon$.*

Proof. Assume *abc*, using its bound with $\varepsilon/2$. A triple of quality above $1 + \varepsilon$ satisfies

$$R^{1+\varepsilon} < c \leq K_{\varepsilon/2} R^{1+\varepsilon/2}.$$

Hence $R^{\varepsilon/2} < K_{\varepsilon/2}$. The radical is bounded. The same conjectural inequality then bounds c, leaving finitely many positive triples.

Conversely, fix $\varepsilon > 0$ and assume the finiteness assertion. Outside its finite exceptional set, $c \leq R^{1+\varepsilon}$. Choose K_ε to be at least one and at least every ratio $c/R^{1+\varepsilon}$ in the exceptional set. This single constant works for all triples. □

This equivalence explains why large searches can be valuable and still be logically limited. A computation can list hits in a finite range. It cannot certify the finiteness of the unsearched set.

1.4 Why the epsilon cannot be removed

The uniform statement $c \leq KR$ for one absolute constant K is false. There is an elementary family that proves this without using *abc*.

Theorem 1.4 (Failure of the zero-epsilon estimate)**.** *The ratios $c/\operatorname{rad}(abc)$ are unbounded among primitive positive additive triples.*

Proof. For $n \geq 0$, take

$$a_n = 1, \quad b_n = 4^{3^n} - 1, \quad c_n = 4^{3^n}.$$

These are primitive, b_n is odd, and $\operatorname{rad}(c_n) = 2$. We claim

$$v_3(b_n) = n + 1.$$

At $n = 0$ this is $v_3(3) = 1$. If $x = 1 + 3^k u$ with $k \geq 1$ and $3 \nmid u$, then

$$x^3 - 1 = 3^{k+1} u \left(1 + 3^k u + 3^{2k-1} u^2\right).$$

The bracket is a unit modulo three, so cubing increases the valuation by exactly one. This proves the claim by induction.

Because b_n contains 3^{n+1}, its powerful part contains 3^n. Thus

$$\operatorname{rad}(b_n) \leq \frac{b_n}{3^n}, \qquad R_n = 2\operatorname{rad}(b_n) \leq \frac{2b_n}{3^n}.$$

Therefore

$$\frac{c_n}{R_n} \geq \frac{3^n c_n}{2b_n} > \frac{3^n}{2} \longrightarrow \infty.$$

□

The displayed lower bound does not contradict the positive-ε conjecture. It does not exhibit infinitely many triples with a fixed positive quality gap. The output grows doubly exponentially with n, while the proved defect lower bound grows exponentially. A large defect lower bound and a fixed positive exponent gap are different assertions.

1.5 A known unconditional bound

Stewart and Yu proved an effective bound of a different scale [3, Theorem 1]: there is an effectively computable absolute $C > 0$ such that every primitive positive triple satisfies

$$c < \exp\left(CR^{1/3}(\log R)^3\right).$$

It controls the height in terms of the radical unconditionally. Its exponential dependence is much larger than the power $R^{1+\varepsilon}$ required by *abc*. It therefore provides a benchmark, without closing the gap this research addresses.

1.6 A published claim and a continuing verification dispute

Mochizuki's four IUT papers were published in 2021. The fourth paper states an application to *abc* [9]. Publication and establishment of the claimed proof are distinct questions. Scholze and Stix documented an objection to the comparison underlying IUT III, Corollary 3.12, and Mochizuki published a response disputing their simplifications [10, 11].

A 2026 Project LANA verification report withholds a verdict on whether IUT proves *abc*. It identifies a needed compatibility between the relevant analytic and anabelian comparison maps [13]. That report also distinguishes its objection from accepting every identification in the earlier criticism. This edition follows that precise status: a published proof claim and a documented, unresolved verification dispute. None of the project proofs below depends on treating IUT as an established proof of *abc*, and none claims to repair its disputed comparison.

2 The polynomial theorem and the missing integer analogue

2.1 Roots play the role of distinct primes

For a nonzero polynomial F over an algebraically closed field, let $\operatorname{rad}(F)$ be the product of its distinct monic linear factors. Its degree counts distinct roots. Write $n_0(F) = \deg \operatorname{rad}(F)$.

The analogy is direct. Prime powers p^e become repeated roots $(X-\alpha)^e$. A derivative lowers the multiplicity by one. The divisor left over after that operation is $F/\operatorname{rad}(F)$, just as $n/\operatorname{rad}(n)$ is the repeated-power part of an integer.

Theorem 2.1 (Mason–Stothers in characteristic zero)**.** *Let k be an algebraically closed field of characteristic zero. Suppose $A, B, C \in k[X]$ are nonzero, pairwise coprime, satisfy $A + B = C$, and are not all constant. Then*

$$\max\{\deg A, \deg B, \deg C\} \leq n_0(ABC) - 1.$$

This is the polynomial *abc* theorem. Snyder gives a short primary proof in a form that also makes the derivative hypothesis explicit [4]. The proof below spells out the cancellation that will motivate the arithmetic Wronskian.

Proof. Set $W = A'B - AB'$. It is nonzero. Indeed, if it vanished, then $(A/B)' = 0$, and characteristic zero would force the rational function A/B to be constant. Coprimality would then force both A and B to be constant, and so also C.

If α is a root of A of multiplicity e, then A' is divisible by $(X-\alpha)^{e-1}$ and AB' by $(X-\alpha)^e$. Thus $(X-\alpha)^{e-1}$ divides W. The same argument applies to roots of B.

Since $C = A + B$,

$$W = A'C - AC'.$$

At a root of C of multiplicity e, this expression is divisible by $(X-\alpha)^{e-1}$. The three root sets are disjoint. Therefore

$$\frac{ABC}{\operatorname{rad}(ABC)} \mid W.$$

Taking degrees gives

$$\deg A + \deg B + \deg C - n_0(ABC) \leq \deg W \leq \deg A + \deg B - 1.$$

Hence $\deg C \leq n_0(ABC) - 1$. To bound A, apply the same argument to the relation $C + (-B) = A$. To bound B, use $C + (-A) = B$. Coprimality and the radical root count are unchanged by these permutations and signs. □

2.2 A sharp example

Take $A = X^n$, $B = 1 - X^n$, $C = 1$ for $n \geq 1$. The root sets are disjoint. There is one distinct root of A and n distinct roots of B, so $n_0(ABC) = n + 1$. The maximal degree is n, attaining the theorem's bound. In this example $W = nX^{n-1}$, and the repeated-root divisor is exactly X^{n-1} up to a nonzero scalar.

The proof needs an actual nonzero derivative. In positive characteristic, $X^p + (1 - X^p) = 1$ has every derivative zero, so the characteristic-zero statement cannot simply be transferred. The correct positive-characteristic forms include a condition excluding common Frobenius powers.

2.3 Where the integer argument stops

An ordinary integer has no distinguished derivative with both an exact Leibniz rule and an automatic relation $D(a) + D(b) = D(c)$. Arithmetic derivatives provide a family of Leibniz maps. Additivity at one triple then imposes a lattice constraint on the prime-coordinate values.

The polynomial theorem gets two benefits simultaneously. Repeated roots force a large divisor of W, while degree controls W by a sharp expression one less than the input degree sum. For integers, a prime-coordinate Wronskian also has a large divisor. Its absolute size depends on the selected derivative packet. Choosing an additive transverse packet of uniformly small height is a separate arithmetic problem.

The subsequent prime-packet construction makes this distinction exact. Its local divisibility is the analogue of the displayed polynomial divisor. Its packet norm, integral synthesis and conductor comparisons are the quantities for which the polynomial proof has no automatic integer replacement.

2.4 A lesson for higher-order methods

High contact has value only when the auxiliary object remains nonzero and its global size is controlled. A determinant can raise contact through actual Taylor cancellation. Merely renaming a coordinate with a larger weight does not change its first nonzero Taylor coefficient.

The Hankel determinant later in this piece addresses the nonzero-contact side with an exact coefficient formula and real positivity. Its rational truncation provides a rigorous test of the other side: after projective denominator clearing, the desired output-prime contact disappears. The two obligations must therefore be proved together for a global witness.

3 Consequences and the scale of the missing estimate

3.1 An asymptotic Fermat consequence

The following implication is elementary and illustrates how repeated powers amplify the conjecture.

Theorem 3.1 (Conditional large-exponent Fermat statement)**.** *Assume Conjecture 1.1. There is an integer n_0 such that, for every integer $n > n_0$, the equation*

$$x^n + y^n = z^n$$

has no positive integer solution.

Proof. Divide a putative solution by its common gcd. The resulting x, y, z are pairwise coprime. Apply *abc* to $a = x^n$, $b = y^n$, $c = z^n$ with the fixed exponent $\varepsilon = 1$. Then

$$z^n \le K_1 \operatorname{rad}(xyz)^2 \le K_1 (xyz)^2 < K_1 z^6,$$

because $x, y < z$. As $z \ge 2$, this implies $2^{n-6} < K_1$ whenever $n > 6$. Any fixed integer larger than $6 + \log_2 \max(1, K_1)$ excludes all such exponents. □

The theorem does not give Fermat's Last Theorem for every $n \ge 3$. It supplies only a sufficiently-large-exponent implication, with a threshold depending on a conjectural constant. Fermat's Last Theorem is already an unconditional theorem. Its proof and the subsequent modularity theorem for elliptic curves form established arithmetic context [15, 16, 17].

Using a smaller fixed ε sharpens the displayed exponent six toward three. It still leaves a constant-dependent threshold. Letting ε vary with n without controlling K_ε would not be a valid way to remove the remaining finite exponents.

3.2 Elliptic curves and the modified Szpiro comparison

The marked Frey curve

$$E : y^2 = x(x - a)(x + b)$$

translates the additive triple into an elliptic curve. Its discriminant sees repeated depth, while its conductor sees prime support, with local model corrections at the small primes. This makes elliptic height estimates a natural companion to *abc*.

The precise equivalent statement uses more than the discriminant alone. For an elliptic curve over $\mathbf{Q}$, let N be its conductor and let c_4, c_6 be the invariants of a global minimal integral Weierstrass equation. The modified Szpiro assertion is

$$\max\{|c_4|^3, c_6^2\} \le C_\varepsilon N^{6+\varepsilon}$$

for every $\varepsilon > 0$, with a constant uniform in the elliptic curve. Oesterlé identifies this form as equivalent to the full *abc* conjecture [5].

This equivalence is cited as an external theorem. We do not derive its full small-prime and minimal-model accounting here. The ordinary statement involving only $|\Delta_{\min}|$ should not be silently substituted for the max-invariant formulation. A proposed project route through the Frey curve must preserve the marked integral model and the normalization of its differential, rather than only its rational isogeny class.

3.3 Why local algebra is substantial but insufficient

Several exact statements can coexist while the final height bound remains open:

1. A prime-power contact module can be defined canonically.
2. Its local carry structure can be computed exactly.
3. A Kummer class can carry nearly all of its finite order globally.
4. A logarithmic character can record the residual coherence.
5. A formal or analytic determinant can have large contact and be nonzero.

None of these statements bounds the archimedean size, denominators, or arithmetic degree of a single realization that retains all the contact. The point where those costs are compared with $R_a R_b$ is the point at which the conjecture's quantitative content enters.

This distinction motivates the research's structural limitations. They eliminate specific ways of overlooking a cost: raw coordinate splitting, a split associated graded source, the full Fermat polarization, a complex period without an integral lattice, or a rational truncation without output contact. Each limitation narrows the requirements of the next valid construction.

3.4 The research question to carry forward

The direct output-sector target is

$$M_c \leq K_\varepsilon R_a R_b R^\varepsilon.$$

It asks for a uniform explanation of output repetition in terms of input support. The coherence route factors M_c exactly and asks for a bound on the residual product WC. The determinant route seeks a nonzero arithmetic witness whose divisibility and global height imply the same inequality.

These are explicit interfaces, not estimates already established. A new local formula is progress within its own scope. A new global gain must reduce the exponent or comparison cost on the correctly normalized defect c/R.

4 Prime-Packet Differentials and Radical Divisibility

4.1 The prime-coordinate module

Let

$$a + b = c, \qquad \gcd(a, b) = 1.$$

The three integers are pairwise coprime. Write

$$S = S_a \sqcup S_b \sqcup S_c = \{p : p \mid abc\}.$$

The first object of the calculus is the free prime-packet module

$$\Omega_S = \bigoplus_{p \in S} \mathbb{Z}\xi_p.$$

First define the global free module over all primes, and restrict it to Ω_S for S-supported integers. For such a positive integer n, define the universal multiplicative differential

$$\boxed{\mathrm{d}n = n \sum_{p \mid n} \frac{v_p(n)}{p}\, \xi_p.}$$

Every coefficient is an integer because $p \mid n$. The map obeys the exact Leibniz law

$$\mathrm{d}(mn) = m\,\mathrm{d}n + n\,\mathrm{d}m.$$

The verification is local. For each prime p,

$$v_p(mn) = v_p(m) + v_p(n),$$

so the p-coordinate of the right side is

$$mn\frac{v_p(m) + v_p(n)}{p}.$$

A scalar packet is an integer vector

$$x = (x_p)_{p \in S} \in \mathbb{Z}^S.$$

Pairing with x gives the arithmetic derivative

$$\boxed{\mathrm{d}_x(n) = n \sum_{p \mid n} \frac{v_p(n)}{p} x_p.}$$

The Leibniz rule survives scalarization.

Arithmetic derivatives and their connection with *abc* have an established literature. The present construction is organized here in prime-packet coordinates and later refined into filtered shell objects [6, 7].

4.2 The additive row

The condition

$$\mathrm{d}_x(a) + \mathrm{d}_x(b) = \mathrm{d}_x(c)$$

is one integer equation. Define

$$\lambda_p = \begin{cases} \dfrac{a v_p(a)}{p}, & p \mid a, \\ \dfrac{b v_p(b)}{p}, & p \mid b, \\ -\dfrac{c v_p(c)}{p}, & p \mid c. \end{cases}$$

Then

$$\boxed{\lambda \cdot x = 0.}$$

The additive derivative lattice is

$$\boxed{\mathcal{T}_{a,b,c} = \ker(\lambda) \cap \mathbb{Z}^S.}$$

Since the row is nonzero, its rank is $|S| - 1$. The problem is not existence of additive derivatives. The problem is finding an additive derivative that remains transverse to the dependence direction introduced by a and b.

4.3 The arithmetic Wronskian

Define

$$\boxed{W_x(a,b) = a\,\mathrm{d}_x(b) - b\,\mathrm{d}_x(a).}$$

Writing

$$\mu_p = \begin{cases} -\dfrac{v_p(a)}{p}, & p \mid a, \\ \dfrac{v_p(b)}{p}, & p \mid b, \\ 0, & p \mid c, \end{cases}$$

we obtain

$$W_x(a,b) = ab\,\mu \cdot x.$$

An additive packet is *transverse* when $W_x(a,b) \neq 0$. It is degenerate when the two input terms remain derivative-proportional.

Over $\mathbb{R}$, the additive projection is

$$P_{\mathrm{add}} = I - \frac{\lambda\lambda^{\mathsf{T}}}{\|\lambda\|_2^2}.$$

The obstruction direction inside the additive hyperplane is

$$\mu_\perp = P_{\mathrm{add}}\mu.$$

When $\mu_\perp \neq 0$, the rank-one obstruction projector is

$$P_{\mathrm{obs}} = \frac{\mu_\perp \mu_\perp^{\mathsf{T}}}{\|\mu_\perp\|_2^2}.$$

This is the first appearance of the projector-resolved pattern. A large ambient prime space is decomposed into one equation direction, one obstruction direction, and the remaining null packets.

4.4 Radical divisibility

Set

$$R = \operatorname{rad}(abc).$$

The central first-order theorem is an exact divisibility law.

Theorem 4.1 (Wronskian divisibility)**.** *For every additive packet $x \in \mathcal{T}_{a,b,c}$,*

$$\boxed{\frac{abc}{R} \mid W_x(a,b).}$$

Proof. Let $p^e \parallel a$. Every term of $\mathrm{d}_x(a)$ is divisible by p^{e-1}. Since $p \nmid b$, the term $b\,\mathrm{d}_x(a)$ is divisible by p^{e-1}. Also $p^e \mid a$, so $a\,\mathrm{d}_x(b)$ is divisible by p^e, hence by p^{e-1}. Therefore

$$p^{e-1} \mid W_x(a,b).$$

The same argument applies to primes dividing b.

Now let $p^e \parallel c$. Additivity gives

$$\mathrm{d}_x(c) = \mathrm{d}_x(a) + \mathrm{d}_x(b).$$

Using $c = a + b$,

$$\begin{aligned} W_x(a,b) &= a\,\mathrm{d}_x(b) - b\,\mathrm{d}_x(a) \\ &= a\big(\mathrm{d}_x(c) - \mathrm{d}_x(a)\big) - b\,\mathrm{d}_x(a) \\ &= a\,\mathrm{d}_x(c) - c\,\mathrm{d}_x(a). \end{aligned}$$

The first term is divisible by p^{e-1} because $\mathrm{d}_x(c)$ is. The second is divisible by p^e. Hence $p^{e-1} \mid W_x(a,b)$.

The prime supports of a, b, and c are disjoint. Multiplying all local divisibilities gives

$$\prod_{p^e \parallel abc} p^{e-1} = \frac{abc}{R} \mid W_x(a,b).$$

□

For a transverse packet,

$$\boxed{|W_x(a,b)| \geq \frac{abc}{R}.}$$

This is the integer analogue of the repeated-root divisor appearing in the polynomial Mason–Stothers theorem.

4.5 The master height inequality

For every positive n,

$$|\mathrm{d}_x(n)| \leq n \,\|x\|_\infty \sum_{p \mid n} \frac{v_p(n)}{p}.$$

Since $p \geq 2$,

$$\frac{1}{p} \leq \frac{1}{2\log 2} \log p.$$

Therefore

$$\sum_{p|n} \frac{v_p(n)}{p} \leq \frac{\log n}{2 \log 2},$$

and

$$|\mathrm{d}_x(n)| \leq \frac{n \log n}{2 \log 2} \|x\|_\infty .$$

Consequently,

$$\begin{aligned} |W_x(a,b)| &\leq a|\mathrm{d}_x(b)| + b|\mathrm{d}_x(a)| \\ &\leq \frac{ab}{\log 2} \|x\|_\infty \log c. \end{aligned}$$

Combining this with radical divisibility gives

$$\boxed{c \leq \frac{R}{\log 2} \|x\|_\infty \log c.}$$

Except for $(1,1,2)$, define the shortest transverse packet

$$\boxed{\Theta(a,b,c) = \min \left\{ \|x\|_\infty : x \in \mathcal{T}_{a,b,c}, W_x(a,b) \neq 0 \right\}.}$$

Then

$$c \leq \frac{R}{\log 2} \Theta(a,b,c) \log c.$$

The exceptional triple $(1,1,2)$ has zero input derivatives and is omitted from Θ. It is absorbed into the constant in the conjecture. A subpolynomial estimate

$$\Theta(a,b,c) = c^{o(1)}$$

would imply the strong *abc* inequality. Later sections explain why this target is too close to the conjecture to be reached by generic lattice geometry alone.

4.6 Three-sector compression

For the empty unit sector, set $g_1 = 0$, $t_1 = 0$, and $\rho_1(0) = 0$. Any nonzero target in that sector is infeasible. Within a nonempty a-sector, define the integer row

$$\lambda^{(a)} = \left(\frac{a v_p(a)}{p} \right)_{p|a}$$

and let

$$g_a = \gcd \left\{ \frac{a v_p(a)}{p} : p \mid a \right\}.$$

Define g_b and g_c similarly. A one-row Smith normal form in each sector produces a unimodular coordinate system in which only one aggregate channel survives. The additive equation becomes

$$\boxed{g_a t_a + g_b t_b - g_c t_c = 0.}$$

The Wronskian becomes

$$\boxed{a g_b t_b - b g_a t_a \neq 0.}$$

All remaining sector coordinates are lift gauges. They matter only because they determine the cost of realizing an aggregate channel by an integral prime packet.

For each term define

$$\rho_a(t) = \min\left\{ \left\| x^{(a)} \right\|_\infty : \lambda^{(a)} \cdot x^{(a)} = g_a t \right\},$$

with analogous functions for b and c. Then

$$\Theta(a,b,c) = \min \max\{\rho_a(t_a), \rho_b(t_b), \rho_c(t_c)\}$$

over aggregate solutions satisfying additivity and transversality.

This reduction separates the global two-dimensional arithmetic geometry from the local lifting cost. It is the template used repeatedly later: compress first, then retain the exact synthesis norm of the discarded coordinates.

4.7 Worked example: $3 + 125 = 128$

The prime packets are 3, 5, and 2. The additive row is

$$\lambda = (1, 75, -448).$$

A shortest transverse packet is

$$x = (-2, 6, 1).$$

It gives

$$\mathrm{d}_x(3) = -2, \qquad \mathrm{d}_x(125) = 450, \qquad \mathrm{d}_x(128) = 448.$$

Thus additivity holds. The Wronskian is

$$W_x(3, 125) = 3(450) - 125(-2) = 1600.$$

The radical is

$$R = 2 \cdot 3 \cdot 5 = 30,$$

and

$$\frac{3 \cdot 125 \cdot 128}{30} = 1600.$$

Hence the radical lower bound is attained exactly.

The minimality is elementary. At height at most five, $|x_3 + 75x_5| \leq 380 < 448$, so additivity forces $x_2 = 0$. Then $x_3 = -75x_5$ and the same height restriction force both coordinates to vanish. Thus no transverse packet has height below six, and the displayed packet proves $\Theta = 6$.

This example becomes a recurring calibration point. Every later refinement must explain why the sharp integer divisor is visible while the corresponding height bound remains nontrivial.

5 Valuation Words, Polyphase Refinement, and Local Lift Trees

5.1 Replacing one exponent by a valuation word

The coefficient $v_p(n)$ compresses a run of valuation shells into one scalar. For a prime power p^e, define the valuation word

$$\boxed{V_e(T) = 1 + T + \cdots + T^{e-1}.}$$

Set $V_0(T) = 0$ for an absent prime. It obeys the concatenation identity

$$V_{e+f}(T) = V_e(T) + T^e V_f(T).$$

The refined partial derivative

$$\partial_{p,T}(n) = \frac{n}{p} V_{v_p(n)}(T)$$

therefore satisfies the twisted product rule

$$\boxed{\partial_{p,T}(mn) = n\partial_{p,T}(m) + mT^{v_p(m)}\partial_{p,T}(n).}$$

At $T = 1$, the ordinary arithmetic partial derivative is recovered.

5.2 Finite polyphase rings

Fix an integer $N \geq 2$ and work in

$$A_N = \mathbb{Z}[T]/(T^N - 1).$$

The residue of $V_e(T)$ records the number of valuation shells in each congruence class modulo N. Multiplication by V_e is represented by a circulant matrix $C_{e,N}$.

Theorem 5.1 (Polyphase Smith normal form)**.** *If* $\gcd(e, N) = 1$, *then*

$$\boxed{\mathrm{SNF}(C_{e,N}) = \mathrm{diag}(1, \ldots, 1, e).}$$

Proof. Let

$$I_N = \ker\big(A_N \xrightarrow{T \mapsto 1} \mathbb{Z}\big)$$

be the augmentation ideal. It is a free abelian group of rank $N - 1$, and multiplication by V_e preserves it. After extension to $\mathbb{C}$, the eigenvalues on $I_N \otimes \mathbb{C}$ are

$$V_e(\zeta) = \frac{1 - \zeta^e}{1 - \zeta}, \qquad \zeta^N = 1, \quad \zeta \neq 1.$$

Because $\gcd(e, N) = 1$, the map $\zeta \mapsto \zeta^e$ permutes the nontrivial Nth roots. Therefore

$$\det\big(V_e \mid I_N\big) = \prod_{\substack{\zeta^N = 1 \\ \zeta \neq 1}} \frac{1 - \zeta^e}{1 - \zeta} = 1.$$

Thus multiplication by V_e is an automorphism of the integral lattice I_N. On the quotient

$$A_N / I_N \cong \mathbb{Z},$$

it acts as multiplication by $V_e(1) = e$. Applying the snake lemma to

$$0 \longrightarrow I_N \longrightarrow A_N \longrightarrow \mathbb{Z} \longrightarrow 0$$

shows that the full map is injective and has cokernel $\mathbb{Z}/e\mathbb{Z}$. Hence its Smith normal form is

$$\operatorname{diag}(1, \ldots, 1, e).$$

□

The refined block has $N - 1$ integrally invertible phase channels and one aggregate multiplicity channel. Every local denominator lies in the aggregate channel.

5.3 Dyadic and odd-adic primary lifts

Factor

$$e = 2^{\alpha} \prod_{\ell \text{ odd}} \ell^{\alpha_\ell}.$$

The cokernel of the valuation block is

$$\mathbb{Z}/e\mathbb{Z} \cong \mathbb{Z}/2^{\alpha}\mathbb{Z} \times \prod_{\ell \text{ odd}} \mathbb{Z}/\ell^{\alpha_\ell}\mathbb{Z}.$$

The primary factors specify the local lifting problems for the aggregate multiplicity channel. Its 2-primary component must be treated 2-adically, while the odd components admit the corresponding odd-adic lifts. The phase channels require no local denominator correction.

This Smith decomposition identifies where separate 2-primary bookkeeping first enters. The additional one-level shift in multiplicative principal-unit coordinates at the prime 2 comes from the local 2-adic logarithm and is established later. It is not a consequence of the Smith theorem alone.

5.4 Coarse and phase derivatives

Choose a polyphase order N coprime to every active valuation exponent. After the Smith decomposition, each refined prime packet has one coarse coordinate and integrally invertible phase coordinates in the common augmentation lattice. The additive equation separates accordingly.

The coarse equation recovers the original exponent-weighted row

$$\sum_{p|a} \frac{a v_p(a)}{p} t_p + \sum_{p|b} \frac{b v_p(b)}{p} t_p - \sum_{p|c} \frac{c v_p(c)}{p} t_p = 0.$$

The phase equation is exponent free:

$$\boxed{\sum_{p|a} \frac{a}{p} y_p + \sum_{p|b} \frac{b}{p} y_p - \sum_{p|c} \frac{c}{p} y_p = 0.}$$

Define the phase derivative

$$D_y(n) = n \sum_{p|n} \frac{y_p}{p}.$$

The same radical divisibility argument gives

$$\frac{abc}{R} \mid aD_y(b) - bD_y(a).$$

Let

$$\sigma(a,b) = \sum_{p|a} \frac{1}{p} + \sum_{p|b} \frac{1}{p}.$$

For primitive triples other than $(1,1,2)$, let $\Phi(a,b,c)$ be the shortest transverse phase packet. Then

$$\boxed{c \le R\,\sigma(a,b)\,\Phi(a,b,c).}$$

This can be sharper than the exponent-weighted inequality on individual triples, but finite evidence still shows positive power growth rather than a subpolynomial law.

5.5 First-order no-free-refinement principle

The shell-refined factor

$$\prod_{r=0}^{e-1} \left(1 + \frac{T_r}{p}\right)$$

has first-order expansion

$$1 + \frac{1}{p} \sum_r T_r + O(T^2).$$

Only the shell sum appears. Every internal phase in the kernel of shell synthesis is invisible to both first-order additivity and the first-order Wronskian.

Theorem 5.2 (First-order no-free-polyphase principle)**.** *Let*

$$S : (T_{p,r})_{p,r} \longmapsto \left(\sum_r T_{p,r}\right)_p$$

be shell synthesis. The first-order additive map and first-order Wronskian factor through S. Hence

$$\operatorname{Ker} S$$

is contained in the kernels of both first-order maps.

Polyphase splitting may lower displayed child coordinates. Exact synthesis multiplies by the corresponding reconstruction norm. At first order, refinement cannot create a new obstruction direction.

5.6 Second-order phase energy

At second order,

$$\prod_{r=0}^{e-1} \left(1 + \frac{T_r}{p}\right) = 1 + \frac{1}{p} \sum_r T_r + \frac{1}{p^2} \sum_{r<s} T_r T_s + O(T^3).$$

Let $\widehat{T}_j$ be the discrete Fourier transform of the shell coordinates. Parseval gives

$$\sum_{r<s} T_r T_s = \frac{1}{2} \left[\widehat{T}_0^2 - \frac{1}{e} \sum_j \widehat{T}_j \widehat{T}_{-j} \right].$$

For real coordinates,

$$\sum_{r<s} T_r T_s = \frac{1}{2}\left[\left(1-\frac{1}{e}\right)|\widehat{T}_0|^2 - \frac{1}{e}\sum_{j\neq 0}|\widehat{T}_j|^2\right].$$

Thus nonzero polyphase energy first becomes visible at second order. Later formal-smoothness results show that this curvature is not itself a lifting obstruction.

5.7 Exact local lift gauges

Let

$$r_u = \mathrm{rad}(u), \qquad P_u = \frac{u}{r_u}.$$

A shell packet $y = (y_{p,r})$ produces

$$D_y(u) = u\sum_{p\mid u}\frac{1}{p}\sum_{r=0}^{v_p(u)-1} y_{p,r} = P_u t_u,$$

where

$$t_u = \sum_{p\mid u}\frac{r_u}{p}\sum_r y_{p,r}.$$

Because

$$\gcd_{p\mid u}\frac{r_u}{p} = 1,$$

every integer t_u is attainable when $u > 1$.

The shell lift gauge is

$$\rho_u^{\mathrm{sh}}(t) = \min\left\{\|y\|_\infty : \sum_{p\mid u}\frac{r_u}{p}\sum_r y_{p,r} = t\right\}.$$

At the aggregate level,

$$P_a t_a + P_b t_b = P_c t_c,$$

and

$$W_y(a,b) = P_a P_b (r_a t_b - r_b t_a).$$

The radical divisor is attained sharply when

$$r_a t_b - r_b t_a = P_c.$$

When $a, b > 1$, all sharp aggregate solutions form a one-parameter family

$$t_a(n) = t_a^{(0)} + n r_a,$$
$$t_b(n) = t_b^{(0)} + n r_b,$$
$$t_c(n) = t_c^{(0)} + n r_c.$$

The unit-sector cases have no moving parameter. If $a = 1$, then

$$(t_a, t_b, t_c) = (0, P_c, P_b),$$

and if $b = 1$, then

$$(t_a, t_b, t_c) = (-P_c, 0, -P_a).$$

For obstruction multiple k, these fixed triples are multiplied by k. Thus the high-dimensional sharp search reduces either to one integer parameter plus three local lift gauges, or to one fixed aggregate triple in a unit-sector case.

6 The Exact Obstruction Complex and Formal Shell Geometry

6.1 A joint exponent-free two-row complex

The Smith obstruction is attached to the exponent-free phase, or shell-synthesis, rows rather than to the exponent-weighted derivative row. For every prime $p \mid abc$, define

$$\Lambda_p = \begin{cases} a/p, & p \mid a, \\ b/p, & p \mid b, \\ -c/p, & p \mid c, \end{cases} \qquad \mathsf{w}_p = \begin{cases} -ab/p, & p \mid a, \\ ab/p, & p \mid b, \\ 0, & p \mid c. \end{cases}$$

Consider

$$\mathsf{M}_{a,b,c} = \begin{pmatrix} \Lambda \\ \mathsf{w} \end{pmatrix} : \mathbb{Z}^S \longrightarrow \mathbb{Z}^2.$$

The first row is the exponent-free additive phase row. The second is its integer Wronskian row. Set

$$Q = \frac{abc}{R}.$$

Theorem 6.1 (Smith obstruction theorem)**.** *Let $a + b = c$ be primitive, with at least two of a, b, c greater than one. Then*

$$\boxed{\mathrm{SNF}(\mathsf{M}_{a,b,c}) = \mathrm{diag}(1, Q).}$$

Consequently,

$$\boxed{\mathrm{coker}(\mathsf{M}_{a,b,c})_{\mathrm{tors}} \cong \mathbb{Z}/Q\mathbb{Z}.}$$

The exceptional triple $(1, 1, 2)$ has only one nonempty prime sector, so the matrix has rank one and is excluded from the two-row statement.

Proof. The gcd of the entries of Λ is one. Indeed, every prime dividing one sector is absent from every other nonempty sector, and an entry from another sector is not divisible by it. Hence the first Smith factor is one.

Rows restricted to one sector are proportional, so same-sector 2×2 minors vanish. If p and q lie in two different nonempty sectors, the corresponding minor is

$$\pm\frac{abc}{pq} = \pm Q\frac{R}{pq}.$$

Every nonzero minor is therefore divisible by Q. Conversely, for each prime $\ell \mid R$, choose $p = \ell$ and choose q in a different nonempty sector. Since R is squarefree, $R/(pq)$ is not divisible by ℓ. Thus the gcd of the reduced minors $R/(pq)$ is one. The gcd of all nonzero minors is exactly Q. The determinantal-divisor description of Smith normal form now gives the result. □

6.2 The exact lattice extension

Throughout the remaining height and exact-sequence constructions, exclude $(1, 1, 2)$. Define

$$L = \ker(\Lambda) \cap \mathbb{Z}^S$$

and

$$L_0 = L \cap \ker(\mathsf{w}).$$

Because Λ is primitive, choose $t \in \mathbb{Z}^S$ with $\Lambda(t) = 1$. Then

$$\mathbb{Z}^S = L \oplus \mathbb{Z}t.$$

The image of $\mathsf{M}_{a,b,c}$ is generated by $(1, \mathsf{w}(t))$ together with $(0, \mathsf{w}(L))$. Its index in $\mathbb{Z}^2$ is therefore the index of $\mathsf{w}(L)$ in $\mathbb{Z}$. Theorem 6.1 shows that this index is Q, so

$$\mathsf{w}(L) = Q\mathbb{Z}.$$

Hence

$$\omega(x) = \frac{\mathsf{w} \cdot x}{Q}$$

is an integer-valued surjection and gives the exact sequence

$$\boxed{0 \longrightarrow L_0 \longrightarrow L \xrightarrow{\omega} \mathbb{Z} \longrightarrow 0.}$$

The extension splits algebraically. The difficult quantity is the norm of a section, not its existence.

For $k \in \mathbb{Z}$, define

$$H(k) = \min \{ \|x\|_\infty : x \in L, \omega(x) = k \}.$$

Three heights must be distinguished:

$$H(1), \qquad \Phi = \min_{k \neq 0} H(k), \qquad \overline{H} = \inf_{k \geq 1} \frac{H(k)}{k}.$$

The shortest nonzero detector may live in a nonprimitive obstruction coset.

6.3 The stable real height

Relax the integral packet to a real vector in the additive hyperplane. The exact continuous width can be computed from the three sector weights. Let

$$\sigma_u = \sum_{p \mid u} \frac{1}{p}$$

and

$$\Delta(a, b, c) = \min \left\{ \sigma_a + \sigma_b, \frac{c}{b}(\sigma_a + \sigma_c), \frac{c}{a}(\sigma_b + \sigma_c) \right\}.$$

Then

$$\boxed{\overline{H} = \frac{c}{R\Delta(a, b, c)}.}$$

The discrete master inequality becomes

$$c \leq R\Delta\Phi.$$

Since

$$\Phi = \overline{H}\, k_* \Gamma$$

for the optimal obstruction multiple k_* and an integrality factor $\Gamma \geq 1$,

$$\boxed{\Delta\Phi = \frac{c}{R} k_* \Gamma.}$$

This is the no-escape identity. Optimizing the same first-order invariant cannot lower the continuous floor c/R.

6.4 The full shell deformation

Introduce one variable for every valuation shell and define

$$\mathscr{J}_n(\mathbf{T}) = \prod_{p^e \| n} \prod_{r=0}^{e-1} (p + T_{p,r}).$$

At $\mathbf{T} = 0$,

$$\mathscr{J}_n(0) = n.$$

For $a + b = c$, set

$$F_{a,b,c}(\mathbf{T}) = \mathscr{J}_a(\mathbf{T}) + \mathscr{J}_b(\mathbf{T}) - \mathscr{J}_c(\mathbf{T}).$$

The shell gradient has entries

$$\frac{a}{p}, \qquad \frac{b}{p}, \qquad -\frac{c}{p}$$

repeated according to valuation depth. The gcd of all entries is one.

Theorem 6.2 (Formal affine-space theorem)**.** *Let d be the number of shell variables. After a unimodular integral change of shell coordinates, the $(\mathbf{T})$-adic completion of*

$$\mathbb{Z}[\mathbf{T}]/(F_{a,b,c})$$

at the zero section is isomorphic to a formal power-series ring $\mathbb{Z}[[U_2, \ldots, U_d]]$.

Proof. A primitive gradient row can be completed to a unimodular matrix. In the new coordinates, one variable has linear coefficient one, which is a unit in $\mathbb{Z}$. The formal implicit-function recursion therefore solves uniquely for that variable at every degree with integral coefficients, giving the stated isomorphism of completed rings. □

Corollary 6.3 (Integral jet extension)**.** *Every integral first-order tangent vector to $F_{a,b,c} = 0$ extends to an integral formal curve through all orders.*

The consequence is decisive. Higher shell curvature exists, but there is no formal lifting obstruction to exploit. The missing arithmetic must be a global size or conductor estimate.

6.5 Aggregate products and the integral boundary

Over $\mathbb{Q}$, on the locus where all shell factors are nonzero, product-preserving directions within a sector form a torus. For $a, b > 1$, the normalized products

$$A = \mathscr{J}_a/a, \qquad B = \mathscr{J}_b/b, \qquad C = \mathscr{J}_c/c$$

satisfy $aA + bB = cC$, leaving a two-dimensional aggregate plane and internal product gauges. A unit input sector has no shell variables and leaves a one-dimensional aggregate line.

This rational description introduces denominators. It is not an assertion that the integral completion is a torus. The preceding primitive-gradient argument is the integral statement: local formal extension exists through all orders. Arithmetic size remains the separate global obligation.

7 Exact Moving-Coset Search and the First-Order Barrier

7.1 Obstruction cosets

Exclude $(1,1,2)$ throughout this section. Choose a section $x_0 \in L$ with $\omega(x_0) = 1$. Then

$$L = \bigsqcup_{k\in\mathbb{Z}} (kx_0 + L_0).$$

For fixed k, the problem is a closest-vector problem to the moving target $-kx_0$ in the lattice L_0. The three-sector compression makes this search much more explicit.

7.2 Aggregate affine line

Write

$$u = P_u r_u, \qquad r_u = \mathrm{rad}(u), \qquad P_u = u/r_u.$$

The sharp aggregate conditions are

$$P_a t_a + P_b t_b = P_c t_c$$

and

$$r_a t_b - r_b t_a = kP_c.$$

Since $\gcd(r_a, r_b) = 1$, choose s, t with

$$sr_a + tr_b = 1.$$

One solution is

$$t_a^{(0)} = -tkP_c, \qquad t_b^{(0)} = skP_c.$$

Every aggregate solution is

$$\boxed{\begin{aligned} t_a(k,n) &= t_a^{(0)} + nr_a, \\ t_b(k,n) &= t_b^{(0)} + nr_b, \qquad n \in \mathbb{Z}. \\ t_c(k,n) &= t_c^{(0)} + nr_c, \end{aligned}}$$

When $a, b > 1$, the global search in one obstruction coset is therefore one-dimensional. If $a = 1$, the empty a-sector forces $t_a = 0$ and fixes the coarse parameter, giving

$$(t_a, t_b, t_c) = (0, kP_c, kP_b).$$

If $b = 1$, the corresponding fixed triple is

$$(t_a, t_b, t_c) = (-kP_c, 0, -kP_a).$$

Thus a unit-sector coset has one aggregate candidate rather than a moving affine line.

7.3 Local CRT normal form

Fix one sector with primes

$$p_1, \dots, p_m$$

and radical

$$r = \prod_i p_i.$$

For a nonempty sector, the local equation is

$$\sum_i \frac{r}{p_i} x_i = t.$$

An empty unit sector has no prime coordinates and is feasible exactly when $t = 0$. Modulo p_i, the coordinate x_i is forced:

$$x_i \equiv t \left(\frac{r}{p_i} \right)^{-1} \pmod{p_i}.$$

Choose the centered residue ξ_i and write

$$x_i = \xi_i + p_i z_i.$$

Substitution gives one quotient equation

$$\sum_i z_i = q(t).$$

For a box radius H, each z_i lies in the interval

$$\ell_i(H) = \left\lceil \frac{-H - \xi_i}{p_i} \right\rceil \le z_i \le \left\lfloor \frac{H - \xi_i}{p_i} \right\rfloor = u_i(H).$$

The local lift exists exactly when

$$\boxed{\ell_i(H) \le u_i(H) \text{ for every } i, \qquad \sum_i \ell_i(H) \le q(t) \le \sum_i u_i(H).}$$

Every coordinate interval must be nonempty. Once that condition holds, the sum of the integer intervals contains every integer between the sums of their endpoints. This proves the criterion. Summing endpoints alone is insufficient when one interval is empty.

7.4 Moving-interval theorem

Using the complete local criterion, including the individual nonemptiness guards, for fixed k and H the three sectors produce three integer intervals for the coarse parameter n. A packet of height at most H exists if and only if those intervals share an integer.

Theorem 7.1 (Moving-interval theorem)**.** *For fixed obstruction multiple k and radius H,*

$$H(k) \le H$$

if and only if

$$I_a(H,k) \cap I_b(H,k) \cap I_c(H,k) \cap \mathbb{Z} \neq \varnothing.$$

The first feasible H gives the exact coset height. The preceding radius is an explicit no-smaller-box certificate.

7.5 Finite global stopping

The continuous lower bound gives

$$H(k) \geq k\overline{H}.$$

If the best exact height found so far is B, then no coset with

$$k\overline{H} \geq B$$

can improve it. Therefore only finitely many obstruction multiples must be searched:

$$\boxed{1 \leq k \leq \left\lfloor \frac{B-1}{\overline{H}} \right\rfloor.}$$

Every unsearched coset is excluded by a proved lower bound.

7.6 Phase and coarse separation

The local target has the form

$$t_u(k,n) = \alpha_u k + r_u n.$$

For a local prime $p \mid r_u$,

$$t_u(k,n) \equiv \alpha_u k \pmod{p}.$$

Thus the obstruction multiple k determines the finite phase packet. The coarse parameter n changes only the quotient load. The exact search is a moving polyphase mechanism rather than a generic mixed-integer program.

7.7 Arithmetic correction and arbitrary precision

Some large high-quality triples exceed signed 64-bit range, and intermediate quantities such as

$$ab, \qquad \frac{abc}{R}, \qquad w \cdot x$$

can overflow even when the input terms individually fit. Every public arithmetic entry point must therefore promote integer-like inputs to arbitrary-precision integers before multiplication. Floating-point values are display-only. All logical comparisons and stopping bounds are exact rational or integer computations.

Several reported ratios are mathematically at least one:

$$\frac{\Phi}{\overline{H}} \geq 1, \qquad \frac{\Phi}{k_*\overline{H}} \geq 1, \qquad \frac{H(1)}{\Phi} \geq 1.$$

Values above one are not probability errors.

7.8 What the exact search proved

The moving search found nonprimitive optima on many finite examples. It also found increasingly deep phase resonances. This showed that a fixed cutoff in k is unjustified. More importantly, the no-escape identity proved that even a complete exact search of the first-order invariant cannot imply *abc* independently. The search became a diagnostic and classification engine rather than the final proof mechanism.

8 Prime-Shell Signals and the Frey Nonlinear Transform

8.1 Prime support as bandwidth and valuation as persistence

For

$$n = \prod_p p^{e_p},$$

define the prime-shell signal

$$\Psi_n(p,r) = \sqrt{\log p}\,\mathbf{1}_{0\le r<e_p}.$$

Its total energy is

$$\|\Psi_n\|_2^2 = \sum_p e_p \log p = \boxed{\log n}.$$

Let P_{act} retain only the first shell in each active prime channel. Then

$$\|P_{\mathrm{act}}\Psi_n\|_2^2 = \sum_{p\mid n} \log p = \boxed{\log \mathrm{rad}(n)}.$$

Thus the radical is activated bandwidth, while the difference

$$\log n - \log \mathrm{rad}(n)$$

is repeated persistence depth.

For a primitive triple, the direct-sum signal

$$\Psi_{a,b,c} = \Psi_a \oplus \Psi_b \oplus \Psi_c$$

has disjoint prime support in the three sectors. The standard quality becomes the exact energy ratio

$$\boxed{q(a,b,c) = \frac{\log c}{\log \mathrm{rad}(abc)} = \frac{\|P_c\Psi_{a,b,c}\|_2^2}{\|P_{\mathrm{act}}\Psi_{a,b,c}\|_2^2}.}$$

High quality means large output persistence relative to activated global bandwidth.

8.2 Unitary transforms cannot create a radical saving

Apply a unitary Fourier, wavelet, or polyphase transform in the shell variable. Parseval preserves total shell energy. Therefore the ratio between total depth energy and activated support energy is unchanged by a unitary shell analysis followed by exact synthesis.

Proposition 8.1 (Unitary no-gain principle)**.** *Let U be unitary on the shell coordinate of every active prime channel. Then*

$$\|U\Psi_n\|_2 = \|\Psi_n\|_2 .$$

Transport the activation projector to $UP_{\mathrm{act}}U^{-1}$. Its projected energy then also agrees exactly with the original activation energy. The same holds after exact synthesis. An arbitrary unitary transform need not preserve the energy under a fixed, untransported first-shell projection.

Signal theory is still useful because it reveals which nonunitary or nonlinear operations would be required. It also separates local persistence from global support before introducing a nonlinear arithmetic carrier.

8.3 The three-spike p-adic coherence signal

Represent the additive triple by

$$\nu_{a,b,c} = \delta_0 + \delta_a + \delta_c.$$

The pairwise differences are a, b, and c. At a prime dividing exactly one of these integers, exactly one pair of spikes is p-adically close. The depth of that edge is the corresponding valuation.

The triple therefore determines a prime-indexed coherence triangle. The radical counts the active triangles. The valuations count how long a pair remains coupled through the p-adic refinement tree. This picture anticipates the later cross-prime logarithmic network.

8.4 The Frey curve

Attach the marked Frey curve

$$\boxed{E_{a,b,c} : y^2 = x(x-a)(x+b).}$$

The roots 0, a, and $-b$ retain the three differences a, b, and c. The basic invariants are

$$\Delta_E = 16a^2b^2c^2$$

and

$$c_4(E) = 16(a^2 + ab + b^2).$$

At every odd prime $p \mid abc$, the curve has multiplicative reduction. The conductor records that the channel is active, while the minimal discriminant records the full depth.

In signal language, the Frey curve is a nonlinear encoder that converts:

$$\begin{aligned} \text{prime support} &\longrightarrow \text{conductor}, \\ \text{valuation depth} &\longrightarrow \text{minimal discriminant depth}, \\ \text{additive relation} &\longrightarrow \text{elliptic geometry}. \end{aligned}$$

This is precisely the kind of nonlinear transform that a unitary shell system could not provide.

8.5 The automorphic eigenchannel

Modularity associates a weight-two newform f_E to the rational isogeny class of E. Its Hecke coefficients

$$a_\ell(E)$$

form a global frequency signal. The conductor is the automorphic bandwidth. The newform, however, remembers the rational isogeny class rather than the marked integral representative.

This information loss is essential. Different integral Frey models can lie in the same rational isogeny or twist channel while having different minimal discriminant depth, Néron differential, and modular synthesis cost. A proof cannot use the rational eigen-signal alone.

8.6 Dyadic Landen descent

Let $u > v > 0$ be coprime odd integers. Consider

$$T_0 = (v^2, u^2 - v^2, u^2)$$

and

$$T_1 = \left(uv, \left(\frac{u-v}{2}\right)^2, \left(\frac{u+v}{2}\right)^2 \right).$$

Both are primitive triples. Their radicals are equal:

$$\mathrm{rad}(T_0) = \mathrm{rad}(T_1) = \mathrm{rad}\big(uv(u-v)(u+v)\big).$$

The second output is strictly smaller:

$$\left(\frac{u+v}{2}\right)^2 < u^2.$$

Hence the second quality is smaller.

The Legendre parameter of the parent is

$$\lambda = t^2, \qquad t = v/u.$$

The degree-two Landen transformation gives

$$\boxed{\lambda \longmapsto \frac{4\sqrt{\lambda}}{(1+\sqrt{\lambda})^2}.}$$

This is exactly the child ratio a_1/c_1. The transformation preserves the automorphic channel up to the corresponding two-isogeny while lowering the integral height. It proves that the integral representative inside an isogeny class is relevant to *abc*.

8.7 Exact automorphic analysis–synthesis identity

Let

$$\phi_E : X_0(N_E) \longrightarrow E$$

be a nonconstant modular parametrization, and let

$$\phi_E^* \omega_E = 2\pi i \, c_\phi f_E(z)\, dz.$$

Here c_ϕ is the differential multiplier of this chosen parametrization and differential. The marked Frey curve need not be the optimal representative of its isogeny class. Define the differential area

$$A(E) = \frac{i}{2} \int_{E(\mathbf{C})} \omega_E \wedge \overline{\omega_E}.$$

Pulling the area form back gives

$$\boxed{\deg(\phi_E) A(E) = 4\pi^2 c_\phi^2 \, \|f_E\|_{\mathrm{Pet}}^2 .}$$

For the Frey model, scaling $x = cX$, $y = c^{3/2}Y$ gives an area factor proportional to $1/c$. Thus the height c can be written as a shape-corrected synthesis gain involving modular degree, Manin normalization, and Petersson energy.

The identity does not prove a conductor-scale upper bound. It identifies the exact quantities that an integral automorphic frame theorem would have to control.

9 Divisor-Polyphase Persistence and the Hecke Filter Bank

9.1 Exact-order divisor packets

Let

$$n = \prod_p p^{e_p}.$$

For every $d \geq 1$, define

$$\boxed{B_d(n) = \prod_{\substack{p \mid n \\ d \mid e_p}} p.}$$

Since

$$\sum_{d \mid e} \varphi(d) = e,$$

we have the perfect reconstruction identity

$$\boxed{n = \prod_{d \geq 1} B_d(n)^{\varphi(d)}.}$$

The first packet is

$$B_1(n) = \mathrm{rad}(n).$$

The powerful part is

$$\boxed{\frac{n}{\mathrm{rad}(n)} = \prod_{d \geq 2} B_d(n)^{\varphi(d)}.}$$

This is a multiplicative polyphase decomposition indexed by exact character orders.

9.2 Quality as input-output persistence imbalance

For a primitive triple,

$$q - 1 = \frac{\log(c/R_c) - \log(R_a R_b)}{\log R}.$$

Using the divisor packets,

$$\boxed{q - 1 = \frac{\sum_{d \geq 2} \varphi(d) \log B_d(c) - \log(R_a R_b)}{\log R}.}$$

A high-quality triple has more repeated-depth persistence in the output than the input radicals supply as activated bandwidth.

Strong *abc* is equivalent to

$$\prod_{d \geq 2} B_d(c)^{\varphi(d)} \leq C_\varepsilon R_a R_b R^\varepsilon.$$

The decomposition makes the target transparent, but not easier by itself. A global channel must explain why the exact-order packets cannot all be large simultaneously.

9.3 Local Tate-depth interpretation

At an odd prime $p^e \parallel c$, the Frey discriminant has local depth $2e$. The divisors of e organize the levels at which prime-power coefficient systems can become insensitive to the local extension. The exact-order divisor packet therefore has a local representation-theoretic interpretation, although local disappearance does not automatically produce a global low-level eigenform.

9.4 Atkin–Lehner Walsh packets

Let

$$W_1, \dots, W_r$$

be commuting Atkin–Lehner involutions. For a sign vector

$$\varepsilon \in \{\pm 1\}^r,$$

define

$$A_\varepsilon = \prod_{j=1}^{r} (I + \varepsilon_j W_j).$$

Then

$$\boxed{A_\varepsilon^2 = 2^r A_\varepsilon,}$$

$$A_\varepsilon A_{\varepsilon'} = 0 \quad (\varepsilon \neq \varepsilon'),$$

and

$$\boxed{\sum_\varepsilon A_\varepsilon = 2^r I.}$$

The unnormalized analysis packets are integral. The common synthesis denominator is 2^r, which is subpolynomial in a squarefree conductor. Local sign separation is therefore not the large persistence carrier.

The automorphic interpretation uses the standard Atkin–Lehner and modular-newform framework. The integral separation problem is related to congruence modules and modular degrees [15, 16, 17, 18, 19].

9.5 The same-sign separation problem

After fixing the local signs, several eigenforms can remain congruent inside the same packet. Let Λ be an integral Hecke lattice and let e_f be the rational projector onto the Frey eigenline. Define

$$\eta_f = \min\{m \geq 1 : m e_f(\Lambda) \subseteq \Lambda\}.$$

This is the integral eigenchannel separation denominator. It is an arithmetic analogue of leakage in a filter bank. The difficulty is not rational spectral isolation. It is isolation inside the integral lattice.

10 Integral Projectors, Cotangent Multiplicity, and Wiles Defect

10.1 Projector denominator as a congruence exponent

Let $V = U \oplus W$ be a rational decomposition and let $\Lambda \subset V$ be an integral lattice. Let e_U be the rational projector onto U. Define

$$\Lambda_U = \Lambda \cap U, \qquad \Lambda_W = \Lambda \cap W,$$

and

$$C_U = \Lambda/(\Lambda_U + \Lambda_W).$$

Theorem 10.1 (Integral projector theorem)**.** *The least positive integer η_U satisfying*

$$\eta_U e_U(\Lambda) \subseteq \Lambda$$

is the exponent of the finite quotient C_U.

Proof. For $x = u + w \in \Lambda$, the condition $me_U(x) = mu \in \Lambda_U$ is equivalent to

$$mx = mu + mw \in \Lambda_U + \Lambda_W.$$

Thus m clears every projector denominator exactly when it annihilates every class in C_U. □

For a primitive rank-one vector $v \in \mathbb{Z}^n$, the orthogonal projector denominator is

$$\langle v, v \rangle,$$

and the congruence quotient is cyclic of that order.

10.2 Why one cyclic quotient is not enough for split persistence

The split persistence module

$$\mathcal{P}_c \cong \bigoplus_{p^e \| c} (\mathbb{Z}/p\mathbb{Z})^{e-1}$$

has large p-rank. The p-torsion of a cyclic target has rank at most one. Therefore every map from $\mathcal{P}_c$ to a cyclic group has a forced kernel of order at least

$$\prod_{p^e \| c} p^{\max(e-2,0)}.$$

This is a true obstruction to the split source. It motivates the Hecke cotangent module

$$\mathfrak{p}/\mathfrak{p}^2,$$

which can retain multiple independent congruence directions before cyclic compression.

10.3 A toy Hecke order

Consider

$$\mathbb{T}(\mathbf{m}) = \{(x_0, x_1, \ldots, x_r) \in \mathbb{Z}^{r+1} : x_i \equiv x_0 \pmod{m_i}\}.$$

For the target coordinate x_0, the cotangent module is

$$\boxed{\mathfrak{p}/\mathfrak{p}^2 \cong \bigoplus_{i=1}^{r} \mathbb{Z}/m_i\mathbb{Z}.}$$

The cyclic congruence quotient is

$$\boxed{\mathbb{Z}/\operatorname{lcm}(m_1,\dots,m_r)\mathbb{Z}.}$$

The quotient of cotangent order by congruence order is

$$\frac{\prod_i m_i}{\operatorname{lcm}(m_i)}.$$

This is a toy Wiles-defect factor. It measures the multiplicity discarded by cyclic compression.

The comparison between cotangent and congruence data, including non-complete-intersection defects and local–global formulas, is developed in the Wiles-defect literature [20, 21].

10.4 Complete-intersection control case

Let $F \in \mathbb{Z}[X]$ be monic and squarefree over $\mathbb{Q}$, with an integer simple root a_0. Put

$$\mathbb{T} = \mathbb{Z}[X]/(F)$$

and let $\lambda : \mathbb{T} \to \mathbb{Z}$ evaluate at a_0. The ring is finite free over $\mathbb{Z}$ and reduced over $\mathbb{Q}$. Then the congruence quotient $\mathbb{Z}/\lambda(\operatorname{Ann} I)$ and cotangent module both have order

$$|F'(a_0)|.$$

To see this directly, write $F = (X - a_0)G$. The kernel I is generated by $X - a_0$. Modulo I^2, the relation is $G(a_0)(X - a_0) = 0$, so $I/I^2 \cong \mathbb{Z}/F'(a_0)\mathbb{Z}$. The annihilator of I is generated by G modulo F, whose evaluation is $G(a_0) = F'(a_0)$. Thus the congruence ideal is $F'(a_0)\mathbb{Z}$ and the defect is one. Extra multiplicity is therefore tied to non-complete-intersection structure rather than being a normalization artifact.

10.5 Two prime indices

The persistence is indexed by conductor primes $q \mid abc$. Congruence depth is indexed by coefficient primes ℓ. The natural local signal is a matrix

$$D_f(\ell, q).$$

Let $E/\mathbb{Q}_q$ have split multiplicative reduction with Tate parameter q_E, and let $\ell \neq q$. Scalarity of the full local Galois representation on $E[\ell^n]$ forces

$$n \leq \min\{v_\ell(q-1), v_\ell(v_q(q_E))\}.$$

In a Tate basis, the cyclotomic graded diagonal and the inertia Kummer extension respectively force $\ell^n \mid (q-1)$ and $\ell^n \mid v_q(q_E)$ [2]. This is a necessary condition, rather than a converse or a condition on Frobenius alone. These off-diagonal channels carry tame congruence information. They do not naturally reproduce the full base-q factor q^{e-1}. This mismatch motivates the later filtered contact correction and the diagonal coefficient-prime channel.

11 Filtered Cusp Contact and Witt Carry Structure

11.1 The three-cusp section

View a primitive triple as the integral section

$$\sigma_{a,b,c} : \operatorname{Spec} \mathbb{Z} \longrightarrow \mathbb{P}^1_{\mathbb{Z}}, \qquad \sigma = [a : c].$$

The pullbacks of the boundary divisors 0, 1, and ∞ are

$$\operatorname{div}(a), \qquad \operatorname{div}(b), \qquad \operatorname{div}(c).$$

For the output cusp, let

$$C = \operatorname{div}(c), \qquad C_{\mathrm{red}} = \operatorname{div}(R_c).$$

Place the two-term contact complex in degrees $-1, 0$ and define

$$\boxed{\mathbf{J}_c = [\mathcal{O}(-C) \longrightarrow \mathcal{O}(-C_{\mathrm{red}})].}$$

On global sections,

$$\mathbf{J}_c = [c\mathbb{Z} \hookrightarrow R_c\mathbb{Z}] \simeq [\mathbb{Z} \xrightarrow{c/R_c} \mathbb{Z}].$$

Theorem 11.1 (Filtered contact quotient)**.**

$$\boxed{H^0(\mathbf{J}_c) = R_c\mathbb{Z}/c\mathbb{Z} \cong \mathbb{Z}/(c/R_c)\mathbb{Z}.}$$

The output persistence is cyclic before grading.

11.2 Associated graded and carry extensions

At $p^e \parallel c$,

$$\mathcal{J}_{p,e} = p\mathbb{Z}_p/p^e\mathbb{Z}_p \cong \mathbb{Z}/p^{e-1}\mathbb{Z}.$$

Its filtration

$$p^j\mathbb{Z}_p/p^e\mathbb{Z}_p$$

has successive quotients $\mathbb{F}_p$. Thus

$$\operatorname{gr} \mathcal{J}_{p,e} \cong (\mathbb{F}_p)^{e-1}.$$

The exact sequences

$$0 \longrightarrow \mathbb{Z}/p^{j-1}\mathbb{Z} \xrightarrow{\times p} \mathbb{Z}/p^j\mathbb{Z} \longrightarrow \mathbb{F}_p \longrightarrow 0$$

are nonsplit for $j \geq 2$. The $j = 1$ sequence has zero left term and is split. These carry extensions are precisely what the split persistence packet forgot.

Corollary 11.2 (Correction of the cyclic-rank barrier)**.** *The p-rank obstruction applies to the associated graded source. It does not obstruct a filtered map from $\mathbb{Z}/p^{e-1}\mathbb{Z}$ to a cyclic target of matching depth.*

11.3 Exact cyclic transducer

Let m, η be positive integers and put

$$C(m) = [\mathbb{Z} \xrightarrow{m} \mathbb{Z}]$$

and

$$C(\eta) = [\mathbb{Z} \xrightarrow{\eta} \mathbb{Z}].$$

Set $g = \gcd(m, \eta)$. Multiplication by m/g in degree -1 and by η/g in degree 0 defines a chain map. On cohomology,

$$\boxed{|\ker| = m/g, \qquad |\operatorname{im}| = g, \qquad |\operatorname{coker}| = \eta/g.}$$

The maximal transferable packet is the common prime-power depth.

For the output source $m = c/R_c$, the corrected algebraic defect against a cyclic target of order η_f is

$$\boxed{\delta_{\mathrm{alg}}(c, f) = \frac{c/R_c}{\gcd(c/R_c, \eta_f)}.}$$

Prime by prime,

$$\delta_{\mathrm{alg}}(c, f) = \prod_{p^e \| c} p^{\max(e-1-v_p(\eta_f), 0)}.$$

The problem is missing depth, not missing rank.

11.4 Witt realization

For $e \geq 2$, the local packet has a natural additive identification

$$\boxed{\mathcal{J}_{p,e} \cong W_{e-1}(\mathbb{F}_p).}$$

Division by p identifies the source with $\mathbb{Z}_p/p^{e-1}\mathbb{Z}_p$, and the standard additive identification $W_n(\mathbb{F}_p) \cong \mathbb{Z}/p^n\mathbb{Z}$ gives the result. For $e = 1$, take $W_0 = 0$. The Verschiebung filtration matches multiplication by successive powers of p. This is the typed local object underlying the carry chain.

At odd p, the p-adic logarithm identifies

$$(1 + p\mathbb{Z}_p)/(1 + p^e\mathbb{Z}_p) \cong p\mathbb{Z}_p/p^e\mathbb{Z}_p.$$

At 2, the additive contact packet remains uniform, while the multiplicative coordinate begins on $1 + 4\mathbb{Z}_2$ and therefore carries the familiar weighted dyadic shift.

11.5 The filtered local-global target

The intended transducer is a filtered chain

$$\mathbf{J}_c \longrightarrow \bigoplus_{p|c} \mathbf{L}_p^{\mathrm{st}} \longrightarrow \mathbf{L}_{R^{\mathrm{glob}}} \xrightarrow{R=\mathbb{T}} \mathbf{L}_{\mathbb{T}_f} \longrightarrow \mathbf{C}_f.$$

It must preserve the Bockstein carry maps. The local contact packet exists canonically. The unproved step is its transport through global deformation and Hecke complexes with small kernel and controlled archimedean size.

Theorem 11.3 (Conditional filtered-transfer criterion)**.** *Suppose every primitive positive triple admits a canonical map*

$$H^0(\Theta_f) : \mathbb{Z}/(c/R_c)\mathbb{Z} \longrightarrow H^0(\mathbf{C}_f)$$

satisfies, for every $\varepsilon > 0$, with constants independent of the triple,

$$|\ker H^0(\Theta_f)| \leq C_\varepsilon R^\varepsilon$$

and

$$|H^0(\mathbf{C}_f)| \leq C'_\varepsilon R_a R_b R^\varepsilon.$$

Then the strong abc conjecture follows.

Proof. The image has order

$$\frac{c/R_c}{|\ker|}$$

and is bounded by the target order. Therefore

$$\frac{c/R_c}{C_\varepsilon R^\varepsilon} \leq C'_\varepsilon R_a R_b R^\varepsilon.$$

Multiplying by R_c and replacing ε by $\varepsilon/2$ gives the strong bound. □

12 Kummer Globalization and the Fermat Character Packet

12.1 The contact modulus

Let

$$c = \prod_{i=1}^{r} p_i^{e_i}, \qquad R_c = \prod_i p_i, \qquad M = \frac{c}{R_c} = \prod_i p_i^{e_i - 1}.$$

The filtered output contact packet is the cyclic group

$$\mathbb{Z}/M\mathbb{Z}.$$

The first global question is whether this packet can be realized by one arithmetic class without losing most of its order.

12.2 The global Kummer map

Kummer theory gives

$$H^1(\mathbb{Q}, \mu_M) \cong \mathbb{Q}^\times/(\mathbb{Q}^\times)^M.$$

Define

$$\boxed{\kappa_c : \mathbb{Z}/M\mathbb{Z} \longrightarrow H^1(\mathbb{Q}, \mu_M), \qquad k \longmapsto [c^k].}$$

The valuation vector of c is

$$(e_1, \ldots, e_r) \in (\mathbb{Z}/M\mathbb{Z})^r.$$

Theorem 12.1 (Exact Kummer kernel)**.** *Let*

$$g_c = \gcd(M, e_1, \ldots, e_r).$$

Then

$$\boxed{|\ker \kappa_c| = g_c, \qquad |\operatorname{im} \kappa_c| = M/g_c.}$$

Proof. The class $[c]^k$ is trivial exactly when every prime valuation ke_i is divisible by M. The least positive such k is

$$\operatorname{lcm}_i \frac{M}{\gcd(M, e_i)} = \frac{M}{\gcd(M, e_1, \dots, e_r)}.$$

This is the order of the image. The kernel size follows. □

Because

$$g_c \le \min_i e_i \le \log_2 c,$$

we obtain the universal near-injectivity estimate

$$\boxed{|\ker \kappa_c| \le \log_2 c.}$$

This closes the global-occupancy problem for the contact packet up to logarithmic loss.

12.3 Balanced Kummer class

The class of c does not remember the additive relation. A better coordinate is

$$\beta = -\frac{b}{a} = 1 - \frac{c}{a}.$$

Let

$$v_q = v_q(b) - v_q(a)$$

for input primes $q \mid ab$. The order of $[\beta]$ in

$$\mathbb{Q}^\times / (\mathbb{Q}^\times)^M$$

is determined by the signed valuation vector, with one possible dyadic sign correction when M is even. If

$$g_{\mathrm{val}} = \gcd(M, \{v_q\}), \qquad m_0 = M/g_{\mathrm{val}},$$

then the order is m_0 unless M is even and m_0 is odd, in which case it is $2m_0$.

The balanced class retains the input-output relation and usually has full contact order. Its kernel remains logarithmically bounded.

12.4 The Kummer–Fermat lift

Choose algebraic numbers u and v satisfying

$$u^M = \frac{a}{c}, \qquad v^M = \frac{b}{c}.$$

Then

$$\boxed{u^M + v^M = 1.}$$

Thus

$$P_M = [u : v : 1]$$

lies on the Fermat curve

$$F_M : X^M + Y^M = Z^M.$$

The construction is canonical up to the action of μ_M^2.

Theorem 12.2 (Exact height compression)**.**

$$\boxed{h_{\mathcal{O}(1)}(P_M) = \frac{\log c}{M}.}$$

Proof. The coordinatewise Mth-power map sends P_M to $[a : b : c]$. Absolute logarithmic projective height multiplies by M under this map. Since the primitive positive triple has height $\log c$, the result follows. □

The contact modulus divides the displayed coordinate height by the full powerful part.

12.5 Fermat character polyphase

The group μ_M^2 acts by

$$(\xi, \eta) \cdot (X, Y) = (\xi X, \eta Y).$$

Let σ and τ denote the two coordinate rotations. For

$$(r, s) \in (\mathbb{Z}/M\mathbb{Z})^2,$$

define

$$P_{r,s} = \frac{1}{M^2} \sum_{i,j=0}^{M-1} \zeta_M^{-ri-sj} \sigma^i \tau^j.$$

Character orthogonality gives

$$P_{r,s} P_{r',s'} = \delta_{r,r'} \delta_{s,s'} P_{r,s}$$

and

$$\boxed{\sum_{r,s} P_{r,s} = I.}$$

This is an exact two-dimensional character filter bank.

12.6 Line-bundle cost of the full Fermat model

For $M \geq 4$, adjunction gives

$$K_{F_M} \cong \mathcal{O}_{F_M}(M - 3).$$

For compatible Weil-height normalizations,

$$h_{K_{F_M}}(P_M) = (M - 3) h_{\mathcal{O}(1)}(P_M) + O(1) = \frac{M - 3}{M} \log c + O(1).$$

If the metric on K_{F_M} is defined by the induced tensor-power metric on $\mathcal{O}(1)^{\otimes(M-3)}$, the displayed equality holds without the $O(1)$ term. Thus the full plane-curve polarization reintroduces a factor of order M into the height accounting.

Proposition 12.3 (Line-bundle cost of the full plane model)**.** *The Kummer lift reduces the $\mathcal{O}(1)$ coordinate height by M, while the canonical line bundle of the complete Fermat curve has degree factor $M - 3$. This rules out a height saving obtained merely by retaining the full plane-curve carrier with its induced polarization. It does not rule out a lower-complexity quotient equipped with a different arithmetic metric.*

A successful route must therefore select and control a much smaller quotient of the character bank. Fermat character decompositions and Jacobi-sum factors provide the established ambient framework for such quotients [27, 22, 28].

13 The Polylogarithmic Three-Point Path Packet

13.1 Root-distribution compression

Let $u^M = z$ and let ζ_M be a primitive Mth root of unity. For $n \geq 1$,

$$\boxed{\sum_{j=0}^{M-1} \mathrm{Li}_n(\zeta_M^j u) = M^{1-n}\,\mathrm{Li}_n(u^M).}$$

Inside the disk of convergence, this follows by summing the defining series and using the root-of-unity filter. Analytic continuation extends the identity with compatible branches.

The inverse synthesis gain at weight n is M^{n-1}. Through weights one, two, and three, the total gain is

$$\boxed{1 \cdot M \cdot M^2 = M^3.}$$

The displayed coordinate rescaling has determinant M^3. It has not been identified here with an actual integral motivic or period-lattice map. That identification would require the missing arithmetic realization.

13.2 Three-point functional closure

Set

$$z = \frac{a}{c}, \qquad 1 - z = \frac{b}{c}, \qquad \beta = 1 - \frac{1}{z} = -\frac{b}{a}.$$

The Rogers dilogarithm

$$L_2(z) = \mathrm{Li}_2(z) + \frac{1}{2}\log z \log(1-z)$$

satisfies

$$\boxed{L_2(z) + L_2(1-z) = \frac{\pi^2}{6}.}$$

At weight three,

$$\boxed{\begin{aligned}&\mathrm{Li}_3(z) + \mathrm{Li}_3(1-z) + \mathrm{Li}_3(\beta)\\ &\qquad = \zeta(3) + \frac{1}{6}\log^3 z + \frac{\pi^2}{6}\log z - \frac{1}{2}\log^2 z \log(1-z).\end{aligned}}$$

For $0 < z < 1$, all three displayed arguments use their real branches. The dilogarithm complement identity proves the Rogers relation. The auxiliary identity

$$\mathrm{Li}_2(1 - 1/z) + \mathrm{Li}_2(1-z) = -\tfrac{1}{2}\log^2 z$$

follows by differentiation and the limit at $z = 1$. Using $\mathrm{d}\,\mathrm{Li}_3(w) = \mathrm{Li}_2(w)\,\mathrm{d}w/w$, the derivative of the displayed trilogarithm sum is

$$\frac{\pi^2}{6z} - \frac{\log z \log(1-z)}{z} + \frac{\log^2 z}{2z(1-z)}.$$

It is also the derivative of the right side. Both sides tend to $\zeta(3)$ as $z \to 1$, proving the formula. The balanced Kummer coordinate is therefore exactly the third argument in the depth-three functional closure.

A fixed path packet is

$$\Pi_{\leq 3}(z) = \left(\log z, \log(1-z), L_2(z), \mathrm{Li}_3(z), \mathrm{Li}_3(1-z)\right).$$

Its coordinate count is independent of M.

13.3 Archimedean envelope

For $0 < z < 1$,

$$|\log z|, |\log(1-z)| \le \log c.$$

The positive dilogarithm and trilogarithm values on $(0,1)$ are bounded by $\zeta(2)$ and $\zeta(3)$. The three-point identity gives

$$\left|\mathrm{Li}_3\left(-\frac{b}{a}\right)\right| \le 3\zeta(3) + \frac{\pi^2}{6}\log c + \frac{2}{3}(\log c)^3.$$

Thus the complex packet has only cubic logarithmic growth. The surrounding motivic and regulator language is informed by the polylogarithmic complexes of Goncharov, the regulator framework discussed by Zagier, and Borel's rank theorem [23, 24, 25].

This is a genuine nonlinear compression of the displayed complex data. It is not yet an integral compression. A fixed-rank integer lattice carrying a large torsion quotient must pay through covolume, denominator, algebraic degree, or a short vector.

13.4 Fixed-rank integral uncertainty

Let $A \in M_d(\mathbb{Z})$ be nonsingular and suppose $D \mid \det A$. If every entry satisfies $|A_{ij}| \le B$, then Hadamard's inequality gives

$$D \le |\det A| \le d^{d/2} B^d.$$

Hence

$$\boxed{B \ge \frac{D^{1/d}}{\sqrt{d}}.}$$

A small complex period packet does not imply a small integral realization.

13.5 Exact finite-place notch

At an output prime $p^e \parallel c$,

$$\beta = 1 - \frac{c}{a}.$$

Since $p \nmid a$,

$$v_p(\beta - 1) = e.$$

For odd p,

$$\boxed{v_p(\log_p \beta) = e.}$$

The corresponding weighted dyadic identity holds on the principal branch when $e \ge 2$.

Writing

$$\beta = -\prod_{q \mid ab} q^{\nu_q},$$

we obtain the cross-prime logarithmic filter relation

$$\boxed{\log_p \beta = \sum_{q \mid ab} \nu_q \log_p \langle q \rangle_p.}$$

The same depth-three path therefore has a small complex envelope and exact deep finite notches.

13.6 Weight-three motivation and limitation

Over $\mathbb{Q}$, Borel's rank theorem gives $\operatorname{rank} K_3(\mathbb{Q}) = 0$ and $\operatorname{rank} K_5(\mathbb{Q}) = 1$, the rational weights two and three. This makes weight three a plausible rational regulator layer. It does not assert the absence of all weight-two periods or of regulator directions over arbitrary number fields. This motivates the depth-three path packet. It does not prove that weight three suffices uniformly. A complete proof may require adaptive higher weights or a different motivic quotient.

13.7 Conditional motivic transfer

Let K_β be the cyclic group generated by the balanced Kummer class. Suppose that for every primitive positive triple there exists an integral finite object $\mathcal{H}_3(a,b,c)$ and a canonical map

$$\Theta_3 : K_\beta \longrightarrow \mathcal{H}_3(a,b,c)$$

such that, for every $\varepsilon > 0$, constants independent of the triple give

$$|\ker \Theta_3| \leq C_\varepsilon R^\varepsilon$$

and

$$|\mathcal{H}_3(a,b,c)| \leq C'_\varepsilon R_a R_b R^\varepsilon.$$

Then the strong *abc* conjecture follows after absorbing the logarithmic Kummer kernel.

The missing object must combine finite Kummer depth, the three-point additive relation, an integral motivic or syntomic lattice, and an archimedean period norm.

14 Cross-Prime Logarithmic Coherence

14.1 The signed input exponent signal

Let

$$S_{ab} = \{q : q \mid ab\}.$$

Define

$$\nu_q = v_q(b) - v_q(a).$$

Then

$$-\frac{b}{a} = -\prod_{q \mid ab} q^{\nu_q}.$$

For every output prime $p \mid c$, the same input exponent packet produces the deep principal unit

$$1 - \frac{c}{a}.$$

The output persistence is therefore a simultaneous cross-prime cancellation phenomenon.

14.2 Odd logarithmic rows

For odd $p^e \parallel c$, define the finite-depth row coefficient

$$\ell_{p,e}(q) = \frac{1}{p} \log_p \langle q \rangle_p \pmod{p^{e-1}}.$$

The balanced relation gives

$$\boxed{\sum_{q\mid ab} \nu_q \ell_{p,e}(q) \equiv 0 \pmod{p^{e-1}}.}$$

Define the truncated valuation on $\mathbb{Z}/p^{e-1}\mathbb{Z}$ by

$$\bar{v}_p(0) = e-1, \qquad \bar{v}_p(x) = \min\{e-1, v_p(\widetilde{x})\}$$

for any integer lift $\widetilde{x}$ of a nonzero residue. Then

$$\boxed{\bar{v}_p(\ell_{p,e}(q)) = \min\{e-1, v_p(q^{p-1}-1)-1\}.}$$

Exceptional entrywise depth is a generalized Wieferich event.

14.3 Weighted dyadic row

For $2^e \parallel c$ with $e \geq 2$ and an odd input prime q, define

$$\ell_{2,e}(q) = \frac{1}{4}\log_2\left((-1)^{(q-1)/2}q\right) \pmod{2^{e-2}}.$$

With the analogous truncated valuation on $\mathbb{Z}/2^{e-2}\mathbb{Z}$,

$$\bar{v}_2(\ell_{2,e}(q)) = \min\{e-2, v_2(q^2-1)-3\}.$$

At $e = 1$, the output contact depth is zero and this row is omitted. A separate sign packet supplies the missing first dyadic layer. This is the weighted dyadic branch in its final logarithmic form.

14.4 Sequential factorization of the powerful part

The factorization is defined prime by prime. For an odd repeated output prime $p^e \parallel c$, set

$$d_p = e-1.$$

For a repeated dyadic prime $2^e \parallel c$, remove one sign layer and set

$$s_2 = 1, \qquad d_2 = e-2.$$

If $2 \nmid M$, take $s_2 = 0$. Put $\epsilon_2 = 2^{s_2}$. Let K_{Kum} be the kernel size of the balanced Kummer class and put

$$r_p = \min\{d_p, v_p(K_{\mathrm{Kum}})\}.$$

The balanced order formula implies

$$K_{\mathrm{Kum}} \mid M/\epsilon_2,$$

so $K_{\mathrm{Kum}} = \prod_{p\mid c} p^{r_p}$. Write $K = K_{\mathrm{Kum}}$. Divide the input exponent vector by K. Define $a_{p,q}$ as the original odd or weighted dyadic coefficient, now reduced modulo $p^{d_p - r_p}$. Dividing the relation vector reduces the needed relation modulus. It does not divide the coefficient row by K. Define

$$w_p = \min\left\{d_p - r_p, \min_{q\mid ab} \bar{v}_p(a_{p,q})\right\}, \qquad \chi_p = d_p - r_p - w_p.$$

The normalized global factors are

$$K = \prod_{p|c} p^{r_p}, \qquad W = \prod_{p|c} p^{w_p}, \qquad C = \prod_{p|c} p^{\chi_p}.$$

The balanced Kummer order formula guarantees that the dyadic sign layer is removed before the principal 2-adic Kummer depth, so these exponents partition the contact depth. Hence

$$\boxed{M = \epsilon_2 KWC.}$$

The order of removal matters. The local logarithmic defect is measured only after the Kummer preload has been divided out. The resulting row is then divided by its common logarithmic depth, leaving a primitive row modulo p^{χ_p}.

14.5 The coherence character

Set $\chi_p = 0$ for absent or squarefree output channels. When the residual modulus is one, use the trivial row and character. After normalization, every output prime with nontrivial residual depth produces a primitive row

$$g_p \in (\mathbb{Z}/p^{\chi_p}\mathbb{Z})^{S_{ab}}$$

satisfying

$$g_p \cdot \nu_0 \equiv 0 \pmod{p^{\chi_p}},$$

where ν_0 is the Kummer-normalized input exponent vector. The Chinese remainder theorem combines the local rows into one primitive row

$$g \in (\mathbb{Z}/C\mathbb{Z})^{S_{ab}}.$$

Define the coherence character

$$\boxed{\chi_{\mathrm{coh}}(x) = \exp\left(\frac{2\pi i}{C} g \cdot x\right).}$$

Its kernel is the adelic coherence lattice

$$\boxed{\Lambda_{\mathrm{coh}} = \{x \in \mathbb{Z}^{S_{ab}} : g \cdot x \equiv 0 \pmod{C}\}.}$$

Since g is primitive modulo C,

$$[\mathbb{Z}^{S_{ab}} : \Lambda_{\mathrm{coh}}] = C$$

and

$$\mathbb{Z}^{S_{ab}}/\Lambda_{\mathrm{coh}} \cong \mathbb{Z}/C\mathbb{Z}.$$

The normalized exponent vector lies in the lattice.

14.6 Worked lattice for $3 + 125 = 128$

Here

$$M = 64$$

and

$$(\epsilon_2, K, W, C) = (2, 1, 1, 32).$$

The input vector is

$$v_0 = (-1, 3)$$

on the prime coordinates $(3, 5)$. The residual row is

$$(29, 31) \pmod{32}.$$

Thus

$$\boxed{\Lambda_{\mathrm{coh}} = \{(x_3, x_5) \in \mathbb{Z}^2 : 29x_3 + 31x_5 \equiv 0 \pmod{32}\}.}$$

Indeed,

$$29(-1) + 31(3) = 64 \equiv 0 \pmod{32}.$$

The five residual dyadic contact layers are exactly the index of this lattice.

14.7 Generalized Wieferich versus collective coherence

There are two qualitatively different sources of large output depth.

1. An entrywise defect can make every coefficient in one row divisible by a large power of p. This is the factor W.
2. Ordinary coefficients can cancel coherently against the specific exponent vector. This is the factor C.

In the recovered 361-row coherence corpus, the recorded W factor is usually one while C can be large. The corpus is selected, rather than a statistical sample of all primitive triples. Its frequencies do not establish a generic population law. For its $W = 1$ rows, the residual depth is represented by collective cancellation rather than an entrywise Wieferich defect.

14.8 Why abstract lattice geometry is insufficient

Let $v \in \mathbb{Z}^s$ be primitive, with $s \geq 2$. For every $C \geq 2$, there exists a primitive row $g \pmod{C}$ satisfying

$$g \cdot v \equiv 0 \pmod{C}.$$

Thus a fixed short vector can lie in primitive congruence lattices of arbitrarily large index.

Theorem 14.1 (Arbitrary-index annihilator)**.** *Abstract lattice geometry cannot bound the coherence index from the length of the known relation vector alone.*

The proof completes v to a unimodular basis and chooses a primitive row annihilating its first basis coordinate modulo C.

The missing theorem must use the arithmetic origin of g: its entries are simultaneous p-adic logarithms of the actual input primes, and the output primes are exactly the divisors of $a + b$.

14.9 Why positive divisibility is circular

If positive integers A and B satisfy $c \mid A + B$, then

$$A + B \geq c,$$

so

$$\max(A, B) \geq c/2.$$

A direct search for a small positive S-unit relation divisible by c cannot produce an *abc* saving. Signed cancellation, a determinant, a regulator, or a functional equation is necessary.

14.10 Exact reformulation of strong *abc*

Since

$$c = R_c \epsilon_2 KWC$$

and $\epsilon_2 \leq 2$ while $K \leq \log_2 c$, the all-ε strong *abc* statement is equivalent, after the usual absorption of bounded and logarithmic factors into an arbitrarily small power, to the target

$$\boxed{WC \leq C_\varepsilon R_a R_b R^\varepsilon.}$$

The logarithmic absorption is proved explicitly in Section 21, without assuming an initial relation $c \ll R$. Conversely, $WC \leq M_c$ makes the reverse implication immediate. In the recorded examples with $W = 1$, this leaves the coherence conductor bound

$$C \leq C_\varepsilon R_a R_b R^\varepsilon.$$

14.11 Adelic coherence regulator object

Package the surviving data as

$$\boxed{\mathfrak{R}_{a,b,c} = \left(U_{ab}, \beta^{1/K}, \{\chi_p\}_{p|c}, \chi_{\mathrm{coh}}, \Lambda_{\mathrm{coh}}, \|\cdot\|_\infty\right).}$$

Here U_{ab} is the input S-unit group, the local characters are the normalized logarithmic rows, and the archimedean norm records input height.

The open theorem is an adelic slope or regulator inequality bounding the finite conductor $\log C$ by the input radical and an arbitrarily small global leakage term. General simultaneous linear-form theory contains the right kinds of local and archimedean quantities, but known general estimates are much too coarse for the R^ε scale.

15 Signed arithmetic COS and the sparse spectral leaf

15.1 What the finite assignment signal records

The spectral branch of the project asks which prime assignments realize a prescribed residue. Its exact object is a finite distribution. Fix a finite abelian group G and s eligible variable support primes. At the jth prime choose one of three permitted contributions, represented by $0, \gamma_j, -\gamma_j$ in additive group notation. The sum of the chosen contributions is the assignment residue.

Let $N(x)$ count assignments producing $x \in G$. Repeated contributions remain separate assignments, so

$$M = \sum_{x \in G} N(x) = 3^s, \qquad \mu(x) = N(x)/M.$$

For a character $\chi \in \widehat{G}$, define

$$\widehat{\mu}(\chi) = \sum_x \mu(x)\chi(x) = \prod_{j=1}^{s} \frac{1 + \chi(\gamma_j) + \chi(-\gamma_j)}{3}.$$

On a cyclic channel $G = \mathbb{Z}/n\mathbb{Z}$, this becomes

$$m(k) = \prod_{j=1}^{s} \frac{1 + 2\cos(2\pi k \gamma_j / n)}{3}.$$

This is the discrete cosine or COS expression used in the project. It is an exact characteristic function. Individual cosine factors may be negative. No positivity of all Fourier samples is assumed.

Character orthogonality gives the target count

$$N(h) = \frac{M}{|G|} \sum_{\chi \in \widehat{G}} \widehat{\mu}(\chi)\overline{\chi(h)}.$$

The identity is finite and exact. It solves a representation question after the arithmetic channel and admissible contributions have been specified. It does not by itself bound the height of an integer triple that generated that channel.

15.2 Sparsity forces transverse energy

Let $E = \{x : N(x) > 0\}$ and $r = |E|$. Always $r \le M$. Parseval yields

$$\sum_{\chi \in \widehat{G}} |\widehat{\mu}(\chi)|^2 = \frac{|G|}{M^2} \sum_x N(x)^2.$$

Cauchy–Schwarz on the occupied support gives $\sum_x N(x)^2 \ge M^2/r$. Hence

Proposition 15.1 (Sparse energy barrier)**.**

$$\sum_{\chi \ne 1} |\widehat{\mu}(\chi)|^2 \ge \frac{|G|}{r} - 1 \ge \frac{|G|}{M} - 1.$$

When few assignment atoms lie in a large group, substantial nontrivial Fourier energy is unavoidable. A failure of uniform mixing is therefore a structural feature of the signal, rather than evidence that the arithmetic assignment is invalid.

15.3 The hypothetical exception regime is sparse

For a primitive triple put $Q = abc/R$. Suppose $c > R^{1+\varepsilon}$ for a fixed $\varepsilon > 0$. Since a, b are positive with sum c, one has $ab \ge c - 1$. Thus, for large c,

$$Q = \frac{abc}{R} \ge \frac{c(c-1)}{R} > \frac{1}{2} R^{1+2\varepsilon}.$$

The unit group modulo Q has order $\varphi(Q)$. For every $\delta > 0$,

$$\frac{Q}{\varphi(Q)} \ll_\delta R^\delta, \qquad 3^{\omega(R)} \ll_\delta R^\delta.$$

Both elementary bounds follow by separating a finite set of small primes. For all sufficiently large primes, $(1 - 1/p)^{-1} \le p^\delta$ and $3 \le p^\delta$. Every prime dividing Q also divides R.

If the assignment model has $M = 3^s$ with $s \le \omega(R)$, choose $\delta = \varepsilon/2$ in both estimates. Then

$$\frac{\varphi(Q)}{M} \gg_\varepsilon R^{1+\varepsilon}.$$

Consequently a sufficiently large hypothetical exponent-gap exception, represented in this joint unit-group model, lies on the sparse side of the energy barrier. Uniform spectral mixing cannot be the universal finishing argument for this regime.

The full unit group need not be cyclic. The preceding energy statement holds for any finite abelian group. The one-variable Prony argument below is specifically a theorem for a cyclic channel. It is not a claim that one character line automatically recovers all correlations in an arbitrary noncyclic product channel.

15.4 Exact innovation rank in a cyclic channel

Let $G = \mathbb{Z}/n\mathbb{Z}$, $\alpha_x = e^{2\pi i x/n}$ and

$$m(k) = \sum_{x \in E} \frac{N(x)}{M} \alpha_x^k.$$

Form the shift matrix $S_d = (m(k+j))_{0 \le k < n,\, 0 \le j \le d}$. It factors as a Fourier-column matrix times a Vandermonde matrix on the distinct occupied nodes. The Fourier columns are independent, and all weights $N(x)/M$ are nonzero. Therefore

$$\boxed{\operatorname{rank} S_d = \min(r, d+1).}$$

Before degree r, a small singular value is a conditioning phenomenon, not an exact reduction in the number of occupied residues.

Theorem 15.2 (Exact cyclic Prony reconstruction)**.** *The distribution on a cyclic channel with r occupied residues is determined by the $2r$ exact consecutive samples $m(0), \dots, m(2r-1)$.*

Proof. Write

$$A(z) = \prod_{x \in E} (z - \alpha_x) = z^r + \sum_{j=0}^{r-1} a_j z^j.$$

For every k,

$$\sum_{j=0}^{r-1} a_j m(k+j) = -m(k+r).$$

The coefficient matrix for $0 \le k < r$ is the Hankel matrix $H = (m(i+j))_{0 \le i,j < r}$. With $V = (\alpha_x^i)_{i,x}$ and $D = \operatorname{diag}(N(x)/M)$, one has $H = VDV^{\mathsf{T}}$. The square Vandermonde V is invertible and every diagonal entry of D is nonzero. Thus H is invertible, and the $2r$ samples determine the monic annihilator A.

Its distinct roots identify the occupied nodes. The first r sample equations then give an invertible Vandermonde system for their weights. Multiplication by the known M recovers every multiplicity. □

This is an exact reconstruction theorem. It does not promise numerical stability from approximate samples. Nor does a subpolynomial number of samples by itself imply a subpolynomial arithmetic height for the reconstructed assignment.

15.5 Polynomial target filters and exact count certificates

For a polynomial $P(z) = \sum_{j=0}^{d} c_j z^j$, define the collision quadratic form

$$\mathcal{E}_d(P) = \frac{n}{M^2} \sum_{x \in E} N(x)^2 |P(\alpha_x)|^2.$$

It can be written as a positive semidefinite Gram form in the coefficients of P. Let

$$\lambda_d(h) = \min\{\mathcal{E}_d(P) : \deg P \le d,\ P(\alpha_h) = 1\}.$$

For every such filter, the target term is present with value $(n/M^2)N(h)^2$, so

$$N(h) \le \frac{M}{\sqrt{n}} \sqrt{\lambda_d(h)}.$$

If the right-hand side is rigorously less than one, the integer count is zero. An approximate decimal below one is not a certificate without a proved error enclosure.

If h is absent, a degree-r polynomial vanishes at every occupied node and takes value one at α_h, so $\lambda_r(h) = 0$. If h is occupied, its degree-$(r-1)$ Lagrange polynomial isolates exactly that node. Thus

$$\lambda_{r-1}(h) = \frac{n}{M^2} N(h)^2.$$

At the innovation degree the count bound is exact. The remaining computational issue is finding the needed filter with controlled exact coefficients and reliable arithmetic, rather than hoping for an unexplained asymptotic gain.

15.6 A Galois coordinate with a controlled denominator

For an occupied target define

$$\Delta_h(E) = \prod_{x \in E \setminus \{h\}} (\alpha_h - \alpha_x).$$

This is a nonzero algebraic integer in $\mathbb{Q}(\zeta_n)$. For every unit u modulo n, its conjugate is $\Delta_{uh}(uE)$. The absolute norm is a nonzero integer. Hence

$$\prod_{u \in (\mathbb{Z}/n\mathbb{Z})^\times} |\Delta_{uh}(uE)| \geq 1.$$

At least one factor is at least one. In that coordinate, the Lagrange numerator has coefficient ℓ^1 norm at most 2^{r-1}, and dividing by the target denominator preserves that upper bound.

This removes a catastrophically small denominator in at least one complex coordinate. It does not supply a uniform radical-scale bound for every conjugate or for the arithmetic height. Even if $r = R^{o(1)}$, the quantity 2^r need not be $R^{o(1)}$.

15.7 What the spectral branch leaves open

The valid endpoint is exact compression and extraction of a sparse arithmetic leaf. A further height theorem must exploit the arithmetic of that leaf, such as its simultaneous logarithmic, norm-form or S-unit constraints. The bounded representation count, exact innovation rank and finite Fourier identities do not prove the required inequality on c/R.

16 Correcting the global target before raising contact

16.1 A missing input factor changes the theorem

The September continuation begins with a correction, rather than a larger contact claim. Set

$$R = \mathrm{rad}(abc), \qquad Q = abc/R.$$

Section 1 proved the exact identity $Q/(ab) = c/R$. A uniform subpolynomial bound on Q alone is not equivalent to *abc*. It is false unconditionally.

Theorem 16.1 (Counterfamily for the unbalanced target)**.** *For every fixed $0 \leq \eta < 1$ and $A \geq 0$, no constant K makes*

$$Q \leq K R^\eta (\log(3c))^A$$

valid for all primitive positive triples.

Proof. Take $a = 1$, $b = 2^k$, $c = 2^k + 1$ for integers $k \geq 1$. The triple is primitive and c is odd. Thus

$$R = 2\,\mathrm{rad}(c) \leq 2c \leq 3 \cdot 2^k, \qquad Q = 2^{k-1}\frac{c}{\mathrm{rad}(c)} \geq 2^{k-1}.$$

Consequently

$$\frac{Q}{R^\eta(\log(3c))^A} \geq \frac{2^{(1-\eta)k-1}}{3^\eta(\log(3(2^k+1)))^A} \longrightarrow \infty.$$

An exponential with positive exponent eventually exceeds every fixed power of k. □

The correction retracts the discarded global target. It does not retract coefficient identities or local valuations proved independently of that target. Those valid statements are developed next, with their exact scope retained.

16.2 A necessary leading cost for powers of Q

A stronger elementary obstruction measures the exponent cost of any all-triple bound on the unbalanced quantity.

Theorem 16.2 (Necessary cost lower bound)**.** *Fix $s > 0$. Suppose $B, A \geq 0$ and $C > 0$ satisfy*

$$Q^s \leq CR^B(\log(3c))^A$$

for every primitive positive triple. Then $B \geq 2s$.

Proof. Fix an integer $d \geq 2$ and take $a = t^d$, $b = (t+1)^d$, $c = t^d + (t+1)^d$ for $t \geq 1$. The inputs are coprime. As $\mathrm{rad}(t^d) \leq t$ and $\mathrm{rad}((t+1)^d) \leq t+1$,

$$R \leq t(t+1)c, \qquad Q = \frac{abc}{R} \geq [t(t+1)]^{d-1}.$$

The assumed inequality implies

$$[t(t+1)]^{s(d-1)} \leq C[t(t+1)c]^B(\log(3c))^A.$$

Take logarithms, divide by $\log t$ and send t to infinity. Since $\log c/\log t \to d$ and $\log\log(3c)/\log t \to 0$, this yields

$$2s(d-1) \leq B(d+2).$$

It holds for every d. Sending d to infinity proves $B \geq 2s$. □

No contact multiplier can make the ratio B/s tend to zero for this false target. A valid certificate must instead include $(ab)^s$ in its global size bound, or directly control M_c relative to R_aR_b.

16.3 Two obligations that a new determinant must meet

The proposed determinant program therefore has two separate obligations. It needs a nonzero contact expression, with an actual first nonzero coefficient and local divisibility. It also needs an arithmetic realization whose global size has the correct $(ab)^s$ normalization. The following local theorems solve parts of the first obligation. They leave the second as an explicit hypothesis.

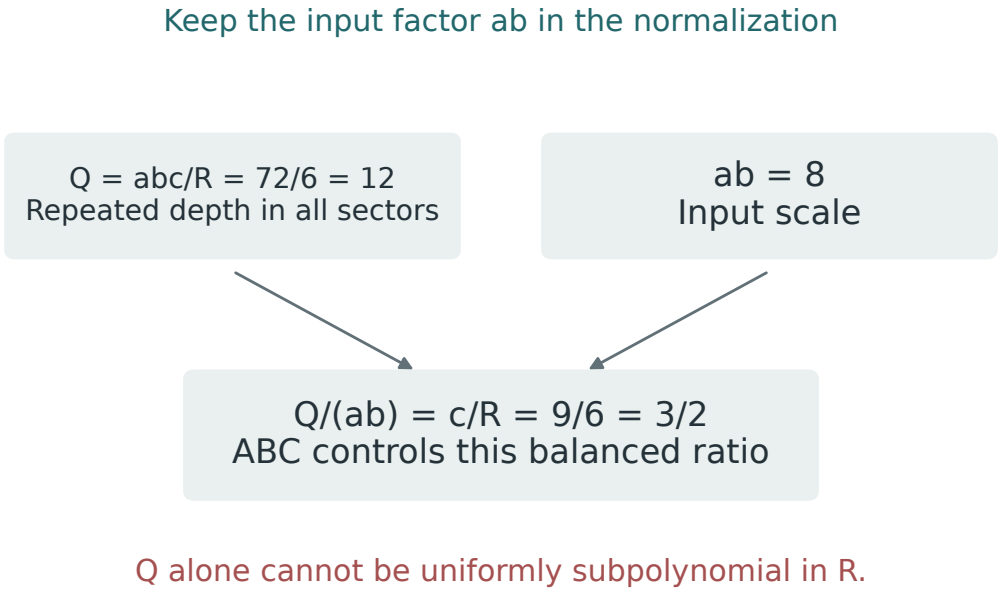


Figure 3: The balanced defect is $c/R = Q/(ab) = M_c/(R_a R_b)$. Removing the input factor ab changes the target into an unconditional false assertion.

17 The universal mixed-weight jet filter

17.1 One polynomial chooses the cancelling coefficients

Let $m \geq 0$ be an integer. Define

$$P_m(X) = X\prod_{j=1}^{m}(jX-1) = \sum_{s=1}^{m+1} c_{m,s}X^s,$$

and the formal series

$$\Phi_m(t) = \frac{1}{(m+1)!}\sum_{s=1}^{m+1} c_{m,s}\,\mathrm{Li}_s(t).$$

Here $\mathrm{Li}_s(t) = \sum_{k\geq 1} t^k/k^s$ is first interpreted in $\mathbb{Q}[[t]]$. It also defines the usual analytic polylogarithm on $|t|<1$ [14]. The label s is its weight. The contact order will follow from its coefficients, rather than from that label.

Theorem 17.1 (Exact jet cancellation)**.**

$$\Phi_m(t) = (-1)^m \sum_{k\geq 1}\binom{k}{m+1}\frac{t^k}{k^{m+2}}.$$

Thus

$$\mathrm{ord}_0\,\Phi_m = m+1, \qquad [t^{m+1}]\Phi_m = \frac{(-1)^m}{(m+1)^{m+2}}.$$

Proof. The coefficient of t^k is $P_m(1/k)/(m+1)!$. Direct substitution gives

$$P_m(1/k) = \frac{1}{k}\prod_{j=1}^{m}\left(\frac{j}{k}-1\right) = (-1)^m\frac{(k-1)\cdots(k-m)}{k^{m+1}}.$$

Divide by $(m+1)!$ and multiply numerator and denominator by k to get the asserted binomial coefficient. It vanishes for $1\leq k\leq m$ and first becomes one at $k=m+1$. The coefficient there is nonzero. □

For $m=0$, this is $\Phi_0 = \mathrm{Li}_1$. For $m=1$,

$$\Phi_1 = (\mathrm{Li}_2 - \mathrm{Li}_1)/2 = -t^2/8 - t^3/9 - \cdots.$$

The sign is part of the exact formula. Changing the weight alone would not produce the cancellation of the linear coefficient.

17.2 Small coefficient norm has a limited meaning

The coefficients of P_m alternate in sign. Its absolute-coefficient polynomial evaluated at one gives

$$\sum_s |c_{m,s}| = \prod_{j=1}^{m}(j+1) = (m+1)!.$$

Also $c_{m,1} = (-1)^m$, so the integer coefficient vector is primitive. After the displayed normalization, its coefficient ℓ^1 norm is one.

These are exact statements about a vector of rational coefficients. The number of participating weights grows with m. A norm of one for the coefficient vector does not bound the denominator, covolume or arithmetic height of a period realization. No such realization is built into the definition of Φ_m.

17.3 The depleted local series

For a prime p, delete the terms whose index is divisible by p:

$$\Phi_{m,p}(t) = (-1)^m \sum_{p \nmid k} \binom{k}{m+1} \frac{t^k}{k^{m+2}} \in \mathbb{Z}_p[[t]].$$

It converges when $v_p(t) > 0$.

Theorem 17.2 (Exact local jet valuation)**.** *If $v_p(t) = e \geq 1$, then*

$$v_p(\Phi_{m,p}(t)) = \begin{cases} (m+1)e, & p \nmid m+1, \\ (m+2)e, & p \mid m+1. \end{cases}$$

Proof. If $p \nmid m+1$, the first nonzero coefficient remains the unit $(-1)^m/(m+1)^{m+2}$. Its term has valuation $(m+1)e$. Every later coefficient is p-integral, so every later term has strictly larger valuation.

If $p \mid m+1$, that first index is deleted. The next index is $m+2$, which is a p-unit. Its coefficient is $(-1)^m/(m+2)^{m+1}$, also a unit. All later terms again have strictly larger valuation. The ultrametric inequality with a unique least-valuation term proves both equalities. □

17.4 An exact differential identity

Let $D = t\,\mathrm{d}/\mathrm{d}t$ be the Euler operator. Termwise formal differentiation is valid. Applying D^{m+2} multiplies the coefficient at t^k by k^{m+2}, so

$$D^{m+2}\Phi_m = (-1)^m \sum_{k \geq 1} \binom{k}{m+1} t^k = (-1)^m \frac{t^{m+1}}{(1-t)^{m+2}}.$$

The final equality is the elementary generating function for the binomial coefficients.

This universal formal identity and the prime-dependent depleted valuations can be studied simultaneously. They do not identify the different $\Phi_{m,p}$ as local values of one algebraic number. A product formula cannot be applied merely because these functions arose from one coefficient pattern.

18 A nonzero Hankel contact determinant

18.1 Independent moment directions replace one filter

For an integer $r \geq 1$, define

$$\mathcal{H}_r(t) = \det\bigl(\mathrm{Li}_{i+j+1}(t)\bigr)_{0\leq i,j<r}.$$

It uses weights one through $2r-1$. The expression is a formal series over $\mathbb{Q}$ and an analytic function for $|t|<1$. No algebraicity at a rational argument is assumed.

For the first two ranks,

$$\mathcal{H}_1 = \mathrm{Li}_1, \qquad \mathcal{H}_2 = \mathrm{Li}_1\,\mathrm{Li}_3 - \mathrm{Li}_2^2.$$

The rank-two determinant compares genuinely different moment directions. Its order-three contact is not inferred from the presence of weight three.

Theorem 18.1 (Cauchy–Binet expansion)**.**

$$\mathcal{H}_r(t) = \sum_{1\leq k_1<\cdots<k_r} t^{k_1+\cdots+k_r}\frac{\prod_{i<j}(k_j-k_i)^2}{\prod_{i=1}^r k_i^{2r-1}}.$$

Proof. Write $v_k = (1, k^{-1}, \ldots, k^{-(r-1)})^{\mathsf{T}}$. The moment matrix is

$$\sum_{k\geq 1}\frac{t^k}{k}v_k v_k^{\mathsf{T}}.$$

For a finite sum, Cauchy–Binet expands its determinant over r-element index subsets. The determinant of the chosen columns is a Vandermonde in k_i^{-1}. Squaring it and including the factors t^{k_i}/k_i gives the stated coefficient. In detail,

$$\prod_{i<j}(k_j^{-1}-k_i^{-1})^2 = \frac{\prod_{i<j}(k_j-k_i)^2}{\prod_i k_i^{2(r-1)}}.$$

Passing to the formal infinite sum is coefficientwise legitimate. At a fixed power of t, only finitely many index subsets occur. □

Every coefficient in this expansion is positive. The formal cancellation implicit in the determinant is now resolved into a manifestly nonzero expansion.

Theorem 18.2 (Exact contact and first coefficient)**.** *Put $s_r = r(r+1)/2$. Then*

$$\operatorname{ord}_0 \mathcal{H}_r = s_r, \qquad [t^{s_r}]\mathcal{H}_r = \prod_{k=1}^{r} k^{-(2k-1)}.$$

Proof. The unique increasing r-tuple of positive indices with least sum is $(1,\ldots,r)$, of sum s_r. Its coefficient is nonzero. Its Vandermonde numerator is $\prod_{j=1}^{r-1}(j!)^2$. Dividing this by $\prod_{k=1}^r k^{2r-1}$ leaves exponent $2k-1$ in the denominator at each k, giving the formula. □

For $r=2$ the first coefficient is $1/8$. For $r=3$ it is $1/(2^3 3^5)$. The contact orders are $1, 3, 6, 10, \ldots$, the triangular numbers.

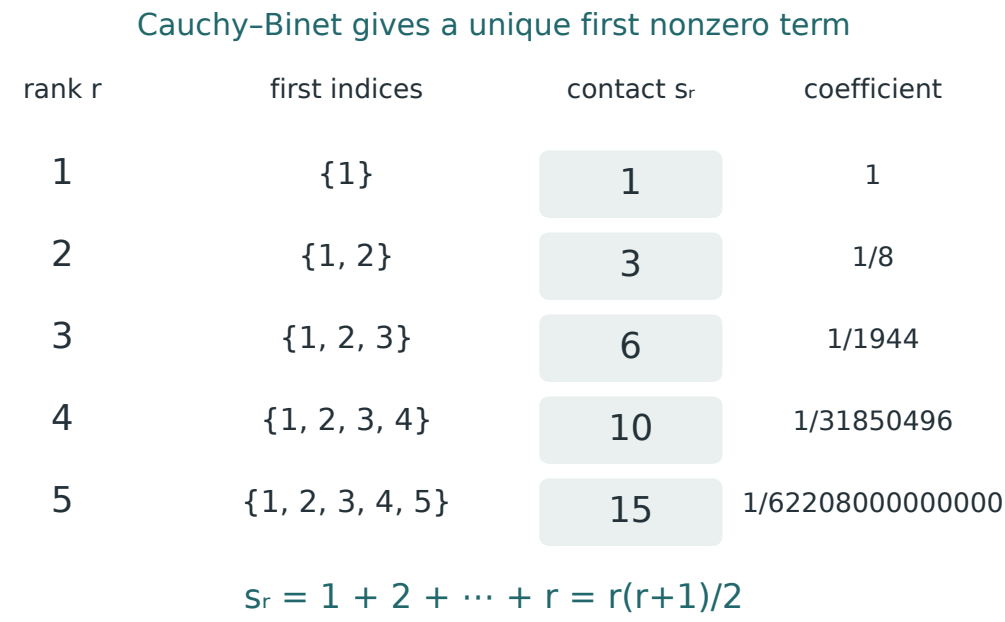


Figure 4: The Hankel rank produces an exact triangular contact order. The first nonzero coefficient is proved and remains separate from the highest polylogarithmic weight.

18.2 Real positivity and a bound uniform in rank

Theorem 18.3 (Positive real determinant)**.** *For every $r \geq 1$ and every real $0 < t < 1$,*

$$0 < \mathcal{H}_r(t) \leq e^{1/4}[-\log(1-t)].$$

Proof. The Cauchy–Binet expansion consists of positive terms and contains the nonzero term $(1, \ldots, r)$, so positivity follows. Equivalently, the real moment matrix is positive definite: a nonzero polynomial of degree below r cannot vanish at all the distinct nodes k^{-1}.

Hadamard's inequality for the positive matrix gives

$$\mathcal{H}_r(t) \leq \prod_{i=0}^{r-1} \mathrm{Li}_{2i+1}(t) \leq [-\log(1-t)] \prod_{i=1}^{r-1} \zeta(2i+1).$$

For the higher weights, $t^k \leq 1$ bounds the series by its zeta value. Since all summands are positive,

$$\sum_{i\geq 1} (\zeta(2i+1) - 1) = \sum_{k\geq 2} \frac{1}{k(k^2-1)} = \frac{1}{4}.$$

The last sum telescopes using

$$\frac{1}{k(k^2-1)} = \frac{1}{2}\left(\frac{1}{k(k-1)} - \frac{1}{k(k+1)}\right).$$

Finally $\log(1+x) \leq x$ for $x \geq 0$ bounds the zeta product by $e^{1/4}$, uniformly in its length. □

At $t = a/c$ and $t = b/c$, the bounds become $e^{1/4}\log(c/b)$ and $e^{1/4}\log(c/a)$. Thus these real period values are nonzero and have a simple logarithmic envelope. This is valuable analytic control. It is not an all-conjugate height theorem for an algebraic scalar.

18.3 The third complex chart has a branch boundary

For fixed rank and a consistent upper or lower boundary branch at $x = c/a > 1$, the classical integer-weight inversion formula expresses each $\mathrm{Li}_j(x)$ as a fixed degree-j polynomial in a chosen $\log(-x)$ plus a sign times $\mathrm{Li}_j(1/x)$ [14]. The latter is bounded by $\zeta(j)$ for $j \geq 2$. For $j = 1$, use $\mathrm{Li}_1(x) = -\log(1-x)$ and $x - 1 = b/a$.

Since $|\log(b/a)| \leq \log c$ and $\log(c/a) \leq \log c$, every entry has a bound by a constant times $(1 + \log c)^j$. A determinant term has total weight

$$\sum_{i=0}^{r-1} (i + \pi(i) + 1) = r^2.$$

Therefore

$$|\mathcal{H}_r(c/a)| \leq C_r (1 + \log c)^{r^2}$$

for each of those consistent branches. The constant may depend on r. No nonvanishing assertion at every argument in this third chart follows from the positive expansion on $0 < t < 1$.

19 Exact local determinant valuation and denominator budget

19.1 Depletion gives an integral local matrix

Fix a prime p and an integer $r \geq 1$. Define

$$F_{s,p}(t) = \sum_{p \nmid k} \frac{t^k}{k^s}, \qquad \mathcal{H}_{r,p}(t) = \det(F_{i+j+1,p}(t))_{0 \leq i,j < r}.$$

These series converge at every $t \in \mathbb{Q}_p$ with $v_p(t) > 0$. They are integral power series. The Cauchy–Binet expansion from Section 18 now ranges only over increasing p-unit indices.

Let $u_1 < \cdots < u_r$ be the first r positive integers not divisible by p. Then

$$u_j = j + \left\lfloor \frac{j-1}{p-1} \right\rfloor, \qquad S_{p,r} = \sum_{j=1}^{r} u_j.$$

For $h \geq 1$, put $s_h = (p-1)p^{h-1}$ and write

$$r = q_h s_h + b_h, \qquad 0 \leq b_h < s_h.$$

Define the finite sum

$$E_{p,r} = \sum_{h \geq 1} \left(s_h \binom{q_h}{2} + b_h q_h \right).$$

The terms are zero once $s_h \geq r$.

19.2 The residue-class energy minimum

For r distinct p-unit integers $k_1, \ldots, k_r$, the valuation of their Vandermonde product is

$$\sum_{i<j} v_p(k_j - k_i) = \sum_{h \geq 1} \#\{i < j : k_i \equiv k_j \pmod{p^h}\}.$$

There are s_h unit residue classes modulo p^h. If their occupation counts are $n_1, \ldots, n_{s_h}$, the number of coincident pairs is $\sum_j \binom{n_j}{2}$.

For a fixed sum r, this quantity is minimized when the counts differ by at most one. If $n_i \geq n_j + 2$, moving one item from class i to class j lowers the pair count by $n_i - n_j - 1 > 0$. At the balanced minimum, b_h classes have count $q_h + 1$ and the others count q_h. The minimum is therefore

$$(s_h - b_h)\binom{q_h}{2} + b_h \binom{q_h + 1}{2} = s_h \binom{q_h}{2} + b_h q_h.$$

The first r positive p-units realize this minimum for every h simultaneously. Ordered unit residues repeat in blocks modulo p^h, and each full block contains each unit class exactly once. A prefix adds at most one occurrence to each class. Hence their Vandermonde valuation is exactly $E_{p,r}$.

Theorem 19.1 (Exact depleted valuation)**.** *If $v_p(t) = e \geq 1$, then*

$$\boxed{v_p(\mathcal{H}_{r,p}(t)) = eS_{p,r} + 2E_{p,r}.}$$

In particular, the depleted determinant is nonzero at every such argument.

Proof. The denominators in the depleted expansion are p-units. Every r-element index subset has Vandermonde valuation at least $E_{p,r}$. The subset $(u_1, \dots, u_r)$ uniquely minimizes the sum of its indices. Its term has valuation $eS_{p,r} + 2E_{p,r}$. Every other subset has a strictly larger index sum and no smaller Vandermonde valuation, so its term has strictly larger valuation.

The expansion converges p-adically. A unique term of minimum valuation cannot be cancelled by the remaining terms. Its valuation is therefore the valuation of the whole determinant. □

For rank two,

$$v_p(\mathcal{H}_{2,p}(t)) = \begin{cases} 3e, & p \neq 2, \\ 4e + 2, & p = 2. \end{cases}$$

At two the first unit indices are $1, 3$, of sum four, and their difference contributes one valuation before squaring. For $p > r$ in arbitrary rank, the first r indices are $1, \dots, r$ and all their differences are units. Thus the general formula reduces to es_r.

19.3 The universal undepleted expression at good primes

The raw $\mathcal{H}_r$ is one universal expression in ordinary polylogarithms. Its local power-series interpretation converges when $v_p(t) > 0$, but its coefficients can have p-denominators. For fixed rank, the possible denominator loss grows only logarithmically with each index, whereas its numerator power of t gives linearly growing valuation. Thus the terms tend to zero and the finite Cauchy–Binet expansions pass to the convergent local limit.

Theorem 19.2 (Exact good-prime valuation)**.** *If $p > 3r - 1$ and $v_p(t) = e \geq 1$, then*

$$v_p(\mathcal{H}_r(t)) = e\, r(r+1)/2.$$

Proof. Write $a_r = 2r - 1$ and $s_r = r(r+1)/2$. A term indexed by $k_1 < \cdots < k_r$ has valuation

$$e \sum_i k_i + 2 \sum_{i<j} v_p(k_j - k_i) - a_r \sum_i v_p(k_i).$$

For every $h \geq 1$, the hypothesis gives $p^h - r > a_r h$. At $h = 1$ this is $p - r > a_r$. Successive increments of p^h exceed a_r, proving the inequality for all h.

If $h_i = v_p(k_i) > 0$, then $k_i - i \geq p^{h_i} - r > a_r h_i$. If $h_i = 0$, then $k_i - i \geq 0$ and there is no denominator loss. Thus every subset other than $(1, \dots, r)$ has valuation strictly above es_r. The first subset has a unit coefficient because its indices and differences are smaller than p. Its term has valuation es_r, and a unique minimum again proves equality. □

The strict prime threshold is retained. A good-prime equality is stronger than the all-prime lower bound below.

19.4 An explicit fixed correction at the remaining primes

Put $a_r = 2r - 1$. For each prime p and $1 \leq i \leq r$, define

$$C_{r,p,i} = \max\left(0, \sup_{h \geq 1} \left\{ a_r h - \left(p^h \left\lceil \frac{i}{p^h} \right\rceil - i \right) \right\}\right).$$

This is a finite integer maximum because p^h eventually grows faster than h. Put $C_{r,p} = \sum_{i=1}^{r} C_{r,p,i}$. The good-prime inequality just proved implies $C_{r,p} = 0$ for $p > 3r - 1$. Hence

$$B_r = \prod_{p \le 3r-1} p^{C_{r,p}}$$

is a fixed positive integer depending only on the rank.

Theorem 19.3 (All-prime contact lower bound)**.** *For every prime p and $v_p(t) = e \ge 1$,*

$$\boxed{v_p(B_r \mathcal{H}_r(t)) \ge e s_r.}$$

Here $v_p(0) = +\infty$. This theorem alone does not establish nonvanishing at the small primes.

Proof. For an ordered index $k_i \ge i$ with $h_i = v_p(k_i) > 0$, the smallest multiple of p^{h_i} at least i is $p^{h_i}\lceil i/p^{h_i}\rceil$. Therefore

$$a_r h_i - (k_i - i) \le C_{r,p,i}.$$

For $h_i = 0$, the same bound holds because its left side is nonpositive. The numerator Vandermonde only improves valuation. Since $e \ge 1$, every term in the convergent expansion has valuation at least

$$e s_r + \sum_i [(k_i - i) - a_r h_i] \ge e s_r - C_{r,p}.$$

The ultrametric inequality gives the same bound for the sum. Multiplication by B_r restores the missing $C_{r,p}$. □

For rank two, $B_2 = 2^7 \cdot 3^3 = 3456$ is a valid budget. For instance, at $p = 2$ the two component maxima, in index order $i = 1, 2$, are three and four. At $p = 3$ they are one and two. The other primes contribute zero. The formula need not be minimal.

The budget is a theorem about the explicitly defined local analytic series. It is not a theorem about the denominator of a motivic class, which has not been constructed here. Even a vanishing correction budget need not imply exact equality at a prime outside the good-prime range. Additional cancellation may raise the valuation.

19.5 The all-place comparison is still absent

These results give exact depleted nonvanishing, universal good-prime equality and universal all-prime lower contact. They do not identify the real value, the various p-adic values, or different sector arguments as the places of one nonzero algebraic number. An arithmetic realization and its norm maps remain separate requirements.

20 A rational truncation that fails the output contact test

20.1 A genuine algebraic candidate can be tested exactly

Fix integers $r \geq 1$ and $K \geq r$. Replace each polylogarithm by its finite series through index K, obtaining a rational polynomial $\mathcal{H}_{r,K}(t)$. Its Cauchy–Binet expansion has the same positive subset coefficients, restricted to $1 \leq k_i \leq K$.

Let

$$L_K = \operatorname{lcm}(1, \dots, K), \quad s_r = r(r+1)/2, \quad d = rK - r(r-1)/2.$$

Then

$$A_{r,K}(t) := L_K^{r^2} \mathcal{H}_{r,K}(t) \in \mathbb{Z}[t], \qquad \operatorname{ord}_0 A_{r,K} = s_r, \qquad \deg A_{r,K} = d.$$

To see integrality, scale row i by L_K^{i+1} and column j by L_K^j. Each entry becomes

$$L_K^{i+j+1} \sum_{k=1}^{K} \frac{t^k}{k^{i+j+1}} \in \mathbb{Z}[t].$$

The total determinant scaling exponent is r^2. The unique smallest subset is $(1, \dots, r)$ and the unique largest is $(K-r+1, \dots, K)$, proving the order and degree.

For a reduced rational argument $t = a/c$, an obvious integral scalar is

$$W_{r,K}(a,c) = c^d A_{r,K}(a/c).$$

It is nonzero when $0 < a < c$, since the truncated positive expansion is nonzero. Thus it is a genuine nonzero arithmetic candidate. The question is whether it retains the required output-prime contact after clearing denominators.

Theorem 20.1 (Loss of output contact)**.** *If $p > K$, $p \mid c$ and $\gcd(a,c) = 1$, then*

$$p \nmid W_{r,K}(a,c).$$

Proof. Expand W as a homogeneous polynomial in a, c. Modulo p, every term below the top degree in a retains a positive power of c and vanishes. The unique top subset is $(K-r+1, \dots, K)$. Its indices, nonzero differences and L_K are all p-units, so its scaled coefficient is a unit. The remaining factor a^d is a unit because $p \nmid a$. The surviving term is therefore nonzero modulo p. □

If an output prime above the cutoff has repeated depth, this witness has no positive valuation there at all. The failed candidate does not pass the local output gate. It is not a theorem excluding every possible determinant realization.

20.2 Rank two exposes the loss in one line

The first two truncated rank-two expressions are

$$\mathcal{H}_{2,2}(t) = t^3/8,$$

$$\mathcal{H}_{2,3}(t) = \frac{t^3}{8} + \frac{4t^4}{27} + \frac{t^5}{216} = \frac{t^3(27 + 32t + t^2)}{216}.$$

Clearing the displayed denominator and the projective denominator in the second gives

$$W(a,c) = a^3(27c^2 + 32ac + a^2).$$

For every $p > 3$ dividing c, this is congruent to a^5 modulo p. Hence it is a unit at that output prime.

The original local series had contact near zero. The algebraic scalar evaluates at a/c and clears powers of c dictated by its top degree. At primes dividing c, it is the top term that survives. Long truncation does not automatically solve this mismatch. A valid construction must prove the contact at the actual local chart and retain it through the exact normalization map.

21 The correctly normalized global bridge

21.1 A sufficient certificate with every hypothesis visible

The jet block supplies formal and local expressions. The next theorem states exactly how a suitable global arithmetic witness would imply *abc*. Its hypotheses have not been established for the present determinant candidate.

Theorem 21.1 (Conditional integral witness criterion)**.** *Let $\mathcal{R}$ be an unbounded set of positive integer ranks, and put $s_r = r(r+1)/2$. Suppose that for each $r \in \mathcal{R}$ there are constants $C_r > 0$, $A_r \geq 0$ and $E_r \geq 0$, independent of the primitive triple, such that every primitive positive triple admits a nonzero integer W_r with*

$$Q^{s_r} \mid W_r, \qquad |W_r| \leq C_r (ab)^{s_r} R^{E_r} (\log(3c))^{A_r}.$$

If $E_r/s_r \to 0$ along $\mathcal{R}$, then Conjecture 1.1 follows.

Proof. Nonzero integer divisibility gives $Q^{s_r} \leq |W_r|$. Divide by $(ab)^{s_r}$ and use $Q/(ab) = c/R$:

$$(c/R)^{s_r} \leq C_r R^{E_r} (\log(3c))^{A_r}.$$

Taking positive roots yields

$$c \leq D_r R^{1+\eta_r} (\log(3c))^{a_r}, \quad D_r = C_r^{1/s_r}, \quad \eta_r = E_r/s_r, \quad a_r = A_r/s_r.$$

Fix the target $\varepsilon > 0$. Choose one rank with $\eta_r < \varepsilon$. Next choose

$$0 < \delta < \frac{\varepsilon - \eta_r}{1+\varepsilon}.$$

For the fixed exponent a_r, there is a constant $L_{r,\delta}$ such that $(\log(3c))^{a_r} \leq L_{r,\delta} c^{\delta}$ for all $c \geq 2$. The elementary fact follows because a fixed logarithmic power divided by any positive power of c tends to zero, and the remainder is bounded on a finite initial interval. Thus

$$c^{1-\delta} \leq D_r L_{r,\delta} R^{1+\eta_r}.$$

Raise to $1/(1-\delta)$. The chosen inequality for δ gives

$$\frac{1+\eta_r}{1-\delta} < 1 + \varepsilon.$$

Absorb the fixed constants into K_ε and use $R \geq 2$ to obtain the desired bound. No prior power bound relating c to R was assumed. □

Only one rank is selected for each target ε. Its constants may be arbitrarily large but must be uniform in the triple. A rank that depends on the triple without uniform constant control would not meet the theorem.

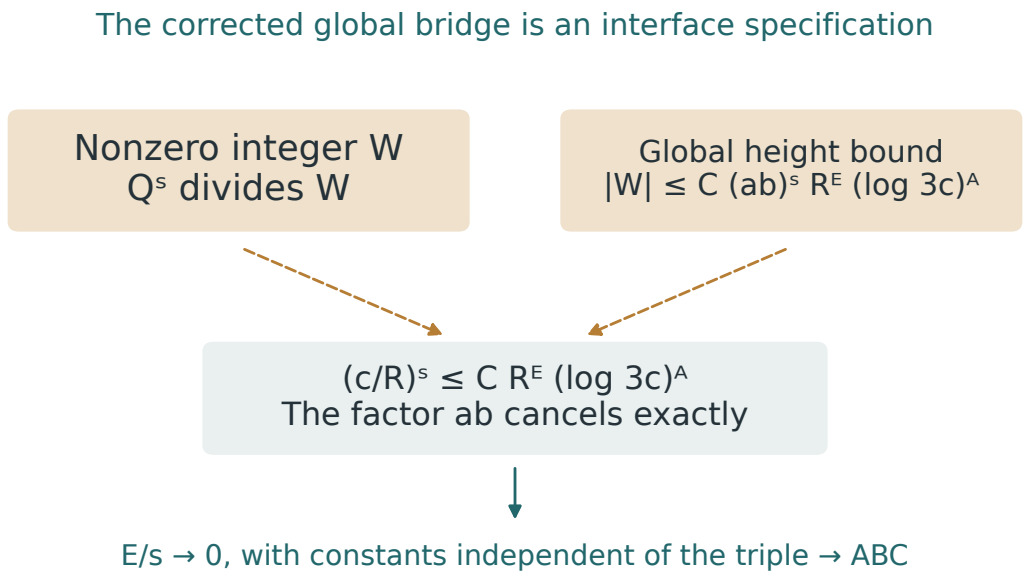


Figure 5: The local determinant results reach a precise global interface. The missing realization, nonvanishing, norm comparison and uniform height hypotheses are shown as obligations, rather than supplied by the diagram.

21.2 The contact determinant has not met these hypotheses

For the proposed Hankel realization, all of the following remain open:

1. An actual global arithmetic object whose local realization is the chosen determinant.
2. A nonzero algebraic scalar, or a nonzero section of a rigorously metrized determinant line.
3. The full local divisibility in the correct sector charts, with every comparison factor included.
4. Bounds on every finite denominator and every archimedean conjugate or norm.
5. A global size estimate with the required $(ab)^{s_r}$ normalization and vanishing relative radical cost.

Real positivity proves nonvanishing of a real analytic value. It does not prove that value algebraic. The depleted exact valuations belong to prime-dependent functions. The raw determinant supplies a common expression but not an authenticated product formula across its analytic realizations. The rational truncation constructs an algebraic number but fails the output-prime contact gate.

A product formula applies to the places of one nonzero algebraic number. It does not apply to an arbitrary list of formally related complex and p-adic periods. A determinant-line alternative would likewise need its own proved arithmetic-degree identity and metric comparisons.

21.3 The coherence route has a parallel endpoint

The inherited exact factorization is

$$M_c = \epsilon_2 KWC, \qquad \epsilon_2 \le 2, \quad K \le \log_2 c.$$

Here the factor denoted W is the local logarithmic defect, distinct from the integer witness W_r in Theorem 21.1. The letters have separate domains.

If for every $\eta > 0$ one proves

$$WC \le C_\eta R_a R_b R^\eta,$$

then $c \le C'_\eta R^{1+\eta} \log(3c)$. The same absorption argument gives abc. Conversely $WC \le M_c$, so the all-ε output-sector abc inequality immediately supplies this bound. The uniform combined WC estimate is therefore the exact endpoint. A bound on C alone suffices only under independent control of W, for example in a stated class where $W = 1$.

21.4 What a useful next rank-two result would be

The strongest immediate target is an explicit arithmetic realization of

$$\mathcal{H}_2(t) = \mathrm{Li}_1(t)\,\mathrm{Li}_3(t) - \mathrm{Li}_2(t)^2,$$

with every local comparison and global norm recorded. A motivic or iterated-integral construction is a possible research direction, not an input assumed here. A different nonzero algebraic determinant could also be useful if it passes the same gates.

A new block should demonstrate a quantitative gain on c/R or $M_c/(R_a R_b)$. Another local valuation, formal transformation or finite successful example is evidence within its declared scope. It does not establish that global gain.

22 Finite evidence, formal checks and reproducibility

22.1 The recovered sources and the new reader edition

The immediate predecessor is the verified prime contact coherence publication from July 2026. Its rendered final edition has 142 pages. The larger complete historical source and the later 19 September jet determinant block are also recovered. Original archives remain unchanged. The new edition rearranges their proof spine, adds the conjecture foundations and spectral leaf treatment, incorporates the corrected normalization, and explicitly revises issues found in the current independent audit.

The source and evidence package records input hashes, executable commands, outputs, audit notes and the new figure data. It distinguishes historical verification records from fresh runs. A prior report is evidence about its stated historical process, rather than a fresh certificate for every sentence of this edition.

22.2 Fresh exact computations

The September jet verifier was replayed unchanged. Its scientific output matched the archive: 149,306 assertion evaluations, including 504 exact modular determinant valuation comparisons. These are evaluation counts, not numbers of independent theorems.

Jet test family	Declared finite range
Coefficient cancellation	$m = 0, \dots, 30$, indices $k = 1, \dots, 80$
Finite Cauchy–Binet	Ranks $1, \dots, 5$, $K = r, \dots, r+4$, arguments $1/3, 2/3, -1/2$
Residue energy minimum	Primes $2, 3, 5, 7$, ranks through six, subsets of the first eleven unit indices
Depleted valuations	Primes through seventeen in the selected seven-prime list, ranks through eight, $e = 1, 2, 3$, three unit choices
Denominator budget	Ranks through eight, ordered indices through three hundred, plus rational determinant checks
Stored triangle rows	All 361 stored moduli and phase rows

The stored triangle replay checks its retained finite identities. It does not reconstruct every upstream phase from a primitive triple. The general theorems in the determinant chapters are proved in the text, rather than inferred from these bounded ranges.

The unchanged canonical mathematical audit reproduced all seven archived output groups. Its phase Smith check covered 152,095 nondegenerate primitive triples with $a \le b$ and $c \le 1000$. It also checked 195 polyphase cases, 2,499 seeded Wronskian packets, 5,001 divisor-packet examples, 391 Kummer examples and 254 diagonal Walsh packets. The Wronskian packets are sampling, and the Walsh model is an algebraic diagonal model, rather than an automorphic construction.

The nine canonical implementation groups also reproduced their archived values, after two disclosed filesystem path adaptations. The mathematical code was unchanged. At $3 + 125 = 128$, the weighted derivative minimum is $\Theta = 6$, while the exponent-free moving-search minimum is $\Phi = 8$. They optimize different packets. The latter has obstruction multiple one and exact master gap one. At $1 + 4374 = 4375$, the moving-search minimum is 62 and its exact master gap is $248/243$.

Some implementation companion checks are numerical. The polylogarithm residuals use eighty-digit arithmetic and fixed tolerances, without certified interval error bounds. They are reproduced numerical support. They are kept separate from exact integer, rational, modular and symbolic identities.

22.3 An independent primitive census

A new exact enumeration covered every primitive positive triple with $a \le b$ and $2 \le c \le 10000$. It contains 15,198,743 triples. A separate Euler totient sieve gives the same count. Exactly 121 satisfy $c > R$, with membership decided by the integer inequality rather than a floating quality threshold.

Within that declared domain, the largest displayed quality is approximately 1.567887 at $1 + 4374 = 4375$, with $R = 210$. The example $2 + 109 \cdot 3^{10} = 23^5$ lies outside the census. This computation supplies neither a uniform conjectural constant nor a conclusion about unsearched triples.

22.4 Two controls on delicate local boundaries

The CRT lift criterion was checked against complete finite brute force in 2,928 cases. Every individual coordinate interval must be nonempty. The sum-of-endpoints condition alone can incorrectly accept a target when one coordinate interval is empty. The preserved im-

plementation already had the individual guard. The new theorem statement now includes it.

The strict good-prime boundary was also tested with justified infinite-series tails. For rank two and $p = 5 = 3r - 1$, the raw determinant at $t = 10$ has valuation four, while its depleted version has valuation three. For $p = 7$ and $t = 14$, both have valuation three. The modular precision levels were 6, 8, 12 and 16. The tail bounds in the computation report authenticate these named valuations. Thus the strict condition $p > 3r - 1$ cannot be weakened to equality in the universal exact-valuation theorem.

At a small prime, a raw lower bound can even coexist with zero. For rank one at $p = 2, t = 2$, the standard convergent logarithm identity gives $\mathrm{Li}_1(2) = -\log_2(-1) = 0$. Finite zero residues support this identity but do not prove it by themselves. The depleted moment at that argument has valuation one. This is why the all-prime budget theorem explicitly uses $v_p(0) = +\infty$ and does not promise raw nonvanishing at every small prime.

22.5 What the seven jet scripts actually formalize

The original jet file has now been built with its exact Lean 4.19.0 and mathlib v4.19.0 versions. All seven theorem axiom checks passed. Its original source produces two style-linter warnings, so a warning-as-error elaboration fails. A separately preserved checked copy changes only the two tactic separators from `<;>` to `;`. That copy builds and elaborates with warnings treated as errors. The statements, coefficients, quantified domains and imported library are unchanged.

The seven statements cover rational cancellation of the balanced factor, two-node and three-node polynomial Gram identities, a rank-two finite truncation formula, its cleared rational expression, its integer remainder and a rational coordinate change. Their standard logical axiom dependencies are recorded individually. There is no custom axiom or unfinished proof in the inspected outputs.

These scripts do not formalize convergence, general infinite Hankel determinants, positivity, the all-rank valuation theorem, the residue-class minimum proof or the global *abc* criterion. The Gram identities alone have no positivity hypotheses. The integer remainder identity is a polynomial equality, rather than a completed prime-divisibility theorem. The formal scope is finite algebra, distinct from the written analytic and arithmetic proofs.

22.6 The separate checked KTG interface

The recovered KTG Block 03 is a different development with a different source history. Its exact Lean 4.34.0-rc2 clean replay passed all 24 build jobs and all 22 direct source elaborations with warnings treated as errors. A fresh axiom inspection covered 42 named theorems. Forty-one use no axioms, and one generic logical independence theorem uses the standard proposition-extensionality axiom.

Its proved theorem is conditional. Given abstract typed data, realization and normalization certificates, a supplied witness-preservation theorem and a monotone observable, a hull containment yields the scalar observable bound. The source witness for an actual IUT instantiation is still missing. A finite Boolean model demonstrates that the earlier abstract fields do not logically imply the missing containment. That is an independence result inside the defined interface, rather than a counterexample to IUT or *abc*.

Separate type fields distinguish roles but do not require pairwise unequal types. Its positive natural-number rank field does not enforce rank greater than one. Its grade-transfer

equations check the chosen abstract definitions, rather than authenticate their interpretation as actual IUT log-volume or normalization objects. Those interpretation obligations remain mathematical work outside compilation.

22.7 Failures are part of the evidence

The historical jet verifier first encountered a float at the rank-one empty product. The exact identity test detected it, and the accepted historical implementation corrected it. The latest replay uses the corrected exact source. The false unbalanced Q target and the failed rational witness are mathematical corrections retained as explicit theorems.

Current source review also corrected the missing CRT nonemptiness condition, unit-sector definitions, model and branch qualifications, and one draft jet coefficient. The runtime recovery had host-specific compiler startup and extraction failures, corrected by a disclosed compatibility layer for the process's own executable-path lookup. Compiler and proof bytes were unchanged. The exact commands, logs, changes and hashes are preserved in the evidence report.

None of these successful finite or formal checks supplies the global arithmetic realization. The final missing implication remains the one stated in Section 21.

23 Sources and theorem boundaries

The external source check distinguishes a stated theorem, a contextual reference and an unresolved comparison. The critical conjecture quantifiers, known unconditional bound, polynomial theorem, modified Szpiro equivalence, modularity, polylogarithm formulas and current IUT status have authenticated primary-source leaves in the evidence package.

The prime contact and determinant calculations are derived in this text. Pasten's arithmetic derivative work supplies an established base for that branch. Its small-derivative conjecture and its weak *abc* equivalences are not silently identified with the stronger all-ε packet target stated here. The Wiles-defect and motivic references explain relevant established frameworks. They do not construct the project-specific persistence-to-Hecke or integral period transfer maps.

Otsubo and Yamazaki's motivic identities have precise coefficient-field and base-field hypotheses. Some Gauss-sum statements are over finite fields, while the multiplication relation extends to fields containing the needed roots of unity. None gives the missing integral radical-scale norm comparison in this work. Borel's rank input is used only for the rational weight motivation explicitly described in the path-packet section.

Bibliographic entries inherited solely as mathematical background are retained for that purpose. The exact source-check report states which leaves were freshly inspected. A failed search or a citation title is not treated as proof of a theorem or of historical novelty.

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